Giant vortices in the Ginzburg-Landau description of superconductivity

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Recent experiments on mesoscopic samples and theoretical considerations lead us to analyze multiply charged ($n>1$) vortex solutions of the Ginzburg-Landau equations for arbitrary values of the Landau-Ginzburg parameter κ. For $n\gg 1$, they have a simple structure and a free energy $\mathcal{F}\sim n$. In order to relate this behavior to the classic Abrikosov result $\mathcal{F}\sim n^2$ when $\kappa\to+\infty$, we consider the limit where both $n\gg 1$ and $\kappa\gg 1$, and obtain a scaling function of the variable κ/n that describes the crossover between these two behaviors of $\mathcal{F}$. It is then shown that a small-n expansion can also be performed and the first two terms of this expansion are calculated. Finally, large and small n expansions are given for recently computed phenomenological exponents characterizing the free energy growth with κ of a giant vortex.

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I. INTRODUCTION

Recent experiments[1–3] have demonstrated that vortices of charge $n>1$, which are unstable in macroscopic type II superconductors,[4] can exist in mesoscopic superconducting samples and can even be favored over configurations with multiple singly charged vortices. The Ginzburg-Landau description of superconductivity[5–7] appears to be an adequate framework to analyze these results. The magnetization response of a mesoscopic sample can be analytically understood by adding a surface energy contribution to the Ginzburg-Landau free energy of a giant-vortex in an infinite system.[8,9] Moreover, full numerical solutions of the Ginzburg-Landau equations accurately reproduce the experimental findings.[10–12]

Motivated by these experimental and theoretical results, we analyze in the present paper "giant" (i.e., $n>1$) vortex solutions of the Ginzburg-Landau equations. Well-known analytical results have been obtained in the London limit when the Ginzburg-Landau parameter $\kappa\to\infty$,[13,7] and at the special dual-point value $\kappa=1/\sqrt{2}$.[7,14] Here, we take advantage of the supplementary parameter n, the vortex charge, and provide a simple analysis of the giant vortices for arbitrary values of κ. We begin in Sec. II, by recalling the Ginzburg-Landau equations and some elementary properties of their solutions. In Sec. III, we consider the large vorticity limit $n\gg 1$. The vortex structure takes the form of a circular normal core separated by a sharp boundary from the outside superconducting medium. As a consequence, the free energy $\mathcal{F}$ of a giant vortex is found to be proportional to its charge n in contrast to the classic Abrikosov's result $\mathcal{F}\sim n^2$ in the London limit. In order to relate the two results, we consider in Sec. IV, the double limit in which both the vorticity n and the Ginzburg-Landau parameter κ are large and we obtain the scaling form $\mathcal{F}/2\pi\sim n\kappa\Phi(\kappa/n)$. The function Φ provides an explicit interpolation between Abrikosov's result for $\kappa\gg n$ and the result of Sec. III which is valid in the opposite limit $n\gg\kappa$. To complete our analysis of giant vortex solutions, we consider in Sec. V, the somewhat more formal $n\to 0$ limit, which has the advantage to be amenable to a simple and systematic expansion scheme; the free energy is obtained up to order n^2. In Sec. VI, we compare our perturbative results (valid for arbitrary values of κ) to the known exact results at the dual point $\kappa=1/\sqrt{2}$. Besides checking consistency, we obtain a perturbative expansion of the free energy in the vicinity of the dual point and the large-n expansion of phenomenological exponents introduced previously and computed numerically.[9] The large n expansion proves to be fairly accurate for n values as small as 3 or 4.

II. THE GINZBURG-LANDAU MODEL OF SUPERCONDUCTIVITY

In the Ginzburg-Landau description of superconductivity[5–7] the two unknown fields are the complex order parameter $\psi=|\psi|e^{i\chi}$ and the potential vector $\vec{A}$ with $\vec{\nabla}\times\vec{A}=\vec{B}$, where $\vec{B}$ is the local magnetic induction. These fields satisfy the following equations:

$$-\left(\vec{\nabla}-i\frac{2\pi}{\phi_0}\vec{A}\right)^2\psi=\frac{1}{\xi^2}\psi(1-|\psi|^2), \tag{1}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}=\frac{|\psi|^2}{\lambda^2}\left(\phi_0\frac{\vec{\nabla}\chi}{2\pi}-\vec{A}\right). \tag{2}$$

Here, the flux quantum ϕ_0 is given by $\phi_0=hc/2e$, and the two characteristic lengths λ (penetration depth or London length) and ξ (coherence length) appear as phenomenological parameters. The Ginzburg-Landau parameter κ is defined as their ratio, $\kappa=\lambda/\xi$. We shall measure lengths in units of $\lambda\sqrt{2}$, the magnetic field in units of $H_c/\sqrt{2}\kappa=\phi_0/4\pi\lambda^2$ and the vector potential in units of $\phi_0/2\sqrt{2}\pi\lambda$. The Ginzburg-Landau free energy in units of $H_c^2\xi^2/4\pi$ is then given by

$$\mathcal{F}=\int\frac{1}{2}B^2+\kappa^2(1-|\psi|^2)^2+|(\vec{\nabla}-i\vec{A})\psi|^2. \tag{3}$$

A giant vortex of vorticity n is a solution of the Ginzburg-Landau equations with cylindrical symmetry $\psi(r,\theta) = f(r)\exp(in\theta)$, where (r,θ) are polar coordinates in the plane. The potential vector $\vec{A}$ can be chosen to lie in the plane and to have only a nonzero angular component $A_\theta(r)$. It is convenient to introduce an auxiliary function g that represents the difference between the flux through a disk of radius r and the total flux,

$$g(r)=rA_\theta(r)-n, \quad \text{i.e.,} \quad B=\frac{1}{r}\frac{dg}{dr}. \tag{4}$$

The free energy can be rewritten in terms of f and g as

$$\frac{1}{2\pi}\mathcal{F}=\int_0^\infty\left\{\left(\frac{df}{dr}\right)^2+\frac{f^2g^2}{r^2}+\kappa^2(1-f^2)^2+\frac{1}{2r^2}\left(\frac{dg}{dr}\right)^2\right\}r\,dr. \tag{5}$$

The allied Ginzburg-Landau equations reduce to

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{1}{r^2}fg^2=-2\kappa^2f(1-f^2), \tag{6}$$

$$\frac{d^2g}{dr^2}-\frac{1}{r}\frac{dg}{dr}=2f^2g. \tag{7}$$

It is useful to note that a very simple form of the free energy[7] is obtained by an efficient use of Eqs. (6) and (7),

$$\frac{1}{2\pi}\mathcal{F}=2\kappa^2\int_0^\infty r\,dr(1-f^2). \tag{8}$$

A vortex of charge n corresponds to functions f and g which satisfy $f(0)=0$ and $g(0)=-n$ at the origin and which obey $f(\infty)=1$ and $g(\infty)=0$ at infinity. Linearization of Eqs. (6) and (7) around $f=1$ and $g=0$ shows that there exists two exponentially growing and two exponentially decaying spatial modes at $r=\infty$. In the same way, linearization for r close to zero shows that there is one diverging mode with $f\sim r^{-n}$ and one neutral mode corresponding to changes in the vortex charge [note that at the level of Eqs. (6) and (7), n appears simply as a parameter and is not constrained to be an integer]. Once one requires the diverging mode at $r=0$ to be absent and $g(0)=-n$, the expansion of f and g around $r=0$ depends on two arbitrary constants

$$f(r)=\left(\frac{r}{\mathcal{R}}\right)^n+\mathcal{O}(r^{n+1}), \tag{9}$$

$$g(r)=-n\left(1-\left(\frac{r}{r_0}\right)^2\right)+\mathcal{O}(r^{2n+2}). \tag{10}$$

The length scales r_0 and $\mathcal{R}$ are uniquely determined by the cancellation of the two divergent modes at $r=\infty$;[15] they cannot be calculated from a local analysis near 0. Their determination requires the behaviors around $r=0$ and $r=\infty$ to be connected. This can be done numerically for arbitrary parameter values or analytically when n is either large or small as shown in the following sections.

One can note from the definition of g [Eq. (4)] that r_0 is simply related to the value of the magnetic field at the position of the vortex:

$$B(0)=\left(\frac{1}{r}\frac{dg}{dr}\right)_{r=0}=\frac{2n}{r_0^2}. \tag{11}$$

III. A GIANT VORTEX IN THE LARGE VORTICITY LIMIT

We first consider the structure of a giant vortex of charge n for $n\gg 1$ and begin with simple estimates. It seems intuitively clear that the vortex core grows with its charge. So, for $n\gg 1$, one expects the vortex core to be much larger than the London penetration length. As a consequence, the magnetic induction should be approximately constant over the core. For a vortex of charge n and core size r_c, one expects therefore $|B|=2n/r_c^2$ in the vortex core (the total flux divided by the core area with the chosen normalization) and $B=0$ outside. The magnetic energy of such a configuration is approximately $\int B^2/2\simeq 2\pi n^2/r_c^2$. Correlatively, one expects $\psi=0$ in the vortex core and $\psi=1$ outside and the corresponding energy contribution $\int\kappa^2(1-|\psi|^2)^2\simeq 2\pi\kappa^2 r_c^2$. The total Ginzburg-Landau energy (3) of such a configuration can therefore be estimated to be

$$\frac{1}{2\pi}\mathcal{F}\simeq\frac{n^2}{r_c^2}+\frac{\kappa^2}{2}r_c^2+\sigma(\kappa)r_c, \tag{12}$$

where the last term on the right-hand side has been added to take into account the interfacial energy of the transition layer between the vortex core and the superconducting bulk. Minimizing Eq. (12) with respect to r_c gives the dominant order estimates for the vortex size r_c and for its free energy $\mathcal{F}$

$$r_c^2\sim\frac{\sqrt{2}n}{\kappa}, \tag{13}$$

$$\frac{1}{2\pi}\mathcal{F}\simeq n\kappa\sqrt{2}+\left(\frac{\sqrt{2}n}{\kappa}\right)^{1/2}\sigma(\kappa)+\mathcal{O}(1). \tag{14}$$

