Magnetic properties of finite superconducting cylinders. II. Nonuniform applied field and levitation force

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We study the current penetration profiles inside a finite type-II superconducting cylinder when it is in the presence of an axially symmetric applied field created by a cylindrical and uniformly magnetized permanent magnet. The results are obtained using the general framework derived in the first paper of this series. The levitation force in such a system is calculated from the current distribution, for the cases of a constant critical current in the superconductor and an exponential dependence of the critical current upon the internal magnetic field. From the obtained results, we study in detail how the levitation force depends upon the system parameters. We conclude from the results that (i) the levitation force is optimized when the magnet and the superconductor have similar dimensions, (ii) an excess of length in the superconductor can yield no significant increase in the force, and (iii) demagnetizing effects can lead to an important enhancement of the levitation force, particularly in the case of thin films, for which the force per unit volume of material is the highest.

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I. INTRODUCTION

When a nonuniform magnetic field is applied to a superconductor, the magnetic forces resulting from the interaction of the applied field with the currents in the superconductor may produce its levitation.[1] If the superconductor is of type II, the pinning of flux lines in the superconductor defects can yield unique properties with regard to the stability of the levitation. Magnetic levitation has therefore become an important topic in superconductivity research, particularly because of the excellent perspectives of application of high-T_c superconductors in the technology.[2,3]

Several experiments were reported on the study of levitation forces in different situations and for different geometries. Most of the experiments use YBaCuO superconductors cooled by liquid nitrogen. A permanent magnet (commonly of NdFeB or SmCo) is normally matched to a moving system (either vertically or horizontally, or both). The force acting at several positions is measured using a torsion system, or by similar methods. The dimensions of the permanent magnet (PM) in such experiments are on the order of centimeters, and its induction on the order of 1 T. As for the superconductor (SC), the radii are of the same order but their lengths range from centimeters to a few microns (thin films). The described experiment is equivalent to having a permanent magnet fixed at some position and the superconductor moving. The latter configuration is, in principle, more complicated in practice, since the superconductor must be immersed in liquid nitrogen. The stability of the system can be studied from the restoring force measured after producing a small displacement of the levitated sample.

Moon *et al.*[4] measured the levitation force of bulk samples of YBaCuO, and showed some of their properties. Several groups extended these measurements to study the dependence of the force upon the orientation of the system,[5,6] the properties of the levitated material,[7] and the shape of the superconductor,[8] sometimes comparing, thin films with bulk samples.[9] There were some attempts at standardization of levitation measurements.[10] The levitation force due to lateral movements was also experimentally studied.[4,11] The stability in these systems has been measured through restoring forces by several groups.[6,12,13]

With regard to the progress in the theoretical understanding of levitation, the first important step was provided by Brandt,[1] when he described the general properties of superconducting levitation, justifying why a type-II superconductor can levitate rigidly over a permanent magnet. However, more detailed theoretical models are either incomplete (calculating only the response in the Meissner state, for example) or have unrealistic approximations (such as neglecting demagnetizing effects). Hellman *et al.*[14] introduced a model based on total flux exclusion. Adler and Anderson[15] presented some results for the suspension force by a flux trapping model. Johansen *et al.*[16] proposed a model considering the granularity of the superconductor. Torng and Chen[17] and Schönhuber and Moon[18] considered the penetration of currents inside the superconductor with an approximate field expression. Badia and Freyhardt[19] presented a formalism for studying a superconducting disk shielding an arbitrary magnetic field, in particular that created by a cylindrical permanent magnet. Yang[20] calculated the levitation forces acting on magnets placed above an infinite superconductor. Coffey[21] studied the levitation force acting on a point magnetic dipole above a semi-infinite type-II superconductor in both Meissner and mixed states. In addition, some numerical calculations based on finite elements were presented.[22–24] Fewer works dealing with stability were carried out. Davis *et al.*[25] studied infinite superconductors, and Hull *et al.*[26] proposed a partial formula without calculations. We studied the levitation force, and the stiffness and damping of both superconducting cylinders without considering demagnetizing effects in Ref. 27, and superconducting thin disks with a more realistic approximation in Ref. 28 (the latter case was also studied by Riise *et al.* in Ref. 29). Finally, lateral forces were theoretically studied using very simplified models.[30,31]

The main difficulty in the development of a complete model for levitation is the presence of the demagnetization effects that appear in finite samples. Demagnetizing fields modify the internal field inside the superconductor and, as a result, the current distribution and the levitation force. In Ref. 32 some analytical results were given including demagnetization corrections through the approximation of considering a constant demagnetization factor. Tsuchimoto *et al.*[22,23] used finite element methods and showed some results on the force resulting when some nonuniform field is applied to a finite superconductor. In spite of all these efforts, a systematic study of the demagnetization effects on the levitation force has not yet been presented yet.

Recently there were important advances in the study of the magnetization of finite superconductors in a uniform field, including demagnetizing effects.[33–36] In the spirit of these works, in Ref. 37 we studied the magnetic response of a cylindrical type-II superconductor in the critical state in the presence of a quasistatically changing uniform applied field. We presented a general framework for calculating current distribution inside the superconducting sample, which accounts for the demagnetizing fields that appear in any finite sample and which allows one to introduce any dependence of the critical current on the internal applied field $J_c(H_i)$.

In this second paper of the series we shall use this general framework to study the magnetic response of a cylindrical type-II superconductor in the critical state in the presence of a cylindrically symmetric applied field created by a uniformly magnetized permanent magnet, and the levitation force that arises from the interaction of the external field with the currents in the superconductor. We will analyze the results in terms of the current penetration and levitation force. Although only the case of a permanent magnet is treated in this work, our method can be readily applied to any other system with cylindrically symmetric fields, such as those created by a pair of coils or a solenoid.

