Magnetization process of random quantum spin chains

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The magnetization process of spin chains with randomness is discussed. We emphasize a close analogy between these systems and disordered particle systems and pay particular attention to the (in)stability of spin-gapped phases against several types of randomness. Generically, spin-gap states (plateaus) and Bose-glass-like localized states compete and stable plateaus are possible only for special magnetizations determined by spin S and the period of a pure system. When no external field is applied, the random-singlet state or the Griffith's phase is possible.

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I. INTRODUCTION

In recent years, peculiar features of low-dimensional quantum spin systems have attracted much attention. Extensive studies have revealed an interesting aspect of quantum spins, which we might regard as a kind of "wave-particle duality" for quantum objects. That is, in some cases, a collection of quantized spins behaves like that of continuous classical angular momenta (or vector spins) and, in other cases, it looks like a system of discrete particles. For example, the low-energy physics of an ordered antiferromagnet is well described by the standard spin-wave theory, which may be derived from the (semi)classical vector-spin picture. On the other hand, one of the most striking phenomena that reflect the latter aspect as "particles" would be plateaus appearing in high-field magnetization processes.

From the theoretical point of view, the appearance of magnetization plateaus is attributed to the formation of a kind of gapped insulating state in a system of "effective" particles. It seems difficult to explain plateaus occurring in a sufficiently high magnetic field by the classical vector-spin picture because in classical systems it will only cost very small energy to increase magnetization infinitesimally. By now, there are several ways to understand how the gap is formed in one dimension. One approach focuses on how a (small) plateau appears in a smooth (i.e., gapless) portion of a magnetization curve. In one dimension (1D), the gapless region of the magnetization curve can be treated[1,2] within the framework of the so-called Tomonaga-Luttinger model; the low-temperature specific heat, the differential susceptibility, the NMR relaxation rate, etc., are described only by two phenomenological parameters contained in the model.

Since the system lives on a discrete lattice, there may be some interactions allowed by a selection rule derived *both* from spatial symmetries of the model and from the value of magnetization. This means that the system is potentially unstable at commensurate values of magnetization. When the magnetization curve is smooth, such (gap-generating) interactions are irrelevant and do not appear in the low-energy physics. However, they may become relevant for some region of parameters to open an excitation gap and thereby form a (small) plateau at that value of magnetization. This weak-coupling argument was successfully applied to the plateau problems.[3,4] Combined with the Lieb-Schultz-Mattis argument,[5] the above selection rule (*commensurability condition*) yields a relation satisfied by the period of the ground state ($Q_{\text{G.S.}}$) and the plateau magnetization m^z_{plat},[3]

$$Q_{\text{G.S.}}(S-m^z_{\text{plat}})\in\mathbf{Z}. \tag{1}$$

From the preceding argument, it is now clear that the periodicity of the underlying lattice plays a crucial role in the plateau formation; it determines the value of a plateau as well as a magnetic (super)structure realized in the plateau phase. Although the situation becomes more complicated, regular lattice structures and commensurability are essential to plateaus even in higher dimensions.[6]

On the other hand, there may be some imperfections or defects in real materials used in experiments and disorder can also be introduced artificially by, say, random substitution. In such cases, the regularity mentioned above no longer exists. In other words, randomness competes with the periodicity crucial to the existence of plateaus. When the effect of randomness overcomes that of commensurability, a gapped plateau phase will be superseded by a certain random phase and the plateau may be smeared out.

The effects of randomness on magnetization processes and on plateaus would be interesting in its own right. As is clear in the weak-coupling argument described above, in many cases magnetization plateaus can be viewed as spin analogs of band or Mott insulators in Fermi and Bose systems. Of course, magnetization translates into particle density and a conducting state of interacting Bose liquids corresponds to a region of smooth magnetization curves. For example, random potentials in particle systems are nothing but random fields in the spin language. Accordingly, there may appear some randomness-dominated phases (like the Bose glass,[7] the Anderson insulators,[8] etc.) also in spin systems. The effects of disorder in gapped systems have been an intriguing problem in strongly correlated systems.[9–12]

Furthermore, the problem of how quenched randomness affects ground states of quantum spin systems attracts much theoretical interest. For example, striking extended states like the random-singlet phase[13,14] with unusual scaling properties and the gapless (but localized) Griffiths-McCoy phase[15] are known to appear in bond-random spin systems[16,17] and quantum Ising spin glasses.[18]

In the study of bond-random systems, it has been realized that spin chains respond to randomness quite differently; some phases of pure systems are topologically stable while others are fragile against any weak randomness. Although there are many analytical[14,17,19–21] and numerical[22–25] studies for the case with zero field and $S=1/2$ or 1, only a few works[26,27] have been done for the magnetization process of free $S=1/2$ random XY chains. Examining the (in)stability of various types of gapped and gapless states of interacting (quasi-) one-dimensional spin systems with or without a field would be interesting also in this respect.

The plan of this paper is as follows. In the next section, we investigate whether plateau states are stable or not by the strong-coupling expansion,[28] which was proved powerful in analyzing spin systems in a strong field. The result is different according to the type of plateau.

Section III is devoted to the derivation of the low-energy effective action for the random spin systems. We use a phenomenological argument based on the Tomonaga-Luttinger (TL) model. This approach relies on a physically reasonable assumption that spin chains in a strong enough field are *generically* described by a single-component TL model. By a careful analysis, we determine the form of the random coupling and then apply the replica trick.

The renormalization-group argument is given in Sec. IV. There we will find that the renormalization-group flow is qualitatively different depending both on the types of randomness and on the value of magnetization (to be more precise, on the value of $S-m^z$). This point was overlooked in previous analyses[19,21] in this field and leads to striking difference between the cases with and without an external field. Exploiting the known exact solutions, we determine strong-coupling phases and discuss the fate of plateau phases in the presence of randomness. One of our main conclusions in this section is that stable intermediate plateaus are possible only for m^z satisfying $Q_{\text{Ham}}(S-m^z)=\text{integer}$ (Q_{Ham} denotes the period of the pure Hamiltonian).

We summarize the main results in the final section. A brief discussion about the effects of randomness on the field-induced long-range order (LRO) is presented as well. Some of the results presented here were reported briefly in Ref. 29.

II. STRONG-COUPLING ARGUMENT

Now we discuss the (in)stability of the plateaus against randomness. From the standard lore[19,30] of the spinless fermion (which is equivalent to the $S=1/2$ problems via the Jordan-Wigner transformation), we may speculate that plateaus in the $S=1/2$ systems are unstable. However, this conclusion is too hasty. Since the relevant (effective) particle and the dominant instability depend both on the interactions and on the value of magnetization, a more careful analysis is needed.

In the pure cases, there are varieties of gapped ground states. For example, the Haldane-gap phase[31] appears in the spin-1 Heisenberg chain and it was recently argued to be robust against randomness.[20,32] Although the notion of topological stability and string order does not seem[33] to allow straightforward extension to higher-S Heisenberg chains, there is at least one family of solvable models called the VBS model,[34,35] whose ground state is completely stable against (bond) randomness for *all* S. On the other hand, a spontaneously dimerized state of a certain $S=1$ chain is argued unstable.[36]

More recently, a series of gapped ground states in a strong magnetic field have been found both theoretically and experimentally. Some of them have their origin in the limit of clusters composed of a few spins and others are attributed essentially to many-body effects. Hence a natural question arises as to which kind of plateau is stable against randomness and which is not.

