Renormalization group and 1/N expansion for the three-dimensional Ginzburg-Landau-Wilson model

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A renormalization-group scheme is developed for the three-dimensional $O(2N)$-symmetric Ginzburg-Landau-Wilson model, which is consistent with the use of a 1/N expansion as a systematic method of approximation. It is motivated by an application to the critical properties of superconductors, reported in a separate paper. Within this scheme, the infrared stable fixed point controlling critical behavior appears at $z=0$, where $z=\lambda^{-1}$ is the inverse of the quartic coupling constant, and an efficient renormalization procedure consists of the minimal subtraction of ultraviolet divergences at $z=0$. This scheme is implemented at next-to-leading order, and the standard results for critical exponents calculated by other means are recovered. An apparently novel result of this nonperturbative method of approximation is that corrections to scaling (or confluent singularities) do not, as in perturbative analyses, appear as simple power series in the variable $y=zt^{\omega\nu}$. At least in three dimensions, the power series are modified by powers of ln y.

I. INTRODUCTION

Renormalization-group techniques have long been established as the standard formal framework for understanding critical phenomena.[1–4] Those based on effective Hamiltonians of the Ginzburg-Landau-Wilson type require some kind of supplementary approximation if concrete information is to be extracted from them, and this is normally provided by perturbation theory, in the form either of the ϵ expansion introduced originally by Wilson and Fisher[5] or of a fixed-dimension calculation.[6] For $O(N)$ symmetric models, a systematic expansion in powers of 1/N provides an alternative method of approximation.[7–10] While the methods that have been used to extract estimates of critical exponents as power series in 1/N are informed by renormalization-group ideas, there seems to exist no method for implementing this approximation scheme within a systematic renormalization-group analysis. The purpose of this paper is to describe a renormalization scheme which does this.

As is well known, the few terms of the 1/N expansions for critical exponents that it has been possible to calculate explicitly do not yield accurate results for the small values of N that characterize physically interesting systems, so the development of a formal technique is not especially useful from this point of view. Our motivation for developing this scheme arises from our efforts to estimate critical-point scaling functions for thermodynamic properties of superconductors in applied magnetic fields, in particular the specific heat, for which experimental data have been available for some time (see, for example, Ref. 11). For this problem, it turns out that perturbation theory does not yield a well-controlled sequence of approximations. As reported in an accompanying paper,[12] the 1/N expansion is advantageous in this respect, and yields qualitatively encouraging results, although quantitative agreement with existing data is still hard to achieve.

In outline, the renormalization-group strategy we propose works as follows. Consider, for example, the order-parameter susceptibility $\chi(t_0,\lambda_0,\Lambda)$, where t_0 is proportional to $T-T_C$, λ_0 is a coupling constant which can be taken as temperature independent in the critical region, and Λ is a momentum cutoff of the order of the inverse of a typical interatomic distance. In the limit $t\rightarrow0$, we expect this susceptibility to diverge as $\chi\sim t_0^{-\gamma}$, where γ is a universal exponent which we would like to determine. In general, a renormalization scheme introduces a renormalized susceptibility χ_R, together with renormalized parameters t and λ, which depend on an arbitrary length scale, say μ^{-1}. The fact that physics is independent of μ leads to a relation of the form

$$\chi_R(t,\lambda,\Lambda/\mu)=P_\chi(l)\chi_R(t(l),\lambda(l),\Lambda/(l\mu)), \qquad (1)$$

where l is an arbitrary number. The prefactor $P_\chi(l)$ and the running variables $t(l)$ and $\lambda(l)$ depend on how the renormalization has been implemented. Now, the function $\chi_R(t,\lambda,\Lambda/\mu)$ diverges when its first argument, t, approaches zero, but the nature of this singularity is hard to determine directly. Suppose, however, that the renormalization scheme has been constructed in such a way that

$$P_\chi(l)\sim l^{-\gamma^*/\nu^*}, \quad t(l)\sim tl^{-1/\nu^*}, \quad \lambda(l)\rightarrow\lambda^* \qquad (2)$$

in the limit $l\rightarrow0$, where γ^*, ν^*, and λ^* (the "fixed-point" coupling) are constants. By setting $l=t^{\nu^*}$, we find in the limit $t\rightarrow0$

$$\chi_R(t,\lambda,\Lambda/\mu)\sim t^{-\gamma^*}\chi_R(1,\lambda^*,\Lambda/(t^{\nu^*}\mu)). \qquad (3)$$

The exponent γ^* will be equal to the true critical exponent γ if $\chi_R(1,\lambda^*,\infty)$ has a finite, nonzero value and the renormalization scheme must be designed to ensure that this is so. The exponent ν^* will then be equal to the critical exponent ν, which characterizes the divergence of the correlation length.

These calculations cannot be carried out exactly, so a renormalization scheme must be supplemented by some systematic means of approximation and its details will depend on this method of approximation. Perturbative methods of

approximation, involving an expansion in powers of λ, rely on the possibility of expressing exponents such as γ as power series in $\epsilon = 4 - d$, where d is the spatial dimensionality (although in practice it proves possible also to implement them directly in three dimensions). Near four dimensions, the bare susceptibility $\chi(t_0, \lambda_0, \Lambda)$ and other thermodynamic functions diverge as $\Lambda \to \infty$, so the principal requirement of a renormalization scheme is to remove these divergences from renormalized functions such as χ_R. One then finds that λ^* is of order ϵ, so the perturbative analysis can be implemented self-consistently.

For the reasons outlined above, we wish to make use of an alternative approximation scheme consisting of an expansion in powers of $1/N$ with $d = 3$. A systematic means of obtaining the $1/N$ expansion is reviewed in Sec. II. It is facilitated by an integral transformation of the Hubbard-Stratonovich type which expresses the partition function as a functional integral over an auxiliary field Ψ, with an effective Hamiltonian $H_{\text{eff}}(\Psi)$. This partition function can be calculated as a power series in $1/N$ by the method of steepest descent. Within this expansion, thermodynamic quantities are most naturally expressed as functions not of the temperature variable t, but rather of the exact inverse susceptibility $\tilde{t}_0 = \chi^{-1}$. The temperature dependence of $\tilde{t}_0$ is given implicitly by solving a constraint equation of the form $t_0 = \Phi(\tilde{t}_0, \lambda_0, \Lambda)$. At lowest order, this constraint equation simply locates the saddle point $\Psi = -i(\tilde{t}_0 - t_0)$ of the effective Hamiltonian $H_{\text{eff}}(\Psi)$.

In Sec. III, a renormalization scheme is exhibited which manifestly extracts the correct critical exponents at leading order. In the case of the susceptibility, the renormalized constraint equation $t = \Phi_R(\tilde{t}, \lambda, \Lambda/\mu) = \Phi_R(\chi_R^{-1}, \lambda, \Lambda/\mu)$ obeys a relation equivalent to Eq. (1), but with the roles of t and χ_R^{-1} reversed. The substantive issue, addressed in Sec. IV, is how this renormalization scheme can properly be extended to higher orders. The usual perturbative strategy fails at this point, because in three dimensions the constraint function Φ and other thermodynamic quantities remain finite as $\Lambda \to \infty$. For this reason, we shall actually omit the cutoff altogether in subsequent sections. We find, however, that the fixed-point coupling strength λ^* is infinite, and that divergences appear in the limit $\lambda \to \infty$. For this reason, we find it convenient to deal with the inverse of the renormalized coupling, $z = \lambda^{-1}$, so that the fixed point appears at $z^* = 0$. In the context of the $1/N$ expansion, then, the primary requirement of a renormalization scheme is not to remove divergences that appear as $\Lambda \to \infty$, but rather to remove those that appear as $z \to 0$. This crucial observation is the first result that we wish to report. The second is a concrete renormalization scheme that actually does remove the divergences at $z = 0$. As shown in Sec. IV, this can be achieved economically by means of a "minimal subtraction" scheme which subtracts powers of $\ln z$.

