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Wigner crystallization of a two-dimensional electron gas in a magnetic field: Single electrons versus electron pairs at the lattice sites

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The ground state energy and the lowest excitations of a two-dimensional Wigner crystal in a perpendicular magnetic field with one and two electrons per cell is investigated. In the case of two electrons per lattice site, the interaction of the electrons *within* each cell is taken into account exactly (including exchange and correlation effects), and the interaction *between* the cells is in second order (dipole) van der Waals approximation. No further approximations are made, in particular Landau level mixing and *in*complete spin polarization are accounted for. Therefore, our calculation comprises a, roughly speaking, complementary description of the bubble phase (in the special case of one and two electrons per bubble), which was proposed by Koulakov, Fogler, and Shklovskii on the basis of a Hartree Fock calculation. The phase diagram shows that in GaAs the paired phase is energetically more favorable than the single electron phase for, roughly speaking, filling factor f larger than 0.3 and density parameter r_s smaller than 19 effective Bohr radii (for a more precise statement see Figs. 3 and 4). If we start within the paired phase and increase magnetic field or decrease density, the pairs first undergo some singlet-triplet transitions before they break.

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I. INTRODUCTION

The Wigner crystal (WC) and the Fermi liquid are states of matter which have been established already for decades. Later on, the Laughlin liquid as a special state in high magnetic fields has been added. In both liquid states, wave functions of the electrons have an appreciable overlap. In the extreme Wigner crystal limit (low density, high magnetic field) this overlap can be neglected. The issue of this paper is if there is a state between crystal and liquid, where a finite number of particles overlap as in the liquid, but interact with the neighboring cluster as in the Wigner crystal (paired or clustered Wigner crystal). The charge density wave (CDW) state, mostly investigated in Hartree Fock (HF) approximation, covers the complementary limit of this state.[1–3] In the CDW state there is a density modulation, but the overlap between the clusters plays an important role. In a sense, the CDW state is the high-density and low-magnetic-field counterpart of the state considered in the present paper. (Observe that in some papers the CDW state is also called WC.) From another point of view, the ''paired Wigner crystal'' can be considered as a derivative of three other states. It is (i) a Wigner crystal[4] with two electrons per cell, (ii) a crystallized paired electron liquid known from superconductivity and theories for the $f=\frac{5}{2}$ state,[5–9] (iii) a state of liquid islands, if the clusters are large.

The first paper which considered the possibility of a paired WC in 3 dimensions (3D) without a magnetic field is Ref. 10 using a cell approximation for the intercell interaction and a variational ansatz for the intracell problem. In a previous paper[11] it was shown, that the model of Ref. 10 can be solved exactly, even if fluctuation corrections in second order van der Waals approximation are included additionally. It turned out that in 3D without a magnetic field, the paired WC is energetically lower than the conventional WC for $r_s < 9$, but in this density region the Fermi liquid is probably already lower than both WCs. More recently, this issue has been investigated in 2D with a magnetic field using the HF approximation.[12,13] The result is that in higher Landau levels (LL) the ground states are phases with circular or stripy clusters. Meanwhile, numerical solutions for finite size systems have been published,[14,15] which confirm these findings qualitatively within their model. All previous work on this issue neglects LL mixing and incomplete spin polarization. In the present approach, we investigated clusters with one and two electrons. In the latter case, the interaction within each cluster is exactly accounted for (including exchange and correlations). In both cases ($N=1$ and 2), the intercell interaction is considered in second order van der Waals approximation (no exchange, Coulomb interaction in dipole approximation). In Sec. II we describe the solution of the Schrödinger equation in more detail and in Sec. III our results are presented. Section IV gives a summary and an overview.