Before embarking on detailed analyses, we investigate the problem of the stability of plateaus in a few extreme limits. In the following, we consider two different types of plateaus. The first one is known to occur in, e.g., the following $S=3/2$ model:[3,37]

$$\mathcal{H}_{\text{large-}D}=J\sum_{j=1}\mathbf{S}_j\cdot\mathbf{S}_{j+1}+D\sum_j (S_j^z)^2-H\sum_j S_j^z . \tag{2}$$

The origin of the plateau at $m^z=\Sigma_j S_j^z/L=1/2$ may be most easily understood in the ''atomic limit'' $J=0$. In this limit, the problem reduces to the single-site one; m^z jumps at $H=0$ and $H=2D$ and between these values it is locked at $m^z=1/2$. The ''plateau'' is shown to persist down to a relatively small value of D/J.[37]

The randomness in the external field is incorporated by disturbing H locally,

$$H_j=H+\Delta_j . \tag{3}$$

The distribution of Δ_j may be taken as a box form:

$$P(\Delta)=\begin{cases}\dfrac{1}{W} & \text{if } |\Delta|\leqslant W/2\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

The steplike magnetization process obtained above is not altered so much apart from that the curve in regions around $H=0$ and $H=2D$ has a finite slope of $1/W$, that is, the (longitudinal) susceptibility $\chi_{//}$ becomes finite due to quenched randomness even in the local limit $J=0$. Furthermore, we can explicitly calculate the transverse susceptibility $\chi_\perp$ (an analog of the superfluid susceptibility) in this limit; in the region of finite $\chi_{//}$ (compressibility, in the language of superfluids), $\chi_\perp$ diverges logarithmically in the zero-temperature limit, whereas it remains finite in the plateau region. We may regard this phase as a spin analog of the Bose glass found in disordered Bose systems.[7]

Then what happens to this plateau if we switch on the coupling J? A crude estimate may be obtained by low-order perturbations in J ($\ll D$). Using the well-known spectral property of random systems,[38] the lower and the upper edges of the $m^z=1/2$ plateau is evaluated at the first order as $H_{\text{lower}}=5J+W/2$ and $H_{\text{upper}}=2D-2J-W/2$; as long as W is smaller than the critical value $2D-7J$ the plateau persists.

The above results may not be so surprising from a common belief that a system having a spectral gap is robust against small perturbations. In this sense, the second example is less trivial. Let us consider the $S=1/2$ zigzag spin ladder

$$\mathcal{H}_{\rm zigzag}=\sum_r \left[J_2(\mathbf{S}_r^{(1)}\cdot\mathbf{S}_{r+1}^{(1)}+\mathbf{S}_r^{(2)}\cdot\mathbf{S}_{r+1}^{(2)})+(1+\delta)\mathbf{S}_r^{(1)}\cdot\mathbf{S}_r^{(2)}\right.$$
$$\left.+(1-\delta)\mathbf{S}_r^{(2)}\cdot\mathbf{S}_{r+1}^{(1)}\right]-\sum_r (H_r^{(1)}S_r^{(1),z}+H_r^{(2)}S_r^{(2),z}). \tag{5}$$

The spins $\mathbf{S}_r^{(1)}$ and $\mathbf{S}_r^{(2)}$ coupled by J_2 bonds constitute $S=1/2$ chains 1 and 2, respectively, which are coupled to each other by the zigzag interaction $1\pm\delta$. It is shown[28,39] theoretically that the magnetization curve of $\mathcal{H}_{\rm zigzag}$ has a single intermediate plateau at $m^z=1/4$. Contrary to the example discussed above, this plateau is attributed to a nontrivial many-body effect and is accompanied by the spontaneous breaking of the translational symmetry.[28] In considering the effects of bond and field randomness, it is convenient to use the strong-coupling expansion around the limit $(1+\delta)\gg|1-\delta|,J_2$, which was successfully applied to the spin ladder in a strong field in Ref. 28 and later extended to more complicated cases.[40]

Random components of the external field are introduced as before:

$$H_r^{(a)}=H+\Delta_r^{(a)} \quad (a=1,2). \tag{6}$$

Of course, the integer $a(=1,2)$ labels two chains. The problem is readily solved for $\delta=1$ to yield four eigenvalues, which depend on $\Delta_r^{(1,2)}$. Provided that $|\Delta_r^{(a)}|$ is small enough compared with $1+\delta$ and H, the strong-coupling expansion is still applicable; we can pick up ‘‘vacancy’’ and ‘‘particle’’ from the four states and calculate the matrix elements of $\mathbf{S}_r^{(1,2)}$ with respect to them to derive an effective Hamiltonian for the particles by a perturbation expansion in $(1-\delta)$ and J_2. Up to the first order in Δ, the result is written down as

$$\mathcal{H}_{\rm eff}=\frac{1}{4}[2J_2-(1-\delta)]\sum_r (b_r^\dagger b_{r+1}+b_{r+1}^\dagger b_r)+\frac{1}{4}[2J_2$$
$$+(1-\delta)]\sum_r n_r n_{r+1}-\frac{1}{2}\sum_r (\Delta_r^{(1)}+\Delta_r^{(2)})n_r. \tag{7}$$

In the above equation, we have tuned the (uniform) external field to the particle-hole symmetric point $H=H_{\rm PH}=(1+\delta)+\frac{1}{4}(2J_2+1-\delta)$. The hardcore boson (or fermion) operator $b_r^\dagger$ (b_r) creates $S^z=1$ (0) state on the rth rung; $n_r=1$ if the rth rung is occupied by $S^z=1$ state and 0 otherwise. Of course, we may rewrite this in terms of $S=1/2$ pseudospins using the Jordan-Wigner transformation; the plateau phase of the original problem is translated into the antiferromagnetically ordered phase of the effective pseudospin (or hard-core boson) model. It is important to note that the random variables $\Delta_r^{(a)}$ appear only in the symmetric form $\Delta_{\rm sym}=\Delta_r^{(1)}+\Delta_r^{(2)}$. The sum $\Delta_{\rm sym}$ acts as a new random field in the effective pseudospin problem. The argument goes in a similar fashion also for the case of weak bond randomness; the original problem reduces to the one of the spin-1/2 XXZ model with *both* weak bond randomness and a (weak) random field.

According to the well-known argument due to Imry and Ma,[41,42] an ordered state like this is unstable against the domain-wall formation and any small random field destroys it in one dimension. Hence we may conclude that the half $(m^z=1/4)$ plateau of the $S=1/2$ zigzag ladder vanishes upon turning on the random field. A similar conclusion is obtained for the bond-random case; the gap due to domain-wall formation collapses and the density of states at $E=0$ becomes finite. For strong enough disorder, we may expect that the Bose-glass-like phase similar to the one occurring to $\mathcal{H}_{\text{large-}D}$ appears.

Now the difference from the first example is clear. In the first example, the gap is purely local in its origin; if we create a cluster with a size l where S^z takes different values from the average (1/2 in the above example), the energy cost is of the order l, while the energy gain due to the random field may be roughly estimated as const$\times\sqrt{l}$. Thus the formation of large clusters is energetically suppressed and the system remains uniform on the whole as is expected from the atomic-limit argument.

III. BOSONIZATION

In the previous section, we have seen that the stability of plateaus against weak (diagonal or off-diagonal) disorder depends upon their types; in the first example, the plateau is robust against randomness while the second plateau is not. In the following, we consider the effects of quenched disorder in the framework of bosonization. We use a phenomenological approach based on the Tomonaga-Luttinger (TL) model so that the argument may be independent of the details of specific models. This approach relies on the (physically plausible) assumption that the low-energy physics of quantum spin chains in a strong magnetic field is described by a single phase (or angular) variable, which corresponds to the azimuthal angle of a vector spin.[43]

Of course, we may start from the so-called composite-spin model[21,44] and investigate how the renormalization-group (RG) flow of $2S$ coupled spin-1/2 chains behaves for low energies. In the presence of disorder, however, such computations are cumbersome and it is difficult to extract the low-energy behavior of the desired spin sector out of many other auxiliary degrees of freedom. Instead, we consider the limit of strong Hund (interchain) coupling from the beginning and incorporate randomness into the relevant spin sector described generically by the single-component TL model. Our approach may be justified when randomness is much weaker than the Hund coupling.

In order to obtain the spin TL model relevant to our purpose, we follow the method in Refs. 4 and 44. That is, we prepare $2S$ copies of $S=1/2$ spin chains and ‘‘fuse’’ them by a strong interchain (Hund) coupling. The desired low-energy mode will be a symmetric combination of the $2S$ boson fields.