We have, of course, checked that the critical exponents obtained from our renormalization scheme agree with those obtained long ago by other methods. However, our purpose in devising this renormalization-group strategy is not simply to reproduce old results by a different method. The application of this formalism that we have mainly in view is the calculation of scaling functions. In particular, the specific heat of a superconductor near its critical point, and in the presence of a magnetic field B, is expected to assume the scaling form $C(T, B) \approx B^{-\alpha/2\nu} \mathcal{C}(t B^{-1/2\nu})$ and we wish to estimate the scaling function $\mathcal{C}(x)$, for which experimental data are available. When the Ginzburg-Landau model is enlarged to include the magnetic field, the renormalized specific heat obeys a relation similar to Eq. (1), namely,

$$C_R(\tilde{t}, z, B) = P_C(l) C_R(\tilde{t}(l), z(l), B l^{-2})$$

$$\xrightarrow{l \to 0} l^{-\alpha/\nu} C_R(l^{-(2-\eta)}\tilde{t}, 0, l^{-2}B). \qquad (4)$$

By setting $\tilde{t}(l) = 1$ and $l = B^{1/2}L$, we can identify the scaling function as $\mathcal{C}(x) = L^{-\alpha/2\nu} C_R(1, 0, L^{-2})$, with L determined as a function of x by solution of the constraint equation, which then has the form $x = L^{1/\nu} \Phi(1, 0, L^{-2})$. Details of this calculation are presented in Ref. 12; here, our principal concern is to explain the renormalization-group formalism that makes it feasible. It happens that special considerations apply to the critical behavior of the specific heat[18] and the impact of these on the renormalization-group formalism is discussed in Sec. V.

While the main purpose of this paper is to expose details of the renormalization scheme that will be applied in Ref. 12, we also wish to report what we believe to be a novel feature of the renormalization group, revealed by the nonperturbative nature of the $1/N$ expansion, that is of some theoretical interest. The leading corrections to asymptotic critical singularities (sometimes described as confluent singularities) appear in the form of a scaling variable $y = (\lambda - \lambda^*)|T - T_C|^{\omega\nu}$. According to conventional wisdom, the order-parameter susceptibility, say, has a singular part which can be expressed as $\chi^{\text{sing}} = |T - T_C|^{-\gamma} X(y)$, where the scaling function $X(y)$ has a power-series expansion in y. In perturbation theory, this is automatically true, but the $1/N$ expansion suggests otherwise. In three dimensions, at least, we find that $X(y)$ contains, in addition to powers of y, singular terms which at next-to-leading order are of the form $y^n \ln y$. This is shown in Sec. IV and further discussed, along with our other principal findings, in Sec. VI.

II. THE 1/N EXPANSION

We consider the standard Ginzburg-Landau-Wilson theory for N complex fields, defined by the Hamiltonian density

$$\mathcal{H} = \sum_{i=1}^{N} |\nabla \phi_i(\mathbf{r})|^2 + t_0 |\phi_i(\mathbf{r})|^2 + \frac{\lambda_0}{4N} \left(\sum_{i=1}^{N} |\phi_i(\mathbf{r})|^2 \right)^2,$$
$$(5)$$

in which t_0 is taken to be linear in temperature ($t_0 \propto T - T_0$, where T_0 is the mean-field transition temperature) and the coupling strength λ_0 to be temperature independent. This is actually an $O(2N)$-symmetric model; we have assembled its $2N$ real fields into N complex fields merely to facilitate the application to a Ginzburg-Landau superconductor which we have ultimately in view. Introducing sources $j_i(\boldsymbol{r})$ and $j_i^*(\boldsymbol{r})$

for the fields $\phi_i(r)$ and $\phi_i^*(r)$, we arrive at the generating functional for correlation functions

$$Z[j_i,j_i^*]=\prod_{i=1}^{N} \int D\phi_i \int D\phi_i^* \exp\left\{ \int d^3r \right.$$

$$\times\left[-\mathcal{H}+\sum_{i=1}^{N} (j_i(r)\phi_i^*(r)+j_i^*(r)\phi_i(r)) \right]\Bigg\}. \tag{6}$$

The $1/N$ expansion is generated by the procedure described, for example, by Brézin *et al.*[13] A Hubbard-Stratonovich transformation

$$\exp\left[-\int d^3r\frac{\lambda_0}{4N}\left(\sum_{i=1}^{N} |\phi_i|^2 \right)^2 \right]$$

$$=\mathcal{N}\int D\Psi \exp\left[-\int d^3r\frac{N}{\lambda_0}\Psi^2-i\Psi\sum_{i=1}^{N} |\phi_i|^2 \right], \tag{7}$$

where $\mathcal{N}$ is a normalization factor whose value is of no consequence, brings the integral over ϕ_i and ϕ_i^* to a Gaussian form. On computing this integral, we obtain

$$Z[j_i,j_i^*]=\mathcal{N}\int D\Psi \exp\left[-NH_{\text{eff}}(\Psi) \right.$$

$$\left. +\int d^3r \int d^3r' \sum_{i=1}^{N} j_i(r)\Delta(r,r';\Psi)j_i^*(r') \right], \tag{8}$$

where the effective Hamiltonian is

$$H_{\text{eff}}(\Psi)=\int d^3r\frac{1}{\lambda_0}\Psi^2(r)-\text{Tr}_{r,r'}\ln\Delta(r,r';\Psi) \tag{9}$$

and the propagator $\Delta(r,r';\Psi)$ is the solution of

$$[-\nabla^2+t_0+i\Psi(r)]\Delta(r,r';\Psi)=\delta(r-r'). \tag{10}$$

The connected correlation functions $G_{i_1\cdots i_n}(r_1\cdots r_n)$ $=\langle\phi_{i_1}^*(r_1)\cdots\phi_{i_n}(r_n)\rangle_c$ are, of course, obtained by differentiation of $Z[j_i,j_i^*]$. For example, the two-point function $G_{ij}(r,r')\equiv\delta_{ij}G^{(2)}(r-r')$ is given by

$$G^{(2)}(r_1-r_2)=\langle\phi_1^*(r_1)\phi_1(r_2)\rangle_c=\frac{\delta^2 \ln Z}{\delta j_1(r_1)\delta j_1^*(r_2)}\Bigg|_{j=j^*=0}$$

$$=\frac{\int D\Psi \,\Delta(r_1,r_2;\Psi)e^{-NH_{\text{eff}}(\Psi)}}{\int D\Psi \,e^{-NH_{\text{eff}}(\Psi)}}. \tag{11}$$

Similarly, we can define a four-point function

$$G^{(4)}(r_1,r_2;r_3,r_4)$$

$$=\frac{\int D\Psi \,\Delta(r_1,r_2;\Psi)\Delta(r_3,r_4;\Psi)e^{-NH_{\text{eff}}(\Psi)}}{\int D\Psi \,e^{-NH_{\text{eff}}(\Psi)}}. \tag{12}$$

Owing to the factor of N that multiplies $H_{\text{eff}}(\Psi)$, the $1/N$ expansion is now generated by the method of steepest descent. It proves convenient to formulate this expansion in terms of a parameter $\tilde{t}_0$, defined by

$$\tilde{t}_0=\Gamma^{(2)}(0), \tag{13}$$

where $\Gamma^{(2)}(p)$ is the inverse of the Fourier transform of $G^{(2)}(r-r')$. We write

$$\Psi(r)=-i\left(\tilde{t}_0-t_0-\frac{1}{N}\delta \right)+(2N)^{-1/2}\psi(r), \tag{14}$$

where δ is defined by imposing the constraint $\langle\psi(r)\rangle=0$. Since δ has contributions of all orders in $1/N$, we set $\delta=\delta_0+N^{-1}\delta_1+\cdots$. The position-independent value $\Psi_0=-i(\tilde{t}_0-t_0)$ locates the saddle point of H_{eff} up to corrections of order $1/N$, and the propagator $\Delta(r,r';\Psi)$ is given by $\Delta(r,r';\Psi)=\Delta(r-r')+O(N^{-1/2})$, where

$$\Delta(r)=\int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot r}}{(p^2+\tilde{t}_0)}. \tag{15}$$

