Elementary excitations in the spin-tube and spin-orbit models

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A bond operator representation for three $S=\frac{1}{2}$ spins is obtained along the same line as the two-spin cases. With the spin and the chirality freedom expressed by the same bond operators, we studied the elementary excitations of the spin-tube and the spin-orbit models. For 3-leg spin ladders and general spin ladders with an odd number of legs and periodic boundary conditions in the rung direction, the spinonlike excitations, which carry $\frac{1}{2}$ spin and chirality freedoms, are calculated by a variational ansatz. The magnonlike excitations, which denote the change of a dimerized bond from spin singlet to spin triplet and/or from chirality triplet to chirality singlet, or from one kind of chirality triplet to another chirality triplet, with different z-component, are also studied. Bound states of spinonlike excitation pairs exist near the vicinity of momentum zone center $\pi/2$. The bond operators are also applied to other general spin-orbit models with spontaneously dimerized ground states, and the elementary excitations are studied.

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I. INTRODUCTION

One-dimensional and quasi-one-dimensional quantum spin systems have been extensively studied in the past years. More recently, a combination of experimental and theoretical efforts has produced significant advances in the realization and understanding of the properties of spin ladders.[1] Various double-chain models[2–7] with $S=\frac{1}{2}$ have been studied in relation to Haldane conjecture.[8] However, when the number of chains is odd, the case becomes different. In the case of open boundary conditions in the rung direction, the spin ladder is massless and effectively equivalent to a spin-$\frac{1}{2}$ chain. While for the periodical boundary conditions in the rung direction, the spin-ladders are frustrating in the rung direction and show different properties such as spin-gap and exponential decay correlation functions. Kawano and Takahashi[9] considered such a three-leg model (it is often called spin tube if the periodic boundary conditions are applied in the rung direction) with the next-nearest neighbor interactions included:

$$H=\sum_{i=1}^{N}\sum_{p=1}^{3}\left(J_0\vec{S}_{ip}\cdot\vec{S}_{ip+1}+J_1\vec{S}_{ip}\cdot\vec{S}_{i+1p}+J_2\vec{S}_{ip}\cdot\vec{S}_{i+2p}\right), \tag{1}$$

where, periodic boundary condition in the rung direction is applied, $p+3=p$, J_0 is the intrarung interaction, and J_1 and J_2 are the nearest- and next-nearest-neighbor interactions along the chain. In the limit of strong intrarung interaction, it can can be mapped to a spin-orbital model[9–11]

$$H_{\text{eff}}=\frac{J_1}{3}\sum_i \mathcal{S}_i\cdot\mathcal{S}_{i+1}[1+4(\tau_i^+\tau_{i+1}^-+\tau_i^-\tau_{i+1}^+)]$$
$$+\frac{J_2}{3}\sum_i \mathcal{S}_i\cdot\mathcal{S}_{i+2}[1+4(\tau_i^+\tau_{i+2}^-+\tau_i^-\tau_{i+2}^+)], \tag{2}$$

where $\mathcal{S}$ and τ are the effective spin and chirality, respectively. At $J_1=2J_2$, the effective Hamiltonian is exactly solvable and has a spontaneously dimerized ground state, similar to what occurs in the single spin-$\frac{1}{2}$ antiferromagnetic chain.[12] By the DMRG alogrithm Kawano and Takahashi also showed that there is no transition between $0\leqslant J_2/J_1\leqslant\frac{1}{2}$. On the other hand, partly stimulated by the progress in the experimental study of the the quasi-one-dimensional spin gap materials $Na_2Ti_2Sb_2O$ and NaV_2O_5,[13] some other spin-tube models, double chain models with biquadratic interactions and spin-orbit models are extensively studied. For example, the SU(8) quantum spin tube,[14] the spin ladder with single-ion anisotropy and bond alternation,[15] have been studied by Bethe ansatz or bosonization technique. As a generalized spin-orbit model and a candidate model of $Na_2Ti_2Sb_2O$ and NaV_2O_5, the coupled XXZ chains with biquadratic interactions with a Hamiltonian

$$H=\sum_i\left[u+\gamma(S_i^+S_{i+1}^-+S_i^-S_{i+1}^+)+J_zS_i^zS_{i+1}^z\right]$$
$$\times\left[v+\alpha(\tau_i^+\tau_{i+1}^-+\tau_i^-\tau_{i+1}^+)+J_z'\tau_i^z\tau_{i+1}^z\right], \tag{3}$$

