# Critical behavior of two-dimensional frustrated spin models with noncollinear order

Pasquale Calabrese[1,*] and Pietro Parruccini[2,†]
[1]Scuola Normale Superiore and INFN, Piazza dei Cavalieri 7, I-56126 Pisa, Italy
[2]Dipartimento di Fisica dell'Università di Pisa and INFN, Via Buonarroti 2, I-56127 Pisa, Italy
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We study the critical behavior of frustrated spin models with noncollinear order in two dimensions, including antiferromagnets on a triangular lattice and fully frustrated antiferromagnets. For this purpose we consider the corresponding $O(N)\times O(2)$ Landau-Ginzburg-Wilson (LGW) Hamiltonian and compute the field-theoretic expansion to four loops and determine its large-order behavior. We show the existence of a stable fixed point for the physically relevant cases of two- and three-component spin models. We also give a prediction for the critical exponent $\eta$ which is $\eta=0.24(6)$ and $\eta=0.29(5)$ for $N=3$ and 2, respectively.

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## I. INTRODUCTION

The critical behavior of frustrated spin systems with noncollinear or canted order has been the object of intensive theoretical and experimental studies. An important and debated issue is the nature of the universality of the phase transition (see, e.g., Refs. 1 and 2 for reviews on this issue in the three-dimensional case).

In physical magnets, noncollinear order is due to frustration that may arise either because of the special geometry of the lattice, or from the competition of different kinds of interactions. Typical examples of systems of the first type are two-dimensional triangular antiferromagnets (AFT), where magnetic ions are located at each site of a two-dimensional triangular lattice. At the chiral transition, they can be described by using short-ranged Hamiltonians for $N$-component spin variables defined on a triangular lattice as

$$\mathcal{H}_{\mathrm{AFT}}=-J\sum_{\langle ij\rangle}\vec{s}_i\cdot\vec{s}_j, \tag{1}$$

where $J<0$ and the sum is over nearest-neighbor pairs.

In these spin systems the Hamiltonian is minimized by noncollinear configurations, showing a 120° spin structure. Frustration is partially released by mutual spin canting, and the degeneracy of the ground state is limited to global O($N$) spin rotations and reflections. As a consequence, at criticality there is a breakdown of the symmetry from O($N$) in the high-temperature phase to O($N$-2) in the low-temperature phase, implying a matrixlike order parameter. Frustration due to interactions may be realized in the frustrated $XY$ model on a square lattice, governed by the Hamiltonian:

$$\mathcal{H}_{\mathrm{FXY}}=-J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j-A_{ij}), \tag{2}$$

where $\theta_i$ is the angle of $\vec{s}_i$ with a datum direction, $A_{ij}$ is the quenched vector potential, and the sum is over nearest-neighbor pairs. The frustration is determined by the sum of $A_{ij}$ around a plaquette: in the fully frustrated case (FF$XY$) this sum is equal to $\pi$. The large number of studies on this Hamiltonian is mainly motivated by the fact that it describes the statistical properties of a superconducting Josephson junction in a transverse magnetic field.

Field theoretical (FT) studies of systems with noncollinear order are based on the O($N$)×O($M$) symmetric Hamiltonian[1,3]

$$\mathcal{H}=\int d^dx\left\{\frac{1}{2}\sum_a\left[(\partial_\mu\phi_a)^2+r\phi_a^2\right]+\frac{1}{4!}u_0\left(\sum_a\phi_a^2\right)^2+\frac{1}{4!}v_0\sum_{a,b}\left[(\phi_a\cdot\phi_b)^2-\phi_a^2\phi_b^2\right]\right\}, \tag{3}$$

where $\phi_a$ $(1\leqslant a\leqslant M)$ are $M$ sets of $N$-component vectors. We will consider the case $M=2$, that, for $v_0>0$, describes frustrated systems with noncollinear ordering such as AFT's. Negative values of $v_0$ correspond to simple ferromagnetic or antiferromagnetic ordering, and to magnets with sinusoidal spin structures.[3] The same LGW Hamiltonian has also been used in other problems such as that of the phase transition of the dipole-locked A phase of helium three.[4]

Even if the critical behavior of the noncollinear two-dimensional $XY$ frustrated systems has been the subject of many recent theoretical and experimental[5] studies, there is still no definite conclusion about the nature of the phase transitions that occur in these systems. It can be shown that these models possess a twofold degenerate ground state according to the chiral degeneracy;[6,1] chirality is invariant under global spin rotation $SO(2)$, while it changes sign under global spin reflection $Z_2$. This situation leads to having an order parameter space (defined as a topological space isomorphic to the set of ordered states[4]) $V=Z_2\times S_1\equiv Z_2\times SO(2)$ different from the one of the normal $XY$ model.

Monte Carlo simulations performed using the AFT model,[7,8,6] the FF$XY$ model,[9–18] or other related models (some exotic realizations of the frustrated $XY$ model are the Coulomb gas with half integer charge,[19,20] the coupled $XY$-Ising model,[21–23] the 19-vertex version,[24,25] the frustrated $XY$ model with zig-zag coupling,[26] the frustrated model with next-nearest-neighbor interaction,[27,28] and the $XXZ$ model[29–31] for particular values of certain parameters) have partially clarified the nature of the transition. A strongly debated question is whether there is only one critical temperature $T_c$, in which both the $SO(2)$ and the $Z_2$ symmetries

 

TABLE I. Monte Carlo results for frustrated $XY$ system. $n^o$ is the number of transitions observed in the simulation. When two transitions are detected the subscripts 1 and 2 are related to the first and to the second one. Where different exponents are reported we use scaling laws to obtain $\nu$ and $\eta$.

| Model (Ref.) | $n^o$ | Exponents |
|---|---|---|
| FF$XY$[9] | | $\alpha=0$ |
| AFT[7] | 1 | $\beta=0.123(3), \gamma=1.73(5)$ |
| AFT[6] | 2 | $\alpha_1=0, \eta_2=1/4$ |
| AFT[8] | 2 | $\nu_1=0.9(2), \eta_1=0.24(11)$ |
| FF$XY$[11] | 2 | $\nu_1=0.898(3), \eta_1=0.216,$ $\sigma_2=0.3069, \eta_2=0.1915$ |
| FF$XY$[14] | 2 | $\nu_1=1$ |
| CG[19] | 1 | |
| CG[20] | 2 | $\nu_1=0.84(5), \eta_1=0.26(4)$ |
| $XY$ Ising[21] | 1 | $\nu=0.82, \eta=0.35$ |
| $XY$ Ising[22] | 1 | $\nu=0.84, \eta=0.30$ |
| FF$XY$[13] | 1 | $\nu=0.80(4), \eta=0.38(3)$ |
| FF$XY$[10] | 1 | $\nu=0.85(3), \eta=0.31(3)$ |
| FF$XY$ tr[10] | 1 | $\nu=0.83(4), \eta=0.28(4)$ |
| FF$XY$[17] | 1 | $\nu=0.852(2), \eta=0.20(1)$ |
| FF$XY$ tr[16] | 2 | $\nu_1=0.83(1), \eta_1=0.25(2)$ |
| $XXZ$[29] | 2 | consistent with Ising |
| FF$XY$[18] | 1 | |
| FF$XY$[12] | 1 | c=1.66(4), $\nu\sim1$, $\eta=0.40(4)$ |
| F$XY$ $zz$[26] ($\rho=1.5$) | 1 | $\nu=0.80(1), \eta=0.29(2)$ |
| F$XY$ $zz$[26] ($\rho=0.7$) | 1 | $\nu=0.78(2), \eta=0.32(4)$ |