For a single $S=1/2$ chain, the low-energy expression of the Hamiltonian and spin operators are given in terms of TL

TABLE I. Expectation values of the pinned field and soft-mode momenta q_{soft} for generic and Ising-like TL model. Note that expectation values which are not invariant under $\phi_{\text{diff}} \mapsto -\phi_{\text{diff}}$ are unlikely for ferromagnetic Hund coupling.

Phases	Expec. value	Soft mode
Generic TL	$\langle \phi_i - \phi_j \rangle \equiv 0 \pmod{2\pi}$	$q_{\text{soft}} = 4Sk_{\text{F}} = 2\pi(S - m^z)$
Ising-like TL	$\langle \tilde{\phi}_i - \tilde{\phi}_j \rangle \equiv 0 \pmod{\pi/\sqrt{2}}$	$q_{\text{soft}} = 2k_{\text{F}} = \pi(1 - m^z/S)$

boson fields ϕ_a $(a = 1, \ldots, 2S)$ by[45]

$$\mathcal{H}_{\text{TL}} = \int dx \frac{v_{\text{S}}}{2\pi} \left[\frac{\pi^2}{R_0^2} :\Pi_a^2: + R_0^2 :(\partial_x \phi_a)^2: \right],$$

$$s_a^z \sim m^z + \frac{1}{\pi} \partial_x \tilde{\phi}_a + \text{const} :\cos[2k_{\text{F}} x - 2\tilde{\phi}_a]:, \tag{8}$$

$$s_a^+ \sim :e^{i\pi x + i\phi_a} \cos[(2k_{\text{F}} - \pi)x - 2\tilde{\phi}_a]: + \text{const} \cos(\pi x) :e^{i\phi_a}:. \tag{9}$$

The Fermi wave number is given by $k_{\text{F}} = \pi(1 - m^z/S)/2$ $(0 \leq m^z \leq S)$. The dynamical variables ϕ_a correspond to (staggered) azimuthal angles of spins, while their duals $\tilde{\phi}_a$ defined by $\Pi_{1,2} = \partial_x \tilde{\phi}_{1,2}/\pi$ are related to the translational mode. The above operators s^z and $s^{\pm}$ are power-law correlated with exponents determined only by a parameter R_0, which, together with the spin-wave velocity v_{S}, is exactly calculable by the Bethe ansatz (see, for example, Ref. 4 for numerical plots of R_0 as a function of m^z).

In general, $(2S-1)$ bosons (we call them ϕ_{diff}) go higher in energy in the limit of strong Hund coupling and the low-energy physics is described only by a single boson ϕ_{sym} (see the Appendix for details). In the strong-coupling regime, there are at least two kinds of gapless phases according to how the ϕ_{diff} fields are renormalized. When ϕ_{diff} are locked, the dominant density wave is of $4Sk_{\text{F}}$-type and the low-energy effective spin operators assume the following forms:

$$s^z \sim m^z + \frac{1}{\pi} \partial_x \tilde{\phi}_{\text{sym}}^{(1)} + \text{const} \cos[4Sk_{\text{F}} x - 2\tilde{\phi}_{\text{sym}}^{(1)}],$$

$$s^{\pm} \sim e^{i\pi x \pm i\phi_{\text{sym}}^{(1)}} + \cdots . \tag{10}$$

On the other hand, if $\tilde{\phi}_{\text{diff}}$ is locked, then the $2k_{\text{F}}$-density wave analogous to that occurring in the so-called Luther-Emery liquid becomes dominant. The expectation value should be invariant under the exchange $\phi_i \leftrightarrow \phi_j$ (see Table I) and accordingly the spin operators are given by

$$s^z \sim m^z + \frac{2S}{\pi} \partial_x \tilde{\phi}_{\text{sym}}^{(2)} + \text{const} \cos[2k_{\text{F}} x - 2\tilde{\phi}_{\text{sym}}^{(2)}],$$

$$(s^{\pm})^{2S} \sim e^{i2S\pi x \pm i\phi_{\text{sym}}^{(2)}} + \ \ldots \ . \tag{11}$$

This type of low-energy theory was discussed in Ref. 44 for the zero-field case and is important in considering the so-called metamagnetic systems. Apart from a factor in front of $\partial_x \tilde{\phi}_{\text{sym}}$, the main difference consists in the form of the characteristic wave number $[4Sk_{\text{F}} = 2\pi(S - m^z)$ or $2k_{\text{F}} = \pi(1 - m^z/S)]$. Therefore it is convenient to introduce

$$K = \begin{cases} \pi(S - m^z) & \text{for generic TL} \\ \dfrac{\pi}{2}\left(1 - \dfrac{m^z}{S}\right) & \text{for Ising-like TL} \end{cases} \tag{12}$$

to write the expressions in a unified manner. Of course, for both cases the Hamiltonian is given by the TL form

$$\mathcal{H} = \int dx \frac{v_{\text{S}}}{2\pi} \left[\frac{\pi^2}{R^2} :\Pi_{\text{sym}}^2: + R^2 :(\partial_x \phi_{\text{sym}})^2: \right] \tag{13}$$

with renormalized parameters v_{S} and R (for example, $R = \sqrt{2S} R_0^*$ in the first case). Since the correlation exponent η_{2K} of the $2K$-oscillating part is given by $\eta_{2K} = 2R^2$, the semiclassical $(\partial_x \tilde{\phi}_{\text{sym}})$ contribution dominates in the long-distance physics for $R > 1$; the feature of discrete particles manifests itself only for smaller values of R.

In what follows, we mainly consider the first case (generic TL phase) and suppress the suffix "sym" of the spin field ϕ_{sym}. The case of the Ising-like TL may be treated similarly. We summarize the expectation values and the soft-mode momenta in Table I.

It would be reasonable to expect that randomness couples to the above low-energy modes $(q = 0, 2K)$ as long as the Hund coupling, which makes the ϕ_{diff} sector gapped, is much larger than randomness and the pinning potential. A similar situation is known to occur in the problem of disordered Hubbard ladders.[46]

We begin by the diagonal disorder. For concreteness, we consider the random magnetic field applied along the symmetry axis as a typical example of this kind of randomness. From the low-energy expression (10) of S^z, it is easily seen that the random field may be incorporated into the low-energy effective action as

$$\mathcal{H}_{\text{R.F.}} = \frac{1}{\pi} \int dx\, \eta(x) \partial_x \tilde{\phi} + \int dx [\xi^*(x) e^{-2i\tilde{\phi}} + \xi(x) e^{+2i\tilde{\phi}}]. \tag{14}$$

In the above equation, the time-independent slowly-varying field $\eta(x)$ and $\xi(x)$ $[\xi^*(x)]$ denote $q = 0$ and $q = 2K(-2K)$ components of the quenched random field, respectively. Since we are considering lattice systems, there are always exceptional cases; for such special values of m^z

as $K=0$ or π modulo 2π (or, $S-m^z=\mathbf{Z}$ or $\mathbf{Z}+1/2$), "random backscattering" $\Xi(x)$ is real and the above expression should be replaced with

$$\mathcal{H}_{\mathrm{R.F.}}=\frac{1}{\pi}\int dx\,\eta(x)\partial_x\tilde{\phi}+2\int dx\,\Xi(x)\cos[2\tilde{\phi}]. \quad (15)$$

For simplicity, we assume Gaussian distributions with zero mean for the δ-correlated random variables

$$\overline{\eta(x)\eta(y)}=\pi^2 D_\eta\delta(x-y), \quad \overline{\xi^*(x)\xi(y)}=\frac{D_\xi}{2}\delta(x-y),$$

$$\overline{\Xi^*(x)\Xi(y)}=\frac{D_\Xi}{2}\delta(x-y), \quad (16)$$

$$\overline{\eta(x)}=0, \quad \overline{\xi^*(x)}=\overline{\xi(x)}=0, \quad \overline{\Xi(x)}=0,$$

where the bars denote quenched disorder averages. These may model the situation of high impurity concentration.