The magnetic field inside the vortex core has the constant value

$$B=\sqrt{2}\kappa\left[1+\frac{\sigma(\kappa)}{2\kappa^2r_c}\right]. \tag{15}$$

That is, in the vortex core B is equal to H_c plus a Gibbs-Thomson correction as expected at a curved normal/superconducting boundary (see, e.g., Refs. 16 and 17). In the next subsections, we first present numerical solutions of the Ginzburg-Landau equations for various values of n which confirm this simple picture of the giant vortex. This picture is then derived from

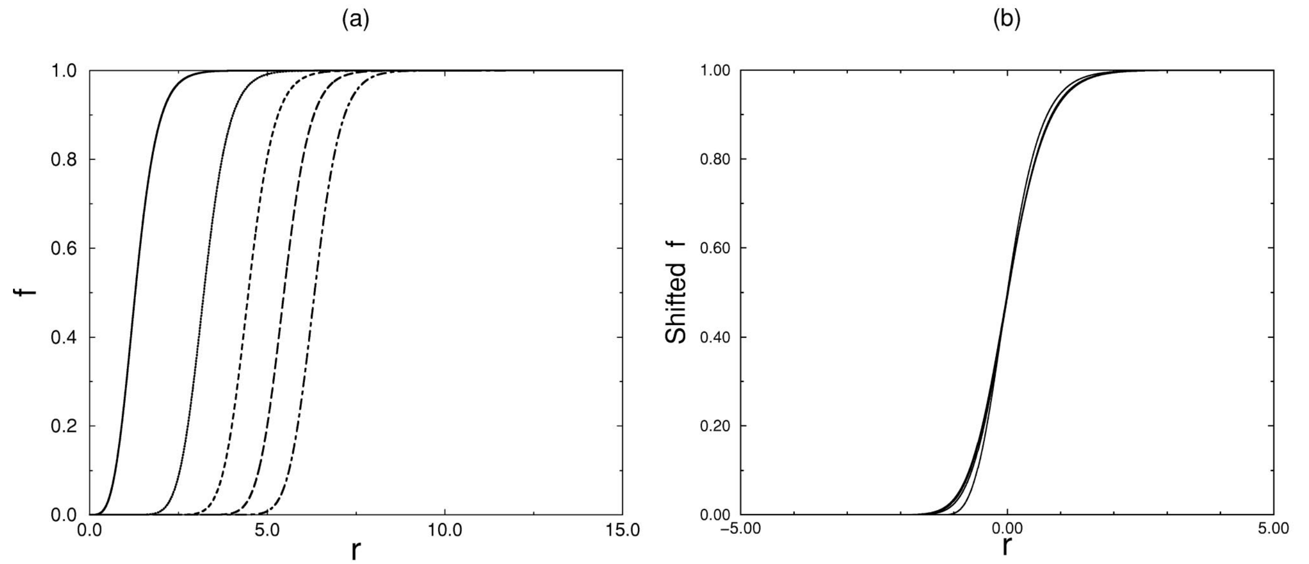

FIG. 1. (a) The function f for $\kappa=1.5$ and $n=4$, 16, 28, 40, and 52. (b) The different front profiles superposed on one another ($f=1/2$ is shifted to $r=0$).

a direct analysis of the Ginzburg-Landau equations in the large n limit.

A. Numerical results

Numerical solution of Eqs. (6) and (7) consists in solving a two-point boundary value problem (at $r=0$ and $r=\infty$) and can be achieved using several methods.[18]

Shooting is generally easy to implement and robust when one mode needs to be cancelled at a boundary: a free parameter is adjusted (for example by dichotomy) at the other boundary until the required cancellation is obtained. In the present case, two diverging modes need to be cancelled at infinity which is more difficult to achieve. We kept a simple one parameter shooting by integrating from a large $r=r_\infty$ toward $r=0$. Of the two free parameters at r_∞ one was used to cancel the diverging mode at $r=0$. The value of g at 0 was then given as a function of the other parameter (which could be adjusted to obtain a particular value of n when desired).

We also implemented a relaxation method. The system of ordinary differential equations is replaced by a set of finite difference equations satisfied on a mesh of points, and the boundary conditions just appear as equations satisfied by the points located at the extremities of the mesh. A multidimensional version of Newton's method provides a solution of these finite difference equations by an iterative procedure. This method requires the inversion of a matrix of size proportional to the number of points in the mesh. Because this matrix is block diagonal, it can be inverted in a very efficient manner. However, the efficiency of the relaxation method depends strongly on the starting point of the iteration: for example, it is useful to take the profile of a n-vortex as an initial guess for the $(n+1)$-vortex solution of the Ginzburg-Landau equations.

Numerical solutions for different values of n are shown in Fig. 1. The plot of f shows a well-defined interface between a normal region (where $f=0$) and a superconducting region ($f=1$) [see Fig. 1(a)]. The interface width does not increase with n but remains of the order of $1/\kappa$. As expected, the interface has a well-defined limiting shape: in Fig. 1(b), the curves represented in Fig. 1(a) are shifted in such a way that they take the value 1/2 at the point $r=0$. The shifted curves superpose on one another almost perfectly. Only the curve for the small value $n=4$ shows a noticeable deviation from the limiting shape.

The simple normal/superconducting interface picture for $n\gg1$ is also confirmed by the graphs of g. In Fig. 2, $B(r)=1/r\ dg/dr$ is plotted for different values of n. The magnetic induction is close to $H_c=\sqrt{2}\kappa$ in the vortex interior and quickly decreases to zero in the interface region. Again, in the large n limit, the curves can be put over one another by

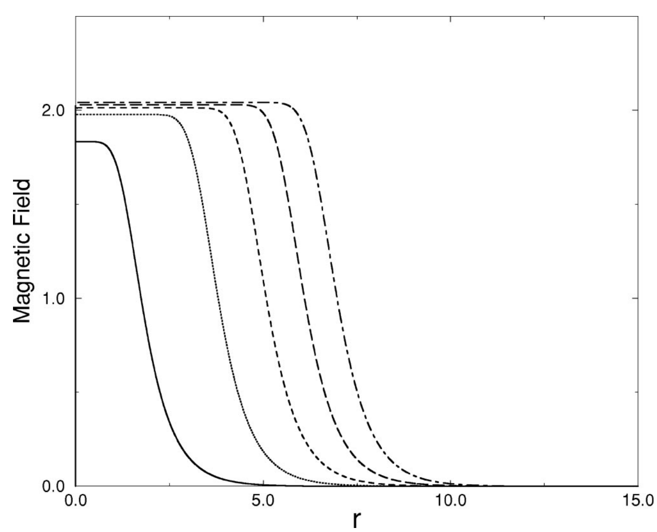

FIG. 2. The magnetic field for $\kappa=1.5$ and $n=4$, 16, 28, 40, and 52.

the same shifts as those used to superpose the different f's in Fig. 1(b).

B. Asymptotic analysis of the Ginzburg-Landau equations for $n \gg 1$

It is instructive to see how the large-n giant vortex structure arises in a direct analysis of the Ginzburg-Landau equations (6) and (7).

1. The vortex core

Since $g = -n$ at $r = 0$ one expects g to be of order n in a whole region near $r = 0$. Therefore, at dominant order in n, Eq. (6) for f reduces to

$$\frac{d^2 f}{dr^2} - \frac{1}{r^2} f g^2 = 0. \tag{16}$$

The solution of Eq. (16) can be obtained in a WKB manner but the important point is that f is exponentially close to zero in the whole core region. Equation (7) for g thus simplifies to

$$\frac{d^2 g}{dr^2} - \frac{1}{r}\frac{dg}{dr} = 0. \tag{17}$$

At dominant order in n, g is therefore given by

$$g(r) = -n\left(1 - \left(\frac{r}{r_0}\right)^2\right). \tag{18}$$

Although Eq. (18) is identical to Eq. (10), it should be noted that it is not an expansion near zero but the full function g in the vortex core at dominant order in n. Of course, Eq. (18) is nothing else but the constancy of B in the vortex core.

Outside the vortex core the medium is in the superconducting phase, $f = 1$ and $g = 0$. In order to match these different solutions of Eqs. (6) and (7), we analyze the interface region (i.e., the boundary layer in the usual matched asymptotics terminology).

It should be noted that r_c the vortex core size and r_0 which is obtained from the second derivative of g at the origin [Eq. (7)] coincide at leading order in n. Their difference is of the order of the planar normal/superconducting interface width and depends on the precise criterion used to measure the vortex core size (i.e., on the precise definition of r_c) as shown below.

