Phase transitions in the quantum Ising and rotor models with a long-range interaction

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(Received 16 April 2001; revised manuscript received 15 June 2001; published 18 October 2001)

We investigate the zero-temperature and finite-temperature phase transitions of quantum Ising and quantum rotor models. We here assume a long-range (falling off as $1/r^{d+\sigma}$, where r is the distance between two spins/rotors in units of lattice spacing) ferromagnetic interaction among the spins or rotors. We find that the long-range behavior of the interaction drastically modifies the universal critical behavior of the system. The corresponding upper critical dimension and the hyperscaling relation and exponents associated with the quantum transition are modified and, as expected, they attain values of short-range system when $\sigma=2$. The dynamical exponent varies continuously as the parameter σ and is unity for $\sigma=2$. The one-dimensional long-range quantum Ising system shows a phase transition at $T=0$ for all values of σ. The most interesting observation is that the phase diagram for $\sigma=d=1$ shows a line of Kosterlitz-Thouless transition at finite temperature even though the $T=0$ transition is a simple order-disorder transition. These finite temperature transitions are studied near the phase boundary using renormalisation group equations and a region with diverging susceptibility is located. We have also studied one-dimensional quantum rotor model which exhibits a rich and interesting transition behavior depending upon the parameter σ. We explore the phase diagram extending the short-range quantum nonlinear σ model renormalisation group equations to the present case.

DOI: 10.1103/PhysRevB.64.184106 PACS number(s): 05.70.Jk, 64.60.Ak, 64.60.Cn, 64.60.Fr

I. INTRODUCTION

Following the experimental studies on the insulating, dipolar Ising spin glass $LiHo_xY_{1-x}F_4$ in a transverse field,[1,2] there has been an upsurge in theoretical investigations of the quantum Ising systems,[3,4] and the n-component quantum rotor models.[3] These models are described by the respective Hamiltonians as

$$H_I=-\sum_{ij} J_{ij}\sigma_i^z\sigma_j^z-\Gamma\sum_i \sigma_i^x \tag{1}$$

or

$$H_R=-\sum_{ij} J_{ij}\mathbf{x}_i^z\cdot\mathbf{x}_j^z+\frac{1}{2\tilde{g}}\sum_i \mathcal{L}_i^2 \quad \text{with} \quad \mathbf{x}_i^2=1. \tag{2}$$

In Eq. (1) σ's are noncommuting Pauli matrices while in Eq. 2 $\mathbf{x}_i$'s are n-component, unit length, vectors occupying the N sites of a d-dimensional hypercubic lattice. The operator $\mathcal{L}_i^2=(1/2)\Sigma_{\mu\nu}(L_i^{\mu\nu})^2$ is the invariant formed from the asymmetric rotor space angular momentum tensor. Γ in Eq. (1) and $(1/\tilde{g})$ in Eq. (2) denotes the strength of quantum fluctuations. In this paper, we assume a ferromagnetic interaction among the spins/rotors of the form[5]

$$J_{ij}=\frac{1}{r_{ij}^{d+\sigma}},$$

with σ always positive (>0) for the thermodynamic stability and r_{ij} denotes the separation between two spins/rotors in units of the lattice spacing. Our aim is to probe the nature of quantum phase transition (QPT) from the ferromagnetic to the paramagnetic phase, driven by quantum fluctuations in the presence of the long-range spatial interaction. One should note here that the classical Ising chain ($\Gamma=0$) with inverse square interaction ($d=\sigma=1$) is relevant to the theory of the Kondo effect in metals.[6]

In Sec. II, we discuss the mean field theory results for the quantum transitions of Eqs. (1) and (2). The essential results are the following: for values of $\sigma\geqslant 2$, the transition is a short-range quantum transition which has been extensively studied for decades.[3] However, for $\sigma<2$, when long-range effects are relevant, the exponents of the zero-temperature transition depend on the parameter σ in a nontrivial fashion. More importantly, the dynamical exponent z associated with the quantum critical point is tuned continuously as σ is varied and picks up the usual value of unity only for $\sigma\geqslant 2$. In the sense of universality, QPT in these models is equivalent to thermal phase transitions in an equivalent $(d+1)$-dimensional classical model with long-range interaction in d-spatial dimensions and short-range ferromagnetic interaction in the $(d+1)$-th (Trotter) dimension. The short-range nature of the interaction in the Trotter dimension modifies the quantum hyperscaling relation in a nontrivial fashion. As mentioned already, one recovers the short-range results for $\sigma=2$. We also evaluate the exponents using renormalisation group (RG) equations up to the one-loop order and comment on the modifications of these results expected at the two-loop level.