Bond randomness can be treated similarly. When $S-m^z\neq\mathbf{Z}$ or $\mathbf{Z}+1/2$, the random part of the Hamiltonian is given by

$$\mathcal{H}_{\mathrm{R.B.}}=\frac{1}{\pi}\int dx\,\eta(x)\partial_x\tilde{\phi}+\int dx[\xi^*(x)e^{iK}e^{-2i\tilde{\phi}}+\xi(x)e^{-iK}e^{+2i\tilde{\phi}}]. \quad (17)$$

In the above expressions, we have dropped less relevant interactions. Note that $\mathcal{H}_{\mathrm{R.B.}}$ [Eq. (17)] is essentially the same as $\mathcal{H}_{\mathrm{R.F.}}$ [Eq. (14)]. Therefore we may expect that the system behaves similarly under *both* types of randomness when the value of $S-m^z$ is generic. In the case where $S-m^z=\mathbf{Z}$ or $\mathbf{Z}+1/2$, the relevant coupling to the random bond is again real and $\mathcal{H}_{\mathrm{R.B.}}$ reads

$$\mathcal{H}_{\mathrm{R.B.}}=\frac{1}{\pi}\int \eta(x)\partial_x\tilde{\phi}(x)+\begin{cases}2\int dx\,\Xi(x)\sin[2\tilde{\phi}] & \text{for } S-m^z\in\mathbf{Z}+1/2\\ 2\int dx\,\Xi(x)\cos[2\tilde{\phi}] & \text{for } S-m^z\in\mathbf{Z}.\end{cases} \quad (18)$$

Equations (15) and (18) are derived for uniform systems. Extension of them to systems with longer periods is straightforward; we have only to replace $S-m^z$ with $Q_{\mathrm{Ham}}(S-m^z)$. For special cases where $m^z=0$ and there is no external (neither uniform nor random) field, the exact "particle-hole symmetry" ($S^z\mapsto -S^z$ and $S^\pm\mapsto S^\mp$),

$$\phi\mapsto-\phi, \quad \tilde{\phi}\mapsto-\tilde{\phi}+S\pi, \quad (19)$$

forbid the first term in Eq. (18) to exist in the action. We will see in Sec. IV B that the RG flow has a special feature for such cases.

Of course, there is a relevant interaction which leads to the plateau

$$\int dx d\tau :\cos[2N\tilde{\phi}]:. \quad (20)$$

Although we have written the interaction using a single cosine, it can be sine or a sum of sine and cosine according to the symmetry constraints such as the site parity. The integer N, magnetization m^z, and the period of the Hamiltonian Q_{Ham} satisfy the commensurability condition[3]

$$NQ_{\mathrm{Ham}}(S-m^z)\in\mathbf{Z}. \quad (21)$$

Of course, this condition is replaced with $NQ_{\mathrm{Ham}}/2(1-m^z/S)\in\mathbf{Z}$ for the Ising-like case. Remember that Q_{Ham} is not always equal to the period of the ground state $Q_{\mathrm{G.S.}}$. To eliminate the random variables, we use the standard replica trick, that is, we first n-plicate the action and then integrate out the Gaussian random fields. Thus we arrive at the multiplicated *nonrandom* action:

$$\begin{aligned}S_{\mathrm{rep}}=&\int dx d\tau\sum_{\alpha=1}^{n}\frac{1}{2\pi}\left[\frac{1}{v_{\mathrm{S}}R^2}(\partial_\tau\tilde{\phi}^{(\alpha)})^2+\frac{v_{\mathrm{S}}}{R^2}(\partial_x\tilde{\phi}^{(\alpha)})^2\right]\\&+U\int dx d\tau\sum_{\alpha=1}^{n}:\cos[2N\tilde{\phi}^{(\alpha)}]:\\&-D_\eta\int dx d\tau d\tau'\sum_{\alpha,\beta=1}^{n}\partial_x\tilde{\phi}^{(\alpha)}(\tau)\partial_x\tilde{\phi}^{(\beta)}(\tau')\\&-D_\xi\int dx d\tau d\tau'\sum_{\alpha,\beta=1}^{n}\cos\{2[\tilde{\phi}^{(\alpha)}(\tau)-\tilde{\phi}^{(\beta)}(\tau')]\}\\&-D_{\xi'}\int dx d\tau d\tau'\sum_{\alpha,\beta=1}^{n}\cos\{2[\tilde{\phi}^{(\alpha)}(\tau)+\tilde{\phi}^{(\beta)}(\tau')]\}.\end{aligned} \quad (22)$$

The resulting low-energy action is similar to that used for the boson localization problem,[7,30] in which only the case where $N=1$ and a region where the locking cos potential (U term) is *irrelevant* was treated. Another important difference is the existence of the $D_{\xi'}$ interaction [the last term in Eq. (22)]. It may seem strange because such an interaction is not generated by the Gaussian average of some random variables. As is easily verified, however, we have to take $D_\xi=D_{\xi'}=D_\Xi$ or $D_\xi=-D_{\xi'}=D_\Xi$ for real random backscattering $\Xi(x)$. This term has not been taken into account in the previous studies[19,21] and the necessity of it is closely related to the particle-hole symmetry. The physical implication of it is discussed below in conjunction with the RG flow.

Considering a statistical selection rule, we can easily see that an interaction like this is actually allowed. Although (typical) random systems do not possess any translational symmetry, the averaged ones recover it and we may impose translational invariance on the *averaged* action S_{rep}. As in the pure cases, we replace the microscopic lattice translation by a discrete shift of the boson fields:[3,4]

$$\tilde{\phi}^{(\alpha)}\mapsto\tilde{\phi}^{(\alpha)}-Q_{\mathrm{Ham}}K \quad (\alpha=1,\ldots,n). \quad (23)$$

If an interaction in the averaged action is invariant under this operation, then it is allowed *in a statistical sense*. The D_ξ interaction is always possible, whereas $D_{\xi'}$ is allowed only for

$$Q_{\text{Ham}}(S-m^z)\in \mathbf{Z} \ \text{ or } \ \mathbf{Z}+1/2. \tag{24}$$

Actually, whenever the $N=2$ pinning potential, which is possible for $Q_{\text{Ham}}(S-m^z)\in\mathbf{Z}$ or $\mathbf{Z}+1/2$, is present, RG generates $D_{\xi'}$ radiatively.

IV. RENORMALIZATION-GROUP ANALYSIS

In this section, we treat the low-energy effective action (22) by the renormalization-group (RG) method and investigate the effects of randomness on various gapped states. However, a weak-coupling RG approach is known to be hampered partly by the lack of the mean-field limit around which the theory is ε-expanded and also by the existence of large strongly correlated rare regions. Hence we first use RG equations to identify flow towards the strong-coupling phases and then investigate these phases with the help of exact results obtained for some special cases.[38]

We compute the beta function using the operator-product expansion following Refs. 10 and 21. After taking the limit $n\to 0$, we arrive at the final result:

$$\frac{dR^2}{d\ln L}=-\frac{\pi^2}{v_S}N^2(R^2)^2U^2-\frac{4\pi}{v_S}(R^2)^2D_\xi,$$

$$\frac{dv_S}{d\ln L}=-4\pi(R^2)D_\xi,$$

$$\frac{dU}{d\ln L}=[(2-N^2R^2)-2\pi^2(R^2)^2N^2D_\eta]U-2\delta_{N,2}D_{\xi'},$$

$$\tag{25}$$

$$\frac{dD_\eta}{d\ln L}=D_\eta+4g^2(R^2)(D_\xi^2-D_{\xi'}^2),$$

$$\frac{dD_\xi}{d\ln L}=(3-2R^2)D_\xi-2\delta_{N,2}g(2R^2)D_{\xi'}U,$$

$$\frac{dD_{\xi'}}{d\ln L}=(3-2R^2)D_{\xi'}-2\delta_{N,2}g(2R^2)D_\xi U - 8\pi^2(R^2)^2D_\eta D_{\xi'}.$$

The effective coupling constant $g(R^2)$ is defined by the integral[10]

$$g(R^2)\equiv\int_{-\infty}^{\infty}dt\,\frac{1}{(1+t^2)^{R^2}}, \tag{26}$$

which is convergent only for $R>1/\sqrt{2}$ to give $\sqrt{\pi}\Gamma(R^2-1/2)/\Gamma(R^2)$ and diverges logarithmically as $R\to 1/\sqrt{2}+0$.

