Resonant scattering and localization in heterogeneous Biot media

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This is the second of a pair of papers considering acoustic multiple scattering and localization in fluid-saturated porous solids. Such systems are represented by the Biot effective medium, and spatially δ-function correlated disorder is incorporated by replica-field methods. In the companion paper it was shown that a preferred form of the wavelength-scale scattering operator, coupling primarily to slow waves, is selected by a universal renormalization-group flow toward strong coupling. The simplification afforded by a single scattering interaction is used here to define both Ginzburg-Landau and coherent-potential descriptions of the localized density of states. It is found that, as long as the Biot medium is the appropriate background, a description of localized slow and extended fast waves is possible, but that fast waves may not localize within the weak-coupling limits of the models. Further, coupling between slow and fast waves contributes an apparent dissipation that can inhibit slow wave localization. Finally, relations are found between fast-wave coherence length and the slow noise spectrum, which make fast waves a potential probe of a slow-localized volume.

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I. INTRODUCTION

The possibility of acoustic Anderson localization[1] has been studied for heterogeneous elastic media, using tools of effective medium theory and multiple-scattering perturbation theory.[2] In three dimensions, the conditions for localization invariably require a medium with dense, strong discontinuities on small scales, and models with dense granular insertions in a resin or fluid background are usually proposed as potential realizations.

An important distinction arises for such models, though, between granular insertions that do not make contact and those that do, if the surrounding saturant is a fluid. In the former case, the material is a suspension, and elastic theory with sometimes-vanishing shear modulus is an adequate model. In the latter, the insertions form a percolating cluster, and the medium is a fluid-saturated porous solid. The importance of percolation is well appreciated in the similar problem of classical electromagnetic localization, where bulk conductivity of connected dielectric spheres would screen the medium entirely.[3] In the acoustic case, the situation is somewhat different. Solid stress percolation does not screen sound from the interior, but rather creates another effective medium, with a richer set of wave types and descriptive parameters than elasticity. The acoustic description of this effective medium is called Biot theory.[4–6]

In a companion article,[7] it was argued that Biot theory is the universal generalization of linear elasticity appropriate for a porous-medium perturbation expansion, and further that all the methods for representing quenched parameter randomness can be applied to the two cases in the same form. A main result of that work was that the complicated set of possible Biot scattering operators,[8] which in the bare theory would be an obstacle to saying anything general about its localizing properties, flow to a universal, fairly simple form, in the long-range renormalized theory. With appropriate caveats about when this universality is useful, the main idea is physically simple: In keeping both fluid and solid degrees of freedom, Biot theory predicts two compressional wave types with different wave speeds and wavelengths. The Rayleigh cross section is generically larger for the one with the shorter wavelength, and a relevant[9] renormalization group (RG) flow tends to select this wave for strong coupling. It was also shown in Ref. 7 that, even when the asymptotic RG is not reached, common examples yield scattering operators that are near the universal form even in the bare theory. It would then naturally be expected that the slower wave would be more susceptible to localization than the faster.

A. Goals and limitations

This paper studies the localizing transition in heterogeneous Biot theory, with the aim of extracting general characteristics. The simplified universal form of the scattering operator from Ref. 7 will be used to represent the generally enhanced coupling of the slower wave type. Both nonperturbative (Ginzburg-Landau)[10] and perturbative (coherent potential approximation),[2,3] approaches to localization will be considered, to derive a difference of treatments of densities of states for the fast and slow wave types, when they experience scattering interconversion, which has no exact analog in acoustic or elastic systems.

As in Ref. 7, the methods used will require an assumption of inviscid fluids, so the results will only apply directly to saturants such as sufficiently cold superfluid helium. One feature of wave interconversion that will be derived is a coherence loss that mimics dissipation in the critical RG flow for diffusivity[3] and inhibits localization, even in the idealized inviscid limit. Any intrinsic absorption due to fluid viscosity would then add to this inhibiting effect.

Another result will be that Biot media have an incoherent resonance effect like that in nonuniformly rough acoustic media,[10] but Biot media preserve it into the uniform limit. The incoherent resonance of nearly-localized slow wave states will arise from the same interconversion that mimics dissipation in the critical RG flow.