In Sec. III, we study the one-dimensional long-range quantum Ising model that exhibits a zero-temperature transition for all values of σ, but a transition at *finite* temperature occurs only if $\sigma<1$. The case $\sigma=1$ is of special interest. The phase diagram of the quantum long-range Ising system (with $\sigma=1$) in the (T-Γ) plane shows a line of Kosterlitz-

Thouless (KT) transitions[7] because any finite temperature transition in the above case is argued to be of KT type even though the zero-temperature transition is a continuous order-disorder transition. We locate a region below the phase boundary where the spatial correlation length and the susceptibility diverges and the width of this region decreases monotonically with increasing Γ. This is our most important observation.

Similarly, the one-dimensional (1D) quantum rotor for $n>2$, discussed in Sec. IV, exhibits a zero-temperature transition for all values of σ except at $\sigma=2$ when the transition is forbidden (as expected from the Mermin-Wagner theorem[8]). However, the quantum rotors show a very rich and interesting phase diagram which we explore extending the renormalization group (RG) equations for the short-range quantum nonlinear sigma (QNLσ) model[9] to the long-range case.

II. MEAN FIELD THEORY AND ONE-LOOP RG

Let us now look at the n-component quantum rotor action in the zero temperature limit in the Fourier (q,ω) representation (ω denoting the continuous Matsubara frequencies),

$$\mathcal{A}=\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\int\frac{d\omega}{2\pi}[\tilde{g}\omega^2+r+aq^\sigma+bq^2]\mathbf{x}_{\mathbf{q}}(i\omega)\mathbf{x}_{-\mathbf{q}}(-i\omega)+u\int\frac{d\omega_1}{2\pi}\cdots\frac{d\omega_4}{2\pi}\int\frac{d^dq_1}{(2\pi)^d}\cdots\frac{d^dq_4}{(2\pi)^d}\delta^d(\mathbf{q}_1+\cdots+\mathbf{q}_4)\delta(\omega_1+\cdots+\omega_4)[\mathbf{x}_{\mathbf{q}_1}(i\omega_1)\cdot\mathbf{x}_{\mathbf{q}_2}(i\omega_2)]\times[\mathbf{x}_{\mathbf{q}_3}(i\omega_3)\cdot\mathbf{x}_{\mathbf{q}_4}(i\omega_4)], \tag{3}$$

where $a,b>0$, r $(=\Gamma-\Gamma_c^{\text{mean field}}$ for $n=1)$ and u are the mass and coupling term, respectively.[3] If $\sigma\geqslant 2$, the leading q^2 term in the action matters and action (3) then describes the short-range quantum models.[3] Clearly, $n=1$ corresponds to the long-range quantum Ising case. Let us now look at the free (Gaussian) propagator (with a set equal to unity)

$$G_0(p,\omega,r)=\frac{1}{q^\sigma+\tilde{g}\omega^2+r}\quad\text{for}\quad\sigma<2. \tag{4}$$

We can now use this propagator to probe the divergence of the mass renormalization term (at the critical point, $r=0$); it gives the lower critical dimension for a QPT for fixed σ as

$$d_l=\frac{\sigma}{2}. \tag{5}$$

The above equation equivalently yields the lower critical range of interaction σ_l in a given dimension d. A one-dimensional quantum Ising chain can be mapped to a two-dimensional classical Ising system with long-range interaction in the spatial direction, and thus it exhibits a transition for all values of σ, even when the spatial interaction is short ranged.

Since the transitions in short-range quantum Ising and rotor systems have been studied extensively over decades,[3] we shall rather concentrate on the case where the interaction is nontrivially long ranged $(\sigma<2)$, and explore the zero-temperature transition behavior. Using the Gaussian propagator given above, one readily finds the mean field long-range quantum critical exponents given as

$$\gamma=1,\quad \nu=\frac{1}{\sigma},\quad \eta=2-\sigma. \tag{6}$$

This mean-field exponents clearly depend on the value of σ and in the limit $\sigma\to 2$, they assume the short-range mean field values.