The two quantities $NH_{\text{eff}}(\Psi)$ and $\Delta(r,r';\Psi)$ can now be expanded in powers of $1/N$. For $NH_{\text{eff}}(\Psi)$, discarding irrelevant terms that are independent of $\psi(r)$, the result is

$$NH_{\text{eff}}=iN^{1/2}\frac{\sqrt{2}}{\lambda_0}[t_0-f(\tilde{t}_0,\lambda_0)]\int d^3r\psi(r)$$

$$+\frac{1}{2}\int d^3rd^3r' \,\psi(r)D^{-1}(r-r')\psi(r')$$

$$-iN^{-1/2}\frac{1}{6\sqrt{2}}\int d^3rd^3r'd^3r''\Delta_3(r,r',r'')$$

$$\times\psi(r)\psi(r')\psi(r'')-N^{-1}\left[\frac{1}{16}\int d^3rd^3r'd^3r''d^3r''' \right.$$

$$\times\Delta_4(r,r',r'',r''')\psi(r)\psi(r')\psi(r'')\psi(r''')$$

$$\left. -\frac{1}{2}\delta_0\int d^3rd^3r'd^3r''\Delta_3(r,r',r'')\psi(r)\psi(r') \right]$$

$$+O(N^{-3/2}), \tag{16}$$

where the ψ propagator is given by

$$D^{-1}(r-r')=\lambda_0^{-1}\delta(r-r')+\frac{1}{2}\Delta(r-r')\Delta(r'-r) \tag{17}$$

and the vertex functions are

$$\Delta_n(r_1,\ldots,r_n)=\Delta(r_1-r_2)\Delta(r_2-r_3)\cdots\Delta(r_n-r_1). \tag{18}$$

The function $f(\tilde{t}_0,\lambda_0)$ is

$$f(\tilde{t}_0,\lambda_0)=\tilde{t}_0-\frac{1}{N}\delta_0-\frac{1}{2}\lambda_0\Delta(\mathbf{0})-\frac{1}{2}\lambda_0\delta_0\int d^3r\Delta_2(\mathbf{r}).$$

(19)

Although in principle the counterterm δ is determined by the constraint $\langle\psi(\mathbf{r})\rangle=0$ while the definition (13) determines $\tilde{t}_0$ as a function of t_0 and λ_0, we see from the expansion (16) that practical calculations invert this logic. Thus, the term proportional to $N^{-1}\delta_0$ is quadratic in $\psi(\mathbf{r})$, so we arrange for Eq. (13) to be true by adjusting the value of δ_0. On the other hand, the term that is linear in $\psi(\mathbf{r})$ provides a counterterm that is adjusted to make $\langle\psi(\mathbf{r})\rangle$ vanish, yielding a constraint (or gap) equation

$$t_0=f(\tilde{t}_0,\lambda_0)+O(N^{-1})$$

(20)

that implicitly determines $\tilde{t}_0$ as a function of t_0 and λ_0.

For the propagator $\Delta(\mathbf{r},\mathbf{r}';\Psi)$ we have the expansion

$$\Delta(\mathbf{r},\mathbf{r}';\Psi)=\Delta(\mathbf{r}-\mathbf{r}')-iN^{-1/2}\frac{1}{\sqrt{2}}\int d^3r_1\bar{\Delta}_2(\mathbf{r},\mathbf{r}';\mathbf{r}_1)\psi(\mathbf{r}_1)$$

$$-N^{-1}\left[\frac{1}{2}\int d^3r_1d^3r_2\bar{\Delta}_3(\mathbf{r},\mathbf{r}';\mathbf{r}_1,\mathbf{r}_2)\psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\right.$$

$$\left.+\delta_0\int d^3r_1\bar{\Delta}_2(\mathbf{r},\mathbf{r}';\mathbf{r}_1)\right]$$

$$+iN^{-3/2}\left[\frac{1}{2\sqrt{2}}\int d^3r_1d^3r_2d^3r_3\bar{\Delta}_4\right.$$

$$\times(\mathbf{r},\mathbf{r}';\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\psi(\mathbf{r}_3)$$

$$\left.+\sqrt{2}\delta_0\int d^3r_1d^3r_2\bar{\Delta}_3(\mathbf{r},\mathbf{r}';\mathbf{r}_1,\mathbf{r}_2)\psi(\mathbf{r}_1)\right]$$

$$+O(N^{-2}),$$

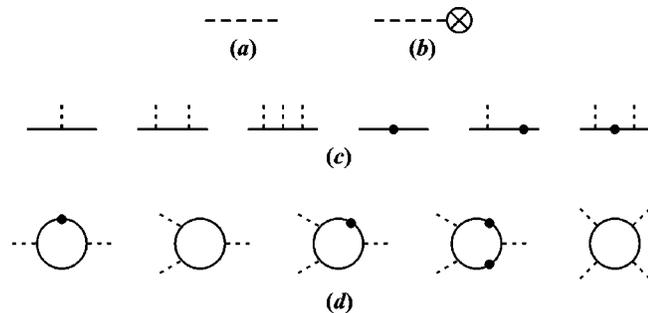(21)

(a)   **(b)**

(c)

(d)

FIG. 1. Elements of Feynman diagrams in the $1/N$ expansion: (a) the ψ propagator, (b) the counterterm $f(\tilde{t}_0,\lambda_0)$, (c) the first few terms in the expansion of the propagator $\Delta(\mathbf{r},\mathbf{r}';\Psi)$, and (d) elementary vertex functions arising from the expansion of NH_{eff}. Solid lines represent the lowest-order ϕ propagator $\Delta(\mathbf{r}-\mathbf{r}')$ and filled circles represent the counterterm δ. In (c) and (d), the dashed lines are the legs to which ψ propagators can be attached; arbitrary sequences of these lines and the dashed legs are allowed.

FIG. 2. Diagrammatic representation of the constraint equation at next-to-leading order.

with

$$\bar{\Delta}_n(\mathbf{r},\mathbf{r}';\mathbf{r}_1,\ldots,\mathbf{r}_{n-1})=\Delta(\mathbf{r}-\mathbf{r}_1)\Delta(\mathbf{r}_1-\mathbf{r}_2)\cdots\Delta(\mathbf{r}_{n-1}-\mathbf{r}').$$

(22)

The two expansions given in Eqs. (16) and (21) can be summarized diagrammatically by the Feynman rules given schematically in Fig. 1 for calculating the correlation functions $G_{i_1\cdots i_n}(\mathbf{r}_1,\ldots,\mathbf{r}_n)$. In practice, it is most convenient to interpret these in momentum space. Then the ϕ and ψ propagators are

$$\Delta(\mathbf{p})=(p^2+\tilde{t}_0)^{-1},$$

(23)

$$D(\mathbf{p})=[\lambda_0^{-1}+\Pi(\mathbf{p})]^{-1},$$

(24)

with

$$\Pi(\mathbf{p})=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{[k^2+\tilde{t}_0][(\mathbf{k}+\mathbf{p})^2+\tilde{t}_0]}$$

$$=\frac{1}{8\pi|\mathbf{p}|}\tan^{-1}\left(\frac{|\mathbf{p}|}{2\tilde{t}_0^{1/2}}\right).$$

(25)

For future use, we note that $\Pi(0)=a\tilde{t}_0^{-1/2}$, with $a=1/16\pi$, while for large momenta we can write

$$\Pi(\mathbf{p})=\frac{b}{|\mathbf{p}|}-\frac{\tilde{t}_0^{1/2}}{p^2}\tilde{\Pi}\left(\frac{\tilde{t}_0}{p^2}\right),$$

(26)

with $b=1/16$ and

$$\tilde{\Pi}(\tau)=\frac{1}{8\pi}\tau^{-1/2}\tan^{-1}(2\tau^{1/2})=4a\left(1-\frac{4}{3}\tau+\cdots\right).$$

(27)

For our immediate purposes, we require explicit expressions at next-to-leading order in $1/N$ for the constraint equation $\langle\psi(\mathbf{r})\rangle=0$, the two-point function $G^{(2)}(\mathbf{p})$, and the truncated four-point function $\Gamma^{(4)}(\mathbf{p}_i)=G^{(4)}(\mathbf{p}_i)/\Pi_{j=1}^4G^{(2)}(\mathbf{p}_j)$, which is one-particle irreducible with respect to the ϕ propagator $\Delta(\mathbf{p})$. These are shown diagrammatically in Figs. 2, 3,

FIG. 3. Diagrammatic representation of the order-parameter two-point function at next-to-leading order.