2. Local equations near the interface

The solution (18) vanishes for $r = r_0$ and therefore the assumption $g \sim n$ under which it was derived breaks down at the interface. A different simplification of the Ginzburg-Landau equations can be made in the vicinity of r_0. We define a different variable x, centered around r_0, such that

$$r = x + r_0. \tag{19}$$

We assume (and will verify at the end) that $r_0 \gg |x|$. Substituting Eq. (19) into Eqs. (6) and (7) we obtain

$$\frac{d^2 f}{dx^2} + \frac{1}{r_0}\frac{df}{dx} - \frac{g^2}{r_0^2} f = -2\kappa^2 f(1 - f^2), \tag{20}$$

$$\frac{d^2 g}{dx^2} - \frac{1}{r_0}\frac{dg}{dx} = 2 f^2 g. \tag{21}$$

Comparing the magnitude of the different terms in these simplified equations, we obtain $f \sim 1$, $g \sim r_0$, $x \sim 1$ and find that the first-order terms df/dx, dg/dx on the LHS of Eqs. (20) and (21) are subdominant. Introducing the rescaled functions $f_0 = f$ and $G_0 = g/r_0$, one derives at dominant order in n (indicated by the subscript 0) the "inner" equations which describe the behavior of the order parameter and the vector potential in the interface region

$$\frac{d^2 f_0}{dx^2} = -2\kappa^2 f_0(1 - f_0^2) + G_0^2 f_0, \tag{22}$$

$$\frac{d^2 G_0}{dx^2} = 2 f_0^2 G_0. \tag{23}$$

As expected, these equations are identical to the Ginzburg-Landau equations in one dimension (i.e., on an infinite line). Matching with the superconducting phase for $x \gg 1$ and the vortex core for $x \ll -1$ imposes the boundary conditions $f_0(+\infty) = 1$, $G_0(+\infty) = 0$, and $f_0(-\infty) = 0$. This last condition implies, using Eq. (23), that G_0 behaves linearly when $x \rightarrow -\infty$

$$G_0 \simeq a x + b. \tag{24}$$

In contrast to Eqs. (6) and (7), the local system (22) and (23) is invariant by translation. Once this is fixed [for example by imposing the arbitrary criterion $f_0(0) = 1/2$] the functions f_0 and G_0 and the constant b are uniquely determined (the value of a is independent of this arbitrary choice). Finding a and b appears to require an explicit solution. This can be avoided, at least for a, because the reduced system (22) and (23) retains the variational structure of the original equations and is invariant by translation in x. The locally conserved quantity $\mathcal{E}$ associated with this continuous symmetry reads (that is, the magnetic field in the vortex core)[5]

$$\mathcal{E} = \frac{1}{2}\left(\frac{df_0}{dx}\right)^2 + \frac{1}{4}\left(\frac{dG_0}{dx}\right)^2 - \frac{1}{2} G_0^2 f_0^2 - \frac{\kappa^2}{2}(1 - f_0^2)^2. \tag{25}$$

Conservation of $\mathcal{E}$ between $-\infty$ and $+\infty$ leads to

$$0 = \mathcal{E}(+\infty) = \mathcal{E}(-\infty) = a^2/4 - \kappa^2/2. \tag{26}$$

Thus, $a = \sqrt{2}\kappa$ and $B = H_c$ in the normal phase at dominant order in n, as expected.

3. Matching the vortex core and the interface region

In order to determine the constant r_0 and the interface position, we have to match the solutions in the vortex core and in the interface region.

Rewriting Eq. (10) in terms of the local variable x leads to

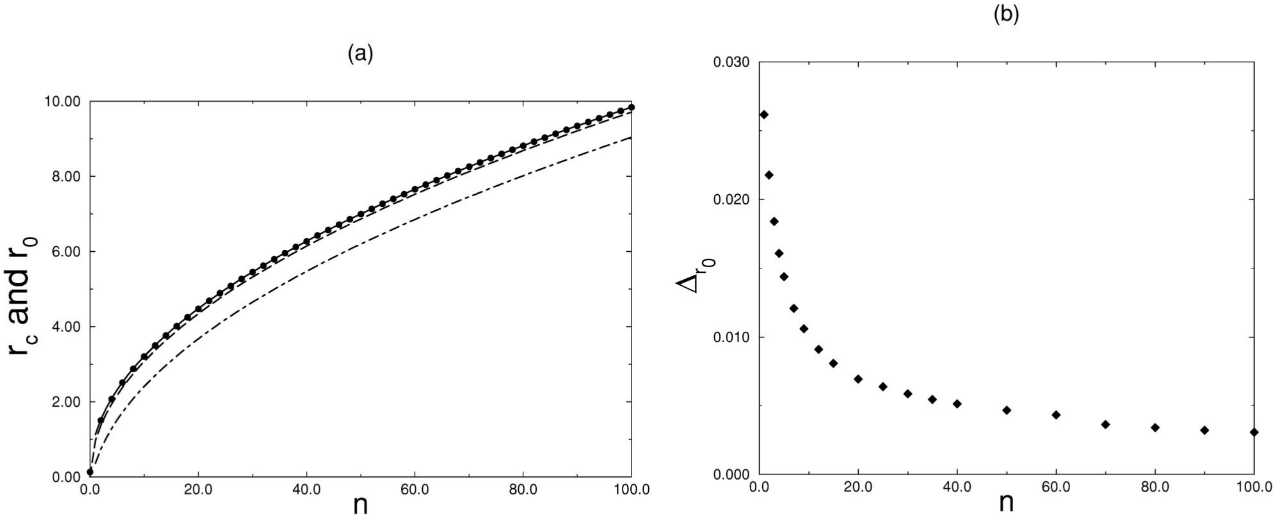

FIG. 3. (a) The vortex-core size r_c (dot-dashed curve) and the length r_0 (solid line curve) are plotted for $\kappa=1.5$ and for values of n ranging from 1 to 100. The dashed curve is the large-n asymptotic estimate (13). The dots represent the next-to-leading order asymptotic expression (38), using the numerical value $\sigma(1.5)\simeq-1.172$. (b) Difference Δ_{r_0} between r_0 and its asymptotic expression (38).

$$g\simeq -n\left(1-\left(1+\frac{x}{r_0}\right)^2\right)\simeq\frac{2n}{r_0}x$$

hence

$$G_0=\frac{g}{r_0}\simeq\frac{2n}{r_0^2}x. \tag{27}$$

The family of solutions of Eqs. (22) and (23) is $f_0(x-x_c),G_0(x-x_c)$ with the asymptotic behavior

$$G_0(x-x_c)=\sqrt{2}\kappa(x-x_c)+b. \tag{28}$$

Identifying the two local expressions (27) and (28) for the function g in their common region of validity $r=r_0+x$ with $|x|\ll r_0$, and $x\to-\infty$ determines r_0 and x_c

$$r_0^2\simeq\sqrt{2}n/\kappa, \tag{29}$$

$$x_c=\frac{b}{\sqrt{2}\kappa}. \tag{30}$$

The large-n asymptotic estimate of r_0 [which agrees with Eq. (13)] is compared with numerical results in Fig. 3(a).

4. Large n limit of the energy

The free energy of a giant vortex can now be calculated for $n\gg 1$. Using Eq. (6) and integrating by parts, the free energy (5) becomes

$$\frac{1}{2\pi}\mathcal{F}=\int_0^\infty r\,dr\left\{\frac{B^2}{2}+\kappa^2-\kappa^2 f^4\right\}. \tag{31}$$

Completing the square with the first two terms inside the brackets and using relation (4) between B and g, leads to

$$\frac{1}{2\pi}\mathcal{F}=n\kappa\sqrt{2}+\int_0^\infty r\,dr\left\{\frac{1}{2}\left(\frac{1}{r}\frac{dg}{dr}-\kappa\sqrt{2}\right)^2-\kappa^2f^4\right\}. \tag{32}$$

The function

$$S(r)=\frac{1}{2}\left(\frac{1}{r}\frac{dg}{dr}-\kappa\sqrt{2}\right)^2-\kappa^2f^4 \tag{33}$$

is nonzero only in the vicinity of r_0. Hence, S can be considered to be as a function of $x=r-r_0$. Replacing the functions $f(r)$ and $g(r)$ in Eq. (33) by their dominant order approximations $f_0(x-x_c)$ and $G_0(x-x_c)$, respectively, solutions of Eqs. (22) and (23), we obtain

$$\begin{aligned}\int_0^\infty rS(r)dr&\simeq r_0\int_{-\infty}^\infty S(x)dx\\&=r_0\int_{-\infty}^\infty\left\{\frac{1}{2}\left(\frac{dG_0}{dx}-\kappa\sqrt{2}\right)^2-\kappa^2f_0^4\right\}dx\\&=r_0\sigma(\kappa).\end{aligned} \tag{34}$$

The last integral is nothing else but the surface energy $\sigma(\kappa)$ of a one-dimensional domain wall between normal and superconducting phases[5,19,20] (see Appendix A). Using Eq. (34) in Eq. (32) gives exactly the large n expression (14) of the free energy of a n giant-vortex with $\sigma(\kappa)$ the surface energy of a one dimensional normal/superconducting interface. The expression (14) is valid for any value of κ in the limit $n\to\infty$. It is compared to the numerical results (for $\kappa=1.5$) in Fig. 4.