This paper is structured as follows. In Sec. II we describe the studied system, introduce the modifications we make in the general framework presented in Ref. 37 to consider the nonuniformity of the applied field, and show how the force can be calculated from the current distribution. In Sec. III we present the calculated current penetration profiles and the levitation force in the case of constant critical current. We study how the nonuniformity of the field and the demagnetizing fields affect both the current distribution and the levitation force and its dependence upon all the relevant parameters of the system. In Sec. IV we extend our study to the case of nonconstant critical current, and discuss the effect of its field dependence upon the levitation force. In Sec. V we compare the obtained results with some simpler analytical models to see when the analytical descriptions are accurate. In Sec. VI, we compare the model results with actual experimental data. Finally, in Sec. VII we present the conclusions of this work.

II. MODEL

In this work we study the levitation properties of a system composed of a permanent magnet and a superconductor. The system is described as follows.

(i) The permanent magnet is a cylinder with radius a and length b, and is uniformly magnetized with axial magnetization M_{PM}. We assume that the presence of the SC does not affect the PM.

(ii) The superconductor is a cylinder of radius R and length L, with the same axis as the PM. Following the modeling of the first paper in this series,[37] we consider the SC to be composed by a set of $n\times m$ linear azimuthal currents separated a distance $\Delta R=R/n$ and $\Delta L=L/m$ in the radial and axial dimension, respectively (see Fig. 1 of Ref. 37). The SC is assumed to be in the critical state, so if there is current in some linear circuit, it must flow with an intensity $J_c(|\mathbf{H}_i|)\Delta R\Delta L$, where $J_c(|\mathbf{H}_i|)$ is the critical current density.

(iii) Initially, the superconductor is zero field cooled, and located very far from the permanent magnet. The distance d between PM and SC is decreased to some value d_1. Then the current distribution is calculated following the energy minimization procedure described in Ref. 37, using the values of the external applied field in the region occupied by the SC. In all the SC movements, the cylindrical symmetry is maintained. As in Ref. 37, the applied fields will always be such that $H_{c1}\ll H_a\ll H_{c2}$.

(iv) When the magnetic energy becomes minimized at d_1, we calculate the magnetic levitation force. The force will have only axial component given by

$$F_z=2\pi\mu_0\sum_{ij}^{nm} I_{ij}H^{a}_{r,ij}\rho_{ij}, \tag{1}$$

where $H^{a}_{r,ij}$ is the radial component of the external magnetic field at the position ij.

(v) The superconductor is now lowered to a new distance d_2 and the minimization process starts again from the previous distribution of currents. The process is repeated until a certain minimum distance d_{min} is reached (ending the initial or descending stage), after which the distance is increased from $d=d_{min}$ to $d\to\infty$ (reversal or ascending stage) in several steps.

It is important to remark that the magnetic field that enters in the force calculation [Eq. (1)] is the external applied field, because all internal forces, such as those created between each pair of current loops, are canceled due to the action-reaction law. This is so even when demagnetizing fields are considered. However, the presence of the demagnetization has an important effect in the value of the force, since demagnetizing fields contribute to the magnetic energy and, therefore, have an influence on the current distribution (and, if J_c depends in H_i, also on the current value).

III. CONSTANT CRITICAL CURRENT

In this section we study the constant critical current case, so $J_c(|\mathbf{H}_i|)=J_c$. The values we will use for the PM are $b=0.01$ m and $M_{\rm PM}=7.95\times10^5$ A/m ($M_{\rm PM}$ corresponds to $\mu_0M_{\rm PM}=1T$). The SC is considered to have a radius $R=0.01$ m. These values are among the typical ones for levitation experiments. We define H_0 as the applied magnetic field at the origin of coordinates (center of the top face of the