B. Approaches to localization

There are two fairly different ways to study localization within the replica formalism: a semiclassical Ginzburg-

Landau (GL) treatment, and the perturbative renormalization of the coherent potential approximation (CPA) sigma model. The GL method is intrinsically nonperturbative (an instanton expansion), and gives an explicit representation of the density of localized states. This is important because the RG of Ref. 7 *never* identifies such states, due to a divergence of the perturbation series, which in fact requires the semiclassical stationary point expansion to be stabilized in the localizing regime.[11] The GL expansion can also be applied to statistically nonuniform heterogeneity, and in fact this was the key to discovering how instantons could arise to stabilize the acoustic problem at all.[10] On the other hand, the GL expansion is intrinsically nonrenormalized, and cannot be applied close to the mobility edge, whenever such exists.

The critical CPA sigma-model renormalization is exactly complementary. It, like the free-field RG, makes no explicit identification of localized states, and furthermore inherently assumes uniform heterogeneity. All states in a uniform system lie on the allowed wave number for propagation, and the entire density of states at a given frequency is only identified as localized or extended as the scale-dependent diffusivity either remains finite or goes to zero at large distances. In the latter case, the length where the diffusivity goes through some characteristic value becomes identified as the localization length, whose scaling with frequency may then be obtained in arbitrary neighborhoods of the mobility edge.

Inputs from both approaches will be used here, to try to piece together a picture of localization in heterogeneous Biot media. A result from the GL expansion will be that, as long as the two wave types have scattering interconversion due to polarization overlap, there is *no* instanton representation of localized states at the allowed (on-shell) fast wave number, as there would be in a simple acoustic medium with the fast wave speed. On the other hand, there are instantons near the slow-wave number, which correspond to the simple acoustic case, but have small admixtures of fast-wave polarizations. The ordering of this species dependence is due precisely to the ordering of wave speeds, and is interpreted to mean that the slow wave localizes first and most strongly, and that if the fast wave localizes at all, it must do so as a secondary transition in the effective theory after slow waves have been integrated out.

A further consequence of the interspecies coupling is deduced from comparison of the GL result with the CPA. Arguing that only the density of states represented by instantons should be renormalized coherently in the slow-wave CPA, the additional coherence attenuation from scattering into fast waves is treated as an apparent intrinsic slow-wave dissipation. The resulting effective theory for slow waves becomes indistinguishable from the scalar-acoustic problem with intrinsic absorption, for which the critical properties have already been derived.[3] An upper bound on the dissipation, to allow coherence-induced localization, becomes in this application a bound on the coupling strength between the wave types.

The piecing together of limited descriptions in this way is likely to be a good approximation when there is a large separation between slow and fast wave speeds (as there often is experimentally), simply because of the difference of Rayleigh cross sections that scale as large powers of the respective wave numbers. In an imaginable opposite limit, where all wave speeds are comparable and all scattering mechanisms *a priori* equivalent, presumably some rapid-mixing argument like that in Ref. 2 should be used. No ideas are presented here, however, about how to tame the unbridled anarchy of forms possible in that limit.

C. Methods and organization

The derivation below reflects the dichotomy and logical precedence of the GL and CPA descriptions. New calculations will almost entirely follow the instanton methods of Ref. 10, which rely on a statistically nonuniform scattering strength, to identify the different treatment of localized states near the fast and slow wave numbers. Once this has been obtained, it will be used to define CPA field theories for fast or slow sectors in the uniform limit, by selective integration out of fields. The resulting effective theories, though, will no longer have the full Biot spectrum of wave types, and so will be described by the scalar sigma models treated elsewhere.[2,3] Thus, no new critical scaling arguments will be computed, and results will simply be applied from the scalar case as needed. In particular, it will follow that when there is a mobility edge ω^*[3] (relevant only in three dimensions), the localization length scales in a neighborhood of it as in the acoustic problem: $l_{\mathrm{loc}} \sim |\omega - \omega^*|^{-1}$.

The language and notation will closely follow the replica adaptation of Biot theory used in the companion paper.[7] Because it is assumed that some aspects of Biot theory may be unfamiliar to localization audiences, canonical results from homogeneous Biot theory are first reviewed in Sec. II. The notation and relations defining replica Green's functions are then presented, using the simplified scattering operator of Ref. 7.

The instanton expansion is derived in Sec. III, and the wave number dependence, polarization, and spatial distribution of localized states computed using nonuniform roughness as a regulator. The incoherent resonance effect corresponding to the acoustic case[10] is also derived in this section.