was studied by the bosonization technique.[11] With different symmetries between the spin and orbital variables, the spin-orbit models have different properties. The SU(4) point ($\gamma=\alpha=J_z/2=J_z'/2$, $u=v=J_z/4$) is Bethe ansatz solvable and critical with three gapless bosonic modes.[16] Many studies focus on this point or near this point.[17,18] A phase diagram of the SU(2)×SU(2) spin-orbit model ($\gamma=J_z/2,\alpha=J_z'/2$) was given by Itoi *et al.*[19] by the bosonization and DMRG technique. When the symmetry of the orbital part is XY-like ($\gamma=J_z/2,J_z'=0$), another phase diagram is given by Pati and Singh.[20] At some special points, the spin-orbit model may be exactly solvable.[21–24] Kolezhuk and Mikeska[21,22] found a family of spin-ladder models which can be exactly solvable by the matrix product approach. For the checkboard-dimer model, they found that the elementary excitation is neither a magnon nor a spinon, but a pair of propagating triplet or singlet solitons connecting the two spontaneously dimerized ground states, belonging to the non-Haldane gap behavior predicted by Nersesyan and Tsvelik.[25] By a strong-coupling expansion and numerical diagonalizations, Cabra *et al.*[26] studied the excitations of the spin-tube model (1) with $J_2=0$, they obtained the magnonlike and spinonlike excitation spectra. In this paper, starting from the three-leg spin ladder model, we obtain a generalized bond operator representation for three spin-$\frac{1}{2}$ spins along the same line to that of two spin cases.[27,28] With the spin and the chirality operators expressed by the same bond operators, the representations are applied to the 3-leg spin ladders and then in general, to spin ladders with any odd number of legs and periodical boundary conditions in the rung direction. The spinonlike and magnonlike elementary excitations are studied by a variational ansatz. These excitations provide another example of non-Haldane spin-liquid properties. The obtained bond operator representations can also be applied to the general spin-orbit models.

II. BOND OPERATOR REPRESENTATION OF THREE SPIN-$\frac{1}{2}$ SPINS

For three $S=\frac{1}{2}$ spins, S_1, S_2 and S_3, which form an equilateral triangle as a rung of the spin ladder, the ground state with energy $-\frac{3}{4}J_0$ is fourfold degenerate, composed of two doublets of spin-$\frac{1}{2}$ excitations, corresponding to the right and left chirality. The excited states are spin-$\frac{3}{2}$ quadruplet with energy $\frac{3}{4}J_0$. Since the quadruplet has much higher energy in the large J_0 limit, we can project out the quadruplet in studying the zero and low-temperature properties. Just as Sachdev and Bhatt[27] did in the case of two $S=\frac{1}{2}$ spins, we can introduce four bosonic bond operators to denote the two doublets:[9–11]

$$\begin{aligned}
|\uparrow L\rangle &= u_l^{\dagger}|0\rangle=\frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle+j|\uparrow\downarrow\uparrow\rangle+j^{-1}|\downarrow\uparrow\uparrow\rangle),\\
|\downarrow L\rangle &= d_l^{\dagger}|0\rangle=\frac{1}{\sqrt{3}}(|\downarrow\downarrow\uparrow\rangle+j|\downarrow\uparrow\downarrow\rangle+j^{-1}|\uparrow\downarrow\downarrow\rangle),\\
|\uparrow R\rangle &= u_r^{\dagger}|0\rangle=\frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle+j^{-1}|\uparrow\downarrow\uparrow\rangle+j|\downarrow\uparrow\uparrow\rangle),\\
|\downarrow R\rangle &= d_r^{\dagger}|0\rangle=\frac{1}{\sqrt{3}}(|\downarrow\downarrow\uparrow\rangle+j^{-1}|\downarrow\uparrow\downarrow\rangle+j|\uparrow\downarrow\downarrow\rangle),
\end{aligned}\tag{4}$$

where $\uparrow$ and $\downarrow$ denote the spin states and L and R the left and right chirality. $|0\rangle$ is the vacuum state and $j=\exp[i(2\pi/3)]$. With these definitions, S_p with $p=1,2,3$ can be expressed as

$$\begin{aligned}
S_p^{+} &= -\frac{1}{3}(u_l^{\dagger}d_l+u_r^{\dagger}d_r)+\frac{2}{3}j^{2p}u_l^{\dagger}d_r+\frac{2}{3}j^{p}u_r^{\dagger}d_l,\\
S_p^{-} &= (S_p^{+})^{\dagger},\\
S_p^{z} &= \frac{1}{6}(u_l^{\dagger}u_l+u_r^{\dagger}u_r-d_l^{\dagger}d_l-d_r^{\dagger}d_r)+\frac{1}{3}j^{2p}(d_l^{\dagger}d_r\\
&\quad -u_l^{+}u_r)+\frac{1}{3}j^{p}(d_r^{\dagger}d_l-u_r^{+}u_l).
\end{aligned}\tag{5}$$

A complete bond operator representation, with the remaining $S=\frac{3}{2}$ eigenstates included, can also be obtained.[29] The restriction that the physical states are either of the doublets leads to the constraint

$$u_l^{\dagger}u_l+u_r^{\dagger}u_r+d_l^{\dagger}d_l+d_r^{\dagger}d_r=1.\tag{6}$$

The total spin of every triangle, $\mathcal{S}=S_1+S_2+S_3$, can be obtained as

$$\begin{aligned}
\mathcal{S}^{+} &= -(u_l^{\dagger}d_l+u_r^{\dagger}d_r),\\
\mathcal{S}^{-} &= -(d_l^{\dagger}u_l+d_r^{\dagger}u_r),\\
\mathcal{S}^{z} &= \frac{1}{2}(u_l^{\dagger}u_l+u_r^{\dagger}u_r-d_l^{\dagger}d_l-d_r^{\dagger}d_r).
\end{aligned}\tag{7}$$