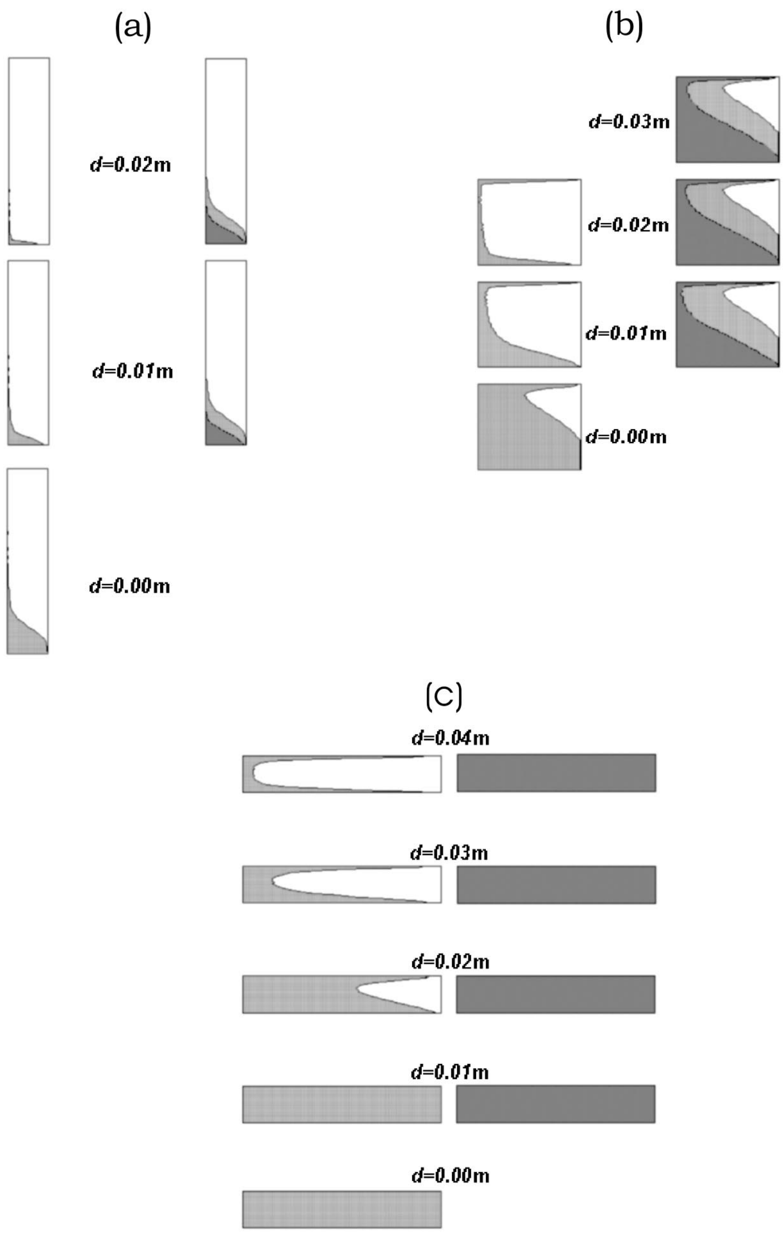

FIG. 1. Calculated current profiles for three differents cylinders of different aspect ratios: (a) $L/R=5$, (b) $L/R=1$, and (c) $L/R=0.1$. The other parameters of the system are reported in the text. The vertical scale for case (c) has been doubled for the sake of clarity.

permanent magnet). Except when explicitly stated, we use a critical current of $J_c=2.81\times 10^7$ A/m^2, which yields $H_p=J_cR=H_0$. H_p is the field, if it were uniform, at which a superconductor with $L\rightarrow\infty$ would become completely penetrated.

When studying the a/R dependence, the value of a will be changed, whereas the value of R will be maintained. The length to radius ratio L/R will be modified by changing only the value of L. Some adequate normalization could reduce the number of parameter needed to completely describe the system (see Refs. 27 and 28). However, in this work we prefer to give the values in absolute terms, because the results could be directly checked with experimental ones.

A. Current profiles

In Fig. 1 we present the current profiles calculated for three cylinders of different aspect ratios, and for different distances d from the PM, for both the descending and the ascending branch (we set $d_{min}=0$). The radius of the PM is $a=0.01$ m. We observe some common facts in all cases: (a) The penetration of currents inside the SC is deeper near both the bottom (nearest to the PM) and the top (farthest from the PM) ends, due to the demagnetization effects as found for constant applied field.[37] (b) The penetration profiles are not symmetric with respect to the central layer of the SC, because of the spatial nonuniformity of the applied field.

Although the exact form of field created by the PM is somehow complicated, as a general trend, the magnitude of the field is larger in the regions close to the PM. Therefore, the SC should shield a field in its bottom region (the closest to the PM) larger than in the top one, so currents have to penetrate deeper at the layers of SC which are close to the PM. In the case of long cylinders [Fig. 1(a)], in the upper layer, we observe no current penetration because the field is almost zero there (actually, there should be some current

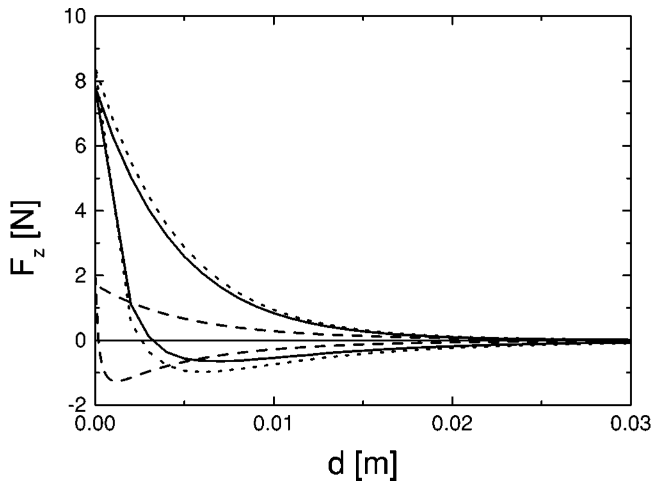

FIG. 2. Levitation force for three samples of Fig. 1, with $L/R=5$ (solid line), $L/R=1$ (dotted line), and $L/R=0.1$ (dashed line), as a function of the vertical distance between the SC and the PM. Other parameters of the system are reported in the text.

very close to the upper layer due to the demagnetization field; however, within the scale and the numerical accuracy used, this current is not seen). For the thin film case [Fig. 1(c)], we observe that the current penetration is almost symmetric with respect to the central layer, as we found for a uniform applied field in Ref. 37. This is because the magnetic field changes slightly in the z direction along the superconductor. There are, however, some differences with the behavior of thin films in a uniform applied field, because in this case the applied field has some radial component and varies through the radial distance.

Another observed fact related to the demagnetization effect is that currents fully penetrate the superconductor at lower applied fields for shorter samples.[38] This is translated here to have a fully penetrated sample at higher d's for short samples, as observed in Fig. 1.

When ascending the superconductor, currents are induced in a sense opposite to those induced in the descending stage. The initial currents are kept frozen in an interior region of the superconductor, whereas the reverse ones penetrate from the surface. The demagnetizing fields affect these reversal currents, producing a deeper penetration into the upper and lower layers of the superconductor. Moreover, since the field is inhomogeneous, an inhomogeneous penetration of reversal currents is again observed. The difference is that, when starting the reversal movement, the applied field and its z variation are both larger than when starting the initial movement. This yields, as seen in the right columns of Fig. 1, a much larger penetration of reversal currents, for a given increment of height, when the SC is close to the reversing point. In the thin film limit [Fig. 1(c)], this behavior is accentuated because the demagnetizing effects are more important.