An interesting observation is that the mean field dynamical exponent z associated with the QPT, is found to be $z=\sigma/2$ (using the dispersion relation $\omega\sim q^z$) and thus the dynamical exponent varies continuously with σ and is always less than unity for $\sigma<2$. Obviously $z=1$ for $\sigma=2$. That the dynamical exponent for $\sigma<2$ is less than unity is a manifestation of the fact that the correlation in the time direction grows much slower than that in the spatial direction due to the long-range nature of interaction.

At or below the upper critical dimension one can always invoke the hyperscaling relation, which is $2-\alpha=\nu(d+z)$, as long as quantum dynamics is conventional (i.e., $\xi_\tau\sim\xi^z$). One thus finds

$$2-\alpha=\nu(d+\sigma/2). \tag{7}$$

Demanding that $\alpha=0$ at the upper critical dimension d_u and using $\nu=1/\sigma$, we obtain

$$d_u=\frac{3\sigma}{2}. \tag{8}$$

Equation (8) yields an upper critical dimension for a given σ and an upper critical range of interaction in a given dimension d and $d_u=3$ for $\sigma=2$.

To evaluate the nontrivial exponents, one has to perform the perturbative RG calculations around the upper critical dimension d_u in the same spirit as in the classical case.[5] Here, we assume dimensionality as a continuous variable and the the one-loop RG equations are written as (with the appropriate redefinition of the variable u)

$$\frac{dr}{dl}=\sigma r+(n+2)\frac{u}{(1+\tilde{g}+r)}, \tag{9}$$

$$\frac{du}{dl}=\epsilon u-(n+8)\frac{u^2}{(1+\tilde{g}+r)^2}. \tag{10}$$

From Eqs. (9) and (10), we find that for $\epsilon<0$ (i.e., for $d>d_u$), the Gaussian fixed point $(r^*=0,u^*=0)$ is stable and the mean field exponents given above hold. The recursion relations for the $\epsilon=0$ case, leads to logarithmic correction on top of the mean field quantum critical behavior as in the classical case.[5] For $\sigma>2$, the exponents always assume short-range values.

For $\epsilon>0$ (i.e., below the upper critical dimension for a given σ), the Gaussian fixed point is unstable with respect to fluctuation (u) and we find a nontrivial fixed point [$u^* \sim \epsilon/(n+8)$]. One can easily linearize the recursion relations near this nontrivial fixed point and thus obtains the correlation length exponent ν given as

$$\nu = \frac{1}{\sigma} + \frac{n+2}{n+8}\epsilon. \tag{11}$$

For systems with long-range interaction, the Fisher exponent η associated with the spatial correlation function is determined by setting the coefficient of the leading q^σ term to unity. One finds up to this order of perturbation $\eta = 2-\sigma$. Invoking the scaling relation for the susceptibility exponent $\gamma = (2-\eta)\nu$, we find

$$\gamma = 1 + \frac{(n+2)}{(n+8)}\frac{\epsilon}{\sigma}. \tag{12}$$

Evaluation of the exponents in the *two*-loop order is quite complicated, and we shall just mention here some important comments regarding the subtleties involved. Up to the one-loop order, we can retrieve the short-range exponents by substituting $\sigma=2$ in Eqs. (11), (12). This is no longer true in the two-loop order as in the corresponding classical case.[5] This is explicitly seen considering the Fisher exponent η. For any value of $\sigma<2$, η sticks to its mean-field value $2-\sigma$ (because the renormalization does not generate new q^σ terms) whereas for $\sigma \to 2$, η picks up its conventional short-range quantum value ($\eta_{\rm SR} = [(n+2)/2(n+8)^2]\epsilon^2$, $\epsilon=(3-d)$, which is also the higher-dimensional classical η for short-range interaction). In the $\sigma \to 2$ limit, the short-range term becomes important and this contribution comes from the renormalization of the short-range coefficient b in Eq. (3). Clearly there is a nonuniformity involved here. Hence, one cannot retrieve the short-range η (and thus the other exponents) by setting $\sigma=2$ in its long-range value in the two-loop order.