For the chirality freedom, defining[9]

$$\begin{aligned}
&\tau^{+}|\cdot L\rangle=0,\quad \tau^{+}|\cdot R\rangle=|\cdot L\rangle,\quad \tau^{-}|\cdot L\rangle=|\cdot R\rangle,\\
&\tau^{-}|\cdot R\rangle=0,
\end{aligned}\tag{8}$$

we can express the chirality operators as

$$\begin{aligned}
\tau^{+} &= u_l^{\dagger}u_r+d_l^{\dagger}d_r,\\
\tau^{-} &= u_r^{\dagger}u_l+d_r^{\dagger}d_l.
\end{aligned}\tag{9}$$

The τ_z can be obtained from the SU(2) algebra. The spin operators and the chirality operators have similar form. With Eqs. (7) and (9), the original spins S_p can be expressed as[11]

$$\begin{aligned}
S_p^{+} &= \frac{1}{3}\mathcal{S}^{+}-\frac{2}{3}j^{2p}\mathcal{S}^{+}\tau^{+}-\frac{2}{3}j^{p}\mathcal{S}^{+}\tau^{-},\\
S_p^{z} &= \frac{1}{3}\mathcal{S}^{z}-\frac{2}{3}j^{2p}\mathcal{S}^{z}\tau^{+}-\frac{2}{3}j^{p}\mathcal{S}^{z}\tau^{-}.
\end{aligned}\tag{10}$$

Although Eqs. (7) and (9) are obtained from the three-leg spin ladders, they are general for all spin ladders with any odd number of legs and periodical boundary conditions in the rung direction. It is remarked that with the same basis states, a fermionic SU(4) representation, similar to Eqs. (7) and (9), was obtained from the group theory anlysis.[18] The concrete representations of S_p [Eqs. (5) and (10)] may be helpful in studying the frustrated Heisenberg models on kagomé and triangular lattices.[30,31] The obtained representations can also be used to study the general spin-orbit models when we regard the chirality freedom as the orbit freedom. In the following, we will use the bond operator representations to study the elementary excitations of the spin-tube and spin-orbit models.

III. ELEMENTARY EXCITATIONS OF SPIN-TUBE AND SPIN-ORBIT MODELS

Substituting the bond operator representations in Eqs. (7) and (9) into the effective Hamiltonian (2) [or substituting Eq. (5) into Hamiltonian (1)], we get

$$H=\sum_i (H_{i,i+1}+\beta H_{i,i+2}), \tag{11}$$

where, we have set $J_1=1$, $J_2/J_1=\beta$, and $H_{i,i+\delta}$ with $\delta=1,2$ are

$$\begin{aligned} H_{i,i+\delta}=&\sum_i \frac{1}{2}\left[\frac{1}{3}(u_{li}^\dagger d_{li}+u_{ri}^\dagger d_{ri})(d_{li+\delta}^\dagger u_{li+\delta}+d_{ri+\delta}^\dagger u_{ri+\delta})+\frac{4}{3}u_{li}^\dagger d_{ri}d_{ri+\delta}^\dagger u_{li+\delta}+\frac{4}{3}u_{ri}^\dagger d_{li}d_{li+\delta}^\dagger u_{ri+\delta}+\text{H.c.}\right]\\ &+\sum_i \left[\frac{1}{12}(u_{li}^\dagger u_{li}+u_{ri}^\dagger u_{ri}-d_{li}^\dagger d_{li}-d_{ri}^\dagger d_{ri})(u_{li+\delta}^\dagger u_{li+\delta}+u_{ri+\delta}^\dagger u_{ri+\delta}-d_{li+\delta}^\dagger d_{li+\delta}-d_{ri+\delta}^\dagger d_{ri+\delta})+\frac{1}{3}(d_{li}^\dagger d_{ri}-u_{li}^\dagger u_{ri})\right.\\ &\left.\times(d_{ri+\delta}^\dagger d_{li+\delta}-u_{ri+\delta}^\dagger u_{li+\delta})+\frac{1}{3}(d_{ri}^\dagger d_{li}-u_{ri}^\dagger u_{li})(d_{li+\delta}^\dagger d_{ri+\delta}-u_{li+\delta}^\dagger u_{ri+\delta})\right]. \end{aligned} \tag{12}$$