B. Levitation forces

1. L/R dependence

Figure 2 shows the levitation force calculated for the three cases of Fig. 1, that is, for three superconductors with the same J_c and in the presence of the same PM, but with different L/R ratios. The force presents the typical hysteretical shape due to the hysteresis in the penetration of currents. When the sample becomes thinner, the force tends to be more hysteretic in the sense that the forces in the descending and ascending branches are more symmetric with respect the value $F_z=0$. This fact was experimentally observed by different authors (see Refs. 9 and 10, for example). The reason for this effect is that, due to the demagnetization, as explained above, a thin superconductor is completely penetrated at higher d's in the descending stage and at an early position in the ascending stage. The result is an almost symmetric behavior of the force versus distance. The levitation force in the case of thin films when the vertical field can be approximated to be independent of the radial distance (when $R \lesssim 0.2a$) was calculated analytically in Ref. 28. In Ref. 29 some calculations and measurements of levitation for thin films were reported.

The force acting over a superconductor with constant critical current density depends upon both the volume filled by the currents and the value of the external field in this volume. At a given distance d (corresponding to a given set of applied field values on the superconductor) thin samples will tend to be more deeply penetrated by currents due to demagnetization fields. So demagnetization fields tends to produce larger forces. However, for thin samples, the absolute volume penetrated by currents is small and so is the force. From the analysis of these two opposite factors, we can draw an important conclusion: not very large samples are needed to produce large levitation forces. In Fig. 2 we see that the force for $L/R=1$ is similar to the case $L/R=5$. This is because [as can be seen in Fig. 1(a)], for the case $L/R=5$, a large portion of material does not contribute to the force, since currents have not penetrated into these regions. For thin films (see the case $L/R=0.1$ in Fig. 2), even considering that the demagnetizing field produces a very large penetration, the volume of material is scarce and the force is lower.

An interesting issue to analyze is the force per unit volume of superconducting material, since this information may be valuable for their use in actual devices. The force F_0 reached at the minimum distance $[F_0=F_z(d=0)]$ per unit volume for the case $L/R=0.1$ is about two times larger than in the case $L/R=1$, and about ten times than in the case $L/R=5$ [see Fig. 3(a)]. This indicates that, with relation to the volume of the material, thin films are the best candidates to produce larger forces.

It is also of interest to study the levitation force attained at a given height d as a function of the shape of the superconductor. In Fig. 3 we show both the levitation force and the force per volume at $d=0$, for different values of the L/R parameter and different H_p/H_0 ratios.[39] It can be observed that the force tends linearly toward zero as L/R decreases. This linear behavior arises from the fact that the force per volume unit is constant when L/R is low enough. When L/R increases the force increases, whereas the force per volume unit decreases. When achieving $L \simeq R$ (the exact value depends on the parameters of the permanent magnet) we ob-

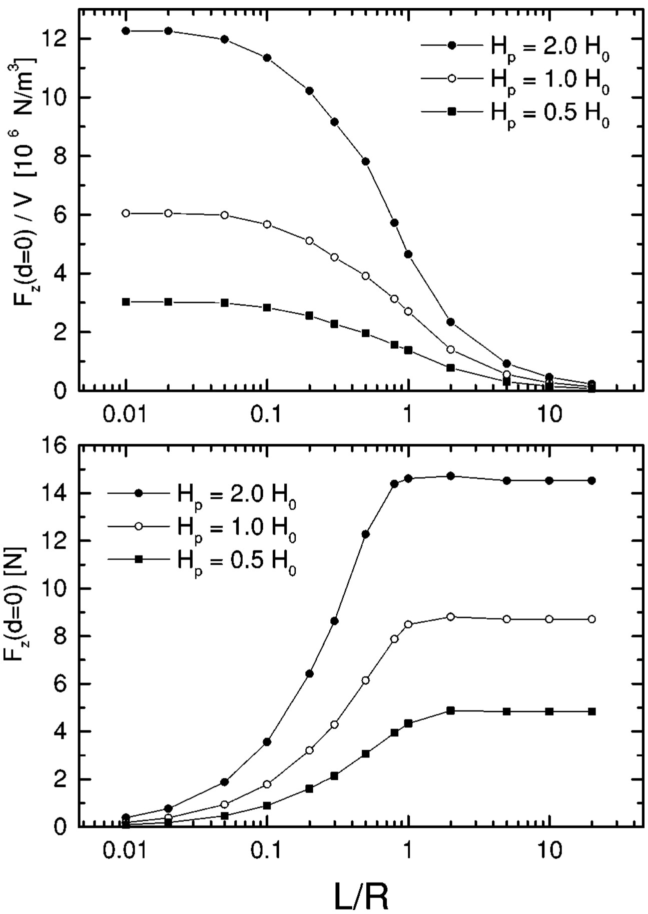

FIG. 3. (Top) Maximum levitation force per unit volume (at $d=0$) as a function of L/R. (b) Maximum levitation force (at $d=0$) as a function of L/R.

serve that the force tends to saturate. This is due to the already discussed fact that for long enough samples there is a superconducting region that does not contribute to the force. As a consequence, for a given PM, the force tends to saturate for large values of L/R, producing a force per unit volume that decreases as $1/L$.

2. a/R dependence

An important factor in the study of the levitation force is the relative size between the SC and the PM. We now discuss the dependence of the force upon the a/R value.

Figure 4 shows the levitation force at $d=0$, for different values of the ratio a/R varying the radius of the permanent magnet. We have plotted the results for a superconductor of $L=0.01$ m. For the three curves, the value of the critical current is such that $H_p/H_0=J_cR/H_0=0.5$, 1, and 2, respectively.

For all values of H_p/H_0 the curves present a similar behavior. When $a/R\rightarrow 0$ the force tends toward zero. F_0 increases until a maximum is reached, and then decreases toward zero for a large enough permanent magnet. The larger H_p/H_0 (which means a larger J_c, for a given PM) is, the larger the force F_0 is, for a given a/R value.