When the value $S-m^z=\mathbf{Z}$ or $\mathbf{Z}+1/2$, the action reduces to a special case $D_\xi^{(0)}=D_{\xi'}^{(0)}=D_\Xi^{(0)}$ or $D_\xi^{(0)}=-D_{\xi'}^{(0)}=D_\Xi^{(0)}$, respectively, as is seen in Eqs. (15) and (18). In this case, whether the forward-scattering D_η exists or not crucially affects the low-energy physics. From the RG flow equations, it follows that if the forward-scattering D_η is zero initially, then it remains so in the RG process. That is, the random forward-scattering D_η is *not* generated radiatively when $S-m^z=\mathbf{Z}$ or $\mathbf{Z}+1/2$ and the initial relation $D_\xi=D_{\xi'}$ or $D_\xi=-D_{\xi'}$ remains true at least in the weak-coupling regime where the β functions (25) are valid. From the RG viewpoint, only the inversion ($S^z\mapsto -S^z$) symmetric case, that is, the case with $D_\eta^{(0)}=0$ and $(D_\xi^{(0)})^2=(D_{\xi'}^{(0)})^2$, is special. Otherwise, the RG process generates terms which do not exist initially and neither $(D_\xi)^2=(D_{\xi'})^2$ nor $D_\eta=0$ holds in general. This fact is important in treating the case of bond randomness when $S-m^z=\mathbf{Z}$ or $\mathbf{Z}+1/2$.

In order to consider the strong-coupling phases, it would be convenient to introduce the (dynamical) spin stiffness $\mathcal{D}_{\text{spin}}$, which is a spin analog of the Drude or superfluid weight, and the (differential) susceptibility $\chi_{//}$. In the Tomonaga-Luttinger language, they are written as

$$\mathcal{D}_{\text{spin}}=\frac{v_S R^2}{\pi},\qquad \chi_{//}=\frac{R^2}{\pi v_S}. \tag{27}$$

The flow equations for these quantities read

$$\frac{d\mathcal{D}_{\text{spin}}}{d\ln L}=-(\pi N^2U^2+8D_\xi)(R^2)^2,$$

$$\frac{d\chi_{//}}{d\ln L}=-\frac{\pi}{v_S}N^2(R^2)^2U^2. \tag{28}$$

Note that the interaction U between magnetic excitations reduces both $\mathcal{D}_{\text{spin}}$ and $\chi_{//}$ while the random backscattering D_ξ decreases only $\mathcal{D}_{\text{spin}}$.

If disorder couplings (D_η,D_ξ) grow much faster than U, the interaction U eventually renormalizes to 0 and we expect the strong-coupling theory to be characterized by $\mathcal{D}_{\text{spin}}=0$ and $\chi_{//}\neq 0$. On the other hand, when both D_η and U renormalize to infinity, the strong-coupling theory will have vanishing $\mathcal{D}_{\text{spin}}$ and $\chi_{//}$. We may regard the first and second cases as spin analogs of the Bose glass[7] and the Mott insulators,[47] respectively.

Taking into account the weak-coupling RG behavior, we have to consider the following three cases: (i) the incommensurate case: $NQ_{\text{Ham}}(S-m^z)\notin\mathbf{Z}$ for small integers N; (ii) the inversion symmetric case: $Q_{\text{Ham}}(S-m^z)\in\mathbf{Z}$ or $\mathbf{Z}+1/2$ and no uniform/random field; and (iii) the commensurate case: $NQ_{\text{Ham}}(S-m^z)\in\mathbf{Z}$ for a not so large integer N. The last case includes situations where $Q_{\text{Ham}}(S-m^z)\in\mathbf{Z}$ or $\mathbf{Z}+1/2$ and a random field is present.

A. Incommensurate cases

Now let us consider what the RG β functions (25) imply to the (in)stability of plateaus. We begin by the simplest case (i) where magnetization m^z takes an incommensurate value. In these cases, we can set $U=0$, $D_{\xi'}=0$ in the RG flow equation and bond and field disorder have almost the same effects as mentioned in the previous section [see Eqs. (14) and (17)]. Then the coupling $\eta(x)$ ($q=0$ component, or random forward scattering) can be eliminated formally[30] by shifting the dual field $\tilde{\phi}$. There are only three couplings, R, v_S, and D_ξ, left and the behavior of this system is well known;[7,30,48] for $R<\sqrt{3/2}$ the system is unstable to an infinitesimal perturbation D_ξ and becomes the Bose glass. On the other hand, for larger values of R the system remains to be the TL liquid as far as the randomness D_ξ is not very strong. In the sense that there is no gap-generating interaction in the absence of disorder, the situation is similar in the so-called gapless spin-fluid region appearing in the absence of an external field. For example, it is suggested[49] that the ferromagnetic region of the spin-S XXZ chain is described by the TL model with R^2 given simply by $R^2=\pi S/(\pi-\cos^{-1}\Delta)$ ($\Delta<0$ denotes the Ising anisotropy) and the TL phase survives the weak (bond/field) randomness when $S\geqslant 1$. Although the Hamiltonian takes the form of the TL model, the ground-state correlation functions get modified substantially and only the $q=0$ component remains in the long-distance asymptotics of $\overline{\langle S^z(x)S^z(0)\rangle}$.

We apply the above results to three examples. In conjunction with the magnetization process, the behavior in the vicinity of the onset of magnetization (i.e., near the edges of a plateau) would be interesting. For a plateau characterized by the order of commensurability N, the parameter R takes an asymptotic value $1/N(\leqslant 1)$ near the edge.[50] Hence when randomness (field or bond) is present, any plateau region, if it exists, is surrounded not by the TL phase but by the Bose glass phase. Near the edge, localization takes place for infinitesimally small randomness (in this sense, Bose glass appears unconditionally). Of course, the existence of such plateaus in the random system is highly nontrivial and we defer the problem to the following subsections.

Whether a region away from the edge belongs to the Bose glass or not is model dependent. For example, we consider a spin chain close to the saturation; it is known that the system is described asymptotically by $R=1/N=1$ theory near saturation (i.e., $m^z\to S-0$) and magnetization approaches the saturated value in a singular manner for clean systems. Analysis of the two-particle scatterings tells us that the spin-S Heisenberg antiferromagnetic chain near the edge is asymptotically described by the nonlinear Schrödinger model[51] with the coupling constant $c=2/(S-1)$.[52] We solved the Bethe ansatz integral equations numerically and found that the region of the above unconditional localization ($R<\sqrt{3/2}$) is confined only in the vicinity of saturation $m^z=S$ for $S>3/2$.[53] A similar argument applies also to the close vicinity of the lower critical field of the integer-spin Heisenberg model. For sufficiently large S, we can use the exact solution of the O(3) nonlinear sigma model[54] and similar results are obtained; R rapidly increases away from the lower critical field and the Bose glass phase is confined only to a narrow region around the critical field.

This is natural in view of the classical limit of antiferromagnetic spin chains; the $T=0$ *classical* antiferromagnetic XXZ chain always has a staggered order in the xy plane regardless of antiferromagnetic bond disorder or a random field in the z direction. Our finding means that as the spin S is increased the classical limit of the Heisenberg model is reached rather quickly except in the vicinity of $m^z=0$ and S.

Another interesting example is the two-leg spin ladders which have been extensively studied in several contexts. Among them, a strong-coupling ladder[28] (i.e., the coupling on rungs is much larger than that in the leg direction) is directly relevant to materials CuHpCl (Ref. 55) and BPCB (Ref. 57). The strong-coupling argument presented in Sec. II is useful also in the incommensurate phase. The region above (below) $m^z=1/2$ corresponds to a positive (negative) external field in the effective XXZ model (7). The exact solution[56] tells us that R is always smaller than $\sqrt{3/2}$ and this fact, when combined with the RG argument,[7,30] implies that the ground state is the Bose glass for any small (bond or field) randomness at least for the repulsive case $2J_2+(1-\delta)>0$. Recent magnetization measurements[58] carried out for Br-substituted CuHpCl $[Cu_2(C_5H_{12}N_2)_2(Cl_{1-x}Br_x)_4]$ show rather different behavior from the pure cases ($x=0$ and $x=1$) and we expect that this is related to the Bose glass formation.