The dynamical exponent z retains its mean-field value $z=\sigma/2$ up to the one-loop order. For a change of length scale by a factor $\exp(\delta l)$ the coefficient $\tilde{g}$ scales as $\tilde{g}' = \tilde{g}\exp^{(\sigma-2z)\delta l}$. Demanding that $\tilde{g}$ scales to a fixed point value one gets $z=\sigma/2$ in the one-loop order. In the two-loop order the $q \to 0$ part of the self-energy term renormalizes the coefficient of ω^2 term $\tilde{g}$ and thus modifies z. For any value of $\sigma<2$, we expect that the dynamical exponent will be of the form $z=\sigma/[2-\tilde{\eta}(\epsilon)]$, In the $\sigma\to 2$, $\tilde{\eta} \to \eta_{\rm SR}$ and this leads to $z=\sigma/(2-\eta_{SR})$ where $\eta_{\rm SR}$ is as given above. On the other hand, as $\sigma \to 2$ the short-range term will be important in defining z and hence z will be unity (ω and q scaling identically). Thus we expect that the similar nonuniformity as seen in η at $\sigma=2$ will also appear in z.

III. ONE-DIMENSIONAL QUANTUM ISING MODEL

Let us now focus on the most interesting part of our study: We have argued already that the one-dimensional transverse Ising model with long-range interaction exhibits a zero-temperature order-disorder transition for all values of σ. Equation (8) for the upper critical range readily indicates that in $d=1$ the quantum transitions with values of $\sigma \leqslant 2/3$ are described by long-range mean-field theory whereas higher values of σ correspond to nontrivial exponents.

On the other hand, the one-dimensional long-range classical Ising chain shows a nonzero T_c only if $\sigma<1$.[6,10,11] For $\sigma=1$ the classical Ising chain undergoes a KT transition[12–16] which is associated with a discontinuous jump in the magnetization. This transition is driven by spin flips or kinks interacting via logarithmic potential. A very recent extensive study[17] is in accurate quantitative agreement with the above RG calculations[14] and moreover provides strong numerical evidence in favor of this one-dimensional KT transition.

Coming back to the quantum chain, the finite-temperature transition of the quantum Ising chain is always determined by the thermal fixed point since the quantum fluctuations are irrelevant near the finite-temperature transition of any pure quantum system. In other words, close to the finite transition transition the spatial correlation length grows (and diverges at the transition point) so that one can neglect the finite Trotter dimension arising due to the quantum term. One can thus neglect quantum effects and apply purely classical theory of phase transition at and near the finite-temperature transition in quantum systems.[3] There is a crossover region [typically given by $T \sim (\Gamma-\Gamma_c)^{z\nu}$, where z,ν are the zero-temperature exponents, not exactly known in the present case] near the phase boundary in the T-Γ plane. In this region, we can treat the systems within a classical framework. The width of this region increases for small Γ and high T.

We can thus conclude that any finite-temperature transition in the present case ($\sigma=1$) is definitely a KT transition at a reduced critical temperature $T_c(\Gamma)$ even though the quantum transition at $T=0$ is a continuous order-disorder transition for all values of σ. These KT transitions should be described in terms of the classical RG equations,[6,12] generalized to a reduced temperature $T(\Gamma)$. If one now probes the critical region at any finite temperature for $\sigma=1$ at a fixed Γ, the corresponding KT transition is then described in terms of the following RG equation:

$$\frac{d\phi}{dl} = -4y^2, \tag{13}$$

$$\frac{dy}{dl} = -y\phi, \tag{14}$$

$$\frac{dh}{dl} = h(1-y^2), \tag{15}$$

where $\phi = [1/T(\Gamma)-1]$ is the temperature field, $y=\exp[-\mu/KT(\Gamma)]$ is the chemical potential field with μ being the chemical potential associated with the local energy to form a kink in absence of the static external magnetic field h.