5. Finite-n correction to H_c

The leading finite-n correction to B in the vortex core can be derived from the previous results. Integrating Eq. (8) by parts leads to

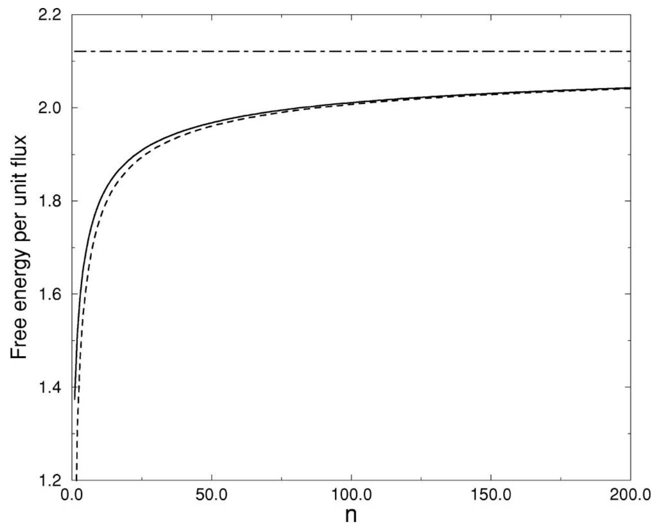

FIG. 4. Free energy $\mathcal{F}/2\pi n$ as a function of n for $\kappa=1.5$ (solid line). The horizontal dot-dashed line is the asymptotic value $\kappa\sqrt{2}\simeq 2.121$. The dashed curve is the large n behavior (14), using the numerical value $\sigma(1.5)\simeq -1.172$.

$$\frac{1}{2\pi}\mathcal{F}=\kappa^2\int_0^\infty r^2\phi(r)dr \quad \text{with} \quad \phi(r)=2f(r)\frac{df}{dr}. \quad (35)$$

The function $\phi(r)$ is localized at the position of the front and its width is of the order $1/\kappa$. Using the local variable $x=r-r_0$ defined in Eq. (19), Eq. (35) can be rewritten, in the large n limit, as

$$\frac{1}{2\pi}\mathcal{F}=\kappa^2 r_0^2+2\kappa^2 r_0\int_{-\infty}^{\infty}x\phi(x)dx+\mathcal{O}(1). \quad (36)$$

The integral in Eq. (36) can be expressed in terms of the interface energy (see Appendix A):

$$2\kappa^2\int_{-\infty}^{\infty}\phi(x)x\,dx=2\kappa^2\int_{-\infty}^{\infty}\left((1-f_0^2)-\frac{1}{\kappa\sqrt{2}}\frac{dG_0}{dx}\right)dx$$
$$=\frac{3}{2}\sigma(\kappa). \quad (37)$$

Comparing Eq. (36) with Eq. (14) provides the large n expansion of r_0 up to the second order

$$r_0=\left(\frac{\sqrt{2}n}{\kappa}\right)^{1/2}-\frac{\sigma(\kappa)}{4\kappa^2}+\mathcal{O}\left(\frac{1}{n^{1/2}}\right). \quad (38)$$

The vortex size r_c, defined by the criterion $f(r_c)=1/2$, can be meaningfully calculated to the same order:

$$r_c=r_0+x_c, \quad (39)$$

with x_c given by Eq. (30).

In order to verify numerically these relations, we first computed the value of the magnetic field at the origin using the following relation, that derives from Eqs. (4) and (7):

$$B(0)=-2\int_0^\infty\frac{f^2g}{r}dr.$$

This allowed us to obtain r_0, from Eq. (11), with a good precision. The vortex size r_c was found from the equation $f(r_c)=1/2$. In Fig. 3(a), the two numerical curves for r_0 and r_c are plotted as a function of n. To the leading order in n, these two curves differ by a constant $x_c\simeq -0.797\ldots$, function of κ only, as predicted by Eq. (39). Solving independently the "inner" Ginzburg-Landau equations (22) and (23), we found that the unique value of b such that $f_0(0)=1/2$ is given by $b\simeq 1.691\ldots$. Hence the relation (30) between x_c and b is verified. We also tested the accuracy of the large n asymptotic expansion of r_0 to the first order (13) and to the second order (38). The difference between r_0 and Eq. (38), hardly visible in Fig. 3(a), is plotted in Fig. 3(b). The agreement is very good.

Substituting this value of r_0 in Eq. (11), we obtain the field value inside the core

$$B(0)=\kappa\sqrt{2}+\frac{\sigma(\kappa)}{2^{3/4}\sqrt{\kappa n}}+\mathcal{O}\left(\frac{1}{n}\right). \quad (40)$$

Hence, when $n\to\infty$, the field at the origin is indeed the thermodynamic critical field $H_c=\kappa\sqrt{2}$ with a Gibbs-Thomson correction as expected [Eq. (15)].

IV. RELATION TO ABRIKOSOV'S FORMULA

The self-energy of a unit vortex was first calculated by Abrikosov[13,7] in the large κ limit

$$\frac{1}{2\pi}\mathcal{F}=\log\kappa+0.081. \quad (41)$$

In the previous section we have studied the case $n\to\infty$ keeping κ fixed and finite. In order to relate our results to the classical calculation of Abrikosov, we consider in this section the double limit $n\to\infty$ and $\kappa\to\infty$ keeping the ratio $u=\kappa/n$ finite and fixed. With these assumptions, we shall see that the Ginzburg-Landau equations decouple and that a scaling form is obtained for the free energy

$$\frac{1}{2\pi}\mathcal{F}=n\kappa\Phi(\kappa/n). \quad (42)$$

The function Φ is plotted in Fig. 5.

A. The double limit $n\gg 1$ and $\kappa\gg 1$

After rescaling the function g by a factor $1/n$, and introducing the ratio $u=\kappa/n$, the Ginzburg-Landau equations become

$$\frac{1}{n^2}\left(\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}\right)=-2u^2f\left(1-\frac{1}{2u^2r^2}g^2-f^2\right), \quad (43)$$

$$\frac{d^2g}{dr^2}-\frac{1}{r}\frac{dg}{dr}=2f^2g; \quad (44)$$

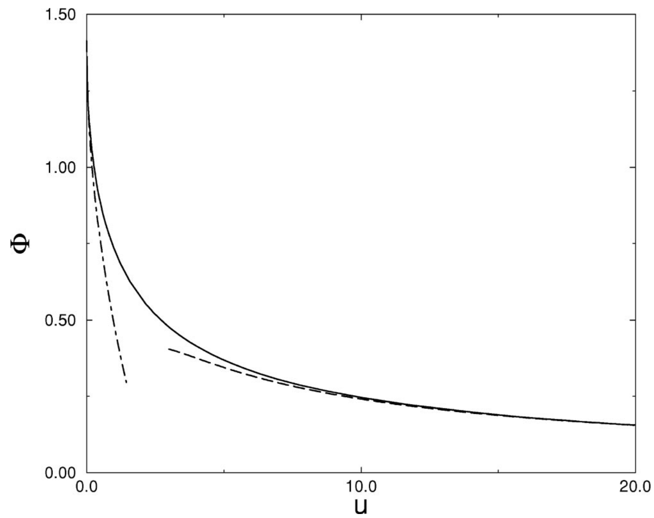

FIG. 5. The scaling function Φ as a function of the variable u (solid line). The dot-dashed curve and the dashed curve represent, respectively, the large u (65) and small u (76) approximations.

the boundary conditions are $f(0)=0$ and $g(0)=-1$ at the origin, and $f(\infty)=1$ and $g(\infty)=0$ at infinity. In the $n\to\infty$ limit, there are two domains where f is slowly varying and the derivative terms in Eq. (43) can be neglected

$$f=0 \quad \text{for} \quad r\leqslant R,$$

$$f=\left(1-\frac{g^2}{2u^2r^2}\right)^{1/2} \quad \text{for} \quad r\geqslant R. \tag{45}$$

As shown below, these two domains match through a boundary layer at $r=R$ where f has a rapid variation but remains small. Matching the two slowly varying expressions (45) of f requires that $f(R)=0$ in the second one and gives

$$g(R)=-\sqrt{2}uR. \tag{46}$$

Substituting Eq. (45) in Eq. (44) leads to the following closed equations for g:

$$\frac{d^2g}{dr^2}-\frac{1}{r}\frac{dg}{dr}=0 \quad \text{for} \quad r\leqslant R, \tag{47}$$

$$\frac{d^2g}{dr^2}-\frac{1}{r}\frac{dg}{dr}=2g\left(1-\frac{g^2}{2u^2r^2}\right) \quad \text{for} \quad r\geqslant R. \tag{48}$$

From Eq. (47), one has

$$g(r)=-1+\left(\frac{r}{r_0}\right)^2 \quad \text{for} \quad r\leqslant R. \tag{49}$$

Continuity of the function g and of its derivative at R, and Eq. (46) imply that

$$g(R)=-1+\left(\frac{R}{r_0}\right)^2=-\sqrt{2}uR, \tag{50}$$

$$g'(R)=2\frac{R}{r_0^2}=\frac{2}{R}-2\sqrt{2}u \tag{51}$$

[the second equality of Eq. (51) is derived using Eq. (50)]. The scaling limit for the free energy when $n\to\infty$ and $\kappa\to\infty$ is obtained from Eqs. (8) and (45)

$$\frac{1}{2\pi}\mathcal{F}=n\kappa\left(uR^2+\frac{1}{u}\int_R^\infty\frac{g^2}{r}dr\right)=n\kappa u\left(R^2+2\int_R^\infty rh^2(r)\right), \tag{52}$$

where we have introduced a function h:

$$g\equiv h\sqrt{2}ur. \tag{53}$$

This provides the explicit expression of the interpolating function Φ (42)

$$\Phi(u)=u\left(R^2+2\int_R^\infty rh^2(r)\right). \tag{54}$$

In order to calculate explicitly Φ one has to determine the position R of the front and the function h as a function of $u=\kappa/n$. In terms of h, Eq. (48) becomes

$$\frac{d^2h}{dr^2}+\frac{1}{r}\frac{dh}{dr}-\frac{1}{r^2}h=2h(1-h^2), \tag{55}$$

and, from Eq. (46), the boundary conditions for h are

$$h(R)=-1 \quad \text{and} \quad h(\infty)=0. \tag{56}$$

Relation (51) implies, however, one more condition on h:

$$h'(R)=\frac{\sqrt{2}}{uR^2}-\frac{1}{R}. \tag{57}$$

The differential equation (55) with the two boundary conditions (56) and the supplementary condition (57) is overdetermined: there is a unique value of R such that all the conditions can be satisfied. Indeed, Eq. (55) with the boundary conditions (56) has a unique solution for any given value of R, i.e., for a given R, the value of $h'(R)$ is unique and can easily be computed numerically by solving Eq. (55) by a shooting method; once $h'(R)$ is known, u is determined as a function of R from Eq. (57)

$$\frac{u}{\sqrt{2}}=\frac{1}{R+R^2h'(R)}. \tag{58}$$

Inverting this relation gives R as a function of u (Fig. 6) and the function Φ using Eq. (54). This calculation can be performed analytically when u is either very large or very small as shown below.