At $\beta=\frac{1}{2}$, Kawano and Takahashi[9] showed that the spin tube has two degenerate spontaneously dimerized ground states, corresponding to the Majumdar-Ghosh point[12] in the single spin-$\frac{1}{2}$ antiferromagnetic chain. They also showed by the DMRG technique that there is no transition between $0\leqslant\beta\leqslant\frac{1}{2}$. Using the bond operator representation, which unifies the spin and chirality freedom together, we can easily describe the ground state as

$$\begin{aligned} \Phi_a&=[1,2][3,4]\cdots[2N-1,2N],\\ \Phi_b&=[2,3][4,5]\cdots[2N-2,2N-1][2N,1], \end{aligned} \tag{13}$$

where $[2i-1,2i]$ denotes a bond with a spin singlet and chirality triplet of $\tau_z=0$: $[2i-1,2i]=\frac{1}{2}(u_{l2i-1}^\dagger d_{r2i}^\dagger-d_{r2i-1}^\dagger u_{l2i}^\dagger+u_{r2i-1}^\dagger d_{l2i}^\dagger-d_{l2i-1}^\dagger u_{r2i}^\dagger)|0\rangle$ with $|0\rangle$ the vacuum state. After a direct calculation, we obtain

$$\langle\Phi_a|H|\Phi_a\rangle=-\frac{5}{4}N,\quad \langle\Phi_b|H|\Phi_b\rangle=-\frac{5}{4}N, \tag{14}$$

where N is the number of the bond. Especially, at $\beta=\frac{1}{2}$, we have

$$H|\Phi_a\rangle=-\frac{5}{4}N|\Phi_a\rangle,\quad H|\Phi_b\rangle=-\frac{5}{4}N|\Phi_b\rangle. \tag{15}$$

The excited state of the spin tube with two spontaneously dimerized ground states are likely to be domain walls between the two ground states. For the checkboard-dimer states,[21,22] the domain wall is represented by a diagonal spin-orbit bond and described by the matrix product. In the present case, the domain wall can be described by a single bond operator, it carries half-integer spin and chirality degrees of freedom. In the following, we will call the domain wall a soliton or spinonlike excitation. Since the spinonlike excitations or the solitons can only be created in pairs, the excitation spectrum is a two-particle continuum. To study the scattering soliton states, we consider the spin tube with $2N+1$ spin triangles and with open boundary conditions along the chain direction. The soliton state at site n can be described as

$$\begin{aligned} |\psi_n\rangle=&[1,2][3,4]\cdots[2n-3,2n-2]a_{2n-1}^\dagger\\ &\times[2n,2n+1]\cdots[2N,2N+1], \end{aligned} \tag{16}$$

where a denotes any one of the four operators of u_l, u_r, d_l, and d_r. In the calculations we choose operator a as u_l. The scattering spinonlike state is

$$|\psi\rangle=\sum_{n=1}^{N+1} e^{ip(2n-1)}|\psi_n\rangle. \tag{17}$$

The variational soliton energy is then

$$\varepsilon(p)=\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}+\frac{5}{8}(2N+1), \tag{18}$$

where $-\frac{5}{8}(2N+1)$ is the zero energy of the $(2N+1)$-triangle system.

When $n-n'\geqslant 1$, the inner products that appear in Eq. (18) are

$$\begin{aligned} \langle\psi_{n'}|\psi_n\rangle&=\left(-\frac{1}{4}\right)^{n-n'},\\ \langle\psi_{n'}|H|\psi_n\rangle&=\left(-\frac{5}{4}\right)\left(-\frac{1}{4}\right)^{n-n'}[N+\beta+(1-2\beta)(n-n')]. \end{aligned} \tag{19}$$

The norm of the wave function is then

$$\begin{aligned} \langle\psi|\psi\rangle&=\sum_{n,n'} e^{i2p(n-n')}\langle\psi_{n'}|\psi_n\rangle\\ &=N\frac{15}{17+8\cos 2p}+\frac{17}{17+8\cos 2p}\\ &\quad+\frac{30(4\cos 2p+1)}{(17+8\cos 2p)^2}, \end{aligned} \tag{20}$$

and the spectrum of the spinonlike excitation is

$$\varepsilon(p)=\left(\frac{5}{8}+\frac{\beta}{6}\right)+\frac{2}{3}\beta\cos 2p+\frac{2}{3}(1-2\beta)\frac{8+17\cos 2p}{17+8\cos 2p}. \tag{21}$$

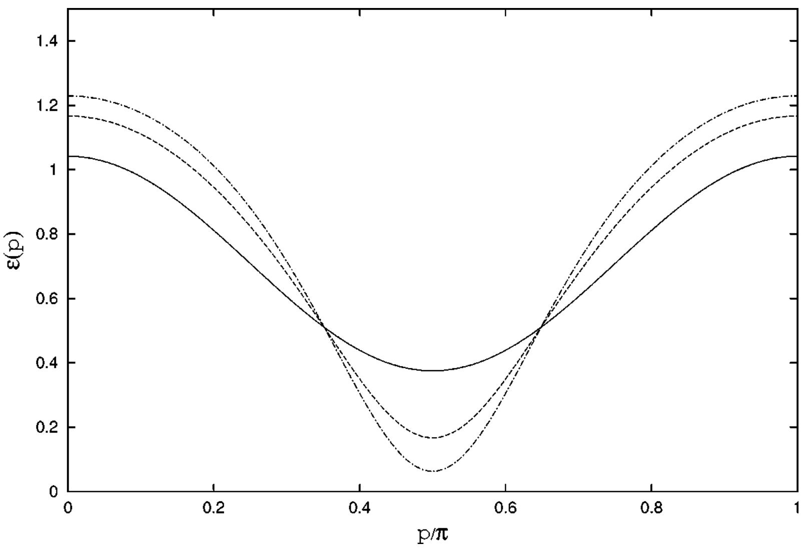

FIG. 1. The spinonlike excitation spectra of three-leg spin ladders at $\beta=\frac{1}{2}$ (full line), $\beta=\frac{1}{4}$ (dashed line), and $\beta=\frac{1}{8}$ (dot-dashed line), respectively. The excitation gap $\varepsilon(\pi/2)$ decreases with decreasing β.

