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Critical exponent of the random flux model on an infinite two-dimensional square lattice and anomalous critical states

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The random flux model (RFM) on a square lattice with random flux $\Phi \in (-p\pi, p\pi), 0 < p \leqslant 1$ per plaquette has been studied by analyisis of high-precision transfer-matrix calculations, as well as the exact numerical diagonalization in the vicinity and at the band center. Interpreting thus obtained results using the recent findings for the Anderson bond-disordered (ABD) model from Phys. Rev. B **62**, 12 775 (2000), the importance of chiral symmetry is further emphasized and localization properties explained in terms of the scaling associated with the central two states of the spectrum. The results support the existence of two critical states at the band center of the infinite system. These states exhibit a different kind of critical behavior, named anomalous, allowing one to calculate the critical exponent which has been done for $p = 0.5$ and 1. Some differences between the ABD and RFM models are also discussed, the most important one being a broken symmetry at the critical point of the former.

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The problem of a quantum particle moving in a random magnetic flux has attracted considerable interest due to its relation to the problem of high-T_c superconductivity,[1] theory of the half-filled lowest Landau level,[2] as well as its unresolved localization properties within the context of the scaling theory of localization.[3,4]

The model of a quantum particle hopping on a square lattice studied here is described by the Hamiltonian:

$$H = -\sum_{\langle i,j \rangle} (t_{i,j}\, c_i^\dagger c_j + t_{i,j}^* c_j^\dagger c_i) + \sum_i \epsilon_i\, c_i^\dagger c_i\,, \qquad (1)$$

where $\langle \ldots \rangle$ denotes the nearest neighbors on a square lattice, c_i is the destruction operator of a particle on the ith site, and * and † are, respectively, complex and Hermitian conjugation. Hoppings $t_{i,j} \equiv \exp(i\phi_{i,j})$, where phases $\phi_{i,j}$ are chosen such that the total flux per plaquette, $\Phi_i \in (-p\pi, p\pi)$, is a uniformly distributed random variable, parametrized by $0 < p \leqslant 1$. The uniformly distributed $\epsilon_i \in (-W/2, W/2)$ represent the on-site disorder, and the random flux model (RFM) refers below to H with $W = 0$ unless explicitly stated differently.

Localization properties of the RFM remained only partially understood despite a significant effort of the community,[5–15] and three incompatible conclusions (with variations) have emerged: (i) all states are localized and the model is in the unitary class;[5,10,13] (iia) there is a critical point at the band center of the RFM (Refs. 11 and 15) due to the chiral symmetry[16–19] of the model (disscussed below), while all the other states are localized, (iib) with an additional Kosterlitz-Thouless-Berezinskii (KTB) transition into a phase of critical states around the band center for $0 < p < 1$;[14] or (iiia) a full metal-insulator transition,[6–9,12] (iiib) even in the presence of (sufficiently weak but finite) on-site disorder.[12]

This paper presents some further evidence that the scenario (iia) is the correct one and that the localization length ξ diverges as $|E|^{-\nu}$ at the band center with a critical exponent ν, which has been calculated for the first time for $p = 0.5$ and 1.

The main source of difficulties in numerical studies of RFM is very large ξ, growing approximately exponentially going from the band edges toward the band center.[5,10,15] The standard numerical method to determine the scaling of ξ is from the finite-size scaling analysis of either the smallest Lyapunov exponent λ_M of the transfer matrix of a long strip of width M (TMM), or the logarithm of the norm of retarded Green's function connecting the two far ends of the strip (GF).[4] In numerical studies of systems with small M, the exponential rise of ξ may give a false impression of a genuine continuous phase transition, as illustrated by Xie *et al.*[14] Particularly difficult is distiguishing between the scenarios (i) and (iia), because the eigenstate at a given energy with very large ξ in a finite-size system looks like a critical state even if it is localized in the infinite system.