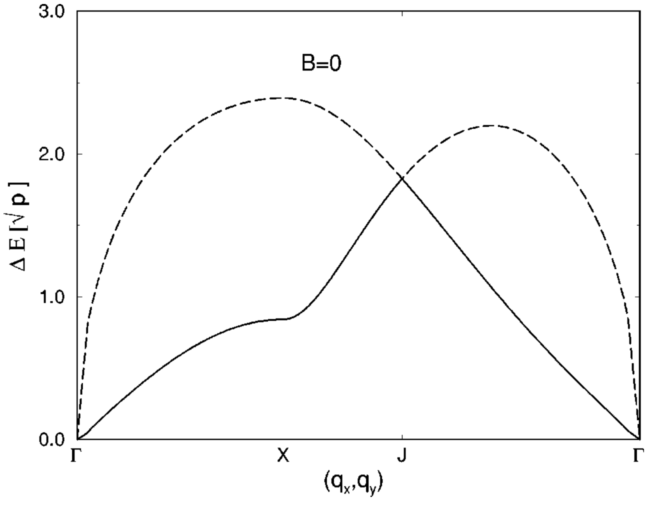

FIG. 1. Phonon dispersion (for $B=0$) of the hexagonal Wigner lattice. (Energies are in units $\sqrt{p}$.)

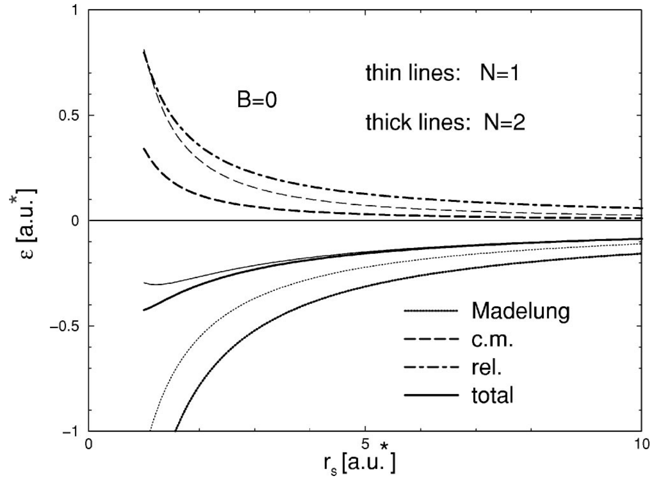

FIG. 2. Comparison of the contributions to the total energy in the single electron ($N=1$) and the paired ($N=2$) Wigner crystal for zero magnetic field.

II. SOLUTION OF THE SCHRÖDINGER EQUATION

The Hamiltonian of a system of N electrons at each lattice site and a positive compensating background of density n_0 reads (atomic units $\hbar=m=e=1$ are adopted)

$$H=T+V^{ee}_{\rm intra}+V^{ee}_{\rm inter}+V^{eb}+V^{bb}, \tag{1}$$

with the kinetic energy

$$T=\sum_n\sum_k\frac{1}{2m^*}\left[\mathbf{p}_{nk}+\frac{1}{c}\mathbf{A}(\mathbf{u}_{nk})\right]^2, \tag{2}$$

the intracell e-e interaction

$$V^{ee}_{\rm intra}=\sum_n\frac{1}{2}\sum_{k\neq k'}\frac{\beta}{|\mathbf{u}_{nk}-\mathbf{u}_{nk'}|}, \tag{3}$$

the intercell e-e interaction

$$V^{ee}_{\rm inter}=\frac{1}{2}\sum_{n\neq n'}\sum_{kk'}\frac{\beta}{|(\mathbf{R}^0_n-\mathbf{R}^0_{n'})+(\mathbf{u}_{nk}-\mathbf{u}_{n'k'})|}, \tag{4}$$

the electron-background interaction

$$V^{eb}=-\sum_n\sum_k\int d\mathbf{r}'\frac{n_0}{|\mathbf{r}_{nk}-\mathbf{r}'|}, \tag{5}$$

and the background-background interaction

$$V^{bb}=\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\frac{n_0n_0}{|\mathbf{r}-\mathbf{r}'|}. \tag{6}$$