When $\beta=\frac{1}{2}$, the spinonlike excitation spectrum is $\varepsilon(p)=\frac{17}{24}+\frac{1}{3}\cos 2p$ and $\varepsilon(\pi/2)=\frac{3}{8}$. When β deviates from $\frac{1}{2}$, Φ_a and Φ_b are no longer the eigenstates of Hamiltonian (2). However, as shown by Kawano and Takahashi,[9] the model is still in the spontaneously dimerized phase. Approximately, the spinonlike excitation and its spectrum can also be described by Eqs. (16), (17), and (18). In Fig. 1, we show the spinonlike excitation spectra of $\beta=\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. The spin gap at $p=\pi/2$ decreases with decreasing β. Around $\beta=\frac{1}{2}$, the decrease is linear. This behavior is similar to that of the single antiferromagnetic spin-$\frac{1}{2}$ chain.[32] At $\beta=\frac{1}{20}$, the gap $\varepsilon(\pi/2)$ goes to zero and we cannot give the results at $\beta=0$. In the view of perturbation, when β deviates from $\frac{1}{2}$, the ground state is the mixture of segments of Φ_a and Φ_b. This mixture will decrease the energy of the spinonlike excitation and make the multi-spinon-like excitation effects become more significant. Cabra *et al.*[26] gave a spectrum of the spinonlike excitation for $\beta=0$ by extrapolating their exact diagonalization results. Our analytical results near $\beta=\frac{1}{2}$ agree well with it.

The continum of the soliton-antisoliton pair is given by

$$\omega(k,q)=\varepsilon\left(\frac{k+q}{2}\right)+\varepsilon\left(\frac{k-q}{2}\right), \tag{22}$$

where, k and q are the total and relative momentum. The lowest boundary of the continum can be obtained by making $\omega(k,q)$ minimal with respect to q. At $\beta=\frac{1}{2}$, $\omega(k)=\frac{17}{12}-\frac{2}{3}|\cos k|$. In comparison, a single spin-$\frac{1}{2}$ antiferromagnetic chain at Majumdar and Ghosh point has a two spinon continum of $\epsilon(k)=\frac{5}{4}-|\cos k|$.[33]

Other possible elementary excitations are magnonlike. They describe the changes of the bond: from spin singlet to spin triplet, from one kind of chirality triplet to another one with different τ_z, and/or from chirality triplet to chirality singlet. A traveling magnonlike excitation can be expressed as

$$|\varphi\rangle_{s(c)}=\frac{1}{\sqrt{N}}\sum_{n=1}^{N}e^{ik(2n-1/2)}|\varphi_n\rangle_{s(c)}, \tag{23}$$

with

$$|\varphi_n\rangle_{s(c)}=[1,2][3,4]\cdots[2n-1,2n]_{s(c)}\cdots[2N-1,2N], \tag{24}$$

where, $[2n-1,2n]_{s(c)}$ denotes a bond with changes of spin and/or chirality. The variational energies of the magnonlike excitations are

$$\omega_{s(c)}(k)=\langle\varphi|H|\varphi\rangle_{s(c)}+\frac{5}{4}N. \tag{25}$$

In the view of bond change, there are totally 15 kinds of magnonlike excitations and at $\beta=\frac{1}{2}$ they are all dispersionless. $[2n-1,2n]_{sc1}=\frac{1}{2}(u^\dagger_{l2n-1}d^\dagger_{r2n}-d^\dagger_{r2n-1}u^\dagger_{l2n}-u^\dagger_{r2n-1}d^\dagger_{l2n}+d^\dagger_{l2n-1}u^\dagger_{r2n})$, $[2n-1,2n]_{sc2}=(1/\sqrt{2})(u^\dagger_{l2n-1}u^\dagger_{r2n}-u^\dagger_{r2n-1}u^\dagger_{l2n})$ and $[2n-1,2n]_{sc3}=(1/\sqrt{2})(d^\dagger_{l2n-1}d^\dagger_{r2n}-d^\dagger_{r2n-1}d^\dagger_{l2n})$ denote the bonds with spin triplet and chirality singlet, the corresponding magnonlike excitations have energies of $\epsilon_{sc1}=1$.

$[2n-1,2n]_{c1}=(1/\sqrt{2})(u^\dagger_{l2n-1}d^\dagger_{l2n}-d^\dagger_{l2n-1}u^\dagger_{l2n})$ and $[2n-1,2n]_{c2}=(1/\sqrt{2})(u^\dagger_{r2n-1}d^\dagger_{r2n}-d^\dagger_{r2n-1}u^\dagger_{r2n})$, represent the bond with spin singlet and chirality triplets with $\tau_z=\pm1$. Their energy is $\epsilon_{c1}=1$.

