Decay of superfluid turbulence at a very low temperature: The radiation of sound from a Kelvin wave on a quantized vortex

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In an earlier paper [Phys. Rev B **61**, 1410 (2000)] concerned generally with grid turbulence in superfluid ^{4}He, we discussed the mechanisms by which such turbulence might decay at a very low temperature. One such mechanism involves generation of Kelvin waves on quantized vortex lines that are present in the turbulent superfluid component, and the loss of energy from these waves by radiation of sound. An inadequate theory of this radiative process is replaced here by a more rigorous treatment. The speculative conclusions in our earlier paper are not substantially changed.

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I. INTRODUCTION

Turbulence in the superfluid phase of liquid ^{4}He has been studied experimentally and theoretically since the 1950s. Turbulence in the superfluid component must take the form of an irregular configuration of quantized vortex lines,[1,2] and turbulence of a more conventional type can exist in the normal component. Early work was concerned largely with superfluid turbulence produced by a counterflow of the two fluids, such as occurs in the presence of a heat current, and which has no classical analog. More recently work has been reported on superfluid turbulence produced by flow through a grid, which is the analog of an important form of classical turbulence, which, when fully developed at high Reynolds number, has a simplicity associated with its being approximately homogeneous and isotropic. The elements of a theory of superfluid grid turbulence were developed recently in Ref. 3 (referred to later as *V*), and the ways in which it is similar to and differs from the classical analog were explored. A particularly interesting aspect of superfluid grid turbulence relates to the mechanisms by which it can decay at very low temperatures, where there is little or no normal fluid. One such mechanism involves the generation on the vortex lines of Kelvin waves, as a result of the occasional close approach of vortices and of vortex reconnections,[4,3] and the subsequent decay of the Kelvin waves by frictional interaction with a small density of normal fluid or, at the very lowest temperatures, by radiation of sound (phonons). The discussion in *V* was based on formulas for the rate of sound radiation for which there was no fully satisfactory proof. In this paper we discuss this radiative process more rigorously, and we use our results to modify, albeit only slightly, the speculative conclusions of our earlier work. We do not attempt a full-scale review of this earlier work.

II. THE RADIATION OF SOUND BY MOVING RECTILINEAR QUANTIZED VORTICES

Any accelerated motion of a quantized vortex will result in the radiation of sound waves. The particular case of two parallel rectilinear vortices of the same sign encircling each other has been treated by Pismen[5] and by Howe.[6] Here we consider two cases involving rectilinear vortices, the second of which reproduces the earlier results, and then extend the treatment to deal with the radiation from a Kelvin wave. The vortices with which we deal have circulation κ and move in a pure superfluid of density ρ, in the absence of any normal fluid (we shall apply the results to temperatures where the normal fluid density is negligible).

A. Radiation from a rectilinear vortex moving in a circle

Consider first a rectilinear vortex parallel to the axis, $r=0$, of a system of cylindrical polar coordinates, r,θ,z. Suppose that the vortex core moves in a circle, $r=b$, at velocity $(0,b\omega,0)$ under the influence of a radial force $(\rho\kappa b\omega,0,0)$. The motion of the vortex perturbs the pressure near it, and this perturbation acts as a source of sound of angular frequency ω and wave number $k=\omega/c$; we shall assume that $kb\ll 1$. The perturbation p_1 in the pressure at the point $P(r,\theta,0)$ can be calculated from the linearized integral of the Euler equation for irrotational flow

$$\frac{\partial\phi}{\partial t}=-\frac{p_1}{\rho}, \tag{2.1}$$

where ϕ is the velocity potential. First we consider the range of r given by $kr\ll 1$, where there are no retardation effects. In terms of the lengths and angles shown in Fig. 1 we find that the perturbation in the velocity potential is given by

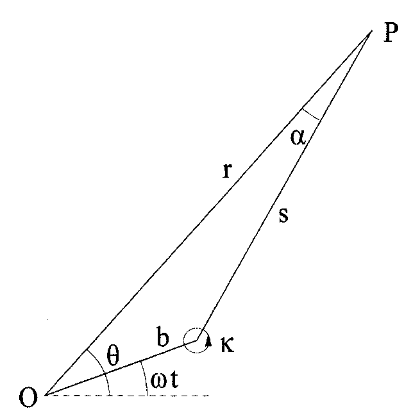

FIG. 1. Illustrating the calculation of the perturbation in the velocity potential when a vortex moves in a circular orbit.