are simultaneously broken, or there are two successive phase transitions at critical temperatures $T_{SO(2)}$ and $T_{Z_2}$. In the case of two transitions, even the order in which they occur and the numerical values of the critical exponents are very controversial (in many papers[14,7,29] it is claimed that the higher temperature transition is a standard Ising one). The main problem is that, if two transitions exist, they have very close critical temperatures and a precise detection of them by a Monte Carlo simulation is very difficult, in fact all the authors that observe a single transition do not exclude the possibility of two very close ones [e.g., a recent upper limit for the difference of the critical temperatures is $\Delta T\sim5\times10^3$ (Ref. 18)]. We summarize all Monte Carlo results in Table I while in Table II we report the results of other kinds of works. At last we want to mention that in some studies[32] it has been argued that the FF$XY$ model is equivalent to a

TABLE II. Other results for frustrated $XY$ systems.

| Model (Ref.) | Method | $n^o$ | Exponents |
|---|---|---|---|
| FF$XY$[15] | pos. space RG | 2 | first nonconsistent with Ising second consistent with KT |
| 19$v$[24] | transfer matrix | 1 | $\eta=0.28(2), \nu=0.81(3)$ $c=1.55(3)$ |
| 19$v$[25] | transfer matrix | 1 | $\eta=0.26(1),$ $\nu=1.0(1), c=1.50(5)$ |

superconformal field theory with central charge $c=3/2$, but also this issue is not fully understood.

The critical behavior of the Heisenberg frustrated antiferromagnet is clearer; many experiments[33] agree with theoretical predictions based on topological considerations,[34–36] Monte Carlo simulations,[34,35,37–40,30] the application of appropriate $NL\sigma$ models,[41–45] and by the use of ERG.[46] The order parameter space of this model is the group of 3D rotations, in fact $O(3)/Z_2\equiv SO(3)\equiv P_3$.[34] Homotopy analysis shows that the system bears a topological stable point defect ($\pi_1[SO(3)]=Z_2$; see, for an exhaustive discussion, Refs. 34 and 1) characterized by a two valued topological quantum number and exhibits a phase transition driven by the dissociation of the vortices. This analysis suggests the occurrence of a KT phase transition mediated by $Z_2$ vortices (while the standard KT transition for $XY$ unfrustrated ferromagnet bears $Z$ vortices[47,4]): this observation strongly suggests that the frustrated noncollinear magnets might exhibit a novel phase transition, possibly belonging to a new universality class.[34] A strong difference between the standard KT transition and the frustrated Heisenberg model is that the last one presents a mass gap also in the low temperature phase.[34,35] Monte Carlo simulations seem to confirm this scenario with the occurrence of a novel phase transition,[34,40,30] although some authors[36,39] think that it is a standard KT transition.

The $NL\sigma$ model is not able to describe a defect mediated transition,[48] in fact the perturbative $\beta$ function that we obtain from a $NL\sigma$ model for a system which in the high temperature phase has a symmetry described by the group $G$ and in the low temperature phase has the symmetry of $H$, subgroup of $G$, depends only on the local structure of the coset $G/H$. For the AFT model the order parameter space $[SO(3)]$ is locally isomorphic to the order parameter space of the $O(4)$ vector model (the four-dimensional sphere $S^3$), but these two spaces have different first homotopy groups:

$$\pi_1(SO(3))=Z_2, \quad \pi_1(S^3)=0.$$

For this reason the AFT model and the $O(4)$ vector model have the same $\beta$ function,[45] even if topological excitations take to a different energy spectra. When these excitations are activated, the $NL\sigma$ model approach leads to wrong predictions. Instead the perturbative $\beta$ function of the LGW model depends only on the representation of $G$ and could be sensitive to the topological degrees of freedom. Let us emphasize that there is no general consensus on the effectiveness of the perturbative treatment of the LGW model with these topological degrees of freedom. We can only mention the fact that the estimate for $g^*$ (the critical four point renormalized coupling) obtained in Ref. 49 for the $XY$ unfrustrated model from the five-loop perturbative expansion directly in two dimensions, and the resummation of the same observable in the framework of the $\epsilon$ expansion at the order $\epsilon^3$ (Refs. 50 and 51) are in perfect agreement with all the other results of nonperturbative methods.[52]

The LGW Hamiltonian (3) has been extensively studied in the framework of $\epsilon=4-d$ expansion,[1,3,53,54] in the $1/N$ expansion,[1,3,54] and at fixed dimension $d=3$.[55,56] The existence and the stability properties of the fixed points depend on the number $N$ and on the spatial dimensionality (see, for an exhaustive discussion, Refs. 1 and 2). In the $\epsilon$ expansion, for sufficiently large values of $N$, there are four fixed points: the Gaussian one, the $O(2N)$ ($v=0$) one, and two new fixed points situated in the region $u,v>0$, called chiral and antichiral. In fixed dimension $d=3$, for the physically relevant cases of $N=2,3$,[56] the six-loop perturbative analysis shows the existence of the four fixed points cited above, of which the chiral is the stable one.

In this paper we present a field-theoretic study based on an expansion performed directly in two dimensions, as proposed for the $O(N)$ models by Parisi.[57]

The paper is organized as follows. In Sec. II we derive the perturbative series for the renormalization-group functions at four loops and discuss the singularities of the Borel transform. The results of the analysis are presented in Sec. III. In Sec. IV we draw our conclusions and we discuss some future and unsolved issues.