B. No external field

Before discussing the fate of plateaus in the presence of randomness, we pause to investigate the special case where no external field ($H=0$ and $m^z=0$) is applied and only the bond randomness exists. That is, we consider systems invariant under a simultaneous spin inversion: $S_j^z\mapsto -S_j^z$ (or, $\phi\mapsto -\phi$, $\tilde{\phi}\mapsto S\pi-\tilde{\phi}$). In random systems without any kind of spatial symmetry, internal symmetries are expected to play important roles in classifying the universality classes. As is expected from the RG flow in Sec. IV, the situation in the inversion-symmetric case is strikingly different from those in other cases and should be treated separately. Inversion symmetry enforces $D_\eta=0$ and either $D_\xi+D_{\xi'}=0$ (for $S\in\mathbf{Z}+1/2$) or $D_\xi-D_{\xi'}=0$ (for $S\in\mathbf{Z}$) holds.

For clarity of argument, we take the $S=1/2$ spin chains with bond randomness, where $D_\xi+D_{\xi'}=0$ holds. In this case, several interactions which lead to gapped ground states (i.e., plateaus) are possible. Among them, we consider two types:

$$U\int dx\,\sin[2\tilde{\phi}(x)] \quad (N=1) \tag{29}$$

and

$$U\int dx\,\cos[4\tilde{\phi}(x)] \quad (N=2). \tag{30}$$

The first interaction corresponds to bond alternation and the second one leads either to the SDW or (spontaneously) dimerized phases[59] according to the sign of U. Cases with $S\geqslant 1$ may be treated similarly; for example, $S=1$ bond-alternating chains correspond to[44] the case $N=1$.

In the first case with $N=1$, both U and $2D_\Xi\equiv D_\xi-D_{\xi'}$ grow under renormalization and the space-time anisotropy develops. It is difficult to obtain the correct low-energy behavior from the low-order RG analysis. Fortunately, exact results are available[38] for this case. That is, if we assume that the precise value of R is irrelevant in the low-energy fixed point, this problem can be fermionized to the so-called random-mass Dirac fermion,

$$\mathcal{H}_{\text{RMDF}}=\int dx\Big[-i\psi_1^\dagger\partial_x\psi_1+i\psi_2^\dagger\partial_x\psi_2 +\left(\frac{1}{2}U+\Xi(x)\right)(\psi_1^\dagger\psi_2+\psi_2^\dagger\psi_1)\Big], \tag{31}$$

discussed in many contexts.[60–62] Note that inclusion of forward-scattering D_η corresponds to adding an imaginary part to $\Xi(x)$. The (averaged) density of states (DOS) $\rho(E)$ is computed exactly[60] and it shows that the gap collapses as soon as the randomness is turned on although a region of very small DOS survives if U is larger than D_Ξ. The special point is that the (averaged) DOS behaves algebraically like

$$\rho(E)\sim E^{1/z'-1} \tag{32}$$

around the band center $E=0$ and the state is localized with a finite localization length $2/U$. As a consequence, the longitudinal susceptibility $\chi_{//}$ has a low-temperature asymptotic form $\chi_{//}\sim(1/T)^{1-1/z'}$, which diverges for $z'>1$. The dynamical exponent z' depends on the strength of disorder D_Ξ, the value of the gap $U/2$, and so on. In our simple case, it is given by $z'=D_\Xi/(2U)$. Namely, the system exhibits the so-called Griffiths-McCoy singularity[15] for low temperatures. This behavior was found also by the real-space decimation method[17] and was confirmed numerically.[63]

It would be interesting to consider how the magnetization increases in this phase. For a weak magnetic field, magnetization per site is proportional to the integrated average density of states $N(E)$, which is also computed exactly as[60]

$$N(E)=\frac{D_\Xi}{\pi^2[J^2_{1/2z'}(2E/D_\Xi)+Y^2_{1/2z'}(2E/D_\Xi)]} \sim\frac{D_\Xi}{\Gamma^2(1/(2z'))}\left(\frac{E}{D_\Xi}\right)^{1/z'} \quad (E\sim 0). \tag{33}$$

In the above equation, J_ν, Y_ν, and Γ denote the Bessel function of the first kind, that of the second kind, and the Γ function, respectively. Hence the ground state is *magnetic* and magnetization increases algebraically from $H=0$ as

$$m^z\sim H^{1/z'}. \tag{34}$$

A similar result has been obtained for the random transverse-field Ising spin chain.[16,64] We show the magnetization curve for a few values of D_Ξ in Fig. 1. In the limit of a uniform chain $U\to 0$, $z'=D_\Xi/(2U)$ diverges and the so-called random singlet phase[14] is realized where an extended state exists only at $E=0$ and a logarithmic singularity appears. Note that contrary to the noninteracting case, the above expression (34) will hold only in a narrow region around $m^z=0$.

It should be stressed that our effective field-theory approach predicts that *all* half-odd-integer spin chains with bond alternation behave similarly under bond randomness; the gap in the DOS due to bond alternation collapses and instead the algebraic behavior at the band center, which leads to the so-called Griffiths-McCoy singularity, sets in.

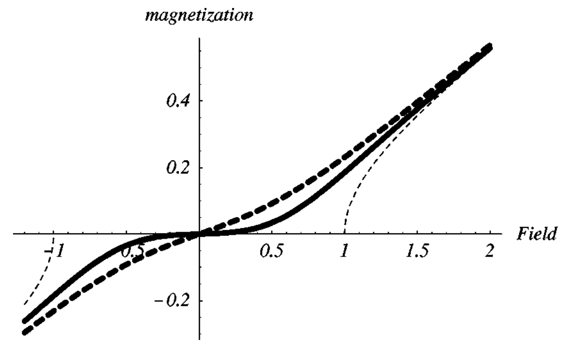

FIG. 1. Magnetization curve obtained for $N=1$ and $U=2$ by using $\mathcal{H}_{\text{RMDF}}$. Solid line: $D_\eta=0.2$, $D_\Xi=0.08$; dashed line: $D_\eta=0.2$, $D_\Xi=0.8$. The origin corresponds to the center of a plateau and magnetization is measured from the plateau magnetization. Also shown is the magnetization curve for the pure case (thin broken line). For the value $R=1$ used here, the system is compressible Bose glass (see bold lines).

In the second case, the order of commensurability is given by $N=2$ and the pure system has two degenerate ground states, both of which break the translational symmetry. For example, in the $S=1/2$ XXZ-like model with next-nearest-neighbor interaction,[59] positive (negative) U corresponds to the dimer [Néel or spin-density wave (SDW)] phase. In general, the sign of U determines whether the site parity (i.e., reflection with respect to a given site) is spontaneously broken or not.

We treat the case $U>0$ (''dimer'' phase) first. Note that dimerization in this case is *spontaneously* generated. Both U and $\tilde{D}_\xi$ grows first and accordingly the value of R gets reduced [see Eq. (25)]. Then the effective coupling constant $g(2R^2)$ takes a large value to increase $\tilde{D}_\xi$ further. Finally, $dU/d\ln L$ turns to negative and U converges to zero. In a sense, this is similar to the field-theoretical interpretation[10] of the Imry-Ma effect. In the vicinity of $R=1$, the low-energy physics will be determined by Eq. (31) with $U=0$ and we may expect that the random-singlet phase[14] takes over the dimer phase.