Another approach was to study the Chern number of eigenstates by analyzing the boundary-phase averaged Hall conductance in the system with generalized periodic boundary conditions.[20] The conductance is quantized by an integer, the Chern number, and states with a zero Chern number are localized. The necessity of averaging the conductance over boundary conditions limits numerical studies to systems of smaller sizes than in other methods, although one may expect that the averaging somewhat reduces the boundary effects due to the finite size. Sheng and Weng[9] have given numerical evidence of scenario (iiia) using this method. Additional calculations on larger systems were done by Yang and Bhatt,[12] who studied the total number of states with a nonzero Chern number, N_c, as a function of L and W, and then used the scaling analysis to determine E_c. They found a critical disorder strength, $W_c > 0$, such that for $0 \leqslant W < W_c$ a band of states with a nonzero Chern number opens up around the band center. What is, however, puzzling about this result is that the calculated N_c, even deep inside the ''metallic'' region, appears to grow as N^α, with $\alpha < 1$, approaching 1

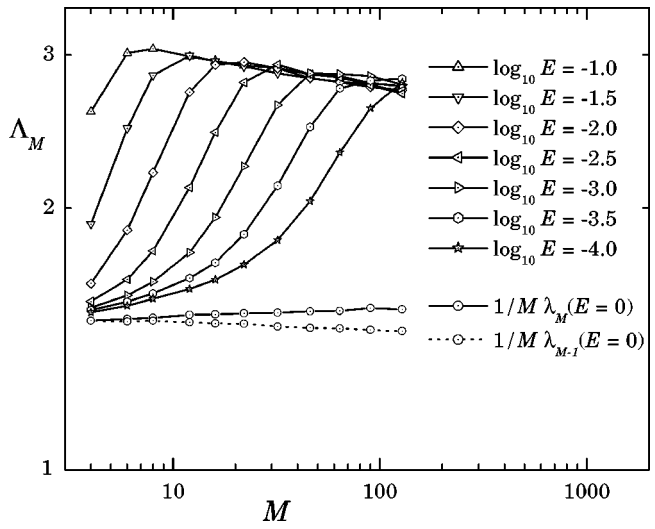

FIG. 1. Largest renormalized inverse Lyapunov exponent Λ_M of the transfer matrix of RFM model near the band center. Λ_M initially increases with strip width M but eventually starts to decrease for sufficiently wide strips, indicating localized behavior.

when $W \to 0$. Hence, numerical data from Ref. 12 seem compatible with the conclusion that the ratio N_c/N goes to zero as $N \to \infty$, suggesting that there is no band of nonzero Chern number states for *any* $W \geqslant 0$ in the infinite system. This is indirectly supported by the TMM calculation of Xie *et al.*[14] who found that the states are localized at the band center for W significantly smaller than W_c.

The most extensive numerical study so far of the RFM was done by Furusaki.[15] He found the one-parameter scaling to be obeyed for $|E| > 2.55$, all states localized for $|E| \geqslant 0.1$, and gave some further evidence for the scenario (iia) and against the scenario (iib). Let us therefore first investigate the scaling properties of $\Lambda_M \equiv 1/\lambda_M M$ in the energy region $|E| \leqslant 0.1$ and $|E| \ll 0.1$. The TMM is applied and scaling of the Λ_M studied for the RFM in a gauge where all the complex phases are zero along the strip of width M and length up to 2×10^7 sites, the largest systems studied so far.

Λ_M decreases with M for $|E| > 0.1$,[15] as it is typical for localized states. Figure 1 shows Λ_M for $E = 10^{-1}$, $10^{-1.5}, \ldots, 10^{-4}, 0$, and $M = 4, \ldots, 128$. For $E \approx 0.1$, there is a suddenn increase of Λ_M for small system sizes before the decrease. This, previously thought of as being a finite-size artifact,[15] marks the beginning of a different scaling regime that is characteristic for the chiral disordered systems near the band center. In this regime, Λ_M increases with increasing of M, implying the existence of some sort of extended states in the infinite system. Figure 1 suggests, however, that this rise is not indefinite and that, for sufficiently wide strips at any given E, Λ_M reaches maximum and then decreases with further increase of M, and numerical evidence for this behavior is seen for $|E| \leqslant 10^{-3}$. This is very reminiscent of the recent TMM calculation[21] for uniformly distributed real $t_{i,j} \in (-1/2, 1/2)$, $W = 0$ Anderson bond-disordered model (ABD), where a similar asympototic behavior was found. Due to much smaller localization lengths for small $|E|$ in the ABD model, it was possible to analyze in more detail the behavior of Λ_M when M is increased, when a restoration of one-parameter scaling, $\Lambda_M(E) = \Lambda[M/\xi(E)]$, was found for sufficiently wide strips, leading to $\xi \sim |E|^{-\nu}$, with $\nu = 0.335 \pm 0.015$. Performing analogous calculation in the case of RFM, however, would require about two orders of magnitude wider strips, which seems to be beyond the present-day computational means.