## II. THE FIXED DIMENSION PERTURBATIVE EXPANSION IN TWO DIMENSIONS

### A. Renormalization of the theory

The fixed-dimension field-theoretical approach[57] represents an effective procedure in the study of the critical properties in the symmetric phase of systems belonging to the $O(N)$ universality class (see, e.g., Ref. 58). So the idea is to extend this procedure to frustrated models where there are two $\phi^4$ couplings with different symmetry.[55,56] One performs an expansion in powers of appropriately defined zero-momentum quartic couplings and renormalizes the theory by a set of zero-momentum conditions for the (one-particle irreducible) two-point and four-point correlation functions:

$$\Gamma^{(2)}_{ai,bj}(p)=\delta_{ai,bj}Z_\phi^{-1}[m^2+p^2+O(p^4)], \tag{4}$$

$$\Gamma^{(4)}_{ai,bj,ck,dl}(0)=Z_\phi^{-2}m\left[\frac{u}{3}S_{ai,bj,ck,dl}+\frac{v}{6}C_{ai,bj,ck,dl}\right], \tag{5}$$

where $\delta_{ai,bj}=\delta_{ab}\delta_{ij}$ and

$$S_{ai,bj,ck,dl}=\delta_{ai,bj}\delta_{ck,dl}+\delta_{ai,ck}\delta_{bj,dl}+\delta_{ai,dl}\delta_{bj,ck},$$

$$C_{ai,bj,ck,dl}=\delta_{ab}\delta_{cd}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\delta_{ac}\delta_{bd}(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk})+\delta_{ad}\delta_{bc}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl})-2S_{ai,bj,ck,dl}. \tag{6}$$

The above relations allow to relate the second-moment mass $m$, and the zero-momentum quartic couplings $u$ and $v$ to the corresponding bare parameters $r$, $u_0$, and $v_0$ of the Hamiltonian (3).

In addition one introduces a renormalization condition for $\Gamma^{(1,2)}$, the one-particle irreducible two-point function with an insertion of the operator $\frac{1}{2}\phi^2$:

$$\Gamma^{(1,2)}_{ai,bj}(0)=\delta_{ai,bj}Z_t^{-1}. \tag{7}$$

From the perturbative knowledge of the functions $\Gamma^{(2)}$, $\Gamma^{(4)}$, and $\Gamma^{(1,2)}$ one can determine the RG functions as series of the two couplings $u$ and $v$.

The fixed points of the model are defined by the common zeros of the $\beta$ functions,

$$\beta_u(u,v)=m\left.\frac{\partial u}{\partial m}\right|_{u_0,v_0}, \quad \beta_v(u,v)=m\left.\frac{\partial v}{\partial m}\right|_{u_0,v_0}, \tag{8}$$

while the stability properties of these points are determined by the eigenvalues $\omega_i$ of the matrix $\Omega$:

$$\Omega=\begin{pmatrix} \frac{\partial\beta_u(u,v)}{\partial u} & \frac{\partial\beta_u(u,v)}{\partial v} \\ \frac{\partial\beta_v(u,v)}{\partial u} & \frac{\partial\beta_v(u,v)}{\partial v} \end{pmatrix}. \tag{9}$$

A fixed point with two positive eigenvalues is stable (totally attractive), and it determines the behavior of the system at criticality. The eigenvalues $\omega_i$ are connected to the leading scaling corrections, which go as $\xi^{-\omega_i}\sim|t|^{\Delta_i}$ where $\Delta_i=\nu\omega_i$, with few exceptions as the two dimensional Ising model (see for a discussion Ref. 59).

The RG functions $\eta_\phi$ and $\eta_t$ are defined in the usual way:

$$\eta_\phi(u,v)=\left.\frac{\partial\ln Z_\phi}{\partial\ln m}\right|_{u_0,v_0}=\beta_u\frac{\partial\ln Z_\phi}{\partial u}+\beta_v\frac{\partial\ln Z_\phi}{\partial v}, \tag{10}$$

$$\eta_t(u,v)=\left.\frac{\partial\ln Z_t}{\partial\ln m}\right|_{u_0,v_0}=\beta_u\frac{\partial\ln Z_t}{\partial u}+\beta_v\frac{\partial\ln Z_t}{\partial v}. \tag{11}$$

These functions are related to the critical exponents that can be obtained using the scaling laws in the following way:

$$\eta=\eta_\phi(u^*,v^*), \tag{12}$$

$$\nu=[2-\eta_\phi(u^*,v^*)+\eta_t(u^*,v^*)]^{-1}, \tag{13}$$

$$\gamma=\nu(2-\eta), \tag{14}$$

where $(u^*,v^*)$ is the position of the stable fixed point.

### B. The four loop series

In this section we report our results on the perturbative expansion of the RG functions (8), (10), and (11) up to four loops. The results are reported in terms of the rescaled couplings

$$u\equiv\frac{8\pi}{3}R_{2N}\overline{u}, \quad v\equiv\frac{8\pi}{3}R_{2N}\overline{v}, \tag{15}$$

where $R_N=9/(8+N)$. We use these rescaled couplings, different from the usual ones,[56] because they have finite fixed point values in the limit $N\to\infty$.

The series are

$$\overline{\beta_{\overline{u}}}=-\overline{u}+\overline{u}^2+\frac{1-N}{(4+N)}\overline{uv}-\frac{1-N}{(8+2N)}\overline{v}^2+\sum_{i+j\geq 3}b^{(u)}_{ij}\overline{u}^i\overline{v}^j, \tag{16}$$

TABLE III. Coefficients $b_{ij}^{(u)}$ of the four-loop expansion of $\overline{\beta_{\bar{u}}}$.

| $i,j$ | $R_{2N}^{-i-j} b_{ij}^{(u)}$ |
|---|---|
| 3,0 | $-0.52318301-0.35762725N-0.05670787N^2$ |
| 2,1 | $+0.27622342+0.20716756N+0.06905585N^2$ |
| 1,2 | $+0.24421150-0.18315862N-0.06105287N^2$ |
| 0,3 | $-0.06173989+0.04630492N+0.01543497N^2$ |
| 4,0 | $+0.63938617+0.52357610N+0.11532026N^2+0.00609697N^3$ |
| 3,1 | $+0.52972245-0.34404956N-0.17236232N^2-0.01331057N^3$ |
| 2,2 | $-0.60971053+0.40445539N+0.19204827N^2+0.01320688N^3$ |
| 1,3 | $+0.25988019-0.18091027N-0.075469945N^2-0.00349997N^3$ |
| 0,4 | $-0.02523851+0.02158825N+0.00431510N^2-0.00066484N^3$ |
| 5,0 | $-1.02844874-0.96463436N-0.27426277N^2-0.02444185N^3-0.00002407294N^4$ |
| 4,1 | $-1.16733461+0.63874480N+0.46960694N^2+0.05903441N^3-0.00005154248N^4$ |
| 3,2 | $+1.57188708-0.87883974N-0.61913519N^2-0.07418884N^3+0.0002766848N^4$ |
| 2,3 | $-0.91454905+0.53282094N+0.34517685N^2+0.03698163N^3-0.000430362N^4$ |
| 1,4 | $+0.21807034-0.13987590N-0.07359806N^2-0.00492708N^3+0.0003307098N^4$ |
| 0,5 | $-0.02243288+0.01659580N+0.006205439N^2-0.0002619636N^3-0.0001063929N^4$ |

$$\overline{\beta_{\bar{v}}}=-\bar{v}+\frac{-6+N}{(4+N)}\overline{v^2}+\frac{6}{(4+N)}\overline{uv}+\sum_{i+j\geq 3} b_{ij}^{(v)}\overline{u^i}\,\overline{v^j}, \tag{17}$$

