Interaction corrections to the Hall coefficient at intermediate temperatures

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We investigate the effect of electron-electron interaction on the temperature dependence of the Hall coefficient of two-dimensional electron gas at arbitrary relation between the temperature T and the elastic mean-free time τ. At small temperature $T\tau \ll \hbar$ we reproduce the known relation between the logarithmic temperature dependences of the Hall coefficient and of the longitudinal conductivity. At higher temperatures, this relation is violated quite rapidly; correction to the Hall coefficient becomes $\propto 1/T$ whereas the longitudinal conductivity becomes linear in temperature.

It is well known[1] that electron-electron interaction at low temperatures ($T\tau \ll \hbar$) leads to the logarithmically divergent correction $\delta\sigma_{xx}$ to the longitudinal conductivity of two-dimensional (2D) electron gas. Whereas for a wide range of temperatures the sign and the magnitude of those corrections are not universal (in particular due to the Fermi-liquid-type interactions in the triplet channel), one property remains intact:[2] there is no logarithmic interaction correction to the Hall conductivity σ_{xy} for any type of the interaction. When the interaction correction is still smaller than the Drude conductivity, the latter fact can be also rewritten as

$$\gamma = 2, \qquad (1)$$

where the parameter γ is defined as

$$\gamma(T) = -\frac{\partial_T \ln \rho_{xy}(T)}{\partial_T \ln \sigma_{xx}(T)}, \qquad (2)$$

with ρ_{xy} being the Hall resistivity.

Equation (1) is not controlled by any symmetries of the system and comes about as a result of the diffusive approximation justified only for $T\tau \ll \hbar$. For the 2D electron gas based on semiconductor heterostructures[3,4] this condition is easily violated. Therefore, one can no longer rely on the relation (1) for $T\tau \gtrsim \hbar$. The theory describing the corresponding temperature behavior of γ is still not available. Our goal in this paper is to derive the corresponding analytic formulas. Temperature dependence of the Hall coefficient much weaker than predicted by Eq. (2) was reported in Ref. 4.

We will use the kinetic equation formalism for the interaction corrections which we briefly summarize below and then apply it to the calculation of the Hall coefficient. The detailed derivation of the formalism and the region of its validity can be found in Ref. 5.

The electric current is expressed in terms of the distribution function $f(t;\epsilon,\vec{r},\vec{n})$ as

$$\vec{J}(t,\vec{r}) = e\nu v_F \int_{-\infty}^{\infty} d\epsilon \langle \vec{n} f(t;\epsilon,\vec{r},\vec{n})\rangle_n . \qquad (3)$$

Here ν is the density of states of the noninteracting system taken at the Fermi surface and v_F is the Fermi velocity, $\vec{n} = (\cos\theta, \sin\theta)$ is the unit vector in the direction of the elec-

tron momentum and angular averaging is $\langle\ldots\rangle_n = \int (d\theta/2\pi)\ldots$. The Boltzmann-like equation for the stationary and homogeneous distribution function is

$$\left[e v_F(\vec{n}\vec{E})\frac{\partial}{\partial\epsilon} + \omega_c \vec{b}\left(\vec{n}\times\frac{\partial}{\partial\vec{n}}\right)\right]f = \mathrm{St}\{f\}, \qquad (4)$$

where $\vec{E}$ is the applied electric field and ω_c is the cyclotron frequency due to the external magnetic field and $\vec{b}$ is the unit vector along the field.

All of the interaction effects are taken into account[5] in the elastic and inelastic parts of the collision integral, $\mathrm{St}\{f\} = \mathrm{St}_{el}\{f\} + \mathrm{St}_{in}\{f\}$. The inelastic part does not contribute to the Hall current. The relevant, elastic part can be written as (in all intermediate formulas we use the units with $\hbar=1$)

$$\mathrm{St}_{el}\{f\} = -\frac{f(\epsilon,\vec{n}) - \langle f(\epsilon,\vec{n})\rangle_n}{\tau} + I_0(\epsilon,\vec{n})\langle f(\epsilon,\vec{n})\rangle_n$$
$$+ n_\alpha I_1^{\alpha\beta}(\epsilon)\langle n_\beta f(\epsilon,\vec{n})\rangle_n ,$$