FIG. 4. Diagrammatic representation of the vertex function $\Gamma^{(4)}$ at next-to-leading order.

and 4 respectively. The constraint equation is

$$t_0 = f(\widetilde{t}_0, \lambda_0) + \frac{\lambda_0}{4N} A + O(N^{-2})$$

$$= \widetilde{t}_0 - \frac{\lambda_0}{2} \int \frac{d^3 k}{(2\pi)^3} \Delta(k) + N^{-1} \left[\frac{\lambda_0}{4} A(\widetilde{t}_0, \lambda_0) - \lambda_0 \delta_0 \Pi(0) \right.$$

$$\left. - \delta_0 \right] + O(N^{-2}), \tag{28}$$

where $A(\widetilde{t}_0, \lambda_0)$ is the integral

$$A(\widetilde{t}_0, \lambda_0) = \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \Delta(k)^2 \Delta(k') D(k+k'). \tag{29}$$

The inverse two-point function $\Gamma^{(2)}(p) = G^{(2)}(p)^{-1}$ is given by

$$\Gamma^{(2)}(p) = p^2 + \widetilde{t}_0 + N^{-1} [\Sigma(p; \widetilde{t}_0, \lambda_0) - \delta_0] + O(N^{-2}), \tag{30}$$

with

$$\Sigma(p; \widetilde{t}_0, \lambda_0) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \Delta(k+p) D(k). \tag{31}$$

Thus, the definition (13) of $\widetilde{t}_0$ allows us to identify

$$\delta_0 = \Sigma(0; \widetilde{t}_0, \lambda_0), \tag{32}$$

and the constraint equation is given explicitly by

$$t_0 = \widetilde{t}_0 - \frac{\lambda_0}{2} \int \frac{d^3 k}{(2\pi)^3} \Delta(k) + N^{-1} \left[\frac{\lambda_0}{4} A(\widetilde{t}_0, \lambda_0) - \Sigma(0; \widetilde{t}_0, \lambda_0) \right.$$

$$\left. \times [1 + \lambda_0 \Pi(0)] \right] + O(N^{-2}). \tag{33}$$

We shall need the four-point function $\Gamma^{(4)}(p_i)$ evaluated at $p_i = 0$. For this case, the multiloop integrals shown in Fig. 4 can be simplified by using the results

$$\int \frac{d^3 k'}{(2\pi)^3} \Delta(k')^2 \Delta(k+k') = -\frac{d}{d\widetilde{t}_0} \Pi(k) = \frac{2a\widetilde{t}_0^{-1/2}}{k^2 + 4\widetilde{t}_0}, \tag{34}$$

$$\int \frac{d^3 k'}{(2\pi)^3} [\Delta(k')^2 \Delta(k+k')^2 + 2\Delta(k')^3 \Delta(k+k')]$$

$$= \frac{d^2}{d\widetilde{t}_0^2} \Pi(k)$$

$$= a\widetilde{t}_0^{-3/2} \left[\frac{1}{k^2 + 4\widetilde{t}_0} + \frac{8\widetilde{t}_0}{(k^2 + \widetilde{t}_0)^2} \right] \tag{35}$$

together with the value of δ_0 given in Eq. (32). We then find

$$\Gamma^{(4)}(0) = N^{-1} D(0) + N^{-2} [B_1(\widetilde{t}_0, \lambda_0) + B_2(\widetilde{t}_0, \lambda_0)$$

$$+ B_3(\widetilde{t}_0, \lambda_0)] + O(N^{-3}), \tag{36}$$

where the remaining integrals are

$$B_1(\widetilde{t}_0, \lambda_0) = \frac{1}{4} a\widetilde{t}_0^{-1/2} D(0)^2 \int \frac{d^3 k}{(2\pi)^3} D(k) \left[\frac{8}{(k^2 + 4\widetilde{t}_0^2)^2} \right.$$

$$\left. - \frac{3}{(k^2 + \widetilde{t}_0)(k^2 + 4\widetilde{t}_0)} \right], \tag{37}$$

$$B_2(\widetilde{t}_0, \lambda_0) = -D(0) \int \frac{d^3 k}{(2\pi)^3} D(k) \frac{1}{(k^2 + \widetilde{t}_0^2)^2}, \tag{38}$$

$$B_3(\widetilde{t}_0, \lambda_0) = -\int \frac{d^3 k}{(2\pi)^3} D(k)^2 \left[\frac{1}{k^2 + \widetilde{t}_0} \right.$$

$$\left. - a\widetilde{t}_0^{-1/2} D(0) \frac{1}{k^2 + 4\widetilde{t}_0} \right]^2, \tag{39}$$

with $D(0) = (\lambda_0^{-1} + a\widetilde{t}_0^{-1/2})^{-1}$.

III. RENORMALIZATION GROUP AT LEADING ORDER

In the limit $N \to \infty$, the one-particle-irreducible four-point function $\Gamma^{(4)}(p_i)$ vanishes, and so do the higher multipoint functions. The theory is therefore effectively Gaussian, all correlations being determined by the two-point function $G^{(2)}(p) = (p^2 + \widetilde{t}_0)^{-1}$. The analysis of critical-point behavior now reduces essentially to determining the dependence of $\widetilde{t}_0$ on the temperaturelike variable t_0 from the constraint Eq. (33). This equation can be written as

$$t_0 - t_{0c} = \widetilde{t}_0 + 2a\lambda_0 \widetilde{t}_0^{1/2}, \tag{40}$$

where t_{0c} is the critical value of t_0, corresponding to $\widetilde{t}_0 = 0$. It represents fluctuation corrections to the mean-field transition temperature and is given formally by the divergent integral

$$t_{0c} = -\frac{\lambda_0}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2}. \tag{41}$$

Near the critical temperature, the leading behavior of $\tilde{t}_0$ is clearly $\tilde{t}_0 \propto (t_0 - t_{0c})^\gamma$, with $\gamma = 2$. According to standard renormalization-group ideas, we should be able to find a fixed-point value of the coupling strength, λ_0^*, for which this power law is exact, and this value is clearly $\lambda_0^* = \infty$. Within a suitable renormalization scheme, we might expect this fixed point to correspond to a finite value of a renormalized coupling λ. The usual motivations for renormalization, namely, the removal of ultraviolet divergencies or the exponentiation of logarithms of $t_0 - t_{0c}$ are absent here, because the constraint equation and correlation functions are finite, and contain no logarithms. Nevertheless, let us define a renormalized temperature variable t, a renormalized coupling λ, and a renormalized inverse susceptibility $\tilde{t}$ by

$$\lambda_0 = \mu Z_\lambda(\lambda)\lambda, \tag{42}$$

$$t_0 = t_{0c} + \mu^2 Z_t(\lambda) t, \tag{43}$$

$$\tilde{t}_0 = \mu^2 \tilde{t}, \tag{44}$$

where, as usual, μ is an arbitrary renormalization scale which serves to make t, $\tilde{t}$, and λ dimensionless. We specify the renormalization factors Z_t and Z_λ by the normalization conditions

$$\Gamma^{(2)}(p^2 = 0, t = 1) = \mu^2, \tag{45}$$

$$\lim_{N \to \infty} N\Gamma^{(4)}(\boldsymbol{p}_i = \boldsymbol{0}, t = 1) = \mu\lambda. \tag{46}$$

In practice, our Feynman rules yield correlation functions as functions of $\tilde{t}$ rather than t, and these conditions are more conveniently expressed as the requirements that $t_0 - t_{0c} = Z_t \mu^2$ and $D(0) = \mu\lambda$ when $\tilde{t} = 1$. We find

$$Z_\lambda(\lambda) = (1 - a\lambda)^{-1}, \tag{47}$$

$$Z_t(\lambda) = \frac{1 + a\lambda}{1 - a\lambda}. \tag{48}$$

Expressed in terms of λ and t, the constraint Eq. (40) is

$$t = (1 + a\lambda)^{-1}[(1 - a\lambda)\tilde{t} + 2a\lambda\tilde{t}^{1/2}]. \tag{49}$$