$$\phi_1=\frac{\kappa}{2\pi}\alpha=\frac{\kappa}{2\pi}\sin^{-1}\left[\frac{b\sin(\theta-\omega t)}{s}\right]\approx\frac{\kappa}{2\pi}\frac{b\sin(\theta-\omega t)}{s}$$

$$=\frac{\kappa}{2\pi}b\sin(\theta-\omega t)\left[\frac{1}{r^2+b^2-2rb\cos(\theta-\omega t)}\right]^{1/2}. \quad (2.2)$$

The perturbation p_1 can be expressed as a power series in b/r, and we shall be interested in only the first two terms in this series. Nonlinear corrections to Eq. (2.1), and higher-order terms in the expansion of α, make no contribution to these terms, while the expansion of Eq. (2.2) leads to

$$\phi_1=\frac{\kappa}{2\pi}\frac{b}{r}\sin(\theta-\omega t)+\frac{\kappa}{4\pi}\frac{b^2}{r^2}\sin(2\theta-2\omega t); \quad (2.3)$$

or, equivalently,

$$\phi_1=\frac{-i\kappa}{2\pi}\frac{b}{r}\exp(i\theta)\exp(-i\omega t)-\frac{i\kappa}{4\pi}\frac{b^2}{r^2}\exp(2i\theta)\times\exp(-2i\omega t). \quad (2.4)$$

It follows that

$$p_1=\frac{\rho\omega\kappa}{2\pi}\frac{b}{r}\exp(i\theta)\exp(-i\omega t)+\frac{\rho\omega\kappa}{2\pi}\frac{b^2}{r^2}\exp(2i\theta)\times\exp(-2i\omega t). \quad (2.5)$$

Inclusion of retardation must lead to an outgoing sound wave with the general form

$$p_s=\sum_{m=1}^{m=\infty}A_m[J_m(kr)+iN_m(kr)]\exp(im\theta)\exp(-i\omega t). \quad (2.6)$$

In the case that we are considering here the dipole $(m=1)$ term dominates the radiated field, so that Eq. (2.6) has the limiting forms

$$p_s\to -i\frac{2A_1}{\pi kr}\exp(i\theta)\exp(-i\omega t) \quad \text{as} \quad kr\to 0; \quad (2.7)$$

$$p_s\to A_1\left(\frac{2}{\pi kr}\right)^{1/2}\exp(i\theta)\exp i\left[kr-\omega t-\frac{3\pi}{4}\right] \quad \text{as} \quad kr\to\infty. \quad (2.8)$$

For $b\ll r\ll 2\pi/k$, the first term in Eq. (2.5) and the form (2.7) must join smoothly, so that

$$A_1=i\frac{\rho\kappa b\omega^2}{4c}. \quad (2.9)$$

It follows that the far sound field is given by

$$p_s=i\rho\kappa b\omega^{3/2}\left(\frac{1}{8\pi cr}\right)^{1/2}\exp(i\theta)\exp i\left[kr-\omega t-\frac{3\pi}{4}\right]. \quad (2.10)$$

The corresponding total mean power radiated by unit length of vortex is then given by

$$\Pi=\int_0^{2\pi}\frac{\langle p_s^2\rangle}{\rho c}r\,d\theta=\frac{\rho\kappa^2b^2\omega^3}{8c^2}. \quad (2.11)$$

B. Radiation from a pair of equal rectilinear vortices encircling each other

The case of a pair of parallel vortices, equal in magnitude and sign, circling around each other under their own velocity fields, is now easily considered. If the separation of the vortices is $2b$, the angular velocity of encircling rotation is $\omega=\kappa/4\pi b^2$. Each vortex radiates as in Sec. II A, but the dipole fields cancel, leaving only a quadrupole field $(m=2)$ with frequency 2ω and wave number $2k$. For each vortex the appropriate limiting forms corresponding to Eqs. (2.7) and (2.8) are then given by

$$p_s\to -i\frac{4A_2}{\pi(2k)^2r^2}\exp(2i\theta)\exp(-2i\omega t) \quad \text{as} \quad kr\to 0 \quad (2.12)$$

$$p_s\to A_2\left(\frac{2}{\pi(2k)r}\right)^{1/2}\exp(2i\theta)\exp i\left[2kr-2\omega t-\frac{5\pi}{4}\right] \quad \text{as} \quad kr\to\infty. \quad (2.13)$$

For $b\ll r\ll 2\pi/k$, the second term of Eq. (2.5) and the form (2.12) must join smoothly, so that

$$A_2=i\frac{\rho\kappa b^2\omega^3}{2c^2}. \quad (2.14)$$

It follows that the far sound field for the pair of vortices is given by

$$p_s=i\frac{\rho\kappa b^2\omega^{5/2}}{c}\left(\frac{1}{\pi cr}\right)^{1/2}\exp(2i\theta)\exp i\left[2kr-2\omega t-\frac{5\pi}{4}\right]. \quad (2.15)$$

The corresponding total mean power radiated at frequency 2ω by unit length of the vortex pair is then given by

$$\Pi=\frac{\rho\kappa^2b^4\omega^5}{c^4}. \quad (2.16)$$

This result seems to agree with those given by Pismen and Howe.