Before considering these limits, we complete the above analysis by giving the boundary-layer equation satisfied by f in the neighborhood of $r=R$. It is convenient to introduce the local variable $x=r-R$. Assuming that f remains small in the neighborhood of $r=R$ and keeping the dominant contribution of each term of Eq. (43), we obtain

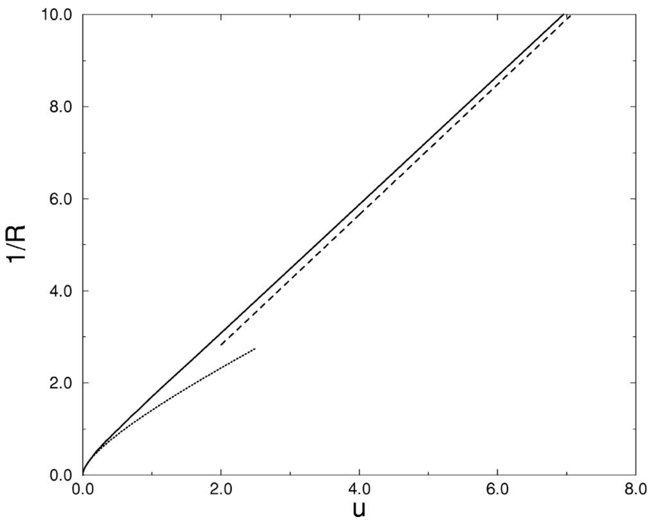

FIG. 6. The inverse of R plotted as a function of the scaling variable u (solid line). The dashed curve represents the linear behavior of $1/R$ for large values of u (64), and the dotted curve shows the asymptotic behavior for small values of u (71).

$$\frac{1}{n^2}\left(\frac{d^2f}{dx^2}+\frac{1}{R}\frac{df}{dx}\right)=2u^2f(-2h'(R)x+f^2). \tag{59}$$

A consistent dominant balance between the different terms of Eq. (59) is obtained when $f/(x^2n^2)\sim fx\sim f^3$. Therefore, as assumed, f evolves on a short scale $x\sim n^{-2/3}$ around $r=R$ but remains small $f\sim n^{-1/3}$. With the rescaled coordinate and function, $\xi=x(4u^2h'(R))^{1/3}n^{2/3}$, $F=(nu)^{1/3}(\sqrt{2}h'(R))^{-1/3}f$, the boundary-layer equation reads

$$\frac{d^2F}{d\xi^2}=-\xi F+F^3. \tag{60}$$

The two different functions of Eq. (45) can be matched through the solution of (the Painlevé) Eq. (60).

B. The large u limit $(\kappa\gg n\gg 1)$

Recalling that R and r_0 are positive, we see from Eq. (51) that $R\leqslant 1/(\sqrt{2}u)$. Therefore, when u tends to infinity, R tends to 0. In the large u limit, Eqs. (47) and (48) reduce to

$$\frac{d^2g}{dr^2}-\frac{1}{r}\frac{dg}{dr}=2g, \tag{61}$$

with the boundary conditions $g(0)=-1$ and $g\to 0$ at infinity. The solution of this equation is readily obtained

$$g(r)=-r\sqrt{2}K_1(r\sqrt{2})$$

and therefore

$$h(r)=-K_1(r\sqrt{2})/u, \tag{62}$$

where K_1 is the modified Bessel function of order 1. Using the behavior of K_1 in the vicinity of zero (see Appendix B), we deduce that

$$h'(R)\simeq\frac{1}{\sqrt{2}uR^2}. \tag{63}$$

Substituting Eq. (63) in Eq. (58) leads to

$$R\simeq\frac{1}{\sqrt{2}u}. \tag{64}$$

Inserting Eqs. (62) and (64) in the expression (52) for the free energy, we obtain

$$\begin{aligned}\frac{1}{2\pi}\mathcal{F}&\simeq\frac{n\kappa}{u}\left[\frac{1}{2}+\int_{1/u}^{\infty}r(K_1(r))^2dr\right]\\&=\frac{n\kappa}{2u}\left\{1+\frac{1}{u^2}\left[K_0\left(\frac{1}{u}\right)K_2\left(\frac{1}{u}\right)-K_1\left(\frac{1}{u}\right)^2\right]\right\}\\&\simeq\frac{n\kappa}{u}(\ln u+\ln 2-\gamma)\simeq n^2\left(\ln\left(\frac{\kappa}{n}\right)+0.116\right),\end{aligned} \tag{65}$$

where the K_ν are modified Bessel functions and γ is the Euler constant.[21] This relation generalizes the classical result (41) of Abrikosov to the case of a giant vortex. Indeed, the structure of the giant vortex is very similar to the $n=1$ case with f varying on a fast scale of order n/κ and g on a slow scale of order one. However, the structure of f is simpler in the large n limit [being 0 for r less than $n/(\sqrt{2}\kappa)$ and linked to g otherwise] than for general n. This is the reason why no new constant needs to be numerically determined in Eq. (65).

C. The small u limit $(n\gg\kappa\gg 1)$

In the small u case, the length R tends to infinity. Defining a local variable $x=r-R$, the differential equation (55) for h becomes

$$\frac{d^2h}{dx^2}+\frac{1}{R+x}\frac{dh}{dx}-\frac{1}{(R+x)^2}h=2h(1-h^2), \tag{66}$$

with the following boundary conditions:

$$h(x=0)=-1 \quad\text{and}\quad h(\infty)=0. \tag{67}$$

When $R\to\infty$, Eq. (66) reduces to

$$d^2h/dx^2=2h(1-h^2). \tag{68}$$

The solution that satisfies the boundary conditions is

$$h(x)=-\sqrt{2}(1-\tanh[\sqrt{2}(x+x_0)])^{1/2} \quad\text{with}$$

$$\tanh(\sqrt{2}x_0)=\frac{1}{\sqrt{2}}. \tag{69}$$

From this expression, we derive that

$$h'(r=R)=h'(x=0)=1. \tag{70}$$

Substituting this result in Eq. (58) leads to the small u behavior of R

$$R \simeq \left(\frac{\sqrt{2}}{u} \right)^{1/2}. \tag{71}$$

The subleading behavior of R as a function of u is found by retaining only the first order term in $1/R$ in Eq. (66):

$$\frac{d^2 h}{dx^2} + \frac{1}{R}\frac{dh}{dx} = 2h(1-h^2), \tag{72}$$

with the same boundary conditions as in Eq. (67). This equation can be solved perturbatively, but we only need to determine the correction to Eq. (70); i.e., we need to calculate the coefficient C such that $h'(r=R)=1+C/R$. We use an argument of energy conservation. Defining

$$\mathcal{E}_2 = \frac{1}{2}\left(\frac{dh}{dx}\right)^2 - h^2 + \frac{h^4}{2},$$

we readily obtain from Eq. (72) that

$$\frac{d\mathcal{E}_2}{dx} = -\frac{1}{R}\left(\frac{dh}{dx}\right)^2.$$

Integrating this relation from 0 to ∞ and evaluating the energy $\mathcal{E}_2$ at both ends leads to

$$C = \int_0^\infty \left(\frac{dh}{dx}\right)^2 dx. \tag{73}$$

Using the 0th-order solution (69) for h in Eq. (73) we obtain

$$h'(r=R) = h'(x=0) = 1 + \left(\frac{2\sqrt{2}-1}{3}\right)\frac{1}{R}. \tag{74}$$

Substituting this result in Eq. (58) and solving for R leads to

$$R \simeq \left(\frac{\sqrt{2}}{u}\right)^{1/2}\left(1 - 2^{-1/4}\frac{\sqrt{2}+1}{3}u^{1/2} + \mathcal{O}(u)\right). \tag{75}$$

Substituting the expressions (75) and (69) for R and h, respectively, in Eq. (52) we obtain the free energy

$$\frac{1}{2\pi}\mathcal{F} \simeq n\kappa\left(uR^2 + 2R\int_0^\infty h^2(x)\,dx\right)$$
$$\simeq n\kappa\sqrt{2}\left(1 - \frac{4}{3}(\sqrt{2}-1)2^{1/4}\left(\frac{\kappa}{n}\right)^{1/2} + \mathcal{O}(u)\right). \tag{76}$$

These results are consistent with those obtained in Sec. III. In fact the small u limit amounts to first taking the limit $n \to \infty$ and then taking $\kappa \to \infty$. For instance, it can be shown that Eq. (38) reduces to Eq. (75) in the $\kappa \to \infty$ limit. Moreover, comparing Eq. (76) with Eq. (14) we retrieve the asymptotic behavior of the surface tension in the large κ limit:[7,20]

$$\sigma(\kappa) \simeq -\frac{4\sqrt{2}}{3}(\sqrt{2}-1)\kappa^2 \simeq -0.7810\kappa^2. \tag{77}$$

V. A VORTEX IN THE SMALL VORTICITY LIMIT

The winding number n must be *a priori* an integer because the phase of the wave function ψ has to be a 2π periodic function. However, as already noted, in the cylindrically symmetric Ginzburg-Landau equations (6) and (7), n appears only as a parameter in the boundary condition at the origin and can be given any real value. In this section, we take n to be close to 0 and determine perturbatively the solutions of Eqs. (6) and (7) and the free energy.

Away from the origin $r=0$, f is close to one and g is small. Thus, we expand f and g as follows:

$$f = 1 + nf_1 + n^2 f_2 + \cdots \quad \text{and} \quad g = ng_1 + n^2 g_2 + \cdots. \tag{78}$$

The functions $f_1, g_1, f_2, g_2 \ldots$ satisfy a hierarchical system of linear differential equations that can be solved recursively by imposing the boundary conditions: $f_i \to 0$ and $g_i \to 0$ when $r \to \infty$, for all $i \geq 1$.