Clearly, on setting $\lambda = \lambda^* = a^{-1}$, we do find the exact power-law behavior $t = \tilde{t}^{1/\gamma}$. More generally, using the scaling fields $z = \lambda^{-1} - a$ and $\tau = (1 + z/2a)t$, we can express the inverse susceptibility $\tilde{t}$ in the scaling form

$$\tilde{t} = \tau^\gamma \mathcal{T}(x), \tag{50}$$

where the scaling variable is $x = z\tau^{\omega\nu}/2a$ with $\omega\nu = 1$ and the universal scaling function is given by

$$\mathcal{T}(x) = (2x)^{-2}[1 + 2x - \sqrt{1 + 4x}] = 1 - 2x + 5x^2 + O(x^3). \tag{51}$$

Formally, the scaling relation (50) and the values of the exponents can be deduced from the renormalization-group equation, which we find convenient to formulate in terms of

the variables $\tilde{t}$ and z. Typically, given a thermodynamic quantity $A(\lambda_0, \tilde{t}_0)$, its renormalized counterpart $A_R(z, \tilde{t})$ is given by

$$A(\lambda_0, \tilde{t}_0) = Z_A(z)\mu^{D_A} A_R(z, \tilde{t}), \tag{52}$$

where D_A is the canonical dimension of A. The criterion for determining the renormalization factor $Z_A(z)$ will be discussed in the next section. The fact that $A(\lambda_0, \tilde{t}_0)$ is independent of the renormalization scale μ is expressed by the renormalization-group equation

$$\left[\beta(z)\frac{\partial}{\partial z} - (2 - \eta(z))\tilde{t}\frac{\partial}{\partial \tilde{t}} + D_A + \sigma_A(z)\right] A_R(z, \tilde{t}) = 0, \tag{53}$$

where

$$\beta(z) = \mu\left(\frac{\partial z}{\partial \mu}\right)_{\lambda_0, \tilde{t}_0}, \tag{54}$$

$$2 - \eta(z) = -\frac{\mu}{\tilde{t}}\left(\frac{\partial \tilde{t}}{\partial \mu}\right)_{\lambda_0, \tilde{t}_0}, \tag{55}$$

$$\sigma_A(z) = \beta(z)\frac{d\ln Z_A(z)}{dz}. \tag{56}$$

Solution of this equation by the method of characteristics yields the relation

$$A_R(z, \tilde{t}) = P_A(l) A_R(z(l), \tilde{t}(l)), \tag{57}$$

where l is an arbitrary number and $z(l)$ is the solution of

$$l\frac{dz(l)}{dl} = \beta(z(l)) \tag{58}$$

with the initial condition $z(1) = z$, while $\tilde{t}(l)$ and the prefactor $P_A(l)$ are given by

$$\tilde{t}(l) = \tilde{t}l^{-2}\exp\left[\int_1^l \frac{dl'}{l'}\eta(z(l'))\right], \tag{59}$$

$$P_A(l) = \exp\left\{\int_1^l \frac{dl'}{l'}[D_A + \sigma_A(z(l'))]\right\}. \tag{60}$$

In the case at hand, taking $A = t_0 - t_{0c}$, we find $\beta(z) = z$, $\eta(z) = 0$, $D_t = 2$, and $\sigma_t(z) = -2a/(z + 2a)$. The characteristic functions are

$$z(l) = zl, \quad \tilde{t}(l) = \tilde{t}l^{-2}, \quad P_t(l) = l\left(\frac{zl + 2a}{z + 2a}\right), \tag{61}$$

and it is easy to verify by substitution into Eq. (49) that the relation (57) is satisfied. To obtain the scaling form (50), we can choose the arbitrary parameter l as the solution of the equation $\tilde{t}(l) = 1$. The neighborhood of the critical point $\tilde{t}$

$=0$ then corresponds [assuming that $\eta(z)<2$] to the limit $l \to 0$. In this limit, we will generally expect that

$$z(l)\sim zl^{\omega}, \qquad (62)$$

$$\tilde{t}(l)\sim \tilde{t}l^{-(2-\eta)}, \qquad (63)$$

$$P_t(l)\sim l^{1/\nu}, \qquad (64)$$

where $\omega=\beta'(0)$, $\eta=\eta(0)$, and $\nu=[2+\sigma_t(0)]^{-1}$ are the usual critical exponents. (Here, of course, we have $\omega=1$, $\eta=0$, and $\nu=1$.) On setting $l\sim\tilde{t}^{1/(2-\eta)}$, the relation (57) becomes

$$t(z,\tilde{t})\sim\tilde{t}^{1/\gamma}t(z\tilde{t}^{\omega\nu/\gamma},1), \qquad (65)$$

with $\gamma=(2-\eta)\nu$, which can be inverted to express $\tilde{t}$ in the scaling form (50).

IV. EXTENSION TO HIGHER ORDERS

It is well known that the exponents obtained in the previous section are modified when N is finite, and can be expressed as power series in $1/N$. The series for correlation functions contain infrared singularities, in the form of logarithms of $\tilde{t}_0$, which must be exponentiated to obtain the correct power laws, and our aim is to find a renormalization prescription that will effect this exponentiation. Experience with perturbation theory and the ϵ expansion suggests that application of normalization conditions such as Eqs. (45) and (46) should achieve this, but a strategy of this kind is unsatisfactory for two reasons. At a purely practical level, one obtains renormalization-group functions $\beta(z)$, etc., that are cumbersome (and, indeed, singular) functions of z. More fundamentally, we have, so far, no reason within the $1/N$ expansion to expect that a renormalization scheme of this kind really will produce the correct exponentiation. We should therefore consider just what kind of renormalization scheme is needed.

A simple criterion becomes apparent, if we assume that the renormalization-group analysis will produce relations of the kind exhibited in Eq. (57). On the left-hand side, the function $A(z,\tilde{t})$ has infrared singularities when its second argument, $\tilde{t}$, approaches zero. On the right-hand side, these singularities are removed by the condition $\tilde{t}(l)=1$, but there remains the danger that they might reappear through the limit $z(l)\to 0$. Evidently, the singularities will be correctly exponentiated into the prefactor $P_A(l)$, provided that $A(z(l),1)$ remains finite and nonzero in the limit $z(l)\to 0$. The primary requirement of a renormalization scheme is to ensure that this is so.

To see how a renormalization scheme might work, let us examine the divergences that might occur in the unrenormalized theory when $\tilde{t}_0$ is fixed to a nonzero value. The Feynman diagrams generated by the $1/N$ expansion are topologically similar to those in a theory of two fields ϕ and ψ, with propagators $\Delta(\boldsymbol{p})=(p^2+\tilde{t}_0)^{-1}$ and $D(\boldsymbol{p})=[z_0+\Pi(\boldsymbol{p})]^{-1}$, respectively, and a single interaction vertex corresponding to

$\phi^2\psi$. Here, we use the notation $z_0=\lambda_0^{-1}$. With $\tilde{t}_0$ fixed, there are no infrared divergences. In the ultraviolet regime $|\boldsymbol{p}| \to\infty$, we have $\Delta(\boldsymbol{p})\sim 1/p^2$ and, provided that $z_0\neq 0$, $D(\boldsymbol{p})\sim$const. Consider, then, a subintegral that is one-particle irreducible with respect to both propagators, that attaches to m external ϕ legs and n external ψ legs, and has L loop integrations. It is simple to show that the superficial degree of ultraviolet divergence of this integral is

$$\Omega_{\mathrm{uv}}(z_0\neq 0)=4-L-m-2n. \qquad (66)$$

There are in fact only two diagrams that diverge, namely, those with $L=1$ and $(m,n)=(0,1)$ or $(m,n)=(2,0)$. These divergences are subtracted by the constraint $\langle\psi\rangle=0$ and by the additive mass renormalization corresponding to t_{0c}. The remaining theory is ultraviolet finite in three dimensions. When $z_0=0$, however, the limiting behavior of the ψ propagator is $D(\boldsymbol{p})\sim|\boldsymbol{p}|$. In that case, we have

$$\Omega_{\mathrm{uv}}(z_0=0)=3-2n-\frac{m}{2}. \qquad (67)$$

There are now additional divergences which will appear as singularities at $z_0=0$, and these must be removed by further renormalization. Unfortunately, the freedom that we have to rescale the original field ϕ and the parameters t_0-t_{0c} and z_0 does not yield a set of independent counterterms corresponding to the divergent subintegrals. Indeed, we can offer no direct proof that this rescaling will suffice to remove all the divergences. An indirect (and admittedly heuristic) argument suggesting that all the divergences can nevertheless be removed is afforded by the observation that the limit $z\to 0$ is equivalent (with $t_{0c}\sim-\mathrm{const}\times\lambda_0$) to a limit $\lambda_0\to\infty$, $t_0\to -\infty$, in which the model studied here should be equivalent to the nonlinear σ model whose renormalizability to all orders of the $1/N$ expansion is proven in Ref. 14.