As mentioned already, the above RG equations (13)–(15) correctly describe the finite temperature transition in the present case and they should be valid in the crossover region discussed above. Moreover, the above set of equations are associated with a region below the phase boundary (discussed below) with diverging spatial correlation length and static susceptibility and quantum effects are truly negligible

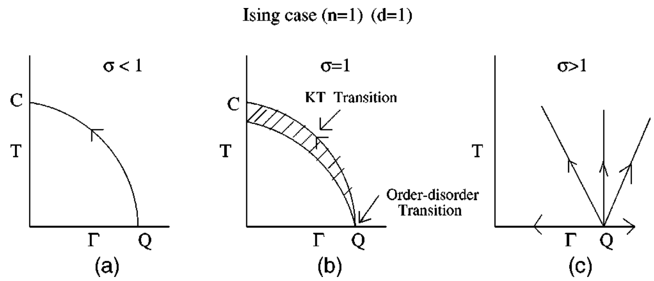

FIG. 1. The schematic phase diagram and RG flow in T-Γ plane of one-dimensional long-range Ising model with different values of σ. For $\sigma<1$ (a) the system has both quantum and thermal transitions. (b) The case $\sigma=1$ where transition at any finite temperature is a KT transition. In the shaded region, the spatial correlation length and the susceptibility diverge. For $\sigma>1$ there is a quantum critical point but no finite temperature transition (c). Q and C denote the quantum and classical transition points respectively.

in this region. We use the above generalized RG Eqs. (13)–(15) to make an approximate estimate of the width of this interesting region. This is a good approximation as mentioned already, in the small Γ and high T region where the crossover region discussed above is wider. The schematic phase diagram of the quantum Ising chain in the Γ-T plane, for different values of σ are shown in Figs. 1(a) and 1(b) corresponds to the especially important case $\sigma=1$ where the phase diagram shows a line of KT transitions.

The RG Eqs. (9)–(11), which are valid strictly at and close to the phase boundary in the Γ-T plane (where quantum effects are negligible), can be integrated at or below the reduced critical temperature $T_c(\Gamma)$.[14] At $T_c(\Gamma)$ the spatial correlation function $G(r)$ falls off as $G(r)\sim 1/\ln r$. This logarithmic fall at the critical point has been supported (over a considerable range of r) in a recent extensive study[17] on the corresponding classical system.

Below but close to the reduced critical temperature we have an algebraic decay given as

$$G(r)\sim\frac{4\sqrt{\Delta t}}{r(4|\Delta t|)^{1/2}},\tag{16}$$

where $\Delta t=[T(\Gamma)-T_c(\Gamma)]/T_c(\Gamma)$. This algebraic form goes over to the logarithmic fall as $T(\Gamma)\rightarrow T_c(\Gamma)$ from below. We evaluate the susceptibility using the fluctuation-dissipation theorem with this algebraic form of the correlation function and, consequently, it is found that for $-1/16<\Delta t<0$ the susceptibility clearly diverges but it is finite and proportional to $1/(4\sqrt{|\Delta t|}-1)$ for $\Delta t<-1/16$. As Δt approaches $(-1/16)$ from below, the behavior becomes $(\Delta t+\frac{1}{16})^{-1}$. In Fig. 1(b) the shaded region with lower boundary given by the equation $[(T_c(\Gamma)-T(\Gamma))]/T_c(\Gamma)=\frac{1}{16}$, corresponds to the critical region discussed above where spatial correlation length and susceptibility clearly diverges. This intermediate region close to the phase boundary in the (Γ-T) plane with algebraically falling spatial correlation function (with exponent determined by T and Γ) and diverging magnetic susceptibility is the most important aspect of this model at finite temperature. We reiterate that our estimation depends on the RG equations we use, which are valid strictly in the vicinity of the phase boundary and especially for small Γ and high T region. We mention that the occurrence of such a region in the classical case ($\Gamma=0$) is proved using rigorous analytical[16] and numerical studies.[17]

Clearly, the width of this region with diverging susceptibility decreases monotonically as Γ increases, and such a region is absent for the $T=0$ transition. Note that unlike the $n=d=2$ short-range rotor case (discussed later), we have true long-range order, i.e., nonzero spontaneous magnetization in the ordered phase. The pure classical KT transition ($\Gamma=0$) is associated with a discontinuous jump in magnetization;[6] one can argue in a similar spirit that this discontinuity decreases monotonically with Γ and the pure quantum transition is a continuous one.