where the second line describes the effect of interaction on elastic collisions and

$$I_0(\epsilon,\vec{n}) = -\frac{8}{\tau}\int \frac{d\omega}{2\pi}\Bigg\{ n_\alpha K_0^{\alpha\beta}(\omega)\langle n_\beta f(\epsilon-\omega,\vec{n})\rangle_n$$
$$+ \frac{n_\alpha L_0^{\alpha\beta}(\omega)}{2} e E_\beta \frac{\partial}{\partial\epsilon}\langle f(\epsilon-\omega,\vec{n})\rangle_n \Bigg\} \qquad (5a)$$

$$I_1^{\alpha\beta}(\epsilon) = -\frac{8}{\tau}\int \frac{d\omega}{2\pi} K_1^{\alpha\beta}(\omega)\langle f(\epsilon-\omega,\vec{n})\rangle_n . \qquad (5b)$$

One can easily see that the elastic collision integral vanishes in equilibrium $f(\epsilon,\vec{n}) = f(\epsilon)$, $E=0$. The explicit definitions of K_0, K_1, and L_0 entering Eqs. (5) are

$$K_1^{\alpha\beta}(\omega) = \mathrm{Im}\int \frac{d^2q}{(2\pi)^2}\mathcal{D}^R(\omega,\vec{q})\Bigg\{ \langle n_\alpha D\rangle\langle D n_\beta\rangle$$
$$- \frac{\delta_{\alpha\beta}}{2}\left(\langle D\rangle\langle D\rangle + i\frac{\partial}{\partial\omega}\langle D\rangle\right)\Bigg\}, \qquad (6a)$$

$$K_0^{\alpha\beta}(\omega) = \mathrm{Im} \int \frac{d^2q}{(2\pi)^2} \mathcal{D}^R(\omega,\vec{q}) \Big\{ \langle n_\alpha D n_\beta \rangle \langle D \rangle$$

$$- \frac{i}{v_F} \frac{\partial}{\partial q_\alpha} \langle D n_\beta \rangle - \langle D n_\alpha \rangle \langle D n_\beta \rangle \Big\}, \quad (6b)$$

$$L_0^{\alpha\beta}(\omega) = -\mathrm{Re} \int \frac{d^2q}{(2\pi)^2} \mathcal{D}^R(\omega,\vec{q}) \Big\{ \langle D \rangle \frac{\partial}{\partial q_\beta} \langle n_\alpha D \rangle$$

$$- \langle D n_\alpha \rangle \frac{\partial}{\partial q_\beta} \langle D \rangle - \Big\langle D n_\alpha \frac{\partial}{\partial q_\beta} D \Big\rangle \Big\}. \quad (6c)$$

Here, $\mathcal{D}^R$ denotes the retarded interaction propagator in singlet channel[5] and the angular averaging is

$$\langle aDb \rangle \equiv \int \frac{d\theta d\theta'}{(2\pi)^2} a(\vec{n}) D(\vec{n},\vec{n}';\omega,\vec{q}) b(\vec{n}'),$$

$$\langle aDbDc \rangle$$

$$\equiv \int \frac{d\theta d\theta' d\theta''}{(2\pi)^3} a(\vec{n}) D(\vec{n},\vec{n}') b(\vec{n}') D(\vec{n}',\vec{n}'') c(\vec{n}'')$$

for arbitrary functions a, b. The function $D(\vec{n},\vec{n}';\omega,\vec{q})$ describes the classical motion of a particle on the energy shell ϵ_F in a magnetic field:

$$\left[-i\omega + iv_F \vec{n}\vec{q} + \omega_c \vec{b} \left(\vec{n} \times \frac{\partial}{\partial \vec{n}} \right) \right] D(\vec{n},\vec{n}')$$

$$+ \frac{1}{\tau}[D(\vec{n},\vec{n}') - \langle D(\vec{n},\vec{n}') \rangle_n] = 2\pi \delta(\theta - \theta'). \quad (7)$$

It is noteworthy, that unlike in the ordinary Boltzmann equation, the Lorentz force affects also the collision processes as one can see from Eqs. (6) and (7).