$$\eta_\phi=\frac{1.83417(1+N)}{(8+2N)^2}\overline{u^2}+\frac{1.83417(1-N)}{(8+2N)^2}\overline{uv}-\frac{1.37563(1-N)}{(8+2N)^2}\overline{v^2}+\sum_{i+j\geq 3} e_{ij}^{(\phi)}\overline{u^i}\,\overline{v^j}, \tag{18}$$

$$\eta_t=-\frac{2(1+N)}{(4+N)}\bar{u}-\frac{(1-N)}{(4+N)}\bar{v}+\frac{13.5025(1+N)}{(8+2N)^2}\overline{u^2}+\frac{13.5025(1-N)}{(8+2N)^2}\overline{uv}-\frac{10.1269(1-N)}{(8+2N)}\overline{v^2}+\sum_{i+j\geq 3} e_{ij}^{(t)}\overline{u^i}\,\overline{v^j}, \tag{19}$$

where

$$\overline{\beta_{\bar{u}}}=\frac{3}{16\pi}R_{2N}^{-1}\beta_u\,,\qquad \overline{\beta_{\bar{v}}}=\frac{3}{16\pi}R_{2N}^{-1}\beta_v\,. \tag{20}$$

The coefficients $b_{ij}^{(u)}$, $b_{ij}^{(v)}$, $e_{ij}^{(\phi)}$, and $e_{ij}^{(t)}$ are reported in the Tables III, IV, and V. Note that due to the rescaling (20), the matrix element of $\Omega$ are two times the derivative of $\bar{\beta}$ with respect to $\bar{u}$ and $\bar{v}$.

We have verified the exactness of our series by the following relations.

For $N=1$ the functions $\beta_{\bar{u}}$, $\eta_\phi$, and $\eta_t$ reproduce the corresponding functions of the standard $O(2)$ model.[49]

For $\bar{v}=0$, the functions $\beta_{\bar{u}}(\bar{u},0)$, $\eta_\phi(\bar{u},0)$, and $\eta_t(\bar{u},0)$ reproduce the corresponding functions of the O$(2N)$-symmetric model.[49]

For $N=M=2$ one can transform the Hamiltonian (3) into two independent $XY$ models via the transformation[3]

$$\phi_{11}=\frac{\phi'_{11}-\phi'_{22}}{\sqrt{2}},\qquad \phi_{12}=\frac{\phi'_{12}+\phi'_{21}}{\sqrt{2}},$$

$$\phi_{21}=\frac{\phi'_{21}-\phi'_{12}}{\sqrt{2}},\qquad \phi_{22}=\frac{\phi'_{22}+\phi'_{11}}{\sqrt{2}}, \tag{21}$$

and with the condition $v_0=-2u_0$. One can easily verify that, calling $x_0=u_0-v_0/2$ the new bare coupling, the RG function $\overline{\beta_{\bar{x}}}$ reduce to the one of the $XY$ model:

$$\overline{\beta_{\bar{x}}}=\frac{5}{6}\overline{\beta_{\bar{u}}}\left(\tfrac{3}{5}\bar{x},-\tfrac{6}{5}\bar{x}\right)-\frac{5}{12}\overline{\beta_{\bar{v}}}\left(\tfrac{3}{5}\bar{x},-\tfrac{6}{5}\bar{x}\right). \tag{22}$$

In the limit $N\to\infty$ the RG functions reduce to

$$\overline{\beta_{\bar{u}}}=-\bar{u}+\overline{u^2}-\overline{uv}+\frac{1}{2}\overline{v^2}, \tag{23}$$

$$\overline{\beta_{\bar{v}}}=-\bar{v}\left(1-\frac{\bar{v}}{2}\right), \tag{24}$$

$$\eta_\phi=0, \tag{25}$$

$$\eta_t=-2\bar{u}+\bar{v}. \tag{26}$$

The zeros of the two $\bar{\beta}$ functions are reported in Fig. 1. There are four fixed points: the Gaussian one (0,0) that is unstable with $\omega_1=\omega_2=-2$, the $O(2N)$ symmetric one that is unstable in the $v$ direction with $\omega_1=-\omega_2=2$, the chiral one that is the only stable with corrections to the scaling given by $\omega_1=\omega_2=2$ and the antichiral fixed point that is unstable with the same eigenvalues of the $\Omega$ matrix of the $O(2N)$ one. We note that the region of attraction of the chi-

TABLE IV. Coefficients $b_{ij}^{(v)}$ of the four-loop expansion of $\overline{\beta}_{\bar{v}}$.

| $i,j$ | $R_{2N}^{-i-j} b_{ij}^{(v)}$ |
|---|---|
| 2,1 | $-1.01710219-0.38232322N-0.03201192N^2$ |
| 1,2 | $+0.76100686+0.17012363N-0.005032022N^2$ |
| 0,3 | $-0.17561980+0.00823505N+0.01303500N^2$ |
| 3,1 | $+1.4980998+0.66351375N+0.06331414N^2-0.0022332645N^3$ |
| 2,2 | $-1.49031358-0.52140551N-0.012396398N^2+0.006202595N^3$ |
| 1,3 | $+0.59692704+0.15592619N-0.006944004N^2-0.002154403N^3$ |
| 0,4 | $-0.07062074-0.00996809N+0.000144371N^2-0.0004443507N^3$ |
| 4,1 | $-2.80757448-1.43132297N-0.19616848N^2-0.004346583N^3-0.0002234496N^4$ |
| 3,2 | $+3.39501758+1.49420844N+0.13249155N^2-0.004523054N^3+0.00067373429N^4$ |
| 2,3 | $-1.9481041-0.78279460N-0.05338075N^2+0.002303844N^3-0.00070912608N^4$ |
| 1,4 | $+0.50147548+0.18762207N+0.01554819N^2+0.001191026N^3+0.0002987006N^4$ |
| 0,5 | $-0.04975526-0.01162418N-0.000186500N^2-0.000278729N^3-0.000045297366N^4$ |

ral fixed point is not all the $(\bar{u},\bar{v})$ plane but only the region with $\bar{u}>0$ and $z=\bar{v}/\bar{u}<2$; if the system is located out of this region it undergoes first order phase transition. The critical indices of the Gaussian and antichiral fixed points are mean field (i.e., $\eta=0$ and $\eta_t=0$), while the $O(2N)$ and the chiral fixed point have nontrivial exponents given by $\eta=0$ and $\eta_t=2$, as it is already known from large $N$ expansion.[1] This is another check of exactness of the series.

### C. Resummations of the series

The field theoretic perturbative expansion generates asymptotic series that must be resummed to extract the physical information about the critical behavior of the real systems.