A few important comments are necessary, following a very recent extensive study[17] on inverse-square classical Ising system. That study as mentioned already, strongly supports the results of the the RG calculation and KT transition scenario that we employ here. The results of that work seem to indicate that the shaded region in Fig. 1(b) should be wider than what is predicted from RG calculations but the qualitative shape of this special region in T-Γ plane remains unaltered. One can easily estimate using the RG Eqs. (13)–(15) the value of temperature $T_1(\Gamma)$, at or below which the exponent describing the power-law correlation is no longer T and Γ dependent. It would be slightly higher than what is predicted in Ref. 17 for $\Gamma=0$ case. Moreover, one has to be careful in our case ($\Gamma\neq 0$) because as we move further down the phase boundary the quantum effects are no longer negligible and the RG equations may not be valid.

We should mention at this point that similar studies can be made for short-range quantum rotor for $d=n=2$ where as well any finite temperature transition is of KT type and one obtains a similar phase diagram as in Fig. 1(b). This can be verified using QNLσ model RG equations[9] (for dimensionless temperature) for $n=d=2$. But one should note the difference that in the short-range XY case the spatial correlation shows an algebraic fall at $T_c(\Gamma)$. Moreover the entire quasi long-range ordered phase is associated with a diverging spatial correlation length and susceptibility. It has a true long-range order only along the $T=0$ line. One should also note here that a similar phase diagram as Fig. 1(b) is also found in the case of a Josephson-Junction array in two spatial dimensions (see Fig. 5 of Ref. 18). In that case, any finite-temperature transition corresponds to a two-dimensional classical KT transition in XY systems but the zero-temperature transition is a $(2+1)$-dimensional classical XY transition. However, the shaded region with diverging response function is a special feature of the model we study, and is not observed in that case.

Following Kosterlitz[12] to describe the finite-temperature transition for $\sigma<1$, one could use a similar set of equations as Eqs. (13)–(15) with the reduced temperature $T(\Gamma)$. The solutions of these equations are nonanalytic as $\sigma\rightarrow 1$ (which is a signature of the KT transition for $\sigma=1$, discussed above). For $\sigma<1$, we have a critical fixed point at $T^*(\Gamma)$ and unstable fixed point at $T(\Gamma)=\infty$. $T^*(\Gamma)$ corresponds to the finite temperature transitions shown in Fig. 1(a) and its value gets smaller with increasing Γ.

IV. ONE-DIMENSIONAL QUANTUM ROTOR MODEL

We shall now consider the transition in the one-dimensional quantum rotor model with $n \geqslant 2$ and long-range interaction. One should note here that the short-range interacting quantum rotor for $n > 2$ does not show a QPT in one dimension[3] whereas the long-range system shows a nontrivial transition even for $d=1$. The exponents at the mean-field level and also close to the upper-critical dimension are already discussed in the Sec. II.

The zero-temperature transition for the one-dimensional case can be investigated extending the one-loop renormalization group equations of the QNLσ model[9] (which is an appropriate field theoretical description of the quantum rotors in low dimensions) to the present case. We obtain for $d=1$ (see the Appendix)

$$\frac{dg}{dl} = -\left(1-\frac{\sigma}{2}\right)g + \frac{1}{4\pi}(n-2)g^2, \tag{17}$$

where g (defined in the Appendix) here corresponds to the dimensionless quantum coupling. Clearly, for $n>2$ and $\sigma<2$, the critical coupling (g_c) is given by the unstable fixed point of Eq. (17) at $g_c = \{2\pi(2-\sigma)\}/(n-2)$. For $n=2$ and $\sigma=2$, g_c happens to be undefined which clearly corresponds to the quantum KT transition of one-dimensional two-component short-range quantum rotor[3] where local defects drive the transition. The transition is forbidden for $\sigma \geqslant 2$. This is easily understood thinking in terms of the equivalent classical Hamiltonian. For $\sigma=2$ the equivalent two-dimensional system is truly short ranged with continuous symmetry and thus transition is forbidden as expected from the Mermin-Wagner theorem. We find from Eq. (17) that the correlation length exponent $\nu = 1/(1-\sigma/2)$ and the Fisher exponent $\eta = 2-\sigma$ (which is in general valid for long-range interacting system[5]). The method we use, is valid for $n>2$ and σ smaller than but close to 2.