$[2n-1,2n]_{sc4}=a^\dagger_{2n-1}a^\dagger_{2n}$ with $a=u_l$, u_r, d_l and d_r, $[2n-1,2n]_{sc5}=(1/\sqrt{2})(u^\dagger_{l2n-1}d^\dagger_{l2n}+d^\dagger_{l2n-1}u^\dagger_{l2n})$ and $[2n-1,2n]_{sc6}=(1/\sqrt{2})(u^\dagger_{r2n-1}d^\dagger_{r2n}+d^\dagger_{r2n-1}u^\dagger_{r2n})$, represent the bond with spin triplet and chirality triplets with $\tau_z=\pm1$. Their energy is $\epsilon_{sc2}=\frac{4}{3}$.

$[2n-1,2n]_{s1}=\frac{1}{2}(u^\dagger_{l2n-1}d^\dagger_{r2n}+d^\dagger_{r2n-1}u^\dagger_{l2n}+u^\dagger_{r2n-1}d^\dagger_{l2n}+d^\dagger_{l2n-1}u^\dagger_{r2n})$, $[2n-1,2n]_{s2}=(1/\sqrt{2})(u^\dagger_{l2n-1}u^\dagger_{r2n}+u^\dagger_{r2n-1}u^\dagger_{l2n})$ and $[2n-1,2n]_{s3}=(1/\sqrt{2})(d^\dagger_{l2n-1}d^\dagger_{r2n}+d^\dagger_{r2n-1}d^\dagger_{l2n})$ are bonds with spin triplet with $S_z=0$, ±1 and chirality triplet with $\tau_z=0$, the corresponding excitation energy is $\epsilon_s=\frac{5}{3}$.

$[2n-1,2n]_{c3}=\frac{1}{2}(u^\dagger_{l2n-1}d^\dagger_{r2n}+d^\dagger_{r2n-1}u^\dagger_{l2n}-u^\dagger_{r2n-1}d^\dagger_{l2n}-d^\dagger_{l2n-1}u^\dagger_{r2n})$, denotes a bond with spin singlet and chirality singlet. The energy is $\epsilon_{c2}=2$. For all of the fifteen kinds of

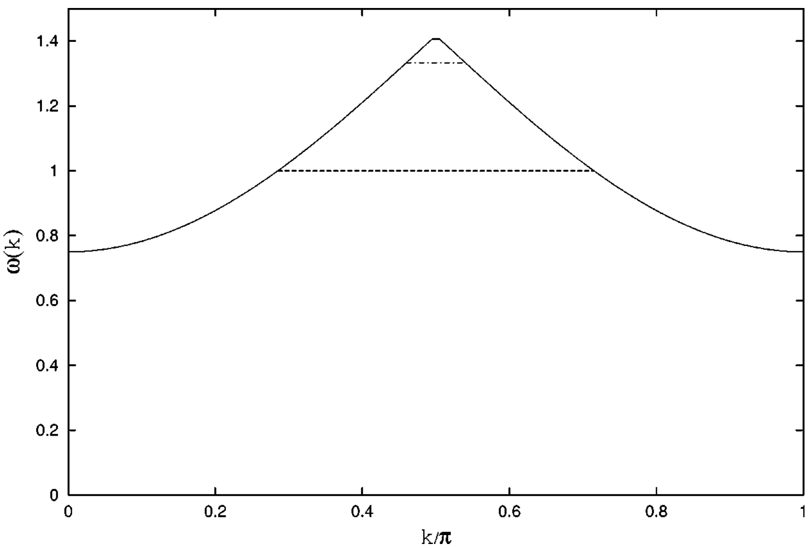

FIG. 2. The threshold of the two spinonlike excitation continuum for three-leg spin ladders. The dashed line and the dot-dashed line are determined by the magnonlike excitation energies of $\epsilon_{sc1}=1$, $\epsilon_{c1}=1$, and $\epsilon_{sc2}=\frac{4}{3}$, indicating the presence of the possible bound states.

excitations, some of them may have lower energies than the two soliton continuum boundary in the vicinity of the zone center $\pi/2$ indicating the possible presence of bound state of soliton-antisoliton pair. In Fig. 2, we show the threshold of the continuum of two spinonlike excitation spectra. The dashed and dot-dashed lines denote the dispersionless energies of magnonlike excitations of $\epsilon_{sc1}=1$, $\epsilon_{c1}=1$, and $\epsilon_{sc2}=\frac{4}{3}$. Kawano and Takahashi[9] studied the low energy spectra of spin-singlet with $\tau^z_{total}=0$ and $\tau^z_{total}=1$ and spin-triplet with $\tau^z_{total}=0$ and $\tau^z_{total}=1$ for $\beta=0.5$, 0.25, and 0 by DMRG, Cabra *et al.*[26] studied the spectra for $\beta=0$ at the sectors of $\Sigma_z=0$, $S_z=1$ and $\Sigma_z=0$, $S_z=0$, by the exact diagonalizations. Compared with these numerical results, it is found that near the momenta of $k=0$ and $k=\pi$, the magnonlike excitaitons are well consistent with the two particle continuum boundary, while near the momentum zone center of $k=\pi/2$, the magnonlike excitations are bound states of two spinonlike excitations, and their energies are largely affected by the interactions between the spinonlike excitations.