III. RADIATION FROM A KELVIN WAVE

Let the Kelvin wave have wave vector $\tilde{k}$. Its angular frequency is given by the approximate dispersion relation[7]

$$\tilde{\omega}=\frac{\kappa\tilde{k}^2}{4\pi}\ln\left(\frac{1}{\tilde{k}\xi_0}\right), \quad (2.17)$$

where ξ_0 is the vortex core parameter. The wave is circularly polarized, so that each element of the vortex is describing a circle with radius b equal now to the amplitude of the wave. The situation is very similar to that described in Sec. II A, each element of the line tending to radiate in the way described by Eq. (2.10); the radial force mentioned in Sec. II A is due now to the line tension in the vortex. However, the Kelvin wave differs from the case in Sec. II A because different parts of the wave will radiate with different phases. Consider radiation emitted at an angle β to the plane normal to the length of the undisplaced vortex. Simple diffraction theory shows that, in the case of a Kelvin wave of uniform amplitude on a rectilinear vortex of infinite length, there is radiation only at values of β equal to $\pm\sin^{-1}[\tilde{k}/k]$, although at these angles the radiated power is similar to that given by Eq. (2.11). Therefore there is no radiation of sound unless $(\tilde{k}/k)<1$. In practice this condition can be satisfied for a quantized vortex in helium only if the wavelength of the Kelvin waves is comparable with ξ_0. As we shall see, the only important Kelvin waves in superfluid turbulence have wave numbers satisfying the condition $\tilde{k}\gg k$.

This total absence of radiation relates only to Kelvin waves on a rectilinear vortex of infinite length. Suppose that the vortex has a finite length $2l_1$. Each element of the Kelvin wave acts as a source of sound, with a certain complex amplitude $\zeta(z)$. The total radiated amplitude in a direction defined by the angle β is given by the Fourier transform of $\zeta(z)$

$$p(q)=\frac{B}{2}\int_{-l_1}^{+l_1}\zeta(z)\exp(-iqz)dz, \tag{2.18}$$

where B is a constant, and $q=k\sin\beta$. For a Kelvin wave of wave number $\tilde{k}$, $\zeta(z)=\zeta_0\exp(i\tilde{k}z)$, where ζ_0 is a constant. For the case of a rectilinear vortex moving in a circle (Sec. II A) $\tilde{k}=0$.

Evaluating Eq. (2.18) for the Kelvin wave we find that

$$p(q)=B\zeta_0 l_1\frac{\sin[(\tilde{k}-q)l_1]}{(\tilde{k}-q)l_1}, \tag{2.19}$$

while for the case of Sec. II A we have

$$p(q)=B\zeta_0 l_1\frac{\sin(ql_1)}{ql_1}. \tag{2.20}$$

The form (2.20) has a sharp peak of height $B\zeta_0 l_1$ centred on $q=0$ $(\beta=0)$, with width $\Delta q\sim\pi/l_1$. The total radiated power into this peak is therefore given roughly by

$$\Pi\approx\pi B^2\zeta_0^2 l_1, \tag{2.21}$$

which can be compared with Eq. (2.11) in Sec. II A. For the case of the Kelvin wave with $\tilde{k}\gg k$, there is no sharp peak, the amplitude oscillating rapidly with the angle β; apart from this oscillation the power is radiated more or less uniformly over the range of β from $-\pi/2$ to $\pi/2(\Delta q=2k)$. The total power radiated is given by

$$\Pi'\approx(B\zeta_0 l_1)^2(\tilde{k}l_1)^{-2}2k=\frac{2B^2\zeta_0^2 k}{\tilde{k}^2}. \tag{2.22}$$

The ratio

$$\frac{\Pi'}{\Pi}\approx\frac{2}{\pi}\frac{1}{\tilde{k}l_1}\frac{k}{\tilde{k}} \tag{2.23}$$

is the factor by which the power (2.11) is reduced by interference effects for the case of the Kelvin wave when $\tilde{k}\gg k$. As we expect, $(\Pi'/\Pi)\to 0$ as $l_1\to\infty$.