The corresponding one-loop RG equation for the temperature in the one-dimensional classical n-vector systems with $n \geqslant 1$ reads (Kosterlitz[12])

$$\frac{dT}{dl} = -(1-\sigma)T + (n-1)\frac{T^2}{2\pi^2}. \tag{18}$$

The above equation yields a nonzero T_c for $\sigma<1$ and $n>1$; the $\sigma=1$ and $n=1$ case corresponds to the KT transition discussed earlier. The phase diagram in the (g,T) plane and corresponding RG flow equations are shown in Fig. 2. It would be definitely intriguing to ask at this point about the nature of the phase diagram and the form of the RG Eq. (17) for a one-dimensional system with $1<n<2$ where Eq. (18) definitely holds. We have mentioned that our QNLσ model approach is valid for $n>2$ and thus no conclusive inference should be drawn for $n<1<2$ with a general $\sigma<2$. However, the short-range ($\sigma=2$) one-dimensional systems with $1<n<2$ should exhibit a quantum transition which is topological in nature. For related studies on short-range classical and quantum systems with $n<2$, we refer to Refs. 19 and 20, respectively.

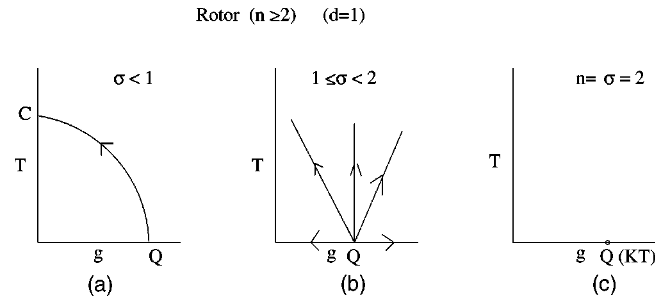

FIG. 2. The schematic phase diagram and RG flow (in the g-T plane) of quantum rotor systems ($n \geqslant 2$) with different values of σ. For $\sigma<1$ (a), the system undergoes both quantum and thermal transitions. For $1 \leqslant \sigma < 2$ (b) there is no finite-temperature transition. The special case $\sigma=n=2$ corresponds to the quantum KT transition of the quantum XY rotor (c).

V. CONCLUSION

In conclusion we have studied very simple quantum systems with nontrivial long-range interactions and find very interesting phase diagrams for different values of the parameter σ. The main results and the methods and the approximations involved are already discussed in the corresponding sections. These studies are important following the recent upsurge in the theoretical and experimental[1–3] investigations of quantum systems and prolonged interest in the inverse-square one-dimensional classical systems.

ACKNOWLEDGMENTS

A. D. acknowledges Ralf Hetzel for carefully reading the manuscript and for his important comments.

APPENDIX

In this section, we shall study one-dimensional long-range quantum rotor models using a QNLσ model approach. Our study will follow Ref. 9 which uses the momentum-shell method for the d-dimensional short-range quantum rotor systems. The essential mathematical steps that lead us to the recursion relation (17) will be briefly indicated here. We consider the effective zero-temperature continuum action of a n-component one-dimensional quantum rotor with long-range interaction in the presence of a magnetic field H, i.e.,

$$S_{\mathrm{eff}}/\hbar = -\frac{J}{\hbar}\int d\tau \int dr \int dr' \frac{[\mathbf{x}(\mathbf{r},\tau) - \mathbf{x}(\mathbf{r}',\tau)]^2}{(r-r')^{1+\sigma}} - \frac{\tilde{g}}{\hbar}\int d\tau \int dr \left(\frac{\partial \mathbf{x}}{\partial \tau}\right)^2 - \frac{H}{\hbar}\int d\tau \int dr\, \sigma(r,\tau), \tag{A1}$$

where we have decomposed vector $\mathbf{x}$'s in the form $\{\pi, \sigma\}$ where π's are $(n-1)$ component vectors. Note that here τ

represents the imaginary time direction and at $T=0$ this direction is infinite. The spatial integrations are cutoff at short-distance by Λ. We can now eliminate σ's from the action using the constraint $\pi^2(r,\tau)+\sigma^2(r,\tau)=1$, which is valid for each r and τ. So the partition function becomes