Here, we adopt the following pragmatic approach. First, we introduce a wave-function renormalization factor $Z_\phi(z)$, so that correlation functions $\Gamma^{(n)}$ are renormalized according to

$$\Gamma^{(n)}(\boldsymbol{p}_i;\lambda_0,t_0)=Z_\phi(z)^{-n/2}\Gamma_R^{(n)}(\boldsymbol{p}_i;z,\tilde{t},\mu). \qquad (68)$$

(These $\Gamma^{(n)}$ are defined as usual by Legendre transformation of the generating functional $\ln Z[j_i,j_i^*]$ and would be one-particle irreducible when calculated in perturbation theory.) The definition (13) then implies that $\tilde{t}_0$ is renormalized according to

$$\tilde{t}_0=Z_\phi^{-1}(z)\mu^2\tilde{t} \qquad (69)$$

and we introduce renormalized parameters z and t defined by

$$z_0=Z_z(z)\mu^{-1}z, \qquad (70)$$

$$t_0=t_{0c}+\mu^2\frac{(z+2a)}{z}Z_t(z)t. \qquad (71)$$

The renormalization factors $Z_\phi(z)$, $Z_z(z)$, and $Z_t(z)$ must be chosen so as to make the two renormalized correlation functions $\Gamma_R^{(2)}(\boldsymbol{p}_i=0)$ and $\Gamma^{(4)}(\boldsymbol{p}=0)$ and the constraint equation

for the renormalized temperature variable t finite when $z \to 0$ with $\tilde{t}$ fixed to some nonzero value. As usual, there are many ways in which this might be achieved. Here, we propose a "minimal subtraction" scheme which, however, may well be susceptible to further refinement for specific purposes. At relative order $1/N$, we show in the Appendix that the singular parts of the relevant Feynman diagrams are of the form $(\ln z) \times$(infinite power series in z). The minimal way of ensuring finite limits as $z \to 0$ is to subtract just the leading singular terms, proportional to $\ln z$. Proceeding in this way, we obtain for the renormalization factors

$$Z_\phi(z) = 1 + \frac{1}{N}\frac{1}{6b}S_3 \ln z + O(N^{-2}), \tag{72}$$

$$Z_t(z) = 1 - \frac{1}{N}\frac{2}{3b}S_3 \ln z + O(N^{-2}), \tag{73}$$

$$Z_z(z) = 1 + \frac{1}{N}\frac{4}{3b}S_3 \ln z + O(N^{-2}) \tag{74}$$

and hence for the renormalization-group functions

$$\beta(z) = \omega z + O(N^{-2}), \tag{75}$$

$$\eta(z) = \eta + O(N^{-2}), \tag{76}$$

$$2 + \sigma_t(z) = \frac{1}{\nu} + \frac{\omega z}{z + 2a} + O(N^{-2}), \tag{77}$$

where the exponents are given by

$$\omega = 1 - \frac{1}{N}\frac{4}{3b}S_3 + O(N^{-2}) = 1 - \frac{32}{3\pi^2 N} + O(N^{-2}), \tag{78}$$

$$\eta = \frac{1}{N}\frac{1}{6b}S_3 + O(N^{-2}) = \frac{4}{3\pi^2 N} + O(N^{-2}), \tag{79}$$

$$\nu = 1 - \frac{1}{N}\frac{2}{3b}S_3 + O(N^{-2}) = 1 - \frac{16}{3\pi^2 N} + O(N^{-2}) \tag{80}$$

in agreement with standard results (bearing in mind that our N complex fields correspond to $2N$ real fields). The characteristic functions that were given at leading order by Eq. (61) become

$$z(l) = zl^\omega, \quad \tilde{t}(l) = \tilde{t}l^{-(2-\eta)}, \quad P_t(l) = l^{1/\nu}\left(\frac{zl^\omega + 2a}{z + 2a}\right). \tag{81}$$

It is at least of formal interest to obtain the scaling form of the inverse susceptibility (50) at next-to-leading order for the following reason. At leading order, the scaling function (51) has a simple series expansion in powers of $x \sim z\tau^{\omega\nu}$. In perturbative realizations of the renormalization group, whether formulated as a systematic expansion in $\epsilon = 4 - d$ or directly in $d = 3$ dimensions, the corresponding corrections to the

leading scaling behavior automatically appear as a power series in the scaling variable $(\lambda - \lambda^*)t^{\omega\nu}$. These corrections have been studied in great detail, for example, in Refs. 15–17. In the $1/N$ expansion studied here, this is not so. After the minimal renormalization that removes terms proportional to $\ln z$ from the correlation functions, there remain weaker singularities of the form $z^n \ln z$, which will give rise to corrections of the form $x^n \ln x$. Keeping only the leading term, proportional to $z \ln z$, the renormalized version of the constraint Eq. (33) is

$$t(z,\tilde{t}) = \frac{1}{z + 2a}\left[2a\tilde{t}^{1/2}\left(1 + \frac{c'}{N} + \cdots\right)\right.$$
$$\left. + z\tilde{t}\left(1 + \frac{c}{N}\ln z + \cdots\right)\right], \tag{82}$$

where

$$c = \left(1 - \frac{8a^2}{b^2}\right)\frac{S_3}{2b} = \frac{4}{\pi^2}\left(1 - \frac{8}{\pi^2}\right), \tag{83}$$

and c' is a number which must be evaluated numerically, though its exact value is immaterial for our present purposes. The ellipses in Eq. (82) indicate both terms of higher order in N^{-1} and higher powers of z. To obtain the scaling form of the inverse susceptibility $\tilde{t}$, we express the right-hand side of Eq. (82) in the form of Eq. (57) using the explicit characteristic functions (81) and choose $l = \tilde{t}^{1/(2-\eta)}$ to obtain

$$t = \frac{\tilde{t}^{1/\gamma}}{z + 2a}\left[2a\left(1 + \frac{c'}{N} + \cdots\right) + z\tilde{t}^{\omega\nu/\gamma}\right.$$
$$\left. \times \left(1 + \frac{c}{N}\ln(z\tilde{t}^{\omega\nu/\gamma}) + \cdots\right)\right]. \tag{84}$$

On defining the scaling field τ and the scaling variable y by

$$\tau = \left(\frac{1 + z/2a}{1 + c'/N + \cdots}\right)t, \quad y = \frac{z\tau^{\omega\nu}}{2a(1 + c'/N + \cdots)}, \tag{85}$$

and the scaling function $\mathcal{T}(y)$ through $\tilde{t} = \tau^\gamma \mathcal{T}(y)$, we find that this scaling function is the solution of

$$\mathcal{T}^{1/\gamma}\left\{1 + y\mathcal{T}^{\omega\nu/\gamma}\left[1 + \frac{c}{N}\ln y + \frac{c}{N}\ln(2a\mathcal{T}^{\omega\nu/\gamma}) + \cdots\right]\right\} = 1. \tag{86}$$

For small y, it has the expansion

$$\mathcal{T}(y) = 1 - \gamma y\left(1 + \frac{c}{N}\ln y\right) + \cdots. \tag{87}$$

The presence of the term proportional to $y \ln y$ seems, to the best of our knowledge, to be a novel result, which is discussed further in Sec. VI.

$$\lambda_0 C = N \bigcirc\!\!\!\cdot\cdot - \lambda_0^{-1}\left[\tfrac{1}{2}\cdot\!\!\bigcirc\!\!\!\cdot\cdot + \cdot\!\cdot\bigcirc\!\cdot\cdot - 2\delta\cdot\!\!\bigcirc\!\!\cdot\cdot - \tfrac{1}{2}\cdot\!\!\bigcirc\!\!\bigcirc\!\!\cdot\cdot\right]$$

FIG. 5. Diagrammatic representation of the specific heat at next-to-leading order. In the first diagram, the cross indicates that the loop contains two ϕ propagators.