The relation to the CPA is developed in Sec. IV, which contains the main results of the paper. The coherent attenuation and noise spectrum are related using the instanton Green's functions, and are then interpreted in passing to the uniform CPA in terms of the free fast and slow densities of states. The proportionality between the fast and slow coherence lengths is derived, thus relating the fast coherence length to the slow noise spectrum (the only noise spectrum in the medium). The argument for different treatment of fast and slow scattering is then made, and the resulting dissipative CPA for the slow wave sector obtained. Results from Ref. 3 are used to derive the slow-wave Ioffe-Regel criterion, and also to quantify a condition that coherence loss from slow-fast wave conversion not eliminate slow wave localization.

The picture that emerges is dominated by the form of the effective coupling. For the generic form of Ref. 7, there is a clear ascending scattering-strength hierarchy: fast-fast, fast-slow=slow-fast, slow-slow. As a result, slow waves can localize with sufficiently strong self-scattering, with slow-fast scattering being only a weaker coherence inhibitor. In contrast, fast wave coherence attenuation is dominated by fast-slow scattering, so both it and the resulting noise spectrum are controlled by slow wave properties at leading order.

Localization in Biot media thus has one more important difference from the acoustic or elastic cases. In the latter, the noise spectrum could not be probed remotely, and was related only to the coherence length of the localized excitations that it comprised. Here, weakly scattered, extended fast waves remain a probe of the bulk medium, and can couple to localized slow waves, as to a distribution of resonators dispersed throughout the medium.

II. HETEROGENEOUS BIOT THEORY

To study localization in porous media with conventional perturbative methods, it is necessary to have a smoothed, effective-medium description that accounts for all of the independent large-scale degrees of freedom. However, since many porous media are granular on scales approaching the acoustic wavelengths of interest, this description must be defined by its symmetries, so as not to rely on a particular homogenization scheme to have a sensible meaning. Biot theory, with the coefficients considered as renormalized effective parameters, satisfies both of these requirements.

A. Biot phenomenology

The Biot effective medium theory[4–6] is the generic homogenized description[12,13] of acoustic excitations in fluid-saturated porous solids. It directly extends the form and generality of linear elasticity to media with two locally-defined, independent, interpenetrating deformational degrees of freedom, one supporting a shear stress and one not. From collocation of two independent compressible media, Biot theory predicts two independent compressional solutions to a Helmholtz-type equation, with different speeds and relative fluid-solid motions (called here ‘‘polarizations;’’ see below). The fast compressional wave will be called here simply the ‘‘fast wave,’’ and the slow compressional wave the ‘‘slow wave.’’ A shear wave is also predicted, which is a direct extension of the elastic shear wave in the porous solid frame. The theory also includes a set of constitutive relations,[14] predicting wave speeds, polarizations, and dissipations from material properties of the frame and saturating fluid that are readily measured in the laboratory.

All three predicted Biot waves have been measured in laboratory experiments (on consolidated, statistically homogeneous porous solids),[15–17] and found to have properties in excellent agreement with those predicted by the constitutive relations. Because the assumption of inviscid flow is so central to the methods used in this paper and Ref. 7, it is significant that Biot theory has been applied to solids saturated with superfluid helium,[18,16,17] and indeed, when the same solids are saturated with liquid helium or water, the wave properties in the two cases agree with the predictions, with no adjustable parameters.[17] The Biot parameter set and constitutive relations are discussed in somewhat greater detail in the first paper.[7]

It was shown in Ref. 7 that both the bulk Biot equations of motion, and the correct boundary conditions for permeable interfaces, are obtained from a simple variational principle directly extending that for linear elasticity. This variational principle is the basis for the definition of scattering operators from formally pointlike fluctuations in the Biot parameters. From the argument that Biot theory is a *universal* homogenized description of porous media,[12] it then follows that this set of parameter fluctuations is also *complete* for describing arbitrary medium heterogeneities at the order of the wave equation in derivatives.

B. Defining relations

Homogeneous Biot theory is defined in terms of two copresent displacement degrees of freedom: u, which will describe a solid, and U, a fluid. β is the porosity (mean volume fraction occupied by the fluid), and the volume-weighted relative fluid displacement is $w \equiv \beta(u-U)$.[19] In general, the averaging procedure defining each of these in terms of the zero-frequency, grain-scale solid and fluid volumes and displacements will be scale and frequency dependent. The perturbative RG (Ref. 7) is used to transform among the descriptions at different scales.