Despite this severe limitation, an alternative approach from Ref. 21 will be used for calculation of ν of the RFM model. The link between the two models is the chiral symmetry. This symmmetry of a lattice Hamiltonian such as H can be defined as the topological property of bipartiteness of the underlying graph whose nodes represent sites and links represent possible hoppings described by the Hamiltonian. On-site disorder, for instance, introduces links in the graph from each site to itself, destroying the bipartiteness.[19] It is well known that the symmetry implies that all eigenstates come in the opposite energy pairs with simply related wave functions. Wegner, to the best of our knowledge, was the first to realize the importance of chiral symmetry in disordered systems.[16] Soukoulis and Economou found in their study[17] of H with real $t_{i,j}$ and $W = 0$ a critical state at the band center using both the TMM and GF method, with results qualitatively suggesting a diverging ξ near the band center. As already mentioned, ν has been recently obtained numerically for the ABD model, using TMM as well as from a finite-size scaling analysis (FSS) related to the two states closest to the band center of a system on a square lattice of linear size L. Since they become zero-energy states when $L \to \infty$ and therefore critical, it was conjectured in Ref. 21 that one should be able to apply a finite-size scaling analysis to the width $E_2(L)$ of the energy interval that they occupy in a system of size L, and from there obtain ν.[23] The equation determining $E_2(L)$ is

$$1 = L^2 \int_{E_2/2}^{0} \rho_L(\epsilon) d\epsilon, \tag{2}$$

where ρ_L is the average density of states (DOS) of the ansamble of disordered systems, normalized to 1.[22,21] The inverse function of $E_2(L)$ defines formally a new length scale, $\xi'(E)$, and the FSS gives that $\xi' \propto |E|^{-\nu}$, allowing one to calculate ν from Eq.(2) by the exact numerical diagonalization of the Hamiltonian for many configurations of disorder.[21] The agreement found in Ref. 21 between the such obtained value of ν and the value obtained from the TMM further supported the validity of this conjecture.

Figure 2 presents the calculated ξ' for $p = 0.5$ and $p = 1$, $W = 0$, on a square lattice with periodic boundary conditions for various even L between 20 and 90 with the number of configurations of disorder ranging from 2×10^3 to 100, respectively. The power law is well obeyed for 1.5 decades in energy, and the extracted values of the critical exponent ν are given in the figure. Although the difference between exponents is small and both are close to 1/2, the values nevertheless support the conclusion that $\nu_{p=1} < \nu_{p=0.5} < 1/2$, implying also a weak dependence of ν on the disorder distribution. The second inequality comes from the fact that ρ_L

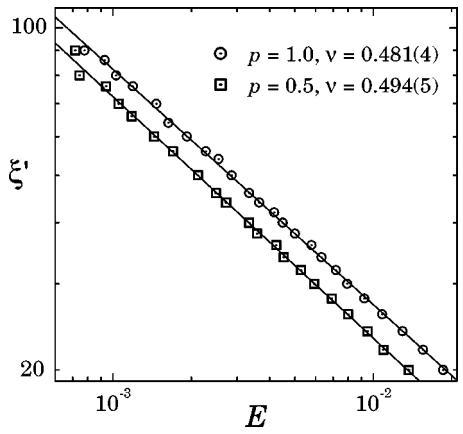

FIG. 2. Scaling of the ξ' length as determined from Eq. (2), for $p=0.5$ and 1. Power law is obeyed well in both cases, and the ν's are determined from two linear fits, represented in the figure by lines.

depends on L near the band center.[15] In the ABD model this dependence is much more pronounced,[21] so that ν is even smaller.

We now discuss the possibility of the KTB transition (iib). Xie *et al.*[14] have found from TMM calculations that Λ_M exhibits a scale-invariant behavior around the band center for $0<W<W_C(p)$, leading to the conclusion (iib). On the other hand, Furusaki has given numerical evidence that the variance of the conductance distribution of RFM quantum wires follows the one of the unitary class for $W>0$ and a different one for $W=E=0$, emphasizing the importance of the chiral symmetry and implying that all the states are localized for $W>0$ or $|E|\neq 0$, for $p\geqslant 0.2$.