The previous results can be understood by viewing the effect of the magnetic field over the superconductor. The fact

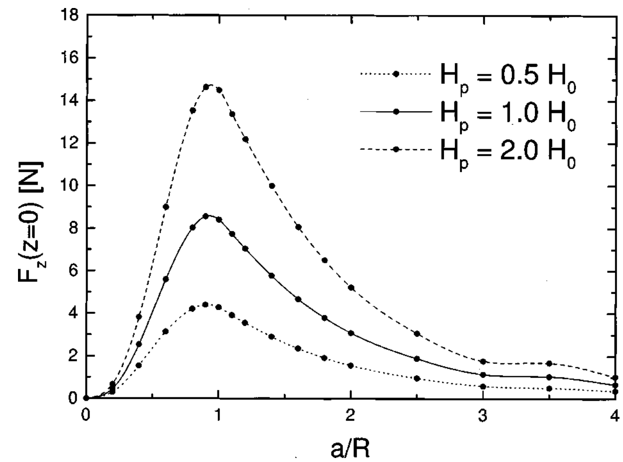

FIG. 4. Levitation force at $d=0$ for as a function of the a/R parameter. Different curves correspond to different values of the H_p/H_0 relation, as shown in the figure. Other parameters of the system are reported in the text.

that the force tends toward zero when $a/R\rightarrow 0$ is obvious, since as the permanent magnet becomes small, the field it produces tends to vanish. This limit for small magnets and for the Meissner state (or high J_c) for the superconductor was studied in Ref. 40. At the opposite limit, when the permanent magnet is much larger than the superconductor ($a/R\gg 1$) the force should also tend toward zero because the magnetic field that the superconductor feels is almost uniform in both radial and axial directions, and uniform fields do not produce levitation forces over a superconductor. Thus there should be, at least, one maximum at some intermediate a/R value. As demonstrated in Fig. 4, F_0 attains its maximum with respect to the a/R value when $a/R\simeq 1$. This means that, regarding the relation between the radius of the permanent magnet and the superconductor, the levitation force is at a maximum when both have similar radii. When studying the field created by a permanent magnet in all its exterior space, one sees that the region where the field is more inhomogeneous is for $\rho\simeq a$. This fact explains the previous conclusion.

3. Dependence on J_c

The levitation force for given d, a/R, and L/R increases with an increasing value of J_c. Actually, the magnetic levitation force is largest when the superconductor is in the Meissner (fully diamagnetic) state, which corresponds to the high-J_c limit. In Fig. 5 we plot the calculated levitation force (in the typical cases $d=0$, $a/R=1$, and $L/R=1$) as a function of the value of the critical current J_c. The force is normalized to the force that a completely shielded superconductor with the same dimensions would support, $F_{\rm Meiss}$.

The observed dependence is explained as follows. When J_c is very low ($J_c\ll H_0/R$), currents easily completely penetrate the SC, yielding a linear dependence of the force upon J_c. However, their value is small, and the force they can produce is small. On the other limit, when J_c is very high ($J_c\gg H_0/R$), the force saturates with respect to J_c (no de-

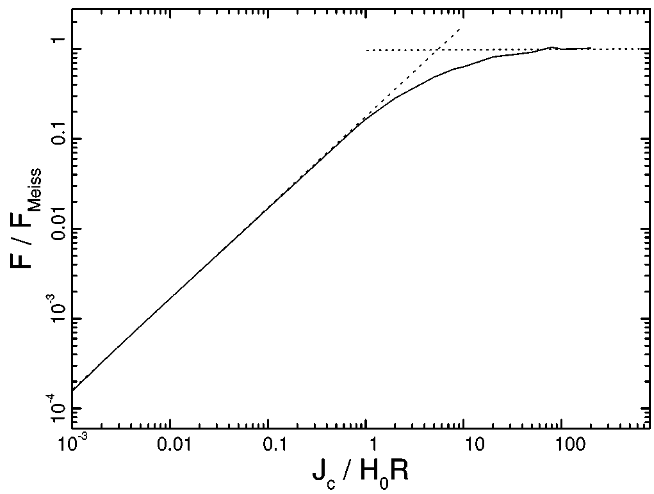

FIG. 5. Levitation force at $d=0$ as a function of the critical current density J_c. The force and the curent density are normalized to an equivalent force in the Meissner state (see the text).

pendence on J_c) since the field is completely shielded by basically surface currents, so no larger magnetization can be achieved. This limit corresponds to the Meissner state limit and, thus, the force should tend to F_{Meiss}. The absolute value of the non-normalized saturated force depends on the particular values of L/R, a/R, and d, although the general described behavior remains the same.

4. Dependence on other parameters

The height of the PM, b, modifies the applied field created by the permanent magnet in no trivial way. The main effect it can produce is that, in some region and for some values of the relation a/b, the magnetic field created by the PM presents inflexion regions (i.e., minima in the variations of some component of the external field with respect to some direction). This could produce a maximum in the force versus distance if a significant volume of the superconductor is in such region. This effect was found in experiments,[29] and was studied in Ref. 27.

In the approximation we have used that the permanent magnet is not affected by the presence of the superconductor and that it has uniform magnetization, the levitation force is proportional to M_{PM}. A more exact treatment should take into account the change in the magnetization of the permanent magnet as the superconductor moves toward or away from it, changing therefore the working point of the permanent magnet.