In these definitions, m^* is the effective mass, β the inverse background dielectric constant, $\mathbf{R}^0_n$ are the lattice sites, and $\mathbf{r}_{nk}=\mathbf{R}^0_n+\mathbf{u}_{nk}$ the electron coordinates. We use the symmetric gauge $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{u}$. Latin letters n and n' denote the lattice sites and k and k' the electrons within a cell (or cluster). If the distance between the cell centers is large compared with the average distance between the electrons within a cell, the intercell interaction can be expanded in a multipole series up to second order

$$V^{ee}_{\rm inter}=V^{ee(0)}_{\rm inter}+V^{ee(2)}_{\rm inter}, \tag{7}$$

with the zero and second order terms

$$V^{ee(0)}_{\rm inter}=N^2\frac{1}{2}\sum_{n\neq n'}\frac{\beta}{|\mathbf{R}^0_n-\mathbf{R}^0_{n'}|}, \tag{8}$$

$$V^{ee(2)}_{\rm inter}=\frac{1}{N}\frac{1}{2}\sum_{nn'}\sum_{kk'}\mathbf{u}_{nk}\cdot\mathbf{C}_{nk,n'k'}\cdot\mathbf{u}_{n'k'}, \tag{9}$$

the force constant matrix

$$\mathbf{C}_{nk,n'k'}=+\beta N^2\mathbf{T}_0 \quad \text{for} \quad (nk)=(n'k') \tag{10}$$

$$=-\beta N\mathbf{T}_{nn'} \quad \text{else}, \tag{11}$$

the dipole tensor

$$\mathbf{T}_{nn'}=\frac{1}{R^5_{nn'}}[3\mathbf{R}_{nn'}\circ\mathbf{R}_{nn'}-R^2_{nn'}\mathbf{I}], \tag{12}$$

and the definitions $\mathbf{T}_0=\Sigma_{n\neq0}\mathbf{T}_{0n}$ and $R_{nn'}=\mathbf{R}^0_n-\mathbf{R}^0_{n'}$. By rearrangement of terms we obtain the following form:

$$H=E^{\rm Mad}_N+(T+V^{ee}_{\rm intra})+V^{ee(2)}_{\rm inter}, \tag{13}$$

with the Madelung energy for a lattice with N electrons at each lattice site

$$E^{\rm Mad}_N=N^2\frac{1}{2}\sum_{n\neq n'}\frac{\beta}{|\mathbf{R}^0_n-\mathbf{R}^0_{n'}|}-N\sum_n\int d\mathbf{r}'\frac{n_0}{|\mathbf{R}^0_n-\mathbf{r}'|}$$
$$+\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\frac{n_0n_0}{|\mathbf{r}-\mathbf{r}'|}. \tag{14}$$

By introducing the center of mass (c.m.) coordinate $\mathbf{U}_n$ and some relative coordinates within each cell using a orthogonal coordinate transformation, the last 3 terms of Eq. (13) can be decomposed into a c.m. Hamiltonian $H_{\rm c.m.}$ and a lattice sum of internal (relative coordinate) Hamiltonians $H_{\rm rel}=\Sigma_nH_{{\rm rel},n}$, where $H_{{\rm rel},n}$ contains only the $(N-1)$ relative coordinates in cell n. Thus, only $H_{\rm c.m.}$ shows a coupling between the cells. This fact is completely analogous to the treatment of interacting quantum dot lattices considered in Ref. 16. Consequently, the total Hamiltonian (13) decomposes into 3 contributions

$$H=E^{\rm Mad}_N+H_{\rm c.m.}+H_{\rm rel}, \tag{15}$$

which are considered in turn.

A. Madelung energy

Generally, we obtain $E^{\rm Mad}_N$ from the conventional Madelung energy $(N=1)$ with the same lattice constant by $E^{\rm Mad}_N=N^2E^{\rm Mad}_{N=1}$. Here, however, we compare phases with the same mean density. In the *hexagonal* lattice, the Madelung