Thus the first order terms f_1 and g_1 are solutions of

$$\frac{d^2 f_1}{dr^2} + \frac{1}{r}\frac{df_1}{dr} = 4\kappa^2 f_1, \tag{79}$$

$$\frac{d^2 g_1}{dr^2} - \frac{1}{r}\frac{dg_1}{dr} = 2g_1. \tag{80}$$

Implementing the boundary and the matching conditions, we obtain

$$f_1 = AK_0(2\kappa r), \quad g_1 = B\sqrt{2}rK_1(\sqrt{2}r), \tag{81}$$

where A and B are two undetermined constants at this stage.

The expansion (78) breaks down in the vicinity of the origin since $f(0)=0$. The expansion of f near $r=0$ is given by Eq. (9)

$$f(r) = \left(\frac{r}{\mathcal{R}}\right)^n + \mathcal{O}(r^{n+1}). \tag{82}$$

The two expressions (81) and (82) can be matched in the parameter range where $r \ll 1$ but $n|\log(r)| \ll 1$. When $r \ll 1$, the correcting terms in Eq. (82) are negligible and when in addition $n|\log(r)| \ll 1$, one obtains

$$f = 1 + n\ln\frac{r}{\mathcal{R}} + \mathcal{O}(n^2, r). \tag{83}$$

In the same parameter range, expansion of Eq. (81) gives

$$f = 1 + nA(-\ln(\kappa r) - \gamma + \mathcal{O}(r^2 \ln r)). \tag{84}$$

Consistency of these two expressions requires $A=-1$ and

$$\mathcal{R} = \frac{1}{\kappa}\exp(-\gamma) \simeq \frac{0.561}{\kappa}, \tag{85}$$

γ being the Euler constant.

We can proceed in a similar way for the function g. Near $r=0$, the expansion of g is

$$g=n\left(-1+\left(\frac{r}{r_0}\right)^2\right)-\frac{r^2}{2(n+1)}\left(\frac{r}{\mathcal{R}}\right)^{2n}+\mathcal{O}(r^{2n+4}). \tag{86}$$

For $r\ll 1$ and $n|\log(r)|\ll 1$, this gives

$$g=-n+r^2\left(-\frac{1}{2}+\frac{n}{r_0^2}+\frac{n}{2}-n\ln\frac{r}{\mathcal{R}}+\mathcal{O}(n^2,r)\right). \tag{87}$$

Again, an alternative expression is obtained from the small r behavior of Eq. (81)

$$g=nB\left(1+r^2\ln r+\frac{r^2}{2}(2\gamma-1-\ln 2)+\mathcal{O}(r^2\ln r)\right). \tag{88}$$

Comparing Eqs. (87) and (88) gives $B=-1$ and

$$r_0^2=2n-2n^2\ln(2\kappa^2). \tag{89}$$

Having determined the small-n expression for f, the free energy limit can be calculated from Eq. (8). Actually, the expression (81) for f is sufficient for this purpose (the range where it is not valid gives an exponentially small contribution) and by performing the integral over modified Bessel functions, one obtains $\mathcal{F}/(2\pi)=n+O(n^2)$.

A similar procedure can be carried out for the second order terms. The expressions for f_2 and g_2 are more intricate and are given in Appendix B. Using Eq. (8), they provide the free energy in the small n limit up to order n^2

$$\frac{1}{2\pi}\mathcal{F}=n+n^2\log(\kappa\sqrt{2})+\mathcal{O}(n^3). \tag{90}$$

VI. THE DUAL POINT OF THE GINZBURG-LANDAU EQUATIONS

A. The dual point equations

Equation (90) suggests that the value $\kappa=1/\sqrt{2}$ has special properties [for instance, for $\kappa=1/\sqrt{2}$ the functions f_1 and g_1 of Eq. (81) satisfy simple identities such as $df_1/dr=-g_1/r$]. It is indeed well-known that at this "dual" point the second order Ginzburg-Landau equations reduce to first order equations, leading to special relations between the functions f_i and g_i. Moreover, the free energy at the dual point has been calculated exactly and is identical to the topological number n. We recall briefly these special properties.

At the dual point the free energy (5) can be written as follows:

$$\begin{aligned}\frac{1}{2\pi}\mathcal{F}&=\int_0^\infty\left\{\left(\frac{df}{dr}+\frac{fg}{r}\right)^2+\frac{1}{2}\left((1-f^2)-\frac{1}{r}\frac{dg}{dr}\right)^2\right\}r\,dr\\&\quad+\int_0^\infty dr\left\{\frac{d}{dr}[g(1-f^2)]\right\}\\&=\int_0^\infty\left\{\left(\frac{df}{dr}+\frac{fg}{r}\right)^2+\frac{1}{2}\left((1-f^2)-\frac{1}{r}\frac{dg}{dr}\right)^2\right\}r\,dr+n.\end{aligned} \tag{91}$$

The minimal free energy is obtained when the following first order system is satisfied (Bogomol'nyi equations[7,22,14]):

$$\left(\frac{d}{dr}+\frac{g}{r}\right)f=0, \tag{92}$$

$$B=\frac{1}{r}\frac{dg}{dr}=(1-f^2). \tag{93}$$

Substituting Eqs. (92) and (93) in Eq. (91) we obtain the minimal energy

$$\frac{1}{2\pi}\mathcal{F}=n. \tag{94}$$

Equation (93) implies that

$$g=-n+\frac{r^2}{2}-\int_0^r rf^2(r)dr,\quad \text{i.e.,}\quad r_0^2=2n. \tag{95}$$

Another remarkable consequence of the Bogomol'nyi equations is that the surface energy $\sigma(\kappa)$, defined in Eq. (34), vanishes identically and changes its sign at the dual point (this is the reason why the dual point separates type I from type II superconductors[7]):

$$\sigma\left(\kappa=\frac{1}{\sqrt{2}}\right)=0. \tag{96}$$

All the calculations that we have carried out in the previous sections are consistent with these (nonperturbative) properties of the dual point. In the small n case of Sec. V, one can verify, using Eq. (81) and the expressions given in the Appendix, that the equations (92) and (93) are satisfied order by order by the expansions (78) of f and g at the dual point. Moreover, the n^2 correction to the free energy in Eq. (90) and to r_0 in Eq. (89) vanishes identically at $\kappa=1/\sqrt{2}$, as expected. In the large n case, the expression (14) for the free energy reduces to Eq. (94) because the $n^{1/2}$ correction disappears thanks to the vanishing of the surface energy; for the same reason, the formula (38) simplifies to $r_0=(2n)^{1/2}$.

At the dual point, the vortices do not interact and the free energy does not depend upon their location. This makes it possible to obtain exact n-vortices solutions, in the large n limit, for arbitrary locations of the vortex cores.[23] However, when κ deviates from the special value $1/\sqrt{2}$, vortices start interacting and their interaction energy is extremal when all the vortices are located at the same point. We now evaluate the energy of such a configuration with cylindrical symmetry when κ is close to the dual point.

B. Free energy of a giant vortex in the vicinity of the dual point

We apply our results to the case where κ is close to the dual point value $\kappa=1/\sqrt{2}$. In experiments on mesoscopic superconductors, values of κ are close to the dual point value[1,9] and in previous theoretical studies the special case $\kappa=1/\sqrt{2}$ was found to be very useful to study analytically the magnetization of a sample as a function of the applied field. In the vicinity of the dual point, a relevant quantity is the relative growth of the free energy:

$$\alpha(n)=\frac{1}{\kappa\sqrt{2}-1}\,\frac{\mathcal{F}(\kappa,n)-\mathcal{F}(1/\sqrt{2},n)}{\mathcal{F}(1/\sqrt{2},n)}\quad \text{when}\quad \kappa\to\frac{1}{\sqrt{2}}. \tag{97}$$

This relation is a local form of an empirical scaling of the free energy

$$\frac{1}{2\pi}\mathcal{F}(\kappa,n)\simeq n(\kappa\sqrt{2})^{\alpha(n)}, \tag{98}$$

found in Ref. 9, where the exponents $\alpha(n)$ were computed numerically. Differentiating the Ginzburg-Landau free energy (5) with respect to the parameter κ at the dual point we obtain

$$\alpha(n)=\frac{1}{n}\int_0^\infty (1-f^2)^2 r\,dr=\frac{1}{n}\int_0^\infty \frac{dr}{r}\left(\frac{dg}{dr}\right)^2. \tag{99}$$

The last relation was derived from the Bogomol'nyi equation (93).