The solid has first-order strain tensor

$$\epsilon^{ij} \equiv \tfrac{1}{2}(u^{i,j}+u^{j,i}), \tag{1}$$

and the fluid displacement

$$\varepsilon^{ij} \equiv \tfrac{1}{2}(w^{i,j}+w^{j,i}), \tag{2}$$

where commas denote partial differentiation with respect to spatial position x in d dimensions. The trace of solid strain is

$$e \equiv \nabla \cdot u = \mathrm{Tr}(\epsilon) \equiv \epsilon^{ii}, \tag{3}$$

where repeated indices are summed, and that of w defines the so-called ‘‘increment of fluid content:’’

$$\zeta \equiv \nabla \cdot w = \mathrm{Tr}(\varepsilon) \equiv \varepsilon^{ii}. \tag{4}$$

To lowest order in derivatives, Biot theory assumes the existence of three density parameters, ρ, ρ_f, and m, and four modulus parameters H, C, M, and μ. At zero frequency, these can be related to grain and fluid densities, porosity, tortuosity of fluid paths, and compressibilities and stiffness of grains, fluid, and the composite grain framework.[14,20–22] These parameters specify canonical equations of motion[22,23]

$$\rho \ddot{u}^i - \rho_f \ddot{w}^i = \nabla^i[(H-2\mu)e - C\zeta] + \nabla^j(2\mu\epsilon^{ij}) \tag{5}$$

and

$$\rho_f \ddot{u}^i - m\ddot{w}^i = \nabla^i[Ce - M\zeta]. \tag{6}$$

Biot theory is the most general possible to second order in derivatives, because Eqs. (5) and (6) are obtained by varying the action

$$S=\int dt\int d^dx\left\{\frac{1}{2}[\dot{u}\quad \dot{w}]^i[\rho]\begin{bmatrix}\dot{u}\\ \dot{w}\end{bmatrix}^i - \frac{1}{2}[e\quad \zeta][\lambda]\begin{bmatrix}e\\ \zeta\end{bmatrix}-\mu(\epsilon^{ij}\epsilon^{ij})\right\}, \tag{7}$$

which is the most general form for two degrees of freedom with the respective symmetries of solid and fluid. The matrix-valued parameters are straightforward extensions of the Lamé parameters for density

$$[\rho]\equiv\begin{bmatrix}\rho & -\rho_f\\ -\rho_f & m\end{bmatrix} \tag{8}$$

and compressibility

$$[\lambda]\equiv[K]-2[\mu], \tag{9}$$

where

$$[K]\equiv\begin{bmatrix}H & -C\\ -C & M\end{bmatrix} \tag{10}$$

and

$$[\mu]\equiv\begin{bmatrix}\mu & 0\\ 0 & 0\end{bmatrix}. \tag{11}$$

The term in Eq. (7) involving μ could also have been written as a contraction with the matrix (11), whose form identifies u as the unique effective solid degree of freedom.

C. Green's functions, replicas, and ensembles

A variety of Green's functions have been solved for Biot theory in various approximations and cases.[24] For the form needed here in general static media, it will be adequate to work in the frequency domain, and the full action (7) is not needed. Starting at the grain scale, it is convenient to form a nondimensionalized Lagrangian

$$L=\frac{1}{\rho_f l_0^{(d+2)}}\int d^dx\left\{\frac{1}{2}\omega^2[u\quad w]^i[\rho]\begin{bmatrix}u\\ w\end{bmatrix}^i - \frac{1}{2}[e\quad \zeta][\lambda]\begin{bmatrix}e\\ \zeta\end{bmatrix}-\mu(\epsilon^{ij}\epsilon^{ij})\right\} \tag{12}$$

with ρ_f, the fluid density, and l_0, a smallest correlation length on which homogenization is sensibly defined. l_0 is generally expected to be a few grain diameters, and forms the "natural scale" for the perturbative RG flow.[7]