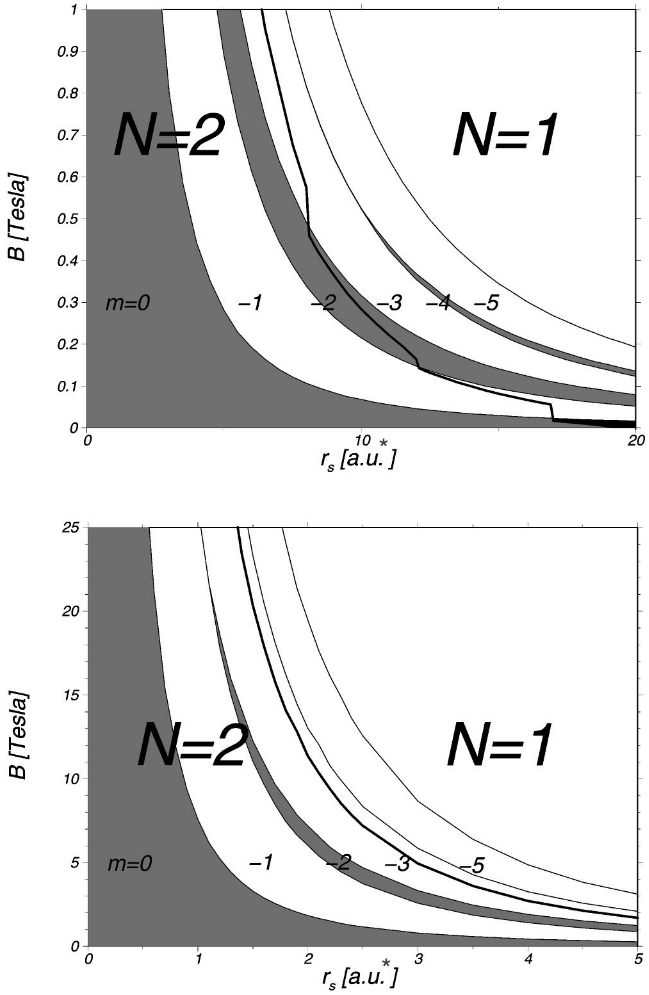

FIG. 3. Phase diagram in the r_s-B plane (see text for details). Upper figure: for small B; lower figure: for small r_s.

energy per electron $\varepsilon_N^{\mathrm{Mad}}$ for a given density parameter $r_s = 1/\sqrt{\pi n_0}$ can be deduced from the data in Ref. 4 providing $\varepsilon_N^{\mathrm{Mad}} = -1.106103\,\beta N^{1/2} r_s^{-1}$.

B. Center-of-mass energy

The c.m. part of the Hamiltonian reads

$$H_{\mathrm{c.m.}} = \frac{1}{N}\left\{ \sum_n \frac{1}{2m^*}\left[\mathbf{P}_n + \frac{N}{c}\mathbf{A}(\mathbf{U}_n)\right]^2 + \frac{N^2}{2}\sum_{n,n'} \mathbf{U}_n\cdot\mathbf{C}_{n,n'}\cdot\mathbf{U}_{n'}\right\}, \tag{16}$$

with

$$\mathbf{C}_{n,n'} = +\beta N^2 \mathbf{T}_0 \quad \text{for} \quad n = n' \tag{17}$$

$$= -\beta N \mathbf{T}_{nn'} \quad \text{else.} \tag{18}$$

After the usual phonon transformation

$$\mathbf{U}_n = \frac{1}{\sqrt{N_c}} \sum_{\mathbf{q}}^{\mathrm{BZ}} e^{-i\mathbf{q}\cdot R_n^0}\mathbf{U}_{\mathbf{q}}, \tag{19}$$

$$\mathbf{P}_n = \frac{1}{\sqrt{N_c}} \sum_{\mathbf{q}}^{\mathrm{BZ}} e^{+i\mathbf{q}\cdot R_n^0}\mathbf{P}_{\mathbf{q}}, \tag{20}$$

where N_c is the number of cells, the Hamiltonian decouples with respect to the lattice sums and we obtain

$$H_{\mathrm{c.m.}} = \sum_{\mathbf{q}}^{\mathrm{BZ}} H_{\mathbf{q}}, \tag{21}$$

with the magnetophonon Hamiltonian

$$H_{\mathbf{q}} = \frac{1}{N}\left\{ \frac{1}{2m^*}\left[\mathbf{P}_{\mathbf{q}} + \frac{N}{c}\mathbf{A}(\mathbf{U}_{\mathbf{q}}^*)\right]^\dagger \cdot \left[\mathbf{P}_{\mathbf{q}} + \frac{N}{c}\mathbf{A}(\mathbf{U}_{\mathbf{q}}^*)\right] + \frac{N^2}{2}\mathbf{U}_{\mathbf{q}}^* \cdot \mathbf{C}_{\mathbf{q}} \cdot \mathbf{U}_{\mathbf{q}}\right\} \tag{22}$$