We now explain how a large n expansion of the exponents $\alpha(n)$ can be derived from our previous results. Integrating Eq. (99) by parts, we find

$$\alpha(n)=\frac{1}{n}\int_0^\infty \frac{r^2}{2}\psi(r)dr \quad \text{with}\quad \psi=4f(1-f^2)\frac{df}{dr}. \tag{100}$$

The function $\psi(r)$ is localized at the position of the front and its width is of the order $1/\kappa$. Hence we can expand Eq. (100) using the local variable $x=r-r_0$ and in the large n limit replace f by f_0 (as in Sec. III B 5):

$$\alpha(n)=\frac{1}{n}\frac{r_0^2}{2}+\frac{r_0}{n}\int_{-\infty}^\infty x\psi(x)dx \quad \text{with}\quad \psi=4f_0(1-f_0^2)\frac{df_0}{dx}. \tag{101}$$

Integrating the last term by parts, we obtain

$$\begin{aligned}\int_{-\infty}^\infty x\psi(x)dx&=\int_{-\infty}^0 dx\{(1-f_0^2)^2-1\}+\int_0^\infty dx(1-f_0^2)^2\\ &=\int_{-\infty}^\infty dx\left\{(1-f_0^2)^2-\frac{dG_0}{dx}\right\}\\ &=-\int_{-\infty}^\infty f_0^2(1-f_0^2)dx.\end{aligned} \tag{102}$$

In the last equality we used the fact that the functions f_0 and G_0 satisfy the following Bogomol'nyi equations at the dual point:[16]

$$\frac{dG_0}{dx}=(1-f_0^2)\quad \text{and}\quad \frac{df_0}{dx}+G_0f_0=0. \tag{103}$$

The function v defined by $f_0=\exp(-v/2)$, satisfies a second order differential equation that can be solved explicitly:[16]

$$\frac{1}{2}\frac{d^2v}{dx^2}=1-\exp(-v)$$

which implies

$$\frac{dv}{dx}=-2(e^{-v}-1+v)^{1/2}. \tag{104}$$

Now changing the variable x to v, we rewrite the last integral in Eq. (102) as

$$\begin{aligned}-\int_0^\infty dv\frac{e^{-v}(1-e^{-v})dv}{(e^{-v}-1+v)^{1/2}}&=-\int_0^\infty dv\,e^{-v}(e^{-v}-1+v)^{1/2}\\ &\simeq-0.5482.\end{aligned} \tag{105}$$

Substituting this result in Eq. (101) and knowing that $r_0^2=2n$ exactly [Eq. (95)], we find an asymptotic expansion for $\alpha(n)$ in the large n limit:

$$\alpha(n)=1-\frac{0.7753}{n^{1/2}}+\mathcal{O}\left(\frac{1}{n}\right). \tag{106}$$

We have not calculated the subleading behavior of the free energy but our numerical results provide an estimate of the higher order term in Eq. (106):

$$\alpha(n)=1-\frac{0.7753}{n^{1/2}}+\frac{\mathcal{K}}{n}+\mathcal{O}\left(\frac{1}{n^{3/2}}\right)\quad \text{with}\quad \mathcal{K}\simeq 0.125. \tag{107}$$

From Eqs. (106) and (97), the expansion of the free energy near $\kappa=1/\sqrt{2}$ is found. This allows us to retrieve, using Eq. (14), the local behavior of the surface energy in the vicinity of the dual point, first calculated by Dorsey:[16]

$$\begin{aligned}\sigma(\kappa)&\simeq-(\kappa\sqrt{2}-1)\int_0^\infty dv(e^{-v}-1+v)^{1/2}e^{-v}\\ &\simeq-0.5482(\kappa\sqrt{2}-1).\end{aligned} \tag{108}$$

TABLE I. Numerical values of the function $\alpha(n)$ obtained from Eq. (99) for different n in the range 1 to 250.

n	$\alpha(n)$	n	$\alpha(n)$	n	$\alpha(n)$
1	0.415	12	0.789	50	0.893
2	0.542	13	0.797	60	0.902
3	0.611	14	0.804	80	0.915
4	0.655	15	0.810	100	0.924
5	0.687	16	0.816	120	0.930
6	0.711	17	0.821	140	0.935
7	0.730	18	0.826	160	0.940
8	0.746	19	0.830	180	0.943
9	0.759	20	0.834	200	0.946
10	0.770	30	0.863	225	0.949
11	0.780	40	0.881	250	0.951

In the same way, from the small n expansion of the free energy up to the second order, we derive that (see Appendix B)

$$\alpha(n)=n-4n^2\int_0^\infty r^2K_0^2(r)K_1(r)\,dr+\mathcal{O}(n^3)$$
$$\simeq n-1.5626n^2+\mathcal{O}(n^3). \tag{109}$$

We have computed numerically from Eq. (97) the function $\alpha(n)$ for n ranging from 1 to 250 and some significant values are given in Table I. The numerically computed and the large n asymptotic expansion (106) of $\alpha(n)$ agree to better than 7% even for n as small as 4. In particular we notice from Eq. (106) that $\alpha(n)\to 1$ when $n\to\infty$. In the $n\to 0$ limit, the second order expansion (109) provides a fairly good approximation for $n\leqslant 0.25$. In Fig. 7, numerically computed values of α are compared to large n and small n expansions.

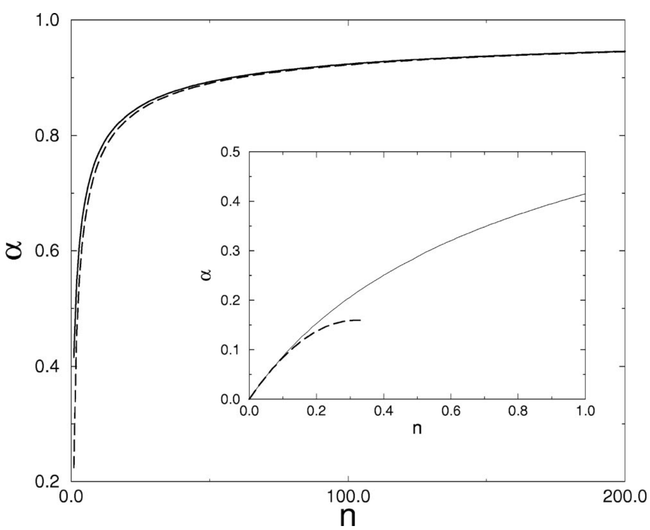

FIG. 7. The function $\alpha(n)$ is plotted for n between 1 and 200, and in inset for values of n between 0 and 1 (solid line curves). The dashed lines represent the asymptotic expansions for large n (106) and small n (109).

VII. CONCLUSION

We have analyzed the structure of a giant vortex of winding number n in an infinite plane for arbitrary values of the parameter κ in contrast to previous analytical results obtained only in the London limit or at the dual point. The vortex and magnetic field profiles are computed by solving the Ginzburg-Landau equations which minimize the free energy of the system. These numerical solutions are compared with analytical results derived in the cases where the vortex multiplicity n is either very large or very small. In particular, a simple structure has been found for large n, its relation to the classic result of Abrikosov has been elucidated and the perturbative expansions are found to agree well with previous numerical computations of a giant vortex free energy. We hope that some of these results will prove useful for the current very active experimental investigations of mesoscopic superconductors.

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APPENDIX A: EXPRESSIONS OF THE NORMAL/SUPERCONDUCTING INTERFACE ENERGY

The energy per unit area of a planar normal/superconducting interface is obtained from the Ginzburg-Landau energy (3) under the form[5]

$$\sigma(\kappa)=\int_{-\infty}^{\infty}\left\{\left(\frac{df_0}{dx}\right)^2+f_0^2G_0^2+\kappa^2(1-f_0^2)^2+\frac{1}{2}\left(\frac{dG_0}{dx}\right)^2-\kappa\sqrt{2}\,\frac{dG_0}{dx}\right\}dx, \tag{A1}$$

where f_0 and G_0 satisfy Eqs. (22) and (23). We notice that for a magnetic field oriented along the z direction, G_0 represents the y component of the potential vector. The expression (A1) can be written in different forms by making use of the identities:

$$\int_{-\infty}^{\infty}\left(\frac{df_0}{dx}\right)^2dx=\int_{-\infty}^{\infty}\{2\kappa^2f_0^2(1-f_0^2)-G_0^2f_0^2\}dx, \tag{A2}$$

$$\int_{-\infty}^{\infty}dx\left\{\left(\frac{dG_0}{dx}\right)^2-\kappa\sqrt{2}\,\frac{dG_0}{dx}\right\}=-2\int_{-\infty}^{\infty}G_0^2f_0^2dx, \tag{A3}$$

$$G_0^2f_0^2=\left(\frac{df_0}{dx}\right)^2+\frac{1}{2}\left(\frac{dG_0}{dx}\right)^2-\kappa^2(1-f_0^2)^2. \tag{A4}$$

The first two identities are obtained (e.g., Ref. 20) from Eqs. (22) and (23) by integration by parts. The third identity results from the conservation law (25). Using Eq. (A2) to eliminate $\int(df_0/dx)^2$ from Eq. (A1) provides the expression (34) of the main text. Similarly, using Eqs. (A2), (A3), and (A4) to eliminate $\int(df_0/dx)^2$, $\int(dG_0/dx)^2$ and $\int f_0^2 G_0^2$ from Eq. (A1) leads to the alternative expression (37) of the main text:

$$\sigma(\kappa)=\frac{4}{3}\kappa^2\int_{-\infty}^{\infty}\left((1-f_0^2)-\frac{1}{\kappa\sqrt{2}}\frac{dG_0}{dx}\right)dx. \tag{A5}$$

Finally, we note that the integral of $x\phi$ in Eq. (37) is transformed into Eq. (A5) by integrating separately by parts the integrals between $-\infty$ and 0 and between 0 and $+\infty$:

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(x)x\,dx&=\int_{-\infty}^{\infty}\left((x+x_c)\frac{df_0^2}{dx}\right)dx\\&=\int_{-x_c}^{\infty}(1-f_0^2)dx-\int_{-\infty}^{-x_c}f_0^2dx\\&=\int_{-\infty}^{\infty}\left((1-f_0^2)-\frac{1}{\kappa\sqrt{2}}\frac{dG_0}{dx}\right)dx.\end{aligned} \tag{A6}$$

The last equality is obtained by adding 0 under the complicated form

$$0=\int_{-x_c}^{+\infty}dx\left[-\frac{1}{\kappa\sqrt{2}}\frac{dG_0}{dx}\right]+\int_{-\infty}^{-x_c}dx\left[1-\frac{1}{\kappa\sqrt{2}}\frac{dG_0}{dx}\right]. \tag{A7}$$