Exploiting the property that these series are Borel summable for $\phi^4$ theories in two and three dimensions,[60] one can resum these perturbative expressions using a Borel transformation combined with a method for the analytical extension of the Borel transform. If the domain of analyticity of the Borel transform is known, one can perform a mapping which maps the domain of analyticity (cut at the instanton singularity) $g_b$ onto a circle, in order to have the maximal analyticity. In the case of the $O(N)$ symmetric model, let $F(g) = \Sigma f_k g^k$ be any Borel summable function, whose expansion in series has to be resummed; exploiting the knowledge of the large order behavior of the coefficients $f_k$ (Ref. 58)

$$f_k \sim k!(-a)^k k^b[1+O(k^{-1})] \quad \text{with} \quad a=-1/g_b \tag{27}$$

(a large order behavior related to the singularity $g_b$ of the Borel transform closest to the origin) one can perform the following mapping[61]

$$y(g)=\frac{\sqrt{1-g/g_b}-1}{\sqrt{1-g/g_b}+1} \tag{28}$$

to extend the Borel transform of $F(g)$ to all positive values of $g$. The singularity $g_b$ depends only on the considered model and can be obtained from a steepest-descent calculation in which the relevant saddle point is a finite-energy solution (instanton) of the classical field equations with negative coupling.[62,63] Instead the coefficient $b$ depends on which Green's function is considered.

Note that the function $F(g)$ can be Borel summable only if there are no singularities of the Borel transform on the positive real axis.

This resummation procedure has worked successfully for the $O(N)$ symmetric theory, for which accurate estimates for the critical exponents and other physical quantities have been obtained.

For this reason, the idea is to extend the resummation procedure cited above to the frustrated model, as it has been done for the three-dimensional case,[56] considering a double expansion in $\bar{u}$ and $\bar{v}$ at fixed $z=\bar{v}/\bar{u}$, and studying the large order behavior (following the same procedure used in Refs. 64 and 56) of the new expansion in powers of $\bar{u}$ to calculate the singularity of the Borel transform closest to the origin $\bar{u}_b$. The results are

$$\frac{1}{\bar{u}_b}=-aR_{2N} \quad \text{for} \quad 0<z<4,$$

$$\frac{1}{\bar{u}_b}=-aR_{2N}\left(1-\frac{1}{2}z\right) \quad \text{for} \quad z<0, \quad z>4, \tag{29}$$

where $a=0.238659217\ldots$ .

We find that for $z>2$ there is a singularity on the real positive axis which however is not the closest one to the origin for $z<4$. Thus, for $z>2$ the series are not Borel summable.

An important issue in the fixed dimension approach to critical phenomena concerns the analytic properties of the $\beta$ functions. As shown in Ref. 59 for the $O(N)$ model, the presence of confluent singularities in the zero of the perturbative $\beta$ function causes a slow convergence of the resummation of the perturbative series to the correct fixed point value. The $O(N)$ two-dimensional field-theory estimates of physical quantities[61,49] are less accurate than the ones of the three-dimensional case, due to the stronger nonanalyticities

TABLE V. Coefficients $e_{ij}^{(\phi)}$ and $e_{ij}^{(t)}$ of the four-loop expansion of $\eta_\phi$ and $\eta_t$.

| $i,j$ | $R_{2N}^{-i-j}e_{ij}^{(\phi)}$ |
|---|---|
| 3,0 | $-0.00119855-0.00149819N-0.000299637N^3$ |
| 2,1 | $-0.00179782+0.00134837N+0.000449456N^2$ |
| 1,2 | $+0.00179782-0.00157301N-0.000224728N^2$ |
| 0,3 | $-0.000599275+0.000636729N-0.0000374547N^2$ |
| 4,0 | $+0.00633062+0.00891668N+0.00247304N^2-0.000113013N^3$ |
| 3,1 | $+0.01266124-0.00748913N-0.00539814N^2+0.000226025N^3$ |
| 2,2 | $-0.01549651+0.00967682N+0.00598920N^2-0.000169519N^3$ |
| 1,3 | $+0.00722529-0.00531519N-0.00196661N^2+0.0000565063N^3$ |
| 0,4 | $-0.00105507+0.00116599N-0.0000897290N^2-0.0000211899N^3$ |
| $i,j$ | $R_{2N}^{-i-j}e_{ij}^{(t)}$ |
| 3,0 | $-0.13266188-0.17878908N-0.04612721N^2$ |
| 2,1 | $-0.19899281+0.12980200N+0.06919081N^2$ |
| 1,2 | $+0.17738991-0.13415335N-0.04323656N^2$ |
| 0,3 | $-0.05120890+0.04833365N+0.00287525N^2$ |
| 4,0 | $+0.17299878+0.25743030N+0.08514284N^2+0.000711326N^3$ |
| 3,1 | $+0.34599757-0.17713453N-0.16744038N^2-0.00142265N^3$ |
| 2,2 | $-0.39410819+0.21062748N+0.18241372N^2+0.00106699N^3$ |
| 1,3 | $+0.16999405-0.10189909N-0.06869231N^2+0.000597349N^3$ |
| 0,4 | $-0.02028992+0.01767450N+0.00343506N^2-0.000819639N^3$ |

at the fixed point.[59,65,50] In Ref. 59 it is shown that the nonanalytic terms cause a large deviation in the estimate of the right correction to the scaling, i.e., the exponent $\omega$; instead the result for the fixed point value is a rather good approximation of the correct one (the systematic error is always less than 10%). We think that this scenario holds also for frustrated models.

## III. FOUR-LOOP EXPANSION ANALYSIS

### A. The analysis method

The analysis of the four-loop series is performed following the procedure used in Ref. 56: we exploit the knowledge of the value of the singularity of the Borel transform closest to the origin (a value given in the previous section), and we generate a set of approximants to our asymptotic series, varying the two parameters $\alpha$ and $b$.

If

$$R(\bar{u},\bar{v})=\sum_{k=0}\sum_{h=0}R_{hk}\bar{u}^h\bar{v}^k, \tag{30}$$

is our asymptotic expression for one of the functions $\bar{\beta}$ or $\eta$, then our approximants can be written as

$$E(R)_p(\alpha,b;\bar{u},\bar{v})=\sum_{k=0}^{p}B_k(\alpha,b;\bar{v}/\bar{u})\times\int_0^\infty dt t^b e^{-t}\frac{y(\bar{u}t;\bar{v}/\bar{u})^k}{[1-y(\bar{u}t;\bar{v}/\bar{u})]^\alpha}, \tag{31}$$

where

$$y(x;z)=\frac{\sqrt{1-x/\bar{u}_b(z)}-1}{\sqrt{1-x/\bar{u}_b(z)}+1}. \tag{32}$$

The coefficient $B_k$ are determined by the condition that the expansion of $E(R)_p(\alpha,b;\bar{u},\bar{v})$ in powers of $\bar{u}$ and $\bar{v}$ gives $R(\bar{u},\bar{v})$ to order $p$.