V. RENORMALIZATION OF THE SPECIFIC HEAT

The specific-heat exponent is given in three dimensions by

$$\alpha = 2 - 3\nu = -1 + \frac{16}{\pi^2 N} + O(N^{-2}). \tag{88}$$

As discovered long ago by Abe and Hikami,[18] the fact that $t^{-\alpha} = t + O(N^{-1})$ is associated with a kind of degeneracy. When exponents are obtained in the 1/N expansion by exponentiating logarithms of $\bar{t}_0 \equiv t_0 - t_{0c}$, it is essential to account correctly for a regular contribution proportional to $\bar{t}_0$. Specifically, writing

$$C(\bar{t}_0) = C_0 \bar{t}_0^{-\alpha} + C_1 \bar{t}_0, \tag{89}$$

it is argued in Ref. 18 that the coefficients C_0 and C_1, considered as functions of dimension d, have discontinuities at $d = 3$ that give rise to "anomalous" logarithms of $\bar{t}_0$ when the whole expression is expanded in powers of 1/N. Within the renormalization scheme proposed here, there are similar "anomalous" logarithms of z, which prevent the specific heat from being multiplicatively renormalized. To be concrete, we define

$$C(\lambda_0, \bar{t}_0) = N^{-1} \int d^3 r \langle \phi^2(\mathbf{r}) \phi^2(\mathbf{0}) \rangle, \tag{90}$$

where $\phi^2 = \Sigma_i \phi_i^* \phi_i$, and find that it is given to relative order 1/N by the sum of diagrams shown in Fig. 5. We will define a renormalized function $C_R(z, \tilde{t})$ by

$$C(\lambda_0, \bar{t}_0) = C(\lambda_0, 0) + C_1(\lambda_0)\bar{t}_0 + \mu^{-1}\bar{Z}_t^{-2}(z)C_R(z, \tilde{t}), \tag{91}$$

where $\bar{Z}_t(z) = (1 + 2a/z)Z_t(z)$ is the same renormalization factor that appears in Eq. (71). Dimensional analysis tells us that $C_1(\lambda_0) = C_1 \lambda_0^{-3}$, where C_1 is a number. Were we able to supply it, an analysis to all orders in 1/N of the divergences of the theory as $z \to 0$ would tell us whether an appropriate choice of this number (at each order in 1/N) is sufficient to make $C_R(0, \tilde{t})$ finite. This is beyond our present skill, but we have verified that the procedure does work (with $C_1 = -4/b^2$) at next-to-leading order. In perturbative renormalization schemes, as is well known, an additive renormalization is also required to make a quantity analogous to C_R finite in the limit $d \to 4$. In that case, the additive term depends on the renormalization scale μ and, in consequence, the renormalized specific heat obeys an inhomogeneous renormalization-group equation (see, for example, Ref. 13). Here, by contrast, the first two terms on the right-hand side of Eq. (91) are functions only of λ_0 and t_0, so C_R obeys a homogeneous equation of the form (53).

Finally, let us write Eq. (91) in the form

$$\bar{Z}_t^2[C(\lambda_0, \bar{t}_0) - C(\lambda_0, 0)] = \bar{C}_0 t^{-\alpha} + \bar{C}_1 t, \tag{92}$$

with $\bar{C}_0 = C_R(0, 1)$ and $\bar{C}_1 = C_1 \bar{Z}_t^3 \lambda_0^{-3}$. It is somewhat reassuring to find that the amplitude ratio can be written as

$$\frac{\bar{C}_0}{\bar{C}_1} = \frac{\pi^2}{4} - 1 + O(N^{-1}) \tag{93}$$

in agreement with that given in Ref. 18, though $\bar{C}_1$ does not, in general, have a well-defined limit as $z \to 0$.

VI. DISCUSSION

We have proposed a renormalization scheme within which critical exponents and scaling functions for an N-vector Ginzburg-Landau model can be estimated by means of a systematic expansion in 1/N. From the field-theoretic point of view, the essential function of the renormalization group is to relate the values of correlation functions in the critical region, where infrared singularities arise from the vanishing "mass" $t \sim T - T_c$, to their values when t is of order 1, as exhibited in Eq. (57). To ensure that critical singularities are correctly exponentiated, a renormalization scheme must ensure that these singularities do not reappear from the running of other parameters. In the context of the 1/N expansion, this potentially happens when the inverse quartic coupling constant $z \sim \lambda^{-1}$ vanishes, and the essential feature of our renormalization scheme is to ensure that renormalized correlation functions have finite limits when $z \to 0$. In three dimensions, the minimal way of achieving this is to subtract leading powers of $\ln z$, and this procedure does indeed serve to recover the standard results for critical exponents.

The main purpose of this paper is to explain how the scheme works, in preparation for a detailed investigation of the critical properties of high-temperature superconductors to be described in Ref. 12. However, in the course of the formal investigation reported here, we have encountered a general feature that seems to have been unsuspected hitherto. It has usually been taken for granted that the approach to a critical point can be described by expressing the Hamiltonian in the form $\mathcal{H} = \mathcal{H}^* + \Sigma_i g_i \mathcal{O}_i$, where $\mathcal{H}^*$ is an infrared-stable fixed-point Hamiltonian and the $\mathcal{O}_i$ are eigenoperators of the renormalization group at this fixed point (see, for example, Refs. 1 and 4). By expanding correlation functions in powers of the coefficients g_i, one expects to obtain corrections to the asymptotic critical singularities in the form of powers of the scaling variables $g_i \tau^{\Delta_i}$, with exponents Δ_i determined by the eigenvalues of the $\mathcal{O}_i$. In treatments based on perturbative expansions in the coupling constant λ, this expectation is realized automatically in the case of corrections of the form $(\lambda - \lambda^*)\tau^{\omega\nu}$ associated with departures of λ from its fixed-point value λ^*. Within the nonperturbative approxima-

tion scheme afforded by the $1/N$ expansion, however, this does not happen. We see explicitly that the fixed point corresponds to $z=\lambda^{-1}=0$, and that correlation functions do not possess power-series expansions in z. Although scaling is maintained, in the sense that corrections appear in terms of the scaling variable $y\propto z\tau^{\omega\nu}$, scaling functions such as that exhibited in Eq. (87) for the inverse susceptibility are not expressible as power series in y.

At relative order $1/N$, the nonanalyticity of the scaling functions is logarithmic, and this is probably true at higher orders also. These are not, however, the same as the well-known logarithmic corrections that occur at the upper critical dimension $d=4$ (Refs. 19, 20, and 13) where ϕ^4 is a marginal operator. The latter arise from a degeneracy in the renormalization-group equations which destroy the scaling property, yielding logarithms of τ rather than of a scaling variable such as y. In fact, our renormalization scheme is restricted to $d=3$; we do not know in detail how to formulate a similar scheme in general dimensions. Quite possibly, the singularities we encountered appear, like the Abe-Hikami specific-heat anomaly, only at special, rational values of d, and it is not clear whether they need be logarithmic in general. It is also far from clear whether they are of more than academic interest. In principle, they should presumably be present in, for example, the specific heat of ^{4}He near the lambda transition (corresponding to $N=1$), where rather precise measurements of critical properties have been made.[21,22] The data are reasonably consistent with the assumed power-law correction, but may well not be precise enough to detect a logarithmic factor. Unfortunately, while the nonperturbative nature of the $1/N$ expansion is helpful in indicating the presence of these logarithms, its notoriously poor convergence makes it hard to estimate the likely sizes of the coefficients that multiply them.