IV. NONCONSTANT CRITICAL CURRENT

We next discuss the results on the force when the critical current depends on the internal magnetic field. We will use the same geometric values as before, $a=0.01$ m and $L=0.01$ m, and consider a typical system with $L/R=1$ and $a/R=1$. An exponential dependence is used for the SC, as in the previous paper in the series, that is, $J_c=J_{c0}\exp(-|\mathbf{H}_i|/H_{0e})$, with the parameters $p=J_{c0}H_{0e}/R$ and $H_{pe}=H_{0e}\ln(1+p)$. As explained in Ref. 37, when introducing a $J_c(|\mathbf{H}_i|)$ dependence, the important parameters that affects the magnetization are p, which characterizes how J_c depends on the field, and the relation between the applied field and a characteristic field of the superconductor, which can be exemplified in the present case by the relation H_{pe}/H_0. H_{pe} would correspond to the penetration field if the field were uniform and $L/R=\infty$. $H_{pe}/H_0 \ll 1$ means that the applied field is strong enough to completely fill at least a large portion of the superconductor (particularly in thin films). On the other hand, $H_{pe}/H_0 \gg 1$ means that the applied field produces a small penetration in the superconductor.

A. Current penetration profiles

In Fig. 6 we show the calculated current penetration profiles for different distances along the descending stage, and for different values of the p and H_{pe}/H_0 parameters. Lines represent the current flux front at each stage. The value of that current is given, at any point from the surface to the flux front, by the $J_c(|\mathbf{H}_i|)$ relation. We see that the general behavior for all cases is similar, and in all of them we recognize the deeper penetration close to the upper and lower surfaces of the SC due to the demagnetization fields, as well as the nonuniformity of such a penetration due to the nonuniformity of the external applied field, as discussed in Sec. III.

However, there are some particularities arising from the $J_c(|\mathbf{H}_i|)$ dependence. For a given d, the lower p is, the deeper the currents penetrate into the superconductor. This is because the lower p is, for the range of fields involved (see Fig. 4 of Ref. 37), the lower the value of the current is, so the current has to penetrate more deeply to shield a given field. In addition, the dependence on the H_{pe}/H_0 parameter is as expected. When $H_{pe} \gg H_0$ the fields that the superconductor ''feel'' are small, and the penetration of currents is shallow. As H_{pe}/H_0 decreases, the currents penetrate deeper for a given distance and a given p.

B. Levitation forces

In Fig. 7 we represent the calculated levitation force for different values of p and H_{pe}/H_0. All results show a typical hysteretic behavior. When the applied field felt by the superconductor is high ($H_{pe} \ll H_0$), currents penetrate the sample at a very early stage, both in descending and ascending branches, resulting in an almost symmetric behavior of the force. On the other hand, when H_0 is small enough, the behavior of the force tends to be nonhysteretic.

The dependence on p can be understood as follows. When the superconductor starts to descend, the magnitude of the field it feels is less than H_{pe}. In this range of fields, the larger p is the higher the value of J_c, and the higher the force. When descending the superconductor, the applied field increases in magnitude. When the field is high enough, it is possible that in some interior region of the superconductor the internal magnetic field is such that a large p would imply a low J_c (see Fig. 4 of Ref. 37). This could produce a lower force at some heights for larger p's. In the ascending stage the behavior is similar but opposite. When the SC is far from

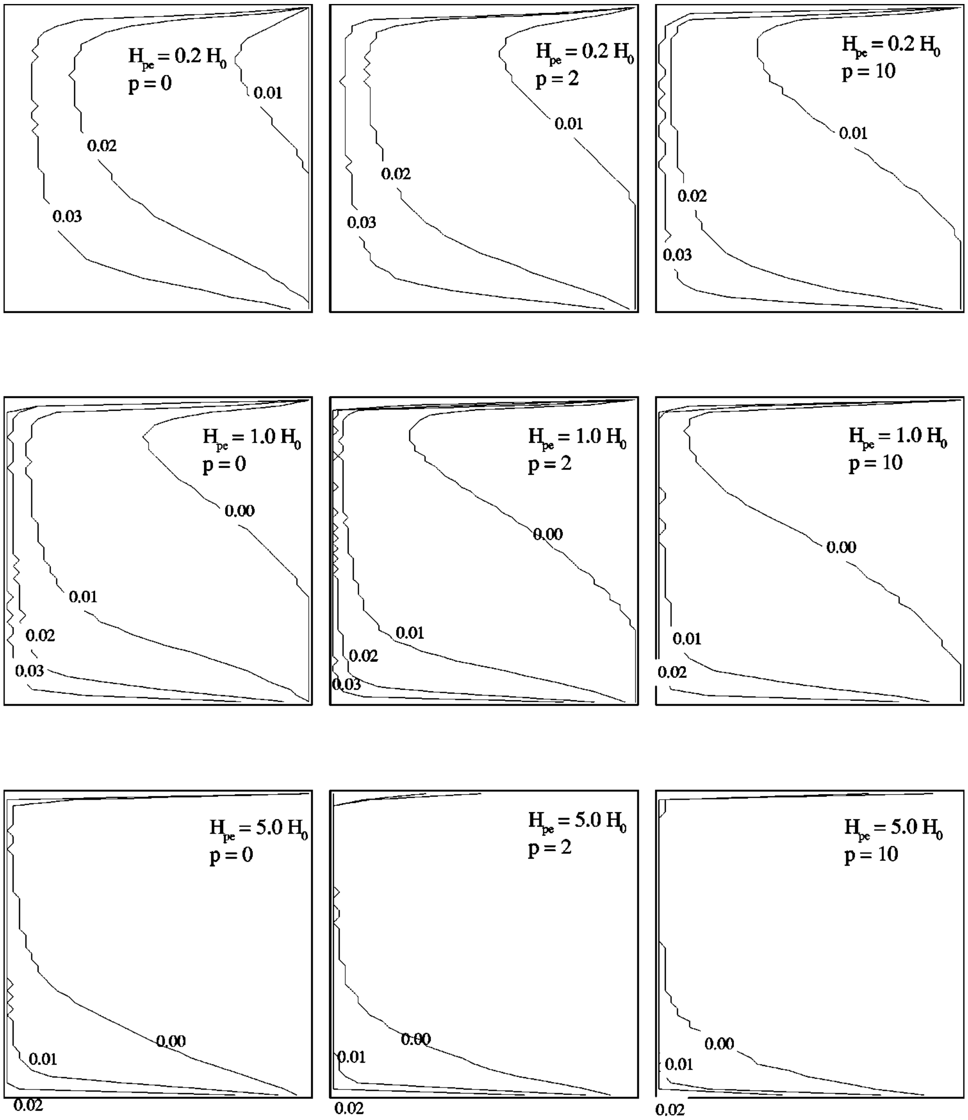

FIG. 6. Calculated current penetration profiles for a cylinder with $L/R=1$, for $p=0$, 2, and 10 (left to right) and $H_{\mathrm{pe}}/H_0=0.2,1$, and 5 (top to bottom). Only a semiplane of constant angle is shown; the cylinder axis is on the right. Numbers represent the distance (in meters) between the PM and the SC for each profile.