Figure 3 gives further evidence that there is no KTB transition (iib) by redoing one of the calculations from Ref. 14 with higher precision. The figure shows Λ_M at the band center of the RFM $p=0.5$ model for different strengths W of the on-site disorder, and $M=16,32,64,128$. The length of strips used in the calculation is, however, 2×10^6 sites, which is

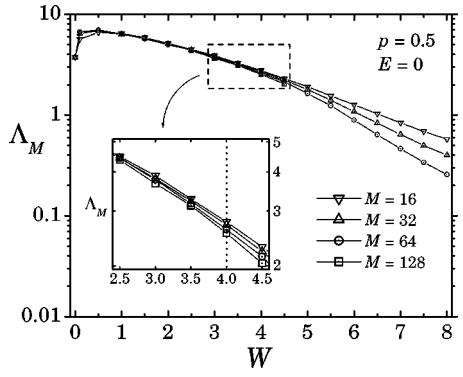

FIG. 3. Scaling of the largest inverse renormalized Lyapunov exponent at the band center, for various strengths of the on-site disorder W. The inset shows magnified region where the KTB transition (vertical dashed line) should be expected according to Ref. 14. Due to higher accuracy of the present calculation, it is clearly seen that the transition is absent, since Λ_M decreases with M, signaling localization.

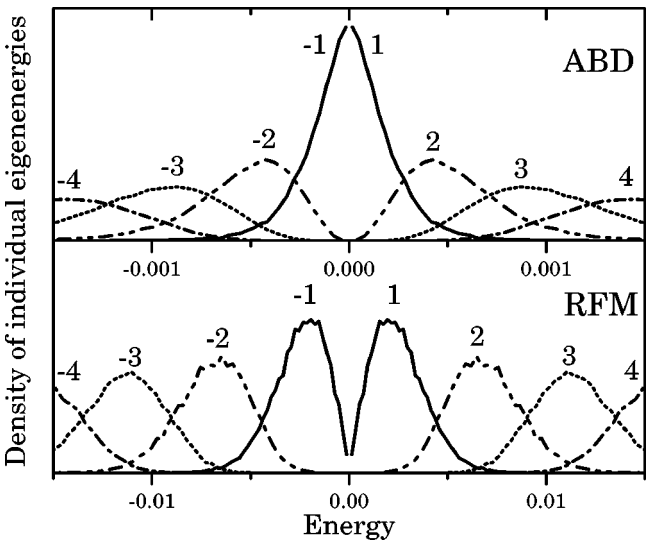

FIG. 4. Distributions of 8 individual eigenenergies, enumerated $-4,-3,\ldots,4$, closest to the band center of the ABD and RFM systems of size $L=40$ for 2×10^4 configurations of disorder. Both systems have symmetric spectrum around $E=0$ due to the chiral symmetry, but the central two states are repelled only in the RFM system.

about an order of magnitude longer than in Ref. 14. This gives more accurate values of Λ_M and it can be noted that, for $W\geqslant 2.5$, Λ_M is systematicaly smaller for $M=128$ than for $M\leqslant 64$, as shown in the inset, indicating clearly localization at the band center for W around 4, and thus the absence of KTB transition seen in Ref. 14.

Let us now turn to the investigation of the $E=0$ states. In the ABD model there exists an additional symmetry among Lyapunov exponents at $E=0$ due to which all of them come in degenerate pairs.[21] This property of chiral disordered models, noticed by Miler and Wang,[11] is naturally expected to hold for the RFM model as well, as it was expected by the authors. Figure 1, however, clearly shows that $\lambda_M<\lambda_{M-1}$, implying that the additional symmetry at the band center is *broken* in the RFM model, in contrast to the ABD model where the two λ's are equal to a high accuracy.

This has its analog in the language of eigenenergies, where the two states closest to the band center are repelled only in the case of RFM. This important difference is shown in Fig. 4, where the distribution of several eigenenergies closest to the band center is presented. The power-law behavior shown in Fig. 2 and discussed above then suggests that the distribution of central two states may be scale invariant and thus $\rho_{L\to\infty}$ equal to 0 in the case of the RFM, as opposed to a divergence in the case of the ABD model.

In the language of eigenstates, the additional symmetry at the band center of the ABD model is $O(2)$, since any linear combination of the two $E=0$ states is also an eigenstate at $E=0$. In the RFM then the broken additional symmetry means that two states closest to the band center in the infinite system are separated by an ''infinitesimal gap,'' which means that they have, as all the other states $E\neq 0$, equal probability density at each of the two sublattices.[21] Nevertheless, it is

still possible to have two critical states at $E=\pm 0$ in the infinite system, because the FSS from Ref. 21 that was used to intepret the scaling seen in Fig. 2 implies

$$\xi(E_2)\sim\xi'(E_2)\sim|E_2|^{-\nu}\sim L.$$

Thus, eigenstates that are not localized in a system of size L for $|E|\lesssim E_2(L)$ will not get localized when L is increased. Since these states exhibit multifractal behavior,[10,21] they will remain multifractal as $L\to\infty$, and this kind of critical behavior, present in both the ABD and RMF, we will refer to as *anomalous critical*.