The above equations are written for the interaction in the singlet channel only. In a situation where both triplet and singlet channels are present, but the distribution function does not have a spin structure (no Zeeman splitting or nonequilibrium spin occupation present), one has to replace in Eq. (6)

$$\mathcal{D}^R \to \mathcal{D}_s^R + 3\mathcal{D}_t^R. \quad (8)$$

We calculate the Hall resistivity ρ_{xy} in the first order in magnetic field. We notice that due to the rotational and reflection symmetries of the system

$$K_i^{\alpha\beta}(\omega) = \delta_{\alpha\beta} K_i(\omega) + \epsilon_{\alpha\beta} \omega_c \tau \tilde{K}_i(\omega), \quad (9)$$

and the same structure for the L kernel. Here $\epsilon_{\alpha\beta}$ is the antisymmetric tensor, $\epsilon_{xy} = 1$.

In order to calculate the conductivity we look for a solution of Eqs. (4) and (5):

$$f(\vec{n},\epsilon) = f_F(\epsilon) + n_\alpha \Gamma_\alpha^{(1)}(\epsilon) + (\omega_c \tau) n_\alpha \epsilon_{\alpha\beta} \Gamma_\beta^{(2)}(\epsilon), \quad (10)$$

where $f_F(\epsilon) = 1/(e^{\epsilon/T}+1)$ is the Fermi distribution function (all the energies are counted from the Fermi level), and the

unknown quantities $\Gamma^{(1,2)}$ are first order in the electric field and zeroth order in the magnetic field.

Substituting Eq. (10) into Eqs. (4), (5), we obtain the following system of equations:

$$ev_F E_\alpha \frac{\partial f_F(\epsilon)}{\partial \epsilon}$$

$$= -\frac{\Gamma_\alpha^{(1)}(\epsilon)}{\tau} - \frac{4}{\tau} \int \frac{d\omega}{2\pi}[K_1(\omega) f_F(\epsilon-\omega)$$

$$\times \Gamma_\alpha^{(1)}(\epsilon) + K_0(\omega) f_F(\epsilon) \Gamma_\alpha^{(1)}(\epsilon-\omega)]$$

$$- \frac{4 f_F(\epsilon)}{\tau} \int \frac{d\omega}{2\pi} L_0(\omega) e E_\alpha \frac{\partial}{\partial \epsilon} f_F(\epsilon-\omega), \quad (11a)$$

$$\Gamma_\alpha^{(1)}(\epsilon) = -\Gamma_\alpha^{(2)}(\epsilon) - 4 \int \frac{d\omega}{2\pi} f_F(\epsilon-\omega)[K_1(\omega) \Gamma_\alpha^{(2)}(\epsilon)$$

$$+ \tilde{K}_1(\omega) \Gamma_\alpha^{(1)}(\epsilon)] - 4 \int \frac{d\omega}{2\pi} f_F(\epsilon)$$

$$\times [K_0(\omega) \Gamma_\alpha^{(2)}(\epsilon-\omega) + \tilde{K}_0(\omega) \Gamma_\alpha^{(1)}(\epsilon-\omega)]$$

$$- 4 f_F(\epsilon) \int \frac{d\omega}{2\pi} \frac{\partial}{\partial \omega_c} \tilde{L}_(\omega e E_\alpha \frac{\partial}{\partial \epsilon} f_F(\epsilon-\omega). \quad (11b)$$

We solve Eqs. (11) by iteration and substitute the result into Eq. (10) [$\Gamma^{(1)}$ does not contribute to the Hall current], and then into Eq. (3). Performing angular averaging and integration over ϵ one finds components of the conductivity tensor.