and the dynamical matrix

$$\mathbf{C}_{\mathbf{q}} = \sum_n e^{i\mathbf{q}\cdot\mathbf{R}_n^0}\mathbf{C}_{n,0} = \beta N \sum_{n\neq 0} (1 - e^{i\mathbf{q}\cdot\mathbf{R}_n^0})\mathbf{T}_{n0}. \tag{23}$$

The eigenvalues of $H_{\mathrm{c.m.}}$ and $H_{\mathbf{q}}$ depend on N only through the N dependence of $\mathbf{C}$. The explicit N dependence in Eqs. (16) and (22) cancels. The last statement is obvious if we consider that the explicit N can be removed by rescaling the displacement vector $\mathbf{U}$ by a factor of $\sqrt{N}$.

The eigenvalues of Eq. (22) are, e.g., given in Ref. 16:

$$E(n_+, n_-) = \left(n_+ + \frac{1}{2}\right)\omega_+ + \left(n_- + \frac{1}{2}\right)\omega_-,$$

$$n_\pm = 0,1,2,\ldots . \tag{24}$$

This provides a ground state energy

$$E_{\mathrm{c.m.}} = \sum_{\mathbf{q}}^{BZ} \frac{1}{2}(\omega_+ + \omega_-), \tag{25}$$

where

$$\omega_\pm = \sqrt{\frac{\omega_c^{*2}}{2} + \tilde{\omega}_0^2 \pm \sqrt{\frac{\omega_c^{*4}}{4} + \omega_c^{*2}\tilde{\omega}_0^2 + \frac{\Delta^2}{4} + C_{12}^2}}, \tag{26}$$

with $\tilde{\omega}_0^2 = \frac{1}{2}(C_{11} + C_{22})$, $\Delta = C_{11} - C_{22}$, $\omega_c^* = B/m^*c$ is the cyclotron frequency with the effective mass, and $C_{i,k}$ are the Cartesian components of $\mathbf{C}_{\mathbf{q}}$. If we define an auxiliary tensor $\mathbf{S}$ through $\mathbf{C}_{\mathbf{q}} = p\,\mathbf{S}_{\mathbf{q}}$ with $p = 2\beta N/a^3$ (a is the lattice constant), then $\mathbf{S}_{\mathbf{q}}$ depends only from the geometry of the lattice, i.e., it is a fixed tensor for the hexagonal and cubic lattice and depends only from the b/a ratio for a rectangular lattice. The explicit form of $\mathbf{S}_{\mathbf{q}}$ is complicated and not of common interest. The sums involved have to be done numerically anyway. The interaction parameter p for the hexagonal lattice reads in terms of the more convenient parameters r_s and n_0

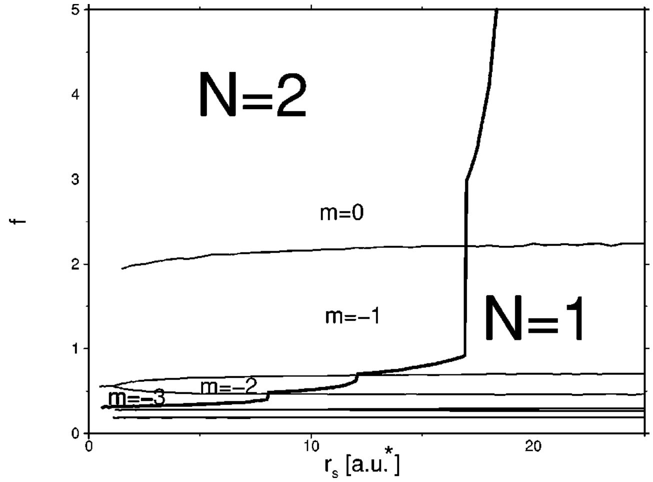

FIG. 4. Phase diagram in the r_s-f plane (see text for details).