In order to find the fixed points of the theory we compute the series (31), with $R=\bar{\beta}_i$, for many values of $\alpha$ and $b$ (and for several values of $N$), obtaining many different estimates of the $\bar{\beta}$ functions in all the plane $(\bar{u},\bar{v})$. We note that for $\alpha\geqslant 3$, the estimates strongly oscillates with varying $b$. For this reason we choose to keep $\alpha$ in the range $0\leqslant\alpha\leqslant 2$. Then we look for the values of $b$, at fixed $\alpha$, which make the series, at small values of $\bar{u}$ and $\bar{v}$ (i.e., $\bar{u},\bar{v}\leqslant 1$), more stable while varying the order $p$. We realize that all values of $b$ between 0 and 20 give reasonable estimates of the $\bar{\beta}$ functions. For the estimates of the functions for larger values of $\bar{u}$ and $\bar{v}$ we take the approximants in this range of $\alpha$ and $b$.

At the end of this exploratory analysis, the strategy to obtain reasonable values and error bars for the fixed points, is to divide the domain $0\leqslant\bar{u}\leqslant 4$, $0\leqslant\bar{v}\leqslant 6$ in $40^2$ rectangles, then to take 18 different approximants with $b=5,7,9,11,13,15$ and $\alpha=0,1,2$ for the $\bar{\beta}$ functions, and to mark all the sites in which at least two approximants for $\bar{\beta}_{\bar{u}}$ and $\bar{\beta}_{\bar{v}}$ vanish. This procedure is applied to the three-loop and four-loop series.

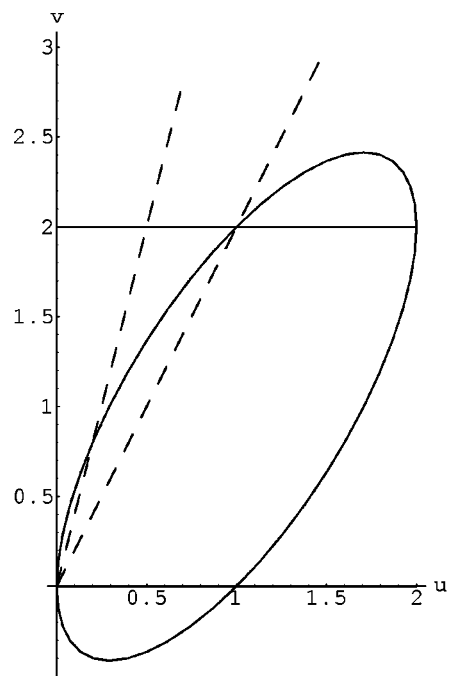


FIG. 1. Zeros of the $\overline{\beta}$ functions for $N=\infty$ in the $(\overline{u},\overline{v})$

We adopt this method also for the values of $z$ for which the series are not Borel summable. It should provide a reasonable estimate if $z<4$, because we take into account the leading large order behavior.

### B. Instability of the symmetric fixed point

To begin with, we present the results related to the instability of the $O(2N)$ fixed point. An accurate estimate of its location already exists from the Padé-Borel analysis of the five-loop series of Ref. 49. In Table VI we report our estimates for

$$\omega_v = 2\frac{\partial \overline{\beta_{\overline{v}}}}{\partial \overline{v}}\Big|_{\overline{u}=\overline{u}^*_{O(2N)},\overline{v}=0},$$

where $\overline{u}^*_{O(2N)}$ is the fixed point value obtained in Ref. 49. The missing $\overline{u}^*_{O(2N)}$ values are computed by the standard resummation procedure of Ref. 61. These results clearly show that this fixed point is always unstable.

### C. Large $N$ analysis ($N\geqslant 4$)

We begin our analysis of the four loop series from large $N$ values, because all these systems are free of topological defects and they are well described by appropriate $NL\sigma$ models. In fact, being the order parameter space $V\equiv O(N)/O(N-2)$, the first homotopy group is $\pi_1(V)=0$ for all $N\geqslant 4$ [this is a consequence of a simple theorem of differential geometry, for which the $\pi_1(G/H)=0$ if $H$ is a continuous subgroup of a Lie group $G$]. We expect that, for these values of $N$, the chiral fixed point exists and it is stable with critical indices and corrections to the scaling equal to the ones found for $N=\infty$. (See Sec. II B.)

We apply the procedure illustrated above for $N$ equal to 32, 16, 8, 6, and 4 (the choice of these particular values of $N$ will be clear in the following). The results for the zeros of the $\overline{\beta}$ functions, obtained from the analysis of three and four-loop series are reported in Figs. 2, 3, 4, 5, and 6. The numerical estimates of the common zeros of the $\overline{\beta}$'s are in Table VII.

For $N=32$ the presence of the four fixed points predicted by the $1/N$ analysis is clear. From Fig. 2 emerges a new characteristic of $\overline{\beta_{\overline{u}}}$: the appearance of a second branch of zeros in the region of upper values of $\overline{v}$. This second branch exists only for the four-loop series, as in the three-dimensional case,[56] but it is present for almost all the considered approximants. For $N=32$ this upper branch has no particular importance, but it will become fundamental for lower values of $N$.

The evaluation of the $\Omega$ matrix shows that the chiral fixed point is stable with corrections to scaling given by $\omega_1=2.2(2)$ and $\omega_2=1.7(2)$. The antichiral fixed point is unstable with crossover exponents $\omega_1=1.7(2)$ and $\omega_2=-1.3(2)$. We use a large set of approximants (all those used for the $\overline{\beta}$'s) to take into account the strong effect of nonanalyticities. In this way we have a big error with respect to the one obtained by using only stability criteria, but our estimates are very close to the correct values $\omega_1=\omega_2=2$.

For $N=16$ the four fixed points of the $1/N$ expansion appear but the curve of zeros of $\overline{\beta_{\overline{v}}}$ cross the upper branch of the zeros of $\overline{\beta_{\overline{u}}}$ in the point $[0.4(2),5.0(3)]$. We call this new fixed point antichiral II fixed point. We cannot exclude the possibility that this fixed point is a numerical artifact of the resummation procedure since it belongs to the region with $z>4$, i.e., the region in which the singularity of the Borel transform closest to the origin is on the positive real axis. This problem is not so relevant because this fixed point is found to be unstable and it does not influence the critical behavior of the system. Figure 4 shows that the shape of the zeros of the $\overline{\beta}$ functions for $N=8$ is similar to the one of $N=16$. The stability properties being the same for these two cases we conclude that the critical behavior should be the same for all values of $N$ between 8 and 16.

For $N=6$ the two antichiral fixed points seem to coalesce, but probably this is a consequence of the large error bar of these zeros. To solve this problem it is necessary to know the $\overline{\beta}$ functions at higher order. However, one eigenvalue of the stability matrix, in all this wide zone, is always negative, so this problem does not influence our understanding of the critical properties of the model. The chiral fixed point is always the only stable one, nevertheless the estimates of the corrections to scaling are now quite different from the exact ones, in fact we obtain $\omega_1=2.4(6)$ and $\omega_2=0.8(4)$ instead of $\omega_1=\omega_2=2$. This is probably due to the nonanalytic terms that, as shown in Ref. 59, have larger effects when $N$ decreases.