For those spin ladders with 5 legs, 7 legs, and in general M (M is an odd number) legs and periodical boundary conditions in the rung directions, an effective Hamiltonian can be obtained in the strong intrarung interaction limit:

$$\mathcal{H}_{\rm eff}=\frac{J_1}{M}\sum_i \mathcal{S}_i\cdot\mathcal{S}_{i+1}[1+\alpha_M(\tau_i^+\tau_{i+1}^-+\tau_i^-\tau_{i+1}^+)] + \frac{J_2}{M}\sum_i \mathcal{S}_i\cdot\mathcal{S}_{i+2}[1+\alpha_M(\tau_i^+\tau_{i+2}^-+\tau_i^-\tau_{i+2}^+)], \tag{26}$$

where, $\alpha_5=\frac{64}{9}$, $\alpha_7=10.483$, $\alpha_9=14.005$ and in the large M limit, $\alpha_M\approx 1.76M$.[9,26] For general α_M, at $\beta=J_2/J_1=\frac{1}{2}$, the effective Hamiltonian has the same ground states to those of $M=3$. The spinonlike excitation is still described by Eqs. (16) and (17). After a similar calculations, we get the spectra of the spinonlike excitations as

$$\varepsilon(p)=\frac{3J_1}{4M}(1+\alpha_M)\left[\frac{1}{2}+\frac{2\beta}{15}(4\cos 2p+1) + \frac{8}{15}(1-2\beta)\frac{8+17\cos 2p}{17+8\cos 2p}\right]. \tag{27}$$

At $\beta=\frac{1}{2}$, the threshold of the two spinonlike excitation continuum is $\omega(k)=(J_1/M)(1+\alpha_M)(\frac{17}{20}-\frac{2}{5}|\cos k|)$. In Fig. 3, we show the spectra of the spinonlike excitations for $M=3$, $M=5$, $M=7$ and the limit result of $M\rightarrow\infty$. For comparison, we also show the spinon spectrum of a single spin-$\frac{1}{2}$ antiferromagnetic chain. The spectra of the magnonlike excitations can also be calculated. These excitation energies are $\epsilon_{sc1}=(1+\frac{1}{2}\alpha_M)(J_1/M)$ for the bond of spin triplet and chirality singlet; $\epsilon_{c1}=(\frac{3}{4}\alpha_M)(J_1/M)$ for the bond of spin singlet and chirality triplet with $\tau_z=\pm 1$; $\epsilon_{sc2}=(1+\frac{3}{4}\alpha_M)(J_1/M)$ for the bond of spin triplet and chirality triplet with $\tau_z=\pm 1$; $\epsilon_s=(1+\alpha_M)(J_1/M)$ for the bond of spin triplet and chirality triplet with $\tau_z=0$; and $\epsilon_{c2}=(\frac{3}{2}\alpha_M)(J_1/M)$ for the bond of spin singlet and chirality singlet. Among these states, the excitations describing the bond of spin triplet and chirality singlet always have the lowest energies, this excitation and those denoting the bonds of spin singlet/triplet and chirality triplet with $\tau_z=\pm 1$, have lower energies than the continuum boundary near the momentum zone center of $k=\pi/2$, indicating that there may be possible bound states.

As we have pointed out before, although the bond operator representation is obtained starting from the three-leg spin ladders, it can also be applied to other spin-orbit models. A more general spin-orbit model with the orbit part XY-like has been studied by Pati and Singh.[20] We rewrite the model as

$$H=\sum_i [J_s\vec{S}_i\cdot\vec{S}_{i+1}+J_o(\tau_i^x\tau_{i+1}^x+\tau_i^y\tau_{i+1}^y)+K(\vec{S}_i\cdot\vec{S}_{i+1}) \times(\tau_i^x\tau_{i+1}^x+\tau_i^y\tau_{i+1}^y)]. \tag{28}$$