This is not restricted to the $S=1/2$ case. For $S\geqslant 1$, spontaneous dimerization is known to occur for a certain family of spin-S chains:[65]

$$\mathcal{H}_{\text{BBK}}=-\sum_j J_{i,i+1}P_0(\mathbf{S}_i,\mathbf{S}_{i+1}), \tag{35}$$

where P_0 denotes the projection operator onto the singlet subspace spanned by $\mathbf{S}_i$ and $\mathbf{S}_{i+1}$ and $J_{i,i+1}$ is assumed positive. The situation for the pure chain is similar[65] to the case $U>0$ described above and we can conclude that the dimerized ground state is fragile and the random-singlet phase appears upon introducing randomness into $J_{i,i+1}$. Actually, the method of real-space decimation[13] is well-defined for the Hamiltonian (35) and leads to the same conclusion.[36,66]

On the other hand, for $U<0$, the divergingly large $g(2R^2)$ implies the contrary; it switches $\tilde{D}_\xi$ from increasing to decreasing [note that U enters the flow equation of $\tilde{D}_\xi$ in the form $+2g(2R^2)\tilde{D}_\xi U$]. The result is that the gap-generating interaction U is diverging. That is, the Néel (or SDW) phase is *robust* against weak bond randomness. Note that the difference in the sign of U leads the system to completely different phases.

C. Case of plateaus

Finally, we discuss stability of plateaus appearing in the presence of a strong field. Here we mean by plateaus gapped states occurring in a finite field. Throughout this subsection, the integer N appearing in Eq. (21) plays an important role. Note that there is no apparent inversion symmetry in these cases and that the special symmetry $|D_\xi|=|D_{\xi'}|, D_\eta=0$ is no longer preserved in the RG process. In fact, we can explicitly verify that nonzero expectation value $\langle S^z\rangle = m^z$ introduces the random forward-scattering $\eta(x)$ through the S^zS^z coupling.

For generic cases where $Q_{\rm Ham}(S-m^z)$ equals neither an integer nor a half-odd integer, the effective low-energy action is given by Eq. (22) with $D_{\xi'}=0$ regardless of whether randomness is concerning the external field or exchange couplings, and the problem reduces to that of the effects of random field at incommensurate magnetization. In these cases, only plateaus with $N\geqslant 3$ are allowed in pure systems. As was pointed out in Ref. 10, when $N\geqslant 2$, the divergingly large $g(R^2)$ prevents the gap caused by U from surviving the disorder. Therefore small randomness smears out plateaus by the Imry-Ma effect when $Q_{\rm Ham}(S-m^z)\neq \mathbf{Z}$ or $\mathbf{Z}+1/2$.

When $Q_{\rm Ham}(S-m^z)\in \mathbf{Z}+1/2$, the lowest possible value of N allowed by Eq. (21) is 2. As in the above case, plateaus are fragile in the presence of disorder like η. Hence three-dimensional effects are necessary to maintain the plateaus of this class. In all the above cases, plateau regions in the pure cases are replaced by localized (but gapless) Bose glass phases.

Now we discuss the only remaining case: $Q_{\rm Ham}(S-m^z)\in \mathbf{Z}$, where the lowest possible commensurability is given by $N=1$. For example, the $m^z=1/2$ plateau of the $S=1$ bond-alternating Heisenberg chain[67,68] and $m^z=1/2$ plateau of $S=3/2$ Heisenberg chain with D term[37] fall into this category. An $N=1$ plateau is in a sense robust against randomness[42,69] because it is not always subject to the Imry-Ma effects. The robustness of this kind of plateau is anticipated also from the strong-coupling argument in Sec. II. As was pointed out by Fujimoto and Kawakami,[10] the low-energy behavior is determined by the competition between U and D_η; for large D_η compared with U, the random forward-scattering D_η collapses a gap and then the resulting gapless system becomes localized (Bose glass) by the "random backscattering" D_ξ. On the contrary, when U is sufficiently large, the gap persists and the system behaves as a Mott insulator ($\mathcal{D}_{\rm spin}=0, \chi_{//}=0$).

Therefore the DOS around the band center is expected to be zero for weak enough randomness and the gap (i.e., plateau) is not smeared out. Strictly speaking, however, this will be valid for such *bounded* randomness as the box distribution. Again, a special point $R=1$ can be treated by adding random forward-scattering $\eta(x)$ to $\mathcal{H}_{\rm RMDF}$ [Eq. (31)].[27,60] Only a smooth crossover from the Anderson localized (finite DOS at $E=0$ but localized) to the Mott localized (vanishing DOS at $E=0$ and localized) behavior occurs around $U\sim D_{\xi/\Xi}$ (see Fig. 1) in our continuum model. We expect that this is an artifact by using an unbounded distribution and that there occurs generically a (first-order) transition in between as seen in Sec. II.

To summarize the above argument, *in strictly one-dimensional systems*, stable plateaus are possible only for magnetization m^z satisfying

$$Q_{\rm Ham}(S-m^z)=\text{integer}. \tag{36}$$

Note that the period appearing in the above equation is that of Hamiltonian. For example, plateaus in Refs. 67 and 68 ($S=1, m^z=1/2, Q_{\rm Ham}=2$) and Ref. 37 ($S=3/2, m^z=1/2, Q_{\rm Ham}=1$) are stable against weak randomness in the above sense. The results obtained here are consistent with those of strong-coupling expansions.

A remark is in order here about the difference between the Bose glass and the TL liquid. In the Bose glass phase, the correlation is *spatially* short ranged and hence the system has a vanishing stiffness $\mathcal{D}_{\rm spin}$, while it has long-ranged correlation in the (imaginary) temporal direction (thus having nonzero compressibility $\chi_{//}$). Because the gap in the DOS vanishes, magnetization increases smoothly *already* in the local limit. As has been shown in Sec. IV A, the vicinity of the edges of a plateau always belongs to the Bose glass (BG) phase; magnetization-onset transitions occurring there is the Bose condensation in the *real* space (not in the momentum space).

On the other hand, the TL liquid has a gapless ground state as a consequence of many-body effects. Accordingly, the ground state is extended and has a finite stiffness. In both cases, the DOS is always finite and the ground state is magnetic (or compressible, in the particle language). In other words, the TL liquid can hardly be distinguished from the Bose glass only by the magnetization process.

V. SUMMARY AND DISCUSSION

In this paper, we investigated magnetization processes of random quantum spin chains. Strong-coupling arguments suggest that (in)stability of various gapped state of 1D quantum spin systems depends on the type of randomness, the existence or nonexistence of spontaneously broken symmetry, and so on. Guided by these findings, we formulated a phenomenological approach to the problem of random spin chains.

We started by mapping a single spin chain without randomness onto a perturbed Tomonaga-Luttinger model. There are many analytical and numerical evidences to support our mapping. Then, we assumed weak randomness and handled it by the so-called replica trick to obtain the renormalization-group (RG) equations. The RG analyses showed us that the weak-coupling RG behavior is qualitatively different according to whether the random forward scattering is present or not. This led us to divide the situations into three categories:

(i) incommensurate cases, (ii) inversion ($S^z \leftrightarrow -S^z$) symmetric cases (with no uniform or random external field), and (iii) commensurate cases (without inversion symmetry) where magnetization plateaus appear.

Case (i). In the first case, there is no relevant interaction competing with randomness and the system is described by a gapless spinfluid when disorder is absent; the weak-coupling RG gives the reliable answer that the system is in the localized Bose-glass (BG)-like phase unconditionally when the (fully renormalized) phenomenological parameter R is smaller than $\sqrt{3/2}$ and finite amount of randomness is necessary to localize the system for $R > \sqrt{3/2}$. In the BG phase, longitudinal susceptibility $\chi_{//} = R^2/(\pi v_S)$ remains finite while the dynamical spin stiffness $\mathcal{D}_{\text{spin}}$ vanishes. A typical example of the incommensurate region is a smooth portion of magnetization processes. In the strong-coupling region of the two-leg spin ladder (see Sec. II), the condition $R < \sqrt{3/2}$ is satisfied all the way from the lower critical field to the saturation field and a single isolated ladder always localizes. The spin-S XXZ chain ($S \geqslant 3/2$), on the other hand, does *not* localize in almost the whole portion of the finitely magnetized region.

The effect of random forward-scattering D_η is most highlighted in the second and third cases.