ACKNOWLEDGMENTS

APPENDIX: SINGULAR PARTS OF FEYNMAN INTEGRALS

To implement our renormalization scheme at next-to-leading order in the $1/N$ expansion, we need to know the singularities of the integrals (29) and (37)–(39) near $z=0$. Each of them can be expressed in terms of integrals of the form

$$I_K=\int\frac{d^3k}{(2\pi)^3}\left[z+\frac{b}{k}-\frac{m}{k^2}\widetilde{\Pi}\left(\frac{m^2}{k^2}\right)\right]^{-1}\frac{1}{k^4}K\left(\frac{m^2}{k^2}\right),$$
(A1)

or dI_K/dz, where $k=|\boldsymbol{k}|$ and $K(m^2/k^2)$ is proportional to k^4 as $k\to0$, but approaches a finite value as $k\to\infty$. The singularities arise from the region of integration where k is large. To evaluate them, we write

$$I_K=S_3\int_1^\infty\frac{dk}{k(zk+b)}\left[1-\frac{(m/k)\widetilde{\Pi}(m^2/k^2)}{zk+b}\right]^{-1}K\left(\frac{m^2}{k^2}\right)+\text{reg},$$
(A2)

where $S_3=1/2\pi^2$ and "reg" denotes contributions that are regular when $z\to0$, arising here from the integration region $0\leqslant k\leqslant1$. Defining

$$f^{(l)}\left(\frac{m}{k}\right)=\left[\left(\frac{m}{k}\right)^2\widetilde{\Pi}\left(\frac{m^2}{k^2}\right)\right]^l K\left(\frac{m^2}{k^2}\right),$$
(A3)

and using the series expansion $f^{(l)}(x)=\Sigma_n f_n^{(l)}(x)$, we have

$$I_K=\sum_{l,n}f_n^{(l)}I_{l,n}+\text{reg},$$
(A4)

where

$$I_{l,n}=S_3\int_1^\infty\frac{dk}{k(zk+b)^{l+1}}\left(\frac{m}{k}\right)^{n-l}=\frac{1}{l!}\left(-\frac{1}{m}\frac{\partial}{\partial z}\right)^l I_n.$$
(A5)

$$I_n=S_3\int_1^\infty\frac{dk}{k(zk+b)}\left(\frac{m}{k}\right)^n.$$
(A6)

The integral I_0 is

$$I_0=S_3\int_1^\infty\frac{dk}{k(zk+b)}=-\frac{S_3}{b}\ln z+\text{reg}$$
(A7)

and for $n\geqslant1$, the recursion relation

$$I_n=\frac{S_3 m}{b}\int_1^\infty\frac{dk}{k}\left(\frac{1}{k}-\frac{z}{zk+b}\right)\left(\frac{m}{k}\right)^{n-1}=\frac{S_3 m^n}{nb}-\frac{mz}{b}I_{n-1}$$
(A8)

is readily solved to yield

$$I_n=\frac{S_3 m^n}{b}\sum_{j=0}^{n-1}\frac{(-1)^j}{(n-j)}\left(\frac{z}{b}\right)^{n-j}+\left(-\frac{mz}{b}\right)^n I_0$$

$$=-\frac{S_3}{b}\left(-\frac{mz}{b}\right)^n\ln z+\text{reg}.$$
(A9)

With these results in hand, we can evaluate I_K as

$$I_K=-\frac{S_3}{b}\sum_{l,n}\frac{1}{l!}f_n^{(l)}\left(-\frac{1}{m}\frac{\partial}{\partial z}\right)^l\left[\left(-\frac{mz}{b}\right)^n\ln z\right]+\text{reg}$$

$$=-\frac{S_3}{b}\ln z\sum_l\frac{1}{l!}\left(-\frac{1}{m}\frac{\partial}{\partial z}\right)^l f^{(l)}\left(-\frac{mz}{b}\right)+\text{reg}$$

$$=-\frac{S_3}{b}\ln z\sum_l\frac{1}{l!}\left(-\frac{1}{m}\frac{\partial}{\partial z}\right)^l\left\{\left[\left(\frac{mz}{b}\right)^2\widetilde{\Pi}\left(\frac{m^2z^2}{b^2}\right)\right]^l\right.$$

$$\left.\times K\left(\frac{m^2z^2}{b^2}\right)\right\}+\text{reg}$$

$$= -\frac{S_3}{b}\ln z \int_{-\infty}^{\infty} dq\, K(q^2) \sum_l \frac{1}{l!}\left[\frac{q^2\Pi(q^2)}{b}\right]^l \left(\frac{\partial}{\partial q}\right)^l$$

$$\times \delta\left(q - \frac{mz}{b}\right) + \text{reg}$$

$$= -\frac{S_3}{b}\ln z \int_{-\infty}^{\infty} dq\, K(q^2)\,\delta\left(q - \frac{mz}{b} + \frac{1}{b}q^2\tilde{\Pi}(q^2)\right) + \text{reg}.$$
(A10)

The factor of $\ln z$ can be extracted because $(\partial/\partial z)^l(z^p \ln z) = \ln z(\partial^l z^p/\partial z^l) + \text{reg}$ when $p \geq 2l$. Carrying out the q integral, we finally obtain

$$I_K^{\text{sing}} = -\frac{S_3 \ln z}{b} K(Q^2)\left\{1 + \frac{1}{b}\frac{d}{dQ}[Q^2\tilde{\Pi}(Q^2)]\right\}^{-1},$$
(A11)

where $Q(z)$ is the solution of

$$bQ + Q^2\tilde{\Pi}(Q^2) = mz.$$
(A12)

Solving for Q as a power series in z, we obtain

$$Q(z) = \frac{mz}{b}\left[1 - \frac{4a}{b^2}mz + \frac{32a^2}{b^4}(mz^2) + O((mz)^3)\right],$$
(A13)

$$I_K^{\text{sing}} = -\frac{S_3}{b}\ln z\left[1 - \frac{8a}{b^2}mz + \frac{96a^2}{b^4}(mz)^2\right.$$

$$\left. + O((mz)^3)\right]K(Q^2).$$
(A14)

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[1] M.E. Fisher, Rev. Mod. Phys. **46**, 587 (1974).

[2] *Phase Transitions and Critical Phenomena*, edited by C. Domb and M.S. Green Vol. 6 (Academic Press, London, 1976).

[3] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, London, 1989).

[4] M.E. Fisher, Rev. Mod. Phys. **70**, 653 (1998).

[5] K.G. Wilson and M.E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).

[6] G. Parisi, J. Stat. Phys. **23**, 49 (1980).

[7] R. Abe, Prog. Theor. Phys. **48**, 1414 (1972).

[8] S.K. Ma, in Ref. 2, p. 249.

[9] I. Kondor and T. Temesvári, Phys. Rev. B **21**, 260 (1980).

[10] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).

[11] N. Overend, M.A. Howson, and I.D. Lawrie, Phys. Rev. Lett. **72**, 3238 (1994).

[12] D.J. Lee and I.D. Lawrie, following paper, Phys. Rev. B **64**, 184506 (2001).

[13] E. Brézin, J.C. le Guillou, and J. Zinn-Justin, in Ref. 2 , p. 125.

[14] I.Ya. Aref'eva, E.R. Nissimov, and S.J. Pacheva, Commun. Math. Phys. **71**, 213 (1980).

[15] C. Bagnuls and C. Bervillier, Phys. Rev. B **32**, 7209 (1985).

[16] C. Bagnuls, C. Bervillier, D.I. Meiron, and B.G. Nickel, Phys. Rev. B **3585**, 3385 (1987).

[17] R. Schloms and V. Dohm, Phys. Rev. B **42**, 6142 (1990).

[18] R. Abe and S. Hikami, Prog. Theor. Phys. **51**, 1041 (1974).

[19] A.I. Larkin and D.E. Khmel'nitskii, Zh. Éksp. Teor. Fiz. **56**, 2087 (1969) [Sov. Phys. JETP **29**, 1123 (1969)].

[20] F.J. Wegner and E. Riedel, Phys. Rev. B **7**, 248 (1973).

[21] J.A. Lipa and T.C.P. Chui, Phys. Rev. Lett. **51**, 2291 (1983).

[22] J.A. Lipa, D.R. Swanson, J.A. Nissen, T.C.P. Chui, and U.E. Israelsson, Phys. Rev. Lett. **76**, 944 (1996).