Rather than writing explicit results for the Hall conductivity we present the expression for the Hall resistivity, $\rho_{xy} = \rho_H^D + \delta\rho_{xy}$,

$$\frac{\delta\rho_{xy}}{\rho_H^D} = -\frac{\delta\sigma_{xy}}{(\omega_c \tau)\sigma_D} - 2\frac{\delta\sigma_{xx}}{\sigma_D},$$

where $\sigma_D = e^2 \nu v_F^2 \tau / 2$ is the Drude conductivity, and ρ_H^D is the classical Hall resistivity ($\rho_H^D = -\omega_c \tau / \sigma_D$). We find

$$\frac{\delta\rho_{xy}}{\rho_H^D} = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\partial}{\partial\omega} \left(\omega \coth\frac{\omega}{2T} \right) \Big[\tilde{K}_1(\omega) - \tilde{K}_0(\omega)$$

$$+ \frac{\tilde{L}_0(\omega)}{v_F \tau} + \frac{L_0(\omega)}{v_F \tau} \Big]. \quad (12)$$

What remains is to find the explicit expressions for the kernels entering into Eq. (12). To do so, we solve Eq. (7) up to the first order in magnetic field:

$$D(\vec{n},\vec{n}') = D_1(\vec{n},\vec{n}') - \omega_c \int \frac{d\theta_1}{2\pi} D_1(\vec{n},\vec{n}_1)$$

$$\times \left(\vec{n}_1 \times \frac{\partial}{\partial \vec{n}_1} \right) D_1(\vec{n}_1,\vec{n}'),$$

$$D_1(\vec{n},\vec{n}') = 2\pi \delta(\theta - \theta') D_0(\vec{n}) + D_0(\vec{n}) D_0(\vec{n}') \frac{C}{C\tau - 1}, \tag{13a}$$

$$C = \sqrt{(-i\omega + 1/\tau)^2 + v_F^2 q^2}, \quad D_0(\vec{n}) = \frac{1}{-i\omega + iv_F \vec{n}\vec{q} + 1/\tau}. \tag{13b}$$

We note, that only the second term on the right-hand side (rhs) of Eq. (13a) contributes to $\tilde{K}_i$ and $\tilde{L}_0$. Substituting Eqs. (13) into Eqs. (6), the resulting kernels into Eq. (12) and performing the angular integration we obtain the results for the correction to the diagonal conductivity, $\delta\sigma_{xx}$, and to the Hall resistivity,

$$\begin{pmatrix} \delta\rho_{xy}/\rho_H^D \\ \delta\sigma_{xx}/\sigma_D \end{pmatrix} = \mathrm{Im} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\partial}{\partial\omega} \left(\omega \coth\frac{\omega}{2T} \right) \int \frac{q\,dq}{4\pi} \mathcal{D}^R(\omega,q)$$

$$\times \begin{pmatrix} B_{xy}(\omega,q) \\ B_{xx}(\omega,q) \end{pmatrix}, \tag{14}$$

where the form-factors are defined as

$$B_{xy} = - \left\{ \frac{2v_F^2 q^2/\tau^2}{C^3(C-1/\tau)^3} + \frac{v_F^2 q^2 [2C - 5(-i\omega + 1/\tau)]}{2\tau^2 C^5 (C-1/\tau)^2} \right.$$

$$\left. + \frac{(-i\omega + 1/\tau)[C - (-i\omega + 1/\tau)]}{\tau^2 C^4 (C-1/\tau)^2} \right\}, \tag{15a}$$

$$B_{xx} = \left\{ \frac{v_F^2 q^2/\tau^2}{C^3(C-1/\tau)^3} + \frac{3v_F^2 q^2}{2\tau C^3 (C-1/\tau)^2} \right.$$

$$+ \frac{2[C - (-i\omega + 1/\tau)]}{C(C-1/\tau)^2}$$

$$\left. + \frac{2C - 1/\tau}{C v_F^2 q^2} \left(\frac{C - (-i\omega + 1/\tau)}{C - 1/\tau} \right)^2 \right\}. \tag{15b}$$

Expression for the form-factor B_{xx} was obtained before in Ref. 5 and cited here for comparison. It is noteworthy that $B_{xy,xx}(\omega, q=0) = 0$ as it is dictated by the gauge invariance.

Equations (15) enable one to anticipate the behavior of parameter γ from Eq. (2) even before the form of the interaction in Eq. (14) is specified. At low temperatures, the integrals are determined by ω, $qv_F \ll 1/\tau$. In this case both formfactors in Eq. (15) are controlled by the first terms in the rhs. As a result, one arrives at Eq. (1). At larger $q \gg 1/v_F \tau$, the behavior of the formfactors is completely different, $B_{xx} \propto 1/q^2$, $B_{xy} \propto 1/q^4$, so the relation (1) can no longer hold.