Case (ii). When $D_\eta = 0$ and no uniform field is applied initially, the RG flow goes to the strong-coupling regime preserving the property $D_\eta = 0$. Although our weak-coupling RG itself is no longer reliable for such strong couplings, we can obtain some insights by mapping the problem to a solvable fermionic model—the random-mass Dirac fermion. A combination of the exact solution and the RG analyses led us to the following conclusion: when the pure system has a gap and no translational symmetry is (spontaneously) broken, the system is in the so-called quantum Griffiths phase and a gapless ground state with a finite localization length proportional to the inverse of the gap. If the translational symmetry is broken *spontaneously*, the situation is more subtle; in some cases the pure gapped ground state is collapsed by the randomness and it is not in other cases. For example, such spontaneously dimerized phases as occur in $\mathcal{H}_{\text{BBK}}$ [Eq. (35)] is easily destroyed by bond randomness while Néel-like charge-density wave (CDW) states are robust. In the former, we may expect that the extended random-singlet phase takes over.

Case (iii). In this case, the system is finitely magnetized and the inversion symmetry which was quite important in case (ii) is already broken. The cases with $m^z = 0$ and $D_\eta \neq 0$ also fall into this category. Then, the treatment of Sec. III shows that diagonal disorder (random field) and off-diagonal disorder (random bond) can be treated in the same manner. We may naively speculate that the gapped systems (plateau systems) are insensitive to disorder. However, at least in one dimension, the stability of the gapped states depends strongly on the type of plateaus, or the degree of commensurability N. As is well known as the Imry-Ma mechanism, the gapped states with $N \geqslant 2$ is fragile to the diagonal disorder like a random field; the gap collapses and the system is localized.

The case with $N = 1$ is, in this sense, not so well established as in the above cases. Strong-coupling arguments (Sec. II) and RG analyses indicate that plateau gaps are robust and transitions into BG phases do occur at finite strength of disorder. Quite recently, the possibility of a new intervening phase (incompressible but conducting) was pointed out[70] in the context of a spinless fermion with disorder. The bosonized action used there is similar to ours ($N = 1$) and there might be analogous phases also in spin systems. We postpone this interesting problem to future works.

Finally, we briefly discuss the effect of disorder on the field-induced LRO occurring generically for incommensurate magnetization.

Recently, examples of long-range 3D ordering in a strong magnetic field have been reported[71–73] for various spin-gapped systems. In the particle language, this LRO in the direction perpendicular to the external field can be viewed as superfluid LRO and the problem is a spin analog of superfluid-insulator transitions driven by disorder.

For example, if an $S = 1/2$ zigzag spin ladder system CuHpCl or BPCB is treated as a strictly one-dimensional model, strong-coupling argument predicts that any weak disorder localizes the incommensurate phase between $m^z = 0$ and $m^z = 1/2$ as has been mentioned in Sec. IV A. The three-dimensional coupling weakens the effect of disorder and favors the 3D LRO. Since the forward scattering $\eta(x)$ makes the $2K$ component of $\overline{\langle S^z(x) S^z(0) \rangle}$ decay exponentially, we have only to take into account the superfluid LRO. We carried out a mean-field analysis similar to that in Ref. 74 for the value $R = 0.9$ (which is appropriate for the midfield region of CuHpCl) and $J_{\text{interchain}}/J_{\text{chain}} = 1/10$ to find that the (mean-field) transition temperature $T_c(D_\xi)/T_c(D_\xi = 0)$ rapidly decreases to zero at around $D_\xi \approx 0.2$ (beyond which no LRO occurs for all temperatures). Therefore for samples with sufficiently weak disorder, we may expect a 3D ordering transition to occur as temperature is varied.

After completion of this work, we became aware of a paper investigating a magnetization process of a random q-merized $S = 1/2$ spin chain.[77] They found a plateau at an *irrational* value of magnetization; the appearance of it is possible only when disorder in exchange couplings exists and completely different from those treated in this paper.

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APPENDIX

In this appendix, we briefly describe the derivation of the low-energy effective model used in Sec. III. In Sec. III, we started from $2S$ coupled Tomonaga-Luttinger models and argued that only one of them is relevant to low-energy physics. On general grounds, we may expect that it is given by a symmetric combination

$$\phi_{\rm sym}=\frac{1}{\sqrt{2S}}\sum_{j=1}^{2S}\phi_j\,. \tag{A1}$$

Considering that $\phi_{\rm sym}$ is directly related to total S^z, we can easily see that it describes the "spin" sector of the spin-S model.

For the sake of clarity, we consider the $S=1$ (i.e., two chain) case to demonstrate how the spin sector is singled out in the limit of a strong Hund coupling. By using these continuum expressions, we can write the relevant part of the interchain interactions[4,44] as

$$\lambda_1:\quad \cos[(G-4k_{\rm F})x+2(\tilde{\phi}_1+\tilde{\phi}_2)]:, \tag{A2}$$

$$\lambda_2:\quad \cos[2(\tilde{\phi}_1-\tilde{\phi}_2)]:, \tag{A3}$$

$$\lambda_3:\quad \cos[(\phi_1-\phi_2)]:. \tag{A4}$$

The scaling dimensions of these operators are

$$x_1=x_2=2R_0^2,\quad x_3=1/(2R_0^2)^2. \tag{A5}$$

In the above equations, the quantity G denotes the reciprocal-lattice vector $G=2\pi/a_0$ and the parameter R_0 is computed using the exact solution.[56] As was described in Sec. III, $k_{\rm F}$ is uniquely determined by magnetization:

$$k_{\rm F}=\frac{\pi}{2}(1-m^z/S).$$

It would be convenient to introduce the two fields ("spin" and "charge" in the language of electron systems)

$$\phi_{\rm sym}=\frac{1}{\sqrt{2}}(\phi_1+\phi_2),\quad \phi_{\rm diff}=\frac{1}{\sqrt{2}}(\phi_1-\phi_2) \tag{A6}$$

in place of ϕ_1 and ϕ_2. For $S\geqslant 3/2$, the decomposition is not unique and it is convenient to consider $\{\phi_{\rm diff}\}$ as a set of independent operators like $\phi_i-\phi_j$.

Note that only the first interaction λ_1 contains the $k_{\rm F}$ (or magnetization)-dependent factor. The remaining two are magnetization independent and drive the $\phi_{\rm diff}$ sector massive regardless of the value of magnetization. Hence we may expect that in the strong-coupling region only the $\phi_{\rm sym}$ boson remains gapless for generic values of magnetization and the system is described by a single-component TL model.

Here we have to comment on another important case with several gapless degrees of freedom. It is known that there are models[75] which are described by multicomponent TL models in a field. The preceding argument is not applicable straightforwardly to these models and we focused only on the single-component case in the text.[76]

As we mentioned in Sec. III, at least two kinds of gapless phases are possible according to how the $\phi_{\rm diff}$ field is renormalized. When $\phi_{\rm diff}$ is locked and $\langle\cos(\sqrt{2}\tilde{\phi}_{\rm diff})\rangle=0$, the $2k_{\rm F}$-oscillating terms vanish and a higher $4k_{\rm F}$-density wave becomes dominant. Therefore the low-energy effective spin operators take the following forms:

$$s^z\sim m^z+\frac{1}{\pi}\partial_x\tilde{\phi}^{(1)}_{\rm sym}+\text{const}\cos[4k_{\rm F}x-2\tilde{\phi}^{(1)}_{\rm sym}]$$

$$s^{\pm}\sim e^{i\pi x\pm i\phi^{(1)}_{\rm sym}}+\cdots. \tag{A7}$$

We included the explicit S dependence in Eq. (10).

On the other hand, if $\tilde{\phi}_{\rm diff}$ is locked, then $\cos\sqrt{2}\tilde{\phi}_{\rm diff}$ has a finite expectation value and the $2k_{\rm F}$-density wave analogous to that occurring in the so-called Luther-Emery liquid becomes dominant. Since the Hund coupling is ferromagnetic, the resulting low-energy theory should be invariant under the interchange $\phi_1\leftrightarrow\phi_2$. This implies that $\langle\tilde{\phi}_{\rm diff}\rangle\equiv 0 \pmod{\pi/\sqrt{2}}$ and we arrive at expressions (11).

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$$\tilde{J}_{1,4}=\frac{2}{(2S+1)^2}\left(\frac{J_{1,2}J_{3,4}}{J_{2,3}}\right)$$

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