IV. APPLICATION TO THE DECAY OF SUPERFLUID TURBULENCE

Turbulence in the superfluid component of liquid ^{4}He must take the form of a more-or-less random array of vortex lines. If the length of line per unit volume is L, the line spacing is roughly $l=L^{-1/2}$. At temperatures close to the absolute zero there is a negligible concentration of normal fluid, so there is no frictional interaction with the normal fluid to damp the motion of the vortex lines. It is known from experiment that the vortex lines can decay under these conditions,[8] and possible mechanisms have been discussed by Svistunov and Vinen. Svistunov[4] suggested that the occasional close approach of two vortex lines, leading to vortex reconnection, would give rise to kinks on the lines. These kinks can be regarded as superpositions of Kelvin waves, and energy loss from these waves can make a significant contribution to the energy loss from the turbulent superfluid. In the absence of frictional interaction with the normal fluid the Kelvin waves can lose energy only by radiation of sound, and the effect of such radiation was discussed tentatively by Vinen (V), using rough estimates of the rate of radiation. We can now extend this earlier discussion by using the more rigorous results derived here.

The ideas in V were openly speculative, and we use them as the basis of this extension. There is clearly a need to review their validity, but we believe that it is premature to do so here, because such a review ought to await the results and interpretation of computer simulations that are presently being carried out by other authors.

In superfluid turbulence the Kelvin waves are formed on vortex lines that form a tangle. The lines are not straight but are typically bent with radii of curvature of order l, the line spacing, the Kelvin waves being superimposed on these bent lines. It turns out, as we shall see, that much of the important sound radiation has a wave number of order l^{-1} (or a little less), the Kelvin waves themselves having a much larger wave number. This means that Kelvin waves from parts of a vortex that are separated by a distance greater than l will not radiate coherently, so that the total power dissipation can be obtained by considering vortex lengths of order l as acting independently of each other. We can therefore use Eqs. (2.11) and (2.23) to estimate the total radiative loss, replacing the length l_1 by the vortex spacing l. It should be added that a review of the ideas of V may lead to the conclusion that for real, randomly irregular, configurations of vortex line it is not appropriate to think of harmonic Kelvin waves (with a range of frequencies) as linearly superimposed on lines that are smoothly curved on the length scale l. In that case the effective correlation length for coherent radiation may be less than l, but l must remain an upper limit.

Equation (7.5) in V, which gives the power radiated per unit length of vortex in the tangle, is replaced by

$$\Pi=\frac{\rho\kappa^3\tilde{\omega}^3b^2}{16\pi^2c^3l}\ln\left(\frac{1}{\tilde{k}\xi_0}\right), \qquad (2.24)$$

where b is now the amplitude of the Kelvin wave. Remembering the energy per unit length in the Kelvin wave is $(\rho\kappa^2\tilde{k}^2b^2/4\pi)\ln(1/\tilde{k}\xi_0)$, we find that the time constant characterizing the rate of loss of energy from the Kelvin wave is therefore given by

$$\tau_s=\frac{c^2}{\kappa\tilde{\omega}^2}\,\frac{16\pi^2cl}{\kappa\ln(1/\tilde{k}\xi_0)}, \qquad (2.25)$$

which replaces Eq. (7.6) in V. The rate of loss of vortex line from the tangle is still given by Eq. (7.12) of V, but the value of the cut-off wave vector $\tilde{k}_2$ is now given by

$$\tilde{k}_2l=\left(\frac{cl}{A^{1/3}\kappa}\right)^{3/4}, \qquad (2.26)$$

where we have dropped factors of order unity as we did in V. Inserting a typical value of l equal to 10^{-5} m, and taking $A=1$, we find that $\tilde{k}_2\sim2\times10^8\,\mathrm{m}^{-1}$ (corresponding to $\tilde{\omega}\sim10^9\,\mathrm{s}^{-1}$). The corresponding value of k is roughly $4\times10^6\,\mathrm{m}^{-1}$, so that we were justified in making the assumption that $\tilde{k}\gg k$; and we confirm that the wavelength of the emitted sound wave is somewhat less than l.

A straightforward modification of the calculations in V allows us to update our calculation of the temperature above which the damping of Kelvin waves in the vortex tangle is determined by mutual friction rather than sound radiation. We find that this temperature is approximately 0.46 K, very similar to one of our earlier estimates based on an inadequate treatment of the radiation.

V. SUMMARY AND CONCLUSIONS

We have derived a fairly reliable expression for the rate of energy loss from a Kelvin wave on a quantized vortex in superfluid ^{4}He due to radiation of sound, and we have used it to improve an earlier discussion of the rate of energy loss from superfluid turbulence at a very low temperature. The conclusions of the earlier discussion are not substantially changed.

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