In summary, the RFM model has been investigated by a high precision TMM and exact numerical diagonalization. The scenario (iia) was found to be valid in the limit of an infinite two-dimensional square lattice, with two *anomalous critical* states at the critical point $E=\pm 0$, and all the other states localized. Also some differences between the two chiral systems with and without time-reversal symmetry are identified and discussed, most striking of which is the broken chiral symmetry at the critical point in the absence of time-reversal symmetry.

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[1] L.B. Ioffe and A.I. Larkin, Phys. Rev. B **39**, 8988 (1988); N. Nagaosa and P.A. Lee, Phys. Rev. Lett. **64**, 2450 (1990); P.A. Lee and N. Nagaosa, Phys. Rev. B **45**, 966 (1992).

[2] B.I. Halperin, P.A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993); V. Kalmeyer and S.C. Zhang, *ibid.* **46**, 9889 (1992); K. Jain, Phys. Rev. Lett. **63**, 199 (1989).

[3] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. B **42**, 673 (1979); P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985); B. Kramer and A. MacKinnon, Rep. Prog. Phys. **56**, 1469 (1993).

[4] A. MacKinnon and B. Kramer, Z. Phys. B: Condens. Matter **53**, 1 (1983).

[5] T. Sugiyama and N. Nagaosa, Phys. Rev. Lett. **70**, 1980 (1993).

[6] Y. Avishai, Y. Hatsugai, and M. Kohmoto, Phys. Rev. **47**, 9561 (1993).

[7] V. Kalmeyer, D. Wei, D. Arovas, and S. Zhang, Phys. Rev. B **48**, 11 095 (1993).

[8] D.Z. Liu, X.C. Xie, S. Das Sarma, and S.C. Zhang, Phys. Rev. B **52**, 5858 (1995).

[9] D.N. Sheng and Z.Y. Weng, Phys. Rev. Lett. **75**, 2388 (1995).

[10] K. Yakubo and Y. Goto, Phys. Rev. B **54**, 13 432 (1996).

[11] J. Miler and J. Wang, Phys. Rev. Lett. **76**, 1461 (1996).

[12] K. Yang and R.N. Bhatt, Phys. Rev. B **55**, R1922 (1997).

[13] J.A. Vergés, Phys. Rev. B **57**, 870 (1998).

[14] X.C. Xie, X.R. Wang, and D.Z. Liu, Phys. Rev. Lett. **80**, 3563 (1998).

[15] A. Furusaki, Phys. Rev. Lett. **82**, 604 (1998).

[16] F. Wegner, Phys. Rev. B **19**, 783 (1979); R. Gade and F. Wegner, Nucl. Phys. **B360**, 213 (1991).

[17] C.M. Soukoulis and E.N. Economou, Phys. Rev. B **24**, 5698 (1981).

[18] R. Gade, Nucl. Phys. **B398**, 499 (1993).

[19] M. Inui, S.A. Trugman, and E. Abrahams, Phys. Rev. B **49**, 3190 (1994).

[20] Q. Niu, D.J. Thouless, and Yong-Shi Wu, Phys. Rev. B **31**, 3372 (1985); D.P. Arovas, R.N. Bhatt, F.D.M. Haldane, P.B. Littlewood, and R. Rammal, Phys. Rev. Lett. **60**, 619 (1988).

[21] V.Z. Cerovski, Phys. Rev. B **62**, 12775 (2000).

[22] V.Z. Cerovski, S.D. Mahanti, T.A. Kaplan, and A. Taraphder, Phys. Rev. B **59**, 13 977 (1999).

[23] Another energy scale that is of the same order as E_2 and thus deserves further consideration is the mean-level spacing between the two states closest to the band center, Δ_0. It is clear that $\Delta_0>E_2$, due to corrections from distributions of eigenenergies not closest to the band center, and we find also $L\propto|\Delta_0(L)|^{-\nu}$, with $\nu_{p=1.0}=0.492(4)$ and $\nu_{p=0.5}=0.497(3)$ (implying that the corrections are quite small).