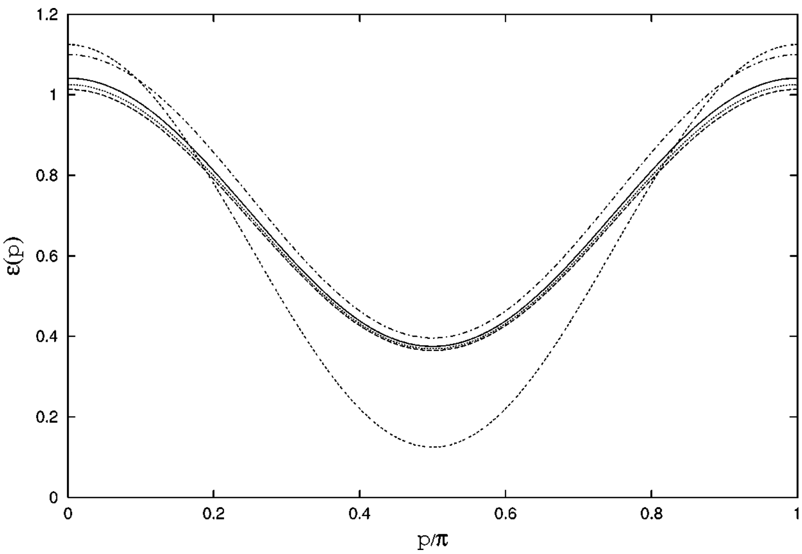

FIG. 3. The spinonlike excitation spectra of 3-leg (solid line), 5-leg (dashed line), and 7-leg (dotted line) spin ladders at $\beta=\frac{1}{2}$. The dot-dashed line denotes the large M limit result with $\alpha_M=1.76M$. For comparison, $\varepsilon(p)=\frac{5}{8}+\frac{1}{2}\cos 2p$, the spinon spectrum of a single spin-$\frac{1}{2}$ antiferromagnetic chain is also shown (broken dashed line).

Compared with the effective Hamiltonian of the spin ladders with odd number of legs and periodical boundary conditions in the rung direction, a nonzero pure orbital part J_o is added. Because of this term, there is no exact solution even when an additional next-nearest neighbor interaction is included. The phase diagram of the above model has been given by Pati and Singh through DMRG,[20] and for suitable parameters, the system is still in a phase with spontaneous dimerization. Therefore, with the ground state and the excited states described by Eqs. (13), (16), and (17), respectively, we can also study the spinonlike excitations by the variational ansatz. The variational ground state energy per bond is $(-\frac{3}{4}J_s+\frac{1}{2}J_o-\frac{3}{8}K)$, and the spinonlike excitation spectrum is

$$\varepsilon(p)=\left(\frac{3}{4}J_s-\frac{1}{2}J_o+\frac{3}{8}K\right)\left(\frac{1}{2}+\frac{8}{15}\,\frac{8+17\cos 2p}{17+8\cos 2p}\right). \tag{29}$$

IV. SUMMARY

In summary, starting from the three-leg spin ladders with periodical boundary conditions in the rung direction, we obtain a bond operator representation for three $S=\frac{1}{2}$ spins along the same line to that of two $S=\frac{1}{2}$ spins. With the spin and the chirality operators expressed by these bond operators, we studied the elementary excitations of the spin-tube and spin-orbit models. For the spin ladders with 3, and in general, with any odd number of legs, when periodical boundary conditions in the rung direction are imposed and suitable next-nearest-neighbor interactions are included, their ground states are spontaneously dimerized with each bound a spin singlet and chirality triplet. The spinonlike excitations, which carry $\frac{1}{2}$ spin and chirality freedoms are calculated by a variational ansatz. The magnonlike excitations, which denote the changes of the bond from spin singlet to spin triplet and/or from chirality triplet to chirality singlet or from one kind of triplet to another one with different τ_z, are also studied. Near the vicinity of the momentum zone center $\pi/2$, bound states of paired spinonlike excitations exist. More general spin-orbit models with spontaneously dimerized ground states are also studied. It should be pointed out that the obtained bond operator representations can also be applied to other systems such as kagomé lattice and triangular-lattice Heisenberg antiferromagnets.

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$$S_p^+=-\tfrac{1}{3}(u_l^\dagger d_l+u_r^\dagger d_r)+\tfrac{2}{3}q_1^+q_{-1}+1/\sqrt{3}\,(q_3^+q_1+q_{-1}^+q_{-3})$$
$$+\tfrac{1}{3}j^{2p}(2u_l^\dagger d_r-u_r^\dagger q_{-1}-q_1^\dagger d_l+\sqrt{3}d_r^\dagger q_{-3}+\sqrt{3}q_3^\dagger u_l)$$
$$+\tfrac{1}{3}j^p(2u_r^\dagger d_l-u_l^\dagger q_{-1}-q_1^\dagger d_r+\sqrt{3}d_l^\dagger q_{-3}+\sqrt{3}q_3^\dagger u_r).$$
$$S_p^z=\tfrac{1}{6}(u_l^\dagger u_l+u_r^\dagger u_r+q_1^\dagger q_1-d_l^\dagger d_l-d_r^\dagger d_r-q_{-1}^\dagger q_{-1})$$
$$+\tfrac{1}{2}(q_3^\dagger q_3-q_{-3}^\dagger q_{-3})+\tfrac{1}{3}j^{2p}(d_l^\dagger d_r+d_r^\dagger q_{-1}$$
$$+q_{-1}^\dagger d_l-u_l^\dagger u_r-u_r^\dagger q_1-q_1^\dagger u_l)+\tfrac{1}{3}j^p(d_r^\dagger d_l$$
$$+q_{-1}^\dagger d_r+d_l^\dagger q_{-1}-u_r^+u_l-q_1^\dagger u_r-u_l^\dagger q_1)$$

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