$$\mathcal{Z}=\int \mathcal{D}\pi(r,\tau)\prod_{r,\tau}\frac{1}{\sqrt{1-\pi^2(r,\tau)}}\times\exp\left\{-\frac{1}{g}\int dx_0\int dx\int dx'\times\left(\frac{[\pi(x,\tau)-\pi(x',\tau)^2]}{(x-x')^{1+\sigma}}+\frac{\sqrt{[1-\pi^2(x,\tau)]}\sqrt{[1-\pi^2(x',\tau)]}}{(x-x')^{1+\sigma}}+\cdots\right)\right\}\times\exp\left[-\frac{1}{g_\tau}\int dx_0\int dx\left(\frac{\partial\pi}{\partial x_0}\right)^2+\frac{[\pi(\partial\pi/\partial x_0)]^2}{1-\pi^2}+\cdots\right]\times\exp\left(h\int dx_0\int dx\sqrt{1-\pi^2}\right), \tag{A2}$$

where

$$g=\frac{2\hbar}{J}\Lambda^{1+z-\sigma}, \tag{A3a}$$

$$g_\tau=\frac{\Lambda^{1-z}}{\tilde{g}}\sim g\Lambda^{\sigma-2z}, \tag{A3b}$$

$$h\sim\Lambda^{-(1+z)}. \tag{A3c}$$

Now x and x_0 are defined to be dimensionless variables. With this choice, the wave-vector cutoff in Eq. (A2) is unity. Note in contrary to the short-range case,[9] the long-range case is anisotropic in space and time. Imaginary time τ here denotes an additional dimension with an effective short-range interaction. Following the discussions in Sec. II, we now set $z=\sigma/2$ (which is correct at least to the first-loop order), so that the additional g_τ appearing in the present case, has the same scaling behavior as g. Henceforth, we shall look at the renormalization of g.

Using the standard technique involved in the QNLσ model, we expand the factors $\sqrt{1-\pi^2}$ and $(1-\pi^2)^{-1}$ in Eq. (A2) and decompose π's in Fourier modes $\pi(q,\omega)$, ω being the continuous Matsubara frequencies. As in the short-range case one can integrate out high wave-vector modes (with $e^{-\delta l}<q\leqslant 1$) to generate an effective action involving low wave-vector modes.[8] With appropriate rescaling of momenta and the spin-field we arrive at the recursion relations given as

$$\left(\frac{1}{g}\right)'=\zeta^2 e^{-(1+\sigma+z)\delta l}\left(\frac{1}{g}+I_{\rm loop}\right), \tag{A4}$$

$$h'=\zeta^2 e^{-(1+z)\delta l}\left(h+\frac{gh}{2}(n-1)I_{\rm loop}\right), \tag{A5}$$

where ζ is the spin-rescaling factor. The loop integral in the present case is given as

$$I_{\rm loop}=\int\frac{d\omega}{2\pi}\frac{dq}{2\pi}\frac{1}{q^\sigma+\omega^2+hg}=\frac{1}{2}\int_{e^{-\delta l}}^{1}\frac{dq}{2\pi}\frac{1}{\sqrt{q^\sigma+hg}}=\frac{1}{4\pi}\frac{1}{\sqrt{1+hg}}\delta l. \tag{A6}$$

One should note here that in the quantum long-range case, we have an additional $[-(g/h)I_{\rm loop}]$ contribution in the recursion relation of h in compared to the Ref. 12. This term arises due to the local quantum term $[\pi(\partial\pi/\partial x_0)]^2$ of action (A2) and hence does not show up in the corresponding classical one-dimensional case.[12] The field rescaling term ζ is determined demanding $h'=\zeta h$ and we find using Eq. (A5)

$$\zeta=e^{(1+z)\delta l}\left(1-\frac{g}{2}(n-1)I_{\rm loop}\right). \tag{A7}$$

Using Eq. (A4) and $z=\sigma/2$, we obtain the recursion relation of coupling $1/g$ given as

$$\left(\frac{1}{g}\right)'=e^{(1-\sigma/2)\delta l}\left(\frac{1}{g}-(n-2)I_{\rm loop}\right). \tag{A8}$$

Expanding the exponential in the $e^{-\delta l}=1-\delta l$, and using the form of $I_{\rm loop}$ from Eq. (A6) we find the differential form of the recursion for the coupling g in the $\delta l\rightarrow 0$ limit (with $h\rightarrow 0$) given as

$$\frac{dg}{dl}=\left(\frac{\sigma}{2}-1\right)g+\frac{(n-2)}{4\pi}g^2. \tag{A9}$$

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