To study the details of temperature behavior of γ, one needs to evaluate the integrals in Eq. (14) for both the singlet and the triplet channel propagator [see Eq. (50) and Eq. (60) in Ref. 5]:

$$\mathcal{D}_t^R = -\frac{1}{\nu} \frac{C - 1/\tau}{i\omega + \frac{F_0^\sigma + 1}{F_0^\sigma}(C - 1/\tau)},$$

where F_0^σ is the interaction constant in the triplet channel, and singlet propagator, $\mathcal{D}_s^R$, is obtained by setting $F_0^\sigma \to \infty$. For completely spin-polarized electrons only the singlet channel correction is present. We find

$$\frac{\delta\rho_{xy}^\rho}{\rho_H^D} = -\frac{e^2}{4\pi^2 \sigma_D} \left\{ -4\mathbf{C} + \frac{1}{24\pi T\tau} \psi' \left(1 + \frac{1}{2\pi T\tau} \right) \right.$$

$$- \frac{11}{12} \psi \left(1 + \frac{1}{2\pi T\tau} \right) - \frac{19}{2} \int_0^1 dx\, x\, \psi \left(1 + \frac{x}{2\pi T\tau} \right)$$

$$\left. + 5 \int_0^1 dx\, x^2 \psi \left(1 + \frac{x}{2\pi T\tau} \right) \right\}, \tag{16}$$

where $\mathbf{C} \approx 0.577$ is Euler's constant, $\psi(x)$ is the di-gamma function. Equation (16) can be approximated by the following interpolation formula:

$$\frac{\delta\rho_{xy}^\rho}{\rho_H^D} = \frac{e^2}{\pi^2 \hbar \sigma_D} \ln \left(1 + \frac{11\pi}{192} \frac{\hbar}{T\tau} \right). \tag{17}$$

This formula reproduces the logarithmic behavior[2] in the diffusive limit ($T\tau \ll \hbar$) in accordance with Eq. (1), and

$$\frac{\delta\rho_{xy}^\rho}{\rho_H^D} = \frac{e^2}{\pi^2 \hbar \sigma_D} \frac{11\pi}{192} \frac{\hbar}{T\tau},$$

for $T\tau \gg \hbar$, in contrast with the linear dependence of the diagonal conductivity in this regime.[5] Finally, at intermediate

$$\left(\frac{\delta\rho_{xy}}{\rho_H^D} \right) \left(\frac{2\pi\hbar\sigma_D}{e^2} \right)$$
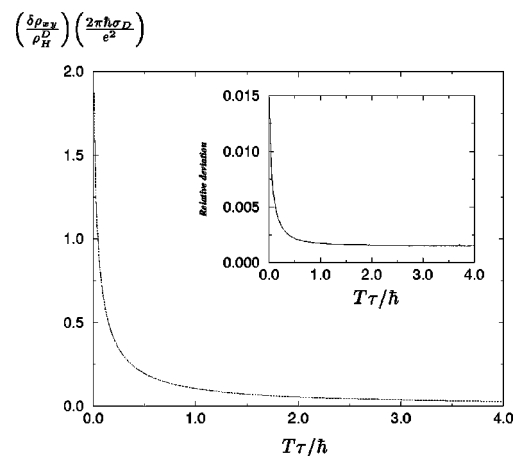