$$p=2\left(\frac{\sqrt{3}}{2\pi}\right)^{3/2}\beta r_s^{-3}N^{-1/2}=2\left(\frac{\sqrt{3}}{2}\right)^{3/2}\beta n_0^{3/2}N^{-1/2}. \quad (27)$$

For $\mathbf{q}=0$, $\omega_\pm$ for $\mathbf{q}=0$ agree with the excitation energies ΔE for long wavelength (infrared) radiation because of the optical selection rules. In the same limit, Kohn's theorem states that $\omega_\pm$ is not affected by the electron-electron interaction, i.e., $\omega_\pm$ agree with the values for a single electron in a magnetic field. (For both statements see also Ref. 16.)

For $B=0$, it can be easily seen that $\omega_\pm=\sqrt{\text{eigenvalues}(\mathbf{C_q})}=\sqrt{p}\sqrt{\text{eigenvalues}(\mathbf{S_q})}$. Because of the universal character of $\mathbf{S_q}$, the excitation energies in units of $\sqrt{p}$ comprise all data on the energies for hexagonal lattices in this limit (see Fig. 1). In connection with formula (27) this means that the phonon frequencies of the paired WC, as compared with the single electron case of the same electron density, are smaller by a factor of $1/\sqrt{2}$. Additionally, the frequencies (and the c.m. contribution to the total energy) for the same N decay with the density parameter like $r_s^{-3/2}$.

For finite B, a universal function comprising all data cannot be defined. However, if we use ω_c^* as energy unit and ω_c^{*2} as unit for the interaction parameter p, then the result depends only on p and shows *no explicit* dependence on B.

For large B, Eq. (26) gives $\omega_+=\omega_c^*+\text{tr}(\mathbf{C_q})/2\omega_c^*$ and $\omega_-=\det(\mathbf{C_q})/\omega_c^*$ (in agreement with Ref. 3), and in the units mentioned above, this reads $\omega_+[\omega_c^*]=1+p[(\omega_c^*)^2]\cdot\text{tr}(\mathbf{S_q})/2$ and $\omega_-[\omega_c^*]=p[(\omega_c^*)^2]\det(\mathbf{S_q})$. Thus the results are determined by the interaction parameter and two universal functions, namely $\text{tr}(\mathbf{S})$ and $\det(\mathbf{S})$. In the limit $B\rightarrow\infty$ we obtain the result of noninteracting electrons $\omega_+=\omega_c^*$ and $\omega_-=0$, as to be expected. Direct comparison of excitation energies with experiments is not the aim of this work. What matters here is only the contribution of the "zero point vibrations" to the total energy.

C. Internal energy for $N=2$

Only for two electrons per cell, the internal (relative coordinate) energy can be calculated easily. (Larger N call for a larger numerical effort and are postponed to a succeeding work.) If we introduce apart from the c.m. $\mathbf{U}_n$ the relative coordinate $\mathbf{u}_n$

$$\mathbf{U}_n=\frac{1}{2}(\mathbf{u}_{n1}+\mathbf{u}_{n2}), \quad (28)$$

$$\mathbf{u}_n=\mathbf{u}_{n2}-\mathbf{u}_{n1}, \quad (29)$$

then all relative Hamiltonians read (the index "n" is omitted)

$$H_{\text{rel}}=2\left\{\frac{1}{2m^*}\left[\mathbf{p}+\frac{1}{2c}\mathbf{A}(\mathbf{u})\right]^2+\frac{1}{2}\mathbf{u}\cdot\mathbf{D}\cdot\mathbf{u}+\frac{\beta}{2u}\right\}, \quad (30)$$

where $\mathbf{p}=-i\boldsymbol{\nabla}_{\mathbf{u}}$ and $\mathbf{D}=(\beta/2)\mathbf{T}_0=\frac{1}{4}pS_0\mathbf{I}$ with S_0 being a lattice sum of the same type as the components of S_{ik}. (Do not mix up the interaction parameter p and the *vector* of the momentum operator $\mathbf{p}$.) It should be emphasized that the electronic *inter*cell interaction contributes a harmonic term to the *intra*cell problem to H_{rel}, namely the last but one in Eq. (30). As to the calculation of the eigenvalues of Hamiltonians of the form (30) we refer to Ref. 17.