In the case of $N=4$ (see Fig. 6), the exact number of the fixed points is not known and it is not clear if they are located on the upper or on the lower branch of zeros of $\overline{\beta_{\overline{u}}}$. From our homotopy analysis, we expect that this model is

TABLE VI. Crossover exponent $\omega_v$ at the $O(2N)$ fixed point.

| $N$ | $\bar{u}^*_{O(2N)}$ | $\omega_v$ |
|---|---|---|
| 2 | 1.70(2) | −0.36(4) |
| 3 | 1.62(1) | −0.7(2) |
| 4 | 1.52(1) | −0.9(2) |
| 6 | 1.407(3) | −1.2(2) |
| 8 | 1.313(3) | −1.3(1) |
| 16 | 1.170(2) | −1.70(4) |

connected with $N=\infty$ and as a consequence the stable zero should be on the lower branch of $\bar{\beta}_{\bar{u}}$. Reasonable estimates of fixed points are not feasible from Fig. 6. Anyway, if we mark the sites in which at least 3, 4, etc. approximants for the $\bar{\beta}$'s vanish, we find two distinct regions located near $\bar{u}\sim 2$ and $\bar{u}\sim 1$. The existence of at least two fixed points is clear, but their estimate is not possible.

Having established the existence of a stable fixed point, we compute the critical exponents from the perturbative series reported in the previous section. We consider a large number of approximants varying $\alpha$ and $b$ in the range $0 \leqslant \alpha \leqslant 5$ and $0 \leqslant b \leqslant 30$ and we choose for the final estimates the ones more stable varying the number of loops. Varying $N$ we find that the exponent $\eta$ takes small values [e.g., $\eta = 0.040(2)$ for $N=32$ and $\eta = 0.11(4)$ for $N=8$] which are close to the ones found for the $O(N)$ models for $N\geqslant 3$, but different from the exact one $\eta=0$, expected for asymptotic free theory. We think that these erroneous predictions are due to nonanalytic terms of $\eta(u,v)$ at $(u^*,v^*)$, as for the $O(N)$ models.

### D. Frustrated Heisenberg model

The zeros of the $\bar{\beta}$ functions for $N=3$ are reported in Fig. 7. It is apparent that the chiral fixed point is located on the upper branch of zeros of $\bar{\beta}_{\bar{u}}$, so this value of $N$ is not connected with $N=\infty$. For this reason the exponent $\eta$ could be different from zero. We have already stressed the strong evidence for a topological phase transition for this frustrated system and so the physical reason for this nontrivial exponent is clear.

The standard structure involving only four common zeros of the $\beta$ functions is restored since we find only one antichiral fixed point. The chiral fixed point is located at

$$\bar{u}^* = 2.3(3), \quad \bar{v}^* = 3.9(5). \tag{33}$$

TABLE VII. Fixed points for $N>4$.

| $N$ | Chiral fixed point $(\bar{u}^*,\bar{v}^*)$ | Antichiral fixed point $(\bar{u}^*,\bar{v}^*)$ |
|---|---|---|
| 32 | [2.15(5), 2.18(8)] | [1.05(5), 2.25(15)] |
| 16 | [2.2(1), 2.40(15)] | [1.05(15), 2.48(8)] |
| 8 | [2.3(1), 2.70(15)] | [1.15(25), 3.08(22)] |
| 6 | [2.2(2), 3.38(22)] | [1.2(3), $>3.6$] |

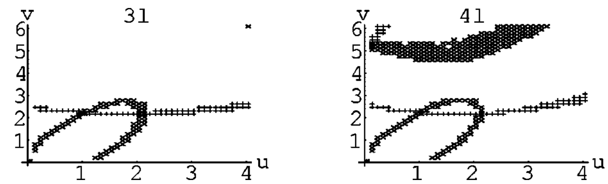


FIG. 2. Zeros of the $\bar{\beta}$ functions for $N=32$ in the $(\bar{u},\bar{v})$ plane. Pluses (+) and crosses (×) correspond to zeros of $\bar{\beta}_{\bar{v}}(\bar{u},\bar{v})$ and $\bar{\beta}_{\bar{u}}(\bar{u},\bar{v})$, respectively.

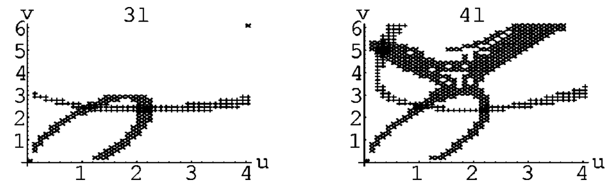


FIG. 3. Zeros of the $\bar{\beta}$ functions for $N=16$ in the $(\bar{u},\bar{v})$ plane. Pluses (+) and crosses (×) correspond to zeros of $\bar{\beta}_{\bar{v}}(\bar{u},\bar{v})$ and $\bar{\beta}_{\bar{u}}(\bar{u},\bar{v})$, respectively.

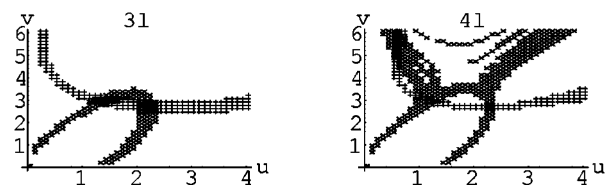


FIG. 4. Zeros of the $\bar{\beta}$ functions for $N=8$ in the $(\bar{u},\bar{v})$ plane. Pluses (+) and crosses (×) correspond to zeros of $\bar{\beta}_{\bar{v}}(\bar{u},\bar{v})$ and $\bar{\beta}_{\bar{u}}(\bar{u},\bar{v})$, respectively.

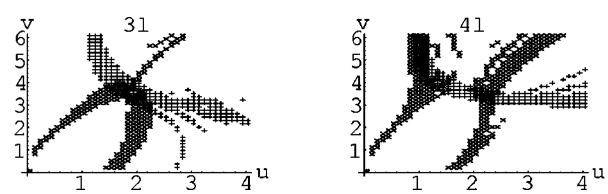


FIG. 5. Zeros of the $\bar{\beta}$ functions for $N=6$ in the $(\bar{u},\bar{v})$ plane. Pluses (+) and crosses (×) correspond to zeros of $\bar{\beta}_{\bar{v}}(\bar{u},\bar{v})$ and $\bar{\beta}_{\bar{u}}(\bar{u},\bar{v})$, respectively.

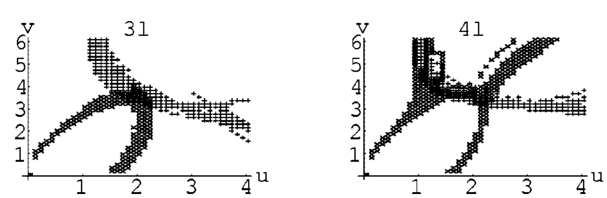


FIG. 6. Zeros of the $\bar{\beta}$ functions for $N=4$ in the $(\bar{u},\bar{v})$ plane. Pluses (+) and crosses (×) correspond to zeros of $\bar{\beta}_{\bar{v}}(\bar{u},\bar{v})$ and $\bar{\beta}_{\bar{u}}(\bar{u},\bar{v})$, respectively.