Quenched randomness with Gaussian fluctuations is modeled by the replica trick,[25] directly following Ref. 7. First, each displacement degree of freedom is promoted to a vector of *n replicas* (given parenthesized superscripts here when needed). Properly normalized Green's functions for physical displacements in a given background may be written as functional-integral expectation values of a *single* replica component [e.g., index (1)] in the n-dimensional theory, when n is taken to zero afterward:

$$\left\langle\begin{bmatrix}u\\ w\end{bmatrix}^i_{x'}[u\quad w]^j_x\right\rangle_\omega = \lim_{n\to 0}\int \mathcal{D}\phi\mathcal{D}\varphi\mathcal{D}A\mathcal{D}a\, e^{-L}\begin{bmatrix}u\\ w\end{bmatrix}^{i(1)}_{x'}[u\quad w]^{j(1)}_x. \tag{13}$$

Spatial subscripts here denote the arguments of the fields, and subscript ω on the expectation value denotes the frequency used in the Lagrangian (12). The outer product (13) gives the general correlation of solid and fluid motions in an arbitrary fixed background of parameters, not necessarily uniform.

The functional measure in Eq. (13) has been written in terms of compressional and shear potentials, conveniently defined from bare theory scale factors, for the solid as

$$u\equiv\sqrt{\rho_f}\,l_0^{(d/2+1)}(\nabla\phi+\nabla\times A), \tag{14}$$

and for the fluid increment as

$$w\equiv\sqrt{\rho_f}\,l_0^{(d/2+1)}(\nabla\varphi+\nabla\times a). \tag{15}$$

Vector potentials A and a include only transverse components, so no gauge condition is needed to fix redundant degrees of freedom.

The conclusion of Ref. 7 was that the dominant long-wavelength scattering operator decouples from vector potentials. Therefore, to simplify this treatment, shear coupling will be ignored from the start, and only compressional correlation functions will be considered. Because $n\to 0$, the functional measure for A and a in Eq. (13) is unity, and shear terms in the action may simply be factored out and dropped. The convenient order in derivatives to consider is $e=\rho_f l_0^{(d/2+1)}\nabla^2\phi$, $\zeta=\rho_f l_0^{(d/2+1)}\nabla^2\varphi$, so the canonical Green's function computed below will be defined as

$$\frac{1}{\rho_f l_0^{d+2}}\left\langle\begin{bmatrix}e\\ \zeta\end{bmatrix}_{x'}[e\quad \zeta]_x\right\rangle_\omega \equiv\mathcal{G}(x',x,\omega). \tag{16}$$

With these simplifications, Eq. (16) has the replica definition

$$\mathcal{G}(x',x,\omega)=\frac{1}{\rho_f l_0^{d+2}}\lim_{n\to 0}\int \mathcal{D}\phi\mathcal{D}\varphi\, e^{-L}\begin{bmatrix}e\\ \zeta\end{bmatrix}^{(1)}_{x'}[e\quad \zeta]^{(1)}_x. \tag{17}$$

For piecewise-uniform parameters, the Lagrangian (12) may be checked to produce the correct boundary conditions for abutting Biot media.[23] The corresponding, correct boundary conditions are automatically generated for pointlike scattering perturbations by dividing the Biot matrices into uniform and fluctuating parts: $[\rho]=[\rho_0]+[\rho']$, $[\lambda]=[\lambda_0]+[\lambda']$, $[\mu]=[\mu_0]+[\mu']$.

It was shown in Ref. 7 that the fixed ray of the long-distance RG flow contains only one relevant fluctuation of the form

$$[\rho']\to\rho^*\sigma^*, \tag{18}$$

with σ^* a fixed, degenerate matrix. [An explicit evaluation of σ^* will be given in Eq. (46), when the necessary notation has been defined.] This form will therefore be assumed from the start, and only the parameter renormalized in relating the bare to the long-wavelength effective theory.

Arbitrary Green's functions are then averaged over the ensemble of instances, by simply averaging the replica functional integrals. A weight function is introduced for fluctuations,

$$\mathcal{Z} \equiv \int \mathcal{D}\rho^* e^{-L^{\text{weight}}}, \tag{19}$$

and the average defined by

$$\overline{\mathcal{G}(x',x,\omega)} \equiv \frac{1}{\mathcal{Z}} \int \mathcal{D}\rho^* e^{-L^{\text{weight}}} \mathcal{G}(x',x,\omega). \tag{20}$$

A choice

$$L^{\text{weight}} \equiv \frac{1}{2G} \int d^d x \rho^{*2} \tag{21}$$

gives Gaussian-random, spatially δ-function correlated density fluctuations:

$$\overline{\rho^*_{x'} \rho^*_x} = G\,\delta^d(x'-x). \tag{22}$$

In the bare theory, the δ-function is regulated as a Kronecker delta on patches of volume l_0^d, which defines the fluctuation magnitude $G|_{l_0} \equiv G_0$. In passing from the bare to the effective theory later, G will be evaluated at the renormalized value, and the δ-function softened to include only wave numbers less than the running cutoff.