III. RESULTS

If not otherwise indicated, all results and parameters are given in effective atomic units for GaAs ($m^*=0.067$,$\beta=1/12$), i.e., energies (and frequencies) in 1 a.u.*$=4.65\times10^{-4}$ double Rydberg$=12.64$ meV and length in 1 a.u.*$=1.791\times10^2$ Bohr radii$=0.9477\times10^2$ Å. (If the units of a quantity is indicated explicitly, then it is given in brackets after the quantity.) Observe that the effective density parameter in current typical experiments is of order 2. The g factor used for the Zeeman energy is 0.44.

A. Contributions to the total energy

For larger r_s, the energy difference between both phases is tiny (see Fig. 2). As an example, at $r_s=10$ (at the right boundary of Fig. 2) it amounts to 2×10^{-4}, calling for high precision computations (of order 10^{-5} a.u.*). This fact is amazing because the Madelung energy and the c.m. energy are both very different (see Fig. 2). This difference is compensated almost completely by the relative energy of the paired phase.

B. Phase diagram for single electron versus paired Wigner crystals

Figure 3 shows the phase diagram for the single electron versus the paired WC phase in the r_s-B plane. In shaded regions, the electron pairs are in singlet state and elsewhere in triplet states. Observe, that the spin configuration of the paired phase is given also in that part of the plot, where the single electron phase is energetically below the paired phases. The thin full lines separate regions with different relative (internal) angular momentum m of the pairs, which has the lowest energy. The value of m is indicated in each region as well. (Observe that even and odd m belongs to the singlet triplet state, respectively.) The thick full line separates the regions where the single electron WC is the ground state, from regions, where the paired WC has the lower energy. We want to emphasize that this figure does not say anything about the issue if any other (e.g., liquid) phase is lower in energy than the phases considered here. The kinks in the phase boundary are connected with changes of the total spin of the paired phase.

First, we discuss the case $B=0$. For r_s above 19, the energetically lower state is the single electron WC and below it is the paired phase. Clearly, in the limit of high densities (small r_s) any liquid state should win. It is physically imaginable, that in increasing the density, the state starts with the single electron WC, then adopts the paired state before it goes over to the liquid. If there are states with 3, 4, . . . electrons involved, will be the focus of further investigations.

For a certain finite *B*, the density, where the transition to the paired ground state occurs, becomes higher. In other words, for higher *B* one needs less dilute electron systems to produce a single electron WC. This is physically obvious, because the magnetic field helps to localize the electrons. In other words, lowering the density cases pair breaking. Now we discuss the pair breaking process as a function of *B* for fixed r_s, say $r_s=2$. If we start with $B=0$ and increase *B*, the paired state undergoes first some spin transitions from singlet to triplet and vice versa before the pairs break in the triplet state. It should be mentioned that pair breaking and spin transition are not coupled together. The reason is that the Zeeman energy is too small to play a decisive role.

Figure 4 shows the phase diagram again, but the magnetic field axis is replaced by the filling factor $f=14.64/(B[T]r_s^2[\mathrm{a.u.}^*])$ (using the parameters of GaAs). This plot has the advantage of showing the physically important parameter *f* directly, but does not allow the discussion of the limit $B=0$, because it is transformed to $f\rightarrow\infty$. Again, thin lines separate regions with different *m* (and therefore total spin) of the paired phase and the thick line separates the single electron WC ground state from the paired WC. As to be expected, the single electron WC is restricted to the low density–small filling factor region. It is clear that in a complete phase diagram, the region, which is here attributed to the paired phase, can be further restricted by the inclusion of all kinds of liquid phases (and possibly WC phases with $N=2,3,\ldots$ electrons per cell).