the PM, the larger the p is the higher J_c is (in absolute value). The result is again a large force (in absolute value) for high p when the SC is away enough from PM.

In Fig. 7(a), a maximum in the levitation force can be seen for the cases $p>0$. For $p=0$, the value of current does not change and, even if the SC is fully penetrated, the force increases as d decreases owing to the increase in the magnetic field . But when $p\neq 0$, the value of the currents decreases and, when the SC is fully penetrated,[41] obviously the force decreases as well. The larger p is, the larger this effect is. The maximum in the force is associated to the minimum that would appear in the magnetization of the superconducting sample.

In Fig. 7(c) we show the force for the case $H_{pe}=5H_0$. When the superconductor is moved in external fields smaller than the penetration field, the previous behavior changes. If $H_{pe}\gg H_0$, the superconductor will be slightly penetrated, regardless of the value of p. The behavior in this range of fields will be almost nonhysteretic for all p values, since the current penetrates slightly and the same happens in the ascending stage. Another point to note is that in this range of fields, for any d, the value of J_c increases with increasing p. This produces, in all descending stages and for a given d, a force increasing with p.

V. COMPARISON WITH PREVIOUS ANALYTICAL MODELS

Some of the results presented in this work can be compared with previous analytical models derived earlier with more restrictive assumptions.[27,28,32] In the simplest model, which we can refer to as the "small sample" model,[27] we assumed that $R\lesssim a$, and neglected demagnetization fields. This allowed us to consider only that the field has only vertical components and is constant along the superconductor length. A particularly relevant case is the thin-film limit ($L \ll R$), in which the demagnetization effects can be included and the model becomes suitable not only to give qualitative behaviors but also to quantitatively fit experimental data.[28] We remark here that a model was presented in Ref. 29, which, different from the vertical applied field assumption of

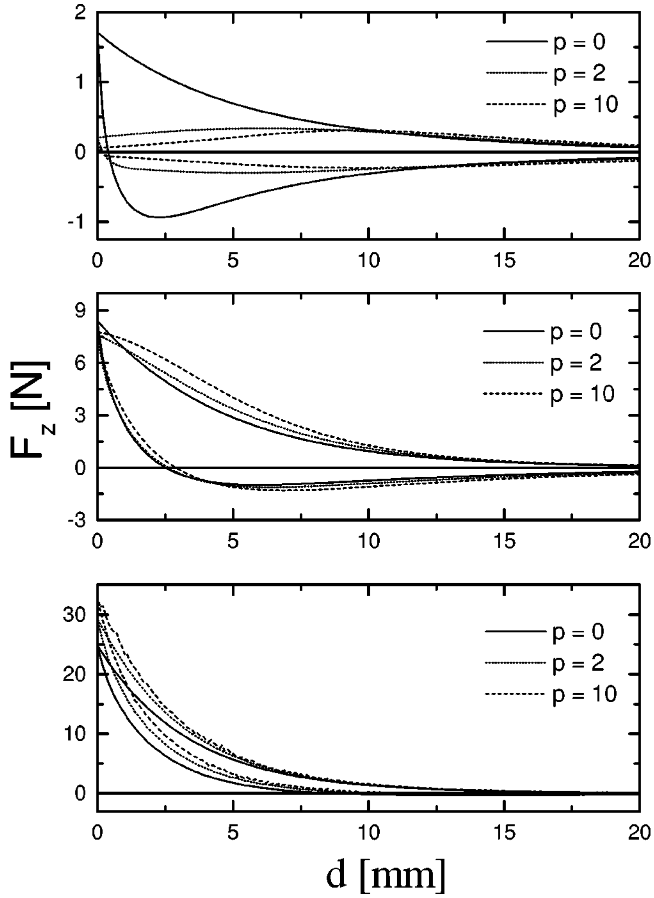

FIG. 7. Levitation force for a cylinder with $L/R=1$, as a function of p for (from top to bottom) $H_{\mathrm{pe}}/H_0=0.2$, 1, and 5.

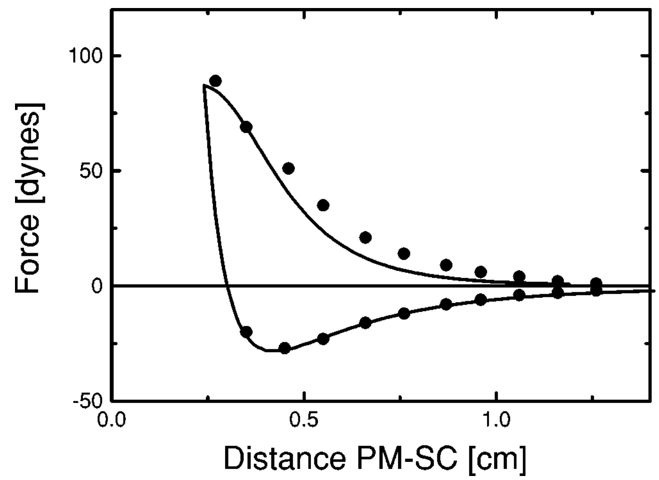

FIG. 8. Fit of the measured levitation force in Ref. 43. The solid line corresponds to our fitting using the exponential dependence. Obtained fitting values are $p=2$ and $H_{pe}=1.5\ 10^6$ A/m. The values of the other parameters are obtained from Ref. 43: $a=0.00315$ mm, $b=0.0063$ mm, $M_{PM}=7.1\times10^5$ A/m, $R=0.005$ m, and $L=0.006$ m.