APPENDIX B: SECOND ORDER COMPUTATION IN THE $n\to 0$ LIMIT AND SOME USEFUL FORMULAS

We first recall some useful asymptotic formulas for the modified Bessel functions:

$$\begin{aligned}K_0(r)&=-\ln\frac{r}{2}-\gamma+\mathcal{O}(r^2\ln r),\\K_1(r)&=\frac{1}{r}+\frac{r}{2}\ln\frac{r}{2}+\frac{r}{4}(2\gamma-1)+\mathcal{O}(r^2\ln r),\\K_2(r)&=\frac{2}{r^2}+\mathcal{O}(r^{-1}).\end{aligned} \tag{B1}$$

$$\frac{dK_0}{dr}=-K_1(r)\ \text{ and }\ \frac{d}{dr}[rK_1(r)]=-rK_0(r),$$
$$\frac{dI_0}{dr}=I_1(r)\ \text{ and }\ \frac{d}{dr}[rI_1(r)]=rI_0(r). \tag{B2}$$

We now explain how the matching procedure is carried out up to the second order in the case $n\to 0$. The functions f_2 and g_2 solve the linear system:

$$\frac{d^2f_2}{dr^2}+\frac{1}{r}\frac{df_2}{dr}=4\kappa^2f_2+\frac{g_1^2}{r^2}+6\kappa^2f_1^2,$$
$$\frac{d^2g_2}{dr^2}-\frac{1}{r}\frac{dg_2}{dr}=2g_2+4f_1g_1.$$

Solving this system by "variation of constants," we obtain

$$\begin{aligned}f_2=&AK_0(2\kappa r)+BI_0(2\kappa r)-I_0(2\kappa r)\\&\times\int_r^{\infty}u\,du K_0(2\kappa u)[2K_1(u\sqrt{2})^2\\&+6\kappa^2K_0(2\kappa u)^2]-K_0(2\kappa r)\\&\times\int_1^r u\,du I_0(2\kappa u)[2K_1(u\sqrt{2})^2+6\kappa^2K_0(2\kappa u)^2],\end{aligned} \tag{B3}$$

$$\begin{aligned}g_2=&CK_0(2\kappa r)+DI_0(2\kappa r)-4\sqrt{2}rI_1(\sqrt{2}r)\\&\times\int_r^{\infty}K_0(2\kappa u)[K_1(\sqrt{2}u)]^2u\,du-4\sqrt{2}rK_1(\sqrt{2}r)\\&\times\int_0^r K_0(2\kappa u)K_1(\sqrt{2}u)I_1(\sqrt{2}u)u\,du.\end{aligned} \tag{B4}$$

Since f_2 and g_2 must be finite at infinity, we have $B=D=0$. The two other coefficients are found by matching with the inner expansions in the vicinity of zero. For this purpose, we need the local expansions of f and g in the vicinity of 0 up to the order n^2. Using Eqs. (82) and (86), we obtain

$$\begin{aligned}f(r)=&1+n\left(\ln\frac{r}{\mathcal{R}}+\mathcal{O}(r\ln r)\right)+\frac{n^2}{2}\left(\left(\ln\frac{r}{\mathcal{R}}\right)^2+\mathcal{O}(r\ln r)\right)\\&+\mathcal{O}(n^3).\end{aligned} \tag{B5}$$

$$\begin{aligned}g(r)=&-n(1+\mathcal{O}(r^2\ln r))+n^2\left(r^2\ln\frac{r}{\mathcal{R}}\left(1-\ln\frac{r}{\mathcal{R}}\right)+\mathcal{O}(r^2)\right)\\&+\mathcal{O}(n^3).\end{aligned} \tag{B6}$$

Because the coefficient of n^2 in the local expansion (B6) of g tends to zero when r vanishes, one must have $g_2\to 0$ when $r\to 0$ and therefore $C=0$. To obtain the coefficient A, we must determine the divergences of f_2 given by Eq. (B3) when $r\to 0$: the term $AK_0(2\kappa r)$ produces a singular term $-A\ln r$ when $r\to 0$; from the expansions of the Bessel functions (B1), we find the diverging part of the first integral in Eq. (B3) to be $-\int_r^{\infty}2u\,du K_0(2\kappa u)K_1(u\sqrt{2})^2$ which is equivalent to $\int_{2\kappa r}^{1}[\ln(u/2)+\gamma]du/u$; thus the first integral gives a singular term equal to

$$-\frac{(\ln r)^2}{2}+(-\gamma-\ln\kappa)\ln r;$$

similarly the singular term due to the second integral in Eq. (B3) is given by

$$(\ln r)^2 - \ln r\left(-\gamma - \ln\kappa + 2\int_0^1 u\,du \times\left(I_0(2\kappa u)K_1(u\sqrt{2})^2 - \frac{1}{2u^2}\right)\right).$$

Hence the total diverging part of f_2 is

$$\frac{(\ln r)^2}{2} - \ln r\left(A + 2\int_0^1 u\,du\left(I_0(2\kappa u)K_1(u\sqrt{2})^2 - \frac{1}{2u^2}\right)\right). \tag{B7}$$

Identifying this expression with the singular part of the n^2 term in the expansion (B5) of *f* and using Eq. (85) provides the value of *A* and leads to the second order terms in the small *n* expansion of *f* and *g*:

$$f_2 = AK_0(2\kappa r) - I_0(2\kappa r)\int_r^\infty u\,du K_0(2\kappa u)(2K_1(u\sqrt{2})^2 + 6\kappa^2 K_0(2\kappa u)^2) - K_0(2\kappa r)\int_1^r u\,du I_0(2\kappa u)(2K_1(u\sqrt{2})^2 + 6\kappa^2 K_0(2\kappa u)^2),$$

$$\text{with}\quad A = -\gamma - \ln\kappa - 2\int_0^1 u\,du \times\left(I_0(2\kappa u)K_1(u\sqrt{2})^2 - \frac{1}{2u^2}\right); \tag{B8}$$

$$g_2 = -4\sqrt{2}rI_1(\sqrt{2}r)\int_r^\infty K_0(2\kappa u)(K_1(\sqrt{2}u))^2 u\,du - 4\sqrt{2}rK_1(\sqrt{2}r)\int_0^r K_0(2\kappa u)K_1(\sqrt{2}u)I_1(\sqrt{2}u)u\,du. \tag{B9}$$

We can now derive the small *n* asymptotic behavior of $\alpha(n)$. From the explicit relation (99), we obtain, up to the second order in *n*,

$$\alpha(n) = \frac{1}{n}\int_0^\infty \frac{dr}{r}\left(\frac{dg}{dr}\right)^2 = n\int_0^\infty \frac{dr}{r}\left(\frac{dg_1}{dr}\right)^2 + 2n^2\int_0^\infty \frac{dr}{r}\left(\frac{dg_1}{dr}\frac{dg_2}{dr}\right) + \mathcal{O}(n^3), \tag{B10}$$

where the functions g_i are calculated at the dual point $\kappa = 1/\sqrt{2}$. Recalling that $g_1 = -\sqrt{2}rK_1(\sqrt{2}r)$ [see Eq. (81)] and using Eq. (B2) we have

$$\int_0^\infty \frac{dr}{r}\left(\frac{dg_1}{dr}\right)^2 = 2\int_0^\infty xK_0^2(x)dx = 1. \tag{B11}$$

From Eqs. (B2) and (B9), we obtain $dg_1/dr = 2rK_0(\sqrt{2}r)$ and

$$\frac{dg_2}{dr} = 4r\left\{-I_0(\sqrt{2}r)\int_{\sqrt{2}r}^\infty K_0(u)(K_1(u))^2 u\,du + K_0(\sqrt{2}r)\int_0^{\sqrt{2}r} K_0(u)K_1(u)I_1(u)u\,du\right\}. \tag{B12}$$

The coefficient of n^2 in Eq. (B10) is therefore given by

$$\begin{aligned} 2\int_0^\infty \frac{dr}{r}\left(\frac{dg_1}{dr}\frac{dg_2}{dr}\right) &= 4\int_0^\infty dr\,K_0(\sqrt{2}r)\frac{dg_2}{dr} \\ &= -8\int_0^\infty dv\,vK_0(v)I_0(v)\int_v^\infty K_0(u)(K_1(u))^2 u\,du \\ &\quad + 8\int_0^\infty dv\,vK_0^2(v)\int_0^v K_0(u)K_1(u)I_1(u)u\,du. \end{aligned} \tag{B13}$$

From Eq. (B2) we can verify that

$$\int dv\,vK_0^2(v) = \frac{v^2}{2}(K_0^2 - K_1^2) \quad\text{and}$$

$$\int dv\,vK_0(v)I_0(v) = \frac{v^2}{2}(K_0I_0 + K_1I_1). \tag{B14}$$

Integrating Eq. (B13) by parts leads to

$$-8\int_0^\infty dv\,vK_0(v)I_0(v)\int_v^\infty K_0(u)(K_1(u))^2 u\,du = -4\int_0^\infty dv\,v^3(K_0I_0 + K_1I_1)K_0K_1^2,$$

$$8\int_0^\infty dv\,vK_0^2(v)\int_0^v K_0(u)K_1(u)I_1(u)u\,du = -4\int_0^\infty dv\,v^3(K_0^2 - K_1^2)K_0K_1I_1. \tag{B15}$$

Summing these two expressions and using $K_0I_1 + K_1I_0 = 1/v$ leads to

$$2\int_0^\infty \frac{dr}{r}\left(\frac{dg_1}{dr}\frac{dg_2}{dr}\right) = -4\int_0^\infty dv\,v^2K_0^2(v)K_1(v). \tag{B16}$$

This concludes the derivation of Eq. (109). The free energy in the small *n* limit up to order n^2 (90) is derived by similar calculations.

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