Completing the square in Eq. (20), and cancelling the factor $\mathcal{Z}$, leaves an averaged Green's function defined entirely in terms of replica fields

$$\overline{\mathcal{G}(x',x,\omega)} = \frac{1}{\rho_f l_0^{d+2}} \lim_{n\to 0} \int \mathcal{D}\phi \mathcal{D}\varphi e^{-L^{\text{ave}}} \begin{bmatrix} e \\ \zeta \end{bmatrix}^{(1)}_{x'} [e \quad \zeta]^{(1)}_x, \tag{23}$$

where

$$L^{\text{ave}} = \frac{1}{2} \int d^d x \nabla^i [\phi \quad \varphi] \Bigg((\omega^2[\rho_0] + \nabla^2[K_0])\,\delta^{ij} - \frac{G\omega^4}{4} \sigma^* \nabla^i \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \nabla^j [\phi \quad \varphi] \sigma^* \Bigg) \nabla^j \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \tag{24}$$

now contains a quartic coupling of strength G. Contraction of Biot fields is indicated by standard two-vector notation. Further, replica contraction is between the same field pairs as Biot contraction, so replica indices may be suppressed, though all fields (ϕ,φ,e,ζ) now represent full replica n-vectors. The forms (23) and (24) remain valid under renormalization because σ^* lies on a fixed ray, so the renormalized value of G, and restricted wave number domains, will be assumed from now on. (It was shown in Ref. 7 that wave function renormalization is irrelevant, and additions to $[\rho_0]$ small until G approaches diverging, so these corrections will be ignored below.)

D. Averaging squared Green's functions

One-particle Green's functions will always be implicitly assumed causal, to allow explicit representation of Biot fields, while minimizing notation. This is sufficient to compute the averaged density of states, but incoherent scattering computations require averaging squared Green's functions, so a supercondensed notation is unavoidable. Causal frequencies will be denoted $\omega + i\eta$, and the fields indexed by them $\phi(\omega \pm i\eta) \equiv \phi^\pm$, $\varphi(\omega \pm i\eta) \equiv \varphi^\pm$. Single-instance Lagrangians such as Eq. (12), computed with $\pm$ fields, will be denoted $L^\pm$. Similarly, Green's functions will be denoted $G^\pm$, and the imaginary regulator on the frequency not written. The Biot column vector of displacement potentials will then be written

$$\begin{bmatrix} \phi \\ \varphi \end{bmatrix}^\pm \equiv \Phi^\pm, \tag{25}$$

and its transpose

$$[\phi \quad \varphi]^\pm \equiv \Phi^{\pm T}. \tag{26}$$

The time dependence of resonant scattering effects will be of interest, so Green's functions and their conjugates must be evaluated at different real frequencies $\omega_\pm$. The mean and difference frequencies will be denoted $\overline{\omega} \equiv (\omega_+ + \omega_-)/2$, $\delta\omega \equiv (\omega_+ - \omega_-)$, so that $\omega_\pm \pm i\eta = \overline{\omega} \pm (\delta\omega + 2i\eta)/2$.

In this notation, Eq. (17) is written

$$\mathcal{G}^\pm(x',x,\omega_\pm) = \lim_{n_\pm \to 0} \int \mathcal{D}\Phi^\pm e^{-L^\pm} \nabla^2_{x'} \Phi^{(1\pm)}_{x'} \nabla^2_x \Phi^{(1\pm)T}_x, \tag{27}$$

and it follows that $\mathcal{G}^-(x',x,\omega) = (\mathcal{G}^+(x',x,\omega))^*$. Carrying out the same averaging steps as for the coherent Green's function yields the two-particle ensemble average

$$\overline{\mathcal{G}^+(x',x,\omega_+)\mathcal{G}^-(x',x,\omega_-)} = \lim_{n_+,n_-\to 0} \int \mathcal{D}\Phi^+ \mathcal{D}\Phi^- e^{-(L_0^+ + L_0^- + L^{\text{int}})} \times \nabla^2_{x'} \Phi^{(1+)}_{x'} \nabla^2_x \Phi^{(1+)T}_x \nabla^2_{x'} \Phi^{(1-)}_{x'} \nabla^2_x \Phi^{(1-)T}_x. \tag{28}$$