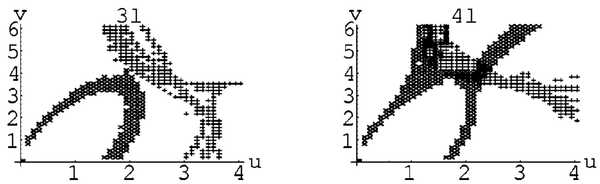


FIG. 7. Zeros of the $\overline{\beta}$ functions for $N=3$ in the $(\overline{u},\overline{v})$ plane. Pluses (+) and crosses (×) correspond to zeros of $\overline{\beta}_{\overline{v}}(\overline{u},\overline{v})$ and $\overline{\beta}_{\overline{u}}(\overline{u},\overline{v})$, respectively.

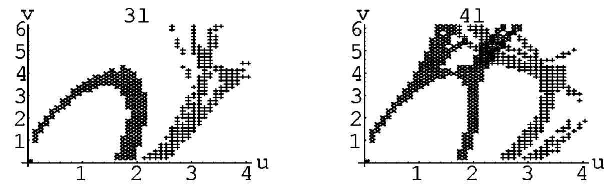


FIG. 8. Zeros of the $\overline{\beta}$ functions for $N=2$ in the $(\overline{u},\overline{v})$ plane. Pluses (+) and crosses (×) correspond to zeros of $\overline{\beta}_{\overline{v}}(\overline{u},\overline{v})$ and $\overline{\beta}_{\overline{u}}(\overline{u},\overline{v})$, respectively.

The eigenvalues $\omega_i$ vary significantly with the two parameters $\alpha$ and $b$ and turn out to be complex in several cases. Reasonable estimates of $\omega_i$ are not feasible, nonetheless the sign of their real part is positive for the majority of the approximants.

The critical behavior of this topological phase transition is expected to be of the KT type, i.e., the correlation length and the susceptibility diverge, for $t>0$ according to the asymptotic law:

$$\xi \sim e^{bt^{-\sigma}}, \quad \chi \sim \xi^{2-\eta}, \tag{34}$$

where $\eta$ is the standard exponent. This means that we cannot define the conventional exponents $\nu$ and $\gamma$ since $\xi$ diverges faster than any power of $t$.

For the $O(2)$ unfrustrated model Kosterlitz and Thouless[47] have shown that $\eta=1/4$ and $\sigma=1/2$. Exact values do not exist for the Heisenberg frustrated model. The result of a very recent and accurate Monte Carlo simulation[40] is $\sigma \sim 0.600$,[66] even if the standard KT form of thermodynamic quantities has been used in the past to fit Monte Carlo data finding a reasonable agreement.[35,39,38]

Our estimate for $\eta$ is

$$\eta = 0.24(6). \tag{35}$$

Note that this value is consistent with the standard KT scenario.

This value of $\eta$, which is very different from zero, seems to confirm that the LGW model is able to describe a topological phase transition.[67] We have already said in the Introduction that there is no general consensus on this issue. We think that this nontrivial result is a possible evidence in favor of this thesis.

### E. Frustrated *XY* model

As discussed in the Introduction, the frustrated *XY* model breaks two symmetries. If there are two transitions we are only able to describe the one that occurs at higher temperature, since we explore the critical properties of the model in the high temperature phase.

Figure 8 shows the zeros of the $\overline{\beta}$ functions. The chiral fixed point is located at

$$\overline{u}^* = 2.3(2), \quad \overline{v}^* = 5.0(5). \tag{36}$$

In the region that is not shown ($v>6$) we find another fixed point that can be identified with the antichiral.

To describe the critical behavior of these systems we have studied the stability of these two fixed points, evaluating the eigenvalues of the $\Omega$ matrix defined in Eq. (9) and varying $\alpha$ and $b$ as explained above. For the shortness of our perturbative series the results obtained are quite unstable to present a reasonable estimate, anyway we find that one of the two eigenvalues calculated at the antichiral fixed point is always negative, while for the chiral one the majority of the results are in agreement with its stability.

The evaluation of the critical exponent $\eta$ at the chiral fixed point gives

$$\eta = 0.29(5). \tag{37}$$

This value is consistent with many Monte Carlo reported in Table I. We note that this estimate is substantially higher than 0.146 obtained by the Padé-Borel analysis of five-loop series for the one-component $\phi^4$ theory.[49] We cannot give a reasonable value for the exponent $\nu$ because of the big oscillations observed when varying the two parameters $\alpha$ and $b$. We think that longer series could solve this problem and partially clarify the nature of this phase transition.

## IV. CONCLUSIONS

In the present paper we have studied the critical behavior of frustrated spin models with noncollinear order by applying the field-theoretic renormalization-group technique directly in two dimensions.

We have begun our analysis from large values of $N$, since these systems are free of topological defects, and they are well described by appropriate $NL\sigma$ models. We have considered $N=\infty$, 32, 16, 8, 6, and 4, always finding the presence of the chiral and antichiral fixed points. In order to verify the stability properties of these fixed points we have evaluated the eigenvalues of the matrix $\Omega$, which demonstrate the full stability of the chiral fixed point. We have also shown the nonstability of the $O(2N)$ fixed point for $N \geqslant 2$, computing the crossover exponent $\omega_v$.

Then we have focused our attention on the frustrated *XY* model ($N=2$) and on the Heisenberg model ($N=3$), which are expected to undergo continuous phase transitions that cannot be described by the use of $NL\sigma$ models.

For the *XY* model we have found the presence of a stable fixed point located in [2.3(2),5.0(5)], in which the critical exponent $\eta$ takes the value 0.29(5).

Even in the case of the frustrated Heisenberg model, we have found the existence of a stable fixed point in [2.3(3),3.9(5)]. We have evaluated the critical exponent $\eta = 0.24(6)$. This nontrivial result confirms the validity of the LGW approach in describing defect mediated transitions.

The issue of the universality class of these systems may be clarified by the computation of the universal quantities

$\overline{u^*}, \overline{v^*}$, related to the four point renormalized coupling constants, with a Monte Carlo simulation. In fact these quantities, which have been estimated in this work, have finite values at the critical point and a precise determination of them may be more simple than the evaluation of exponents.

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*Email address: calabres@df.unipi.it
†Email address: parrucci@df.unipi.it

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[66] In Ref. 40 the ratio between the correlation length of the frustrated Heisenberg model and the one of the defect free $O(4)$ model is evaluated. Since the $O(4)$ model has a regular behavior for finite $T$, we believe that the exponential divergence is only due to the $\xi$ of the frustrated system.

[67] We have resummed the four-loop series for the $O(3)$ model obtaining $\eta = 0.11$ which is substantially lower than the value found in text for the frustrated model.