C. Comparison of Wigner crystal phases with Laughlin liquid

Because the Laughlin liquid is restricted to discrete filling factors, we compare in Fig. 5 the energy [or, more conveniently, $r_s(\varepsilon-\omega_c/2)$] of different phases for fixed $f=1/3$ as a function of r_s. The Zeeman energy is omitted in this figure because the paired phase is triplet over the whole r_s range. Therefore it would have contributed for a given r_s the same amount to all phases, but introduced an unpleasant pole at $r_s=0$ into the curves. (In all calculations, however, the Zeeman energy has to be taken into account because of the different spin configurations in the paired WC.) The curve for the Laughlin liquid from Ref. 19 includes Landau level mixing (as our results do) by adding a variational correlation factor to the Laughlin function. (The value at $r_s=0$ corresponds to the result of the original Laughlin function.) As seen in Fig. 5, the Laughlin liquid and the single electron WC are the ground state for r_s below and above 15, respectively. This is in qualitative agreement with experiment. In our result for $f=1/5$ (not shown), the single electron WC is the ground state over the whole range of densities. This is not quite correct, because experiments suggest that the domain of the WC begins at $f=1/7$. Inclusion of a variational correlation factor to the WC wave function fixes the phase boundary between the Laughlin liquid and the WC.[20,21] This amendment could not be added here, because it would have become too complicated for the paired WC and in comparing both WC phases we needed a description on the same footing.

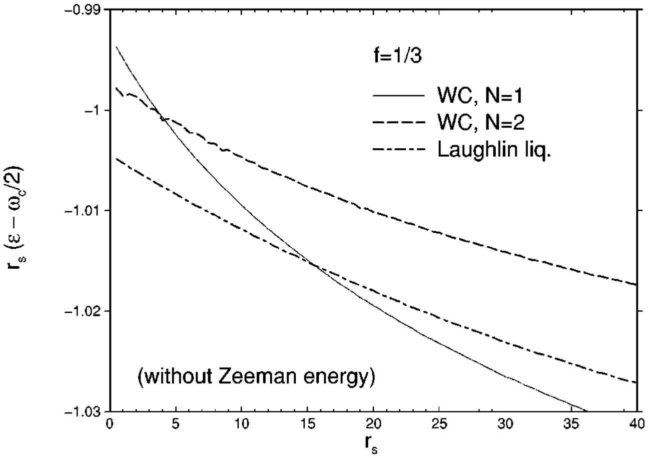

FIG. 5. Comparison of the Wigner crystal states with the Laughlin liquid.

IV. SUMMARY AND OVERVIEW

We have shown that the paired WC is energetically more favorable than the single electron WC for higher densities and lower magnetic fields. If we start with the paired WC, lowering the density and increasing the magnetic field will break the pairs. At least in GaAs, the Zeeman energy plays a less important role than the angular momentum dependence of the internal (relative coordinate) energy. Observe that changing the internal angular momentum (from even to odd) is coupled with a change of the total spin of each cluster due to the Pauli principle. Therefore, the paired phase undergoes first some singlet-triplet transitions before the pairs break and the single electron WC is reached. The singlet-triplet transitions have the same physical origin as the corresponding transitions in quantum dots, because they are driven by the internal energy, which is similar in either system.

In our approach, the interaction within each cluster is exactly accounted for (including exchange and correlations). In both cases, the *inter*cell interaction is considered in second order van der Waals approximation (no exchange, Coulomb interaction in dipole approximation). The validity of the second order approximation has been checked in Ref. 18 by estimating the third order contribution, which amounts to a few percent. We did not include higher order contributions from two reasons. First, they spoil the exact decoupling and make the paired WC intractable. Second, these corrections have most likely the same sign in the single electron and the paired phase and they cancel partly in comparison of both energies. The neglect of intercell exchange (but keeping the Coulomb correlations) is justified in the low density and high magnetic field limit, because the exchange energy drops off exponentially, but the correlation energy like $1/r$. It seems to be worthwhile to elaborate on the differences of this work and Ref. 12. They targeted on high LL's (low magnetic fields), and high electron densities $(N_{LL} \gg r_s^{-1} \gg 1)$. LL mixing and incomplete spin polarization are ruled out and the completely filled LL's are frozen out and influence the upper LL only through a screening in the e-e interaction. Therefore we called both treatments in the abstract ‘‘complementary.’’ What seems to be interesting is that despite approaching reality with different assumptions and virtually from opposite sides, the effect of clusterization of the liquids is found in both treatments.

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