FIG. 1. Singlet channel correction to Hall resistivity $\delta\rho_{xy}^\rho$, given by Eq. (16) (solid line) and interpolation formula (17) (dashed line). On this scale two curves are indistinguishable, and the relative deviation is blown up on the inset.
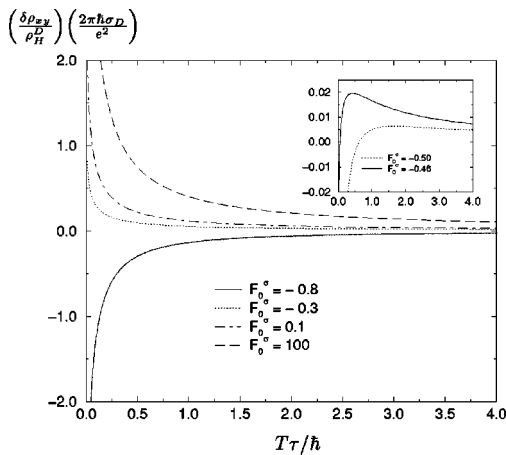
FIG. 2. Total correction to Hall resistivity $\delta\rho_{xy}$ for different values of the Fermi-liquid parameter F_0^σ. For $-0.61 < F_0^\sigma < -0.45$, the temperature dependence is nonmonotonous, though very weak, see inset.

temperatures Eq. (17) provides accurate (for numerical reasons) description of the crossover, see Fig. 1.

The triplet channel correction is calculated analogously and the interpolation formula similar to Eq. (17) reads

$$\frac{\delta\rho_{xy}^\sigma}{\rho_H^D} = \frac{3e^2}{\pi^2\hbar\sigma_D}\left\{1 - \frac{1}{F_0^\sigma}\ln\frac{1+F_0^\sigma+(T\tau/\hbar)g(F_0^\sigma)}{1+T\tau/\hbar}\right\}$$

$$\times\ln\left(1+\frac{11\pi}{192}\frac{\hbar}{T\tau}\right). \tag{18}$$

Once again, Eq. (18) reproduces the asymptotic behavior at high and low temperature and gives numerically accurate results in the intermediate region. The definition of the smooth function $g(x)$ is

$$\ln g(x) = \frac{4}{11}[-5f_3(x) - 12f_2(x) - 3f_1(x) + 4f_0(x)],$$

$$f_j(x) = \frac{1}{x^j}\left(\ln[1+x] + \sum_{n=1}^{j}\frac{(-x)^n}{n}\right), \tag{19}$$

and it has the following asymptotic behavior, $g(x) = 1 + x + \frac{3}{22}x^2 + \ldots$, $x \ll 1$, and $g(x) \to e^{-70/33}$, $x \to -1$. The latter asymptotic is realized when the system is close to the Stoner instability.
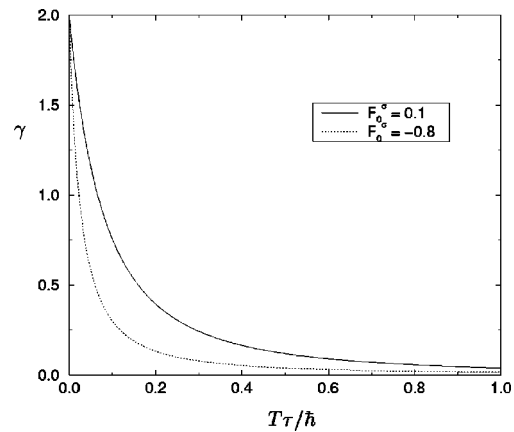

FIG. 3. Temperature dependence of the parameter γ, see Eq. (2), relating the Hall coefficient to σ_{xx}.

The total correction to the Hall coefficient, $\delta\rho_{xy} = \delta\rho_{xy}^\rho + \delta\rho_{xy}^\sigma$, is shown in Fig. 2. Let us note in passing, that if the electron system is polarized by strong in-plane field, only the singlet correction (17) remains.

Finally, we present the temperature dependence of the quantity γ from Eq. (2). The plot shown in Fig. 3 was obtained with the help of Eqs. (17) and (18) of the present paper and Eq. (16) of Ref. 5. We note, that the slope of this curve at zero temperature is always finite. According to the figure, $\gamma(T)$'s deviation from the value of 2 happens already at much lower temperatures than expected: it changes by a factor of 2 at about $T\tau/\hbar \simeq 0.1$ for the weak coupling ($F_0^\sigma \ll 1$) case.

To conclude, we investigated the temperature dependence of the Hall resistivity of 2D electron gas for arbitrary values of $T\tau/\hbar$. It is shown that whereas the relation between the Hall and diagonal resistivity (1) indeed holds for $T\tau \ll \hbar$, it is rapidly violated at higher temperatures, see Fig. 3.

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