Ref. 28, considered a more general expression for the applied field.

A further step was the so-called ''long sample'' model,[32] for which we still considered only vertical components of the applied field but took into account their variation along the SC. The demagnetizing effects could be accounted for by means of a constant demagnetization factor for long enough samples. When comparing the results between the approximate analytical models and the more realistic numerical one we can extract the following conclusions.

(1) The small sample model gives good qualitative results but not very accurate quantitative ones. However, the model is useful to explain and understand the trends of the dependencies of the force upon some of the variables involved in the system. In particular, the hysteresis of the force and its dependence upon J_c, especially in the high-J_c and low-J_c limits, are correctly described by analytical expressions from the simplified small samples model.

(2) The analytical thin disks limit of Ref. 28 is seen to give results very similar to those of the numerical model, as long as $R\lesssim0.2a$. Thus, not only analytical fits of experimental data can be made, but a general study of these systems is easier than using the numerical approach. However, when R is larger the analytical model fails, and a more general model such as that in Ref. 29 can be used.

(3) The long sample approximation describes the descending branch quite well, but in general not the ascending one, because when the SC approaches the PM the radial component of the field it feels is no negligible. In the ascending branch, since reversal currents start to penetrate in a region where both the radial component of the external applied field and the demagnetization fields are important, the calculated force is different when using the long sample approximation than when using the more exact numerical approach.

A detailed treatment of the comparison between the analytical models and the present approach can be found in Ref. 42.

VI. COMPARISON WITH EXPERIMENTAL RESULTS

In this section we compare our calculations with experimental data of Chan *et al.*,[43] as shown in Fig. 8. In the measured sample the demagnetization effects are important ($L/R=1.2$) whereas the sample cannot be regarded as a thin film, so the assumptions of the previous analytical models do not match the experimental situation. In this case we require the use of the general numerical calculations presented in this work.

The experimental values are fitted quite well by using our model with $p=2$ and $H_{pe}=1.5\times10^6$ A/m (these are the only fitting parameters, all the other being provided by the authors). However, some considerations should be made. We have used an exponential model to fit the results, but the real critical current dependence is unknown. Moreover, as indicated in Ref. 43, the measured sample was not cylindrical. This may explain the departure from the experimental data of our calculation in the descending branch.

We have experimentally checked our model using other measurements. For example, the dependence of the force at a given height upon the length of a superconductor was measured by Leblond *et al.*[44] Our results coincide with theirs, showing the linear dependence of the force upon L for short samples and the saturation of the force for long samples. Riise *et al.*[29] measured the levitation force in a thin disk, taking into account the radial components of the applied field. They found a three-parameter function for the dependence of J_c upon the internal field. We have calculated the levitation force, introducing in our model the same J_c function, finding that our calculation coincide with their measurements. Other measurements of Chan *et al.*[45] used permanent magnets much smaller than the superconductor. We have also checked this case, obtaining good qualitative results, although an exact fit could not be done because some of the system parameters were unavailable.

VII. CONCLUSIONS

The macroscopic physics of superconducting levitation has been studied in terms of interactions between the induced currents inside a superconducting material and an external applied field. The general framework presented in this work provides a theoretical basis for explaining the observed characteristics of the magnetic levitation of high-T_c superconductors. In particular, we have calculated the levitation force in a cylindrical symmetric permanent magnet-superconductor system, including the effects of demagnetization in the superconductor and accounting for all components of the applied field. The procedure is based on the model we derived in the first paper of this series for calculating current profiles in a superconductor in the critical state in the presence of an applied field.

This numerical model extends the conventional critical-state model for the superconductor to finite geometries and nonuniform fields, thus allowing a description of the realistic case of superconducting levitation. The results obtained enable us to draw some conclusions, which should be taken into account when designing superconducting levitation systems.

(i) The levitation force reaches the largest value at a given working height (provided that the other parameters are fixed) when the superconductor and the permanent magnet have similar radii.

(ii) For long samples, there can be a region in which the superconducting material yields no significant contribution to the force. As a consequence, not very large samples are needed to achieve large forces, the excess of superconductor resulting in wasted material.

(iii) The force tends to increase for short samples, for which demagnetization effects are important, although the effective volume of material contributing to the force is small. The force per unit volume is shown to be larger in the case of thin films than in the case of longer samples.

The influence of the dependence of the critical current on the internal field upon the levitation force has also been studied. We have discussed a complete set of possibilities, varying the values of parameters p and H_0/H_{pe}. Thus a complete set of results has been obtained, allowing one to carry out the inverse process, that is, to estimate the material parameters p and H_{pe} from the measurement of the levitation force. Although we have always used an exponential dependence for the critical current upon the internal field, the main conclusions are valid for other dependencies as long as the critical current is a decreasing function of the modulus of the internal field.

The model results have been compared with those obtained from previous analytical models derived under more restrictive assumptions. We have discussed in which cases the simplified models are useful, and when they have to be replaced by the numerical procedure presented in this work.

The general framework provided in this work can be easily extended to sources of magnetic field other than a permanent magnet, as long as the cylindrical symmetry is preserved. Finally, the consideration of a noncylindrical geometry would require developments beyond the scope of this work, although the general formalism presented here may still be useful in calculations of situations in which the cylindrical symmetry is being lost, as in the initial response of the levitation force to lateral movements of the superconductor.

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