The homogeneous Lagrangians can be combined into a block-diagonal matrix form mirroring that for the acoustic case[10]

$$L_0^+ + L_0^- = -\frac{1}{2}\int d^d x [\Phi^{+T} \quad \Phi^{-T}]\left(\overline{\omega}^2\begin{bmatrix}[\rho_0] & \\ & [\rho_0]\end{bmatrix} + \nabla^2\begin{bmatrix}[K_0] & \\ & [K_0]\end{bmatrix} + \overline{\omega}(\delta\omega + 2i\eta) \times \begin{bmatrix}[\rho_0] & \\ & -[\rho_0]\end{bmatrix}\right)\nabla^2\begin{bmatrix}\Phi^+ \\ \Phi^-\end{bmatrix}, \tag{29}$$

and the interaction term includes products of + and − replicas:

$$L^{\rm int} = -\frac{G\overline{\omega}^4}{8}\int d^d x \nabla^i [\Phi^{+T} \quad \Phi^{-T}] \times \begin{bmatrix}\sigma^* & \\ & \sigma^*\end{bmatrix}\nabla^i\begin{bmatrix}\Phi^+ \\ \Phi^-\end{bmatrix}\nabla^j[\Phi^{+T} \quad \Phi^{-T}] \times \begin{bmatrix}\sigma^* & \\ & \sigma^*\end{bmatrix}\nabla^j\begin{bmatrix}\Phi^+ \\ \Phi^-\end{bmatrix}. \tag{30}$$

Replica contraction, as before, follows Biot contraction, and now also the +/− two-vector contraction of $\Phi^\pm$ The forms (28)–(30) are direct extensions of those used in acoustics,[10] with scalar potentials replaced by two-component Biot vectors. The term ''polarization'' will apply to these two-vectors (*not* spatial directionality), to describe the relative amplitude and phasing of a collocated deformation of the effective solid and fluid components.

III. STATISTICALLY NONUNIFORM HETEROGENEITY AND INSTANTONS

A self-consistency condition relates the nearly-free Green's function to the ensemble-averaged scattering attenuation in the coherent potential approximation (CPA) for uniform elastic systems,[2] and a similar derivation has been given in quite different form for Biot media.[26] However, the CPA cannot identify the semiclassical configurations that are needed[11] to regulate perturbation theory in the regime of localized states. Therefore it was of mathematical, as well as physical, interest to study statistically nonuniform scattering in acoustic problems.[10] The magnitude and scaling of the density of localized states could then be directly carried over to the uniform limit and CPA treatment.

Because of the essential use of the effective scattering vertex in the current derivation, the applicability of the weak-coupling instanton expansion will be more limited, to strictly nonvanishing nonuniformity. However, as the scaling of the density of states exactly follows that in the acoustic case, and their contribution to the CPA background is determined by a projection matrix that does not depend on the instanton expansion, the continuation of these results to the uniform-limit CPA is expected to hold for Biot theory as well.

The formal structure of coherent and incoherent (two-particle) Green's functions will first be presented, and then explicit forms and existence arguments for semiclassical stationary points given, using the decomposition of the Biot matrices from Ref. 7. The coherent Green's function will be treated first in explicit Biot notation, to give some physical sense of the structure of the instantons. The two-particle Green's function will then be derived in condensed notation, to make contact with the similar replica divergence structure of the acoustic problem.

A. Structure of semiclassical Green's functions

Green's functions must first be transformed to the dual (wave number) representation:

$$\overline{\mathcal{G}(x',x,\omega)} \equiv \int \frac{d^d k'}{(2\pi)^d}\int \frac{d^d k}{(2\pi)^d} e^{-ik'\cdot x' + ik\cdot x}\overline{\mathcal{G}(-k',k,\omega)}, \tag{31}$$

and likewise for products like Eq. (28). The wave number representation of the causal-frequency action (24) becomes

$$L^{\rm ave} = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}[\phi \quad \varphi]_{-k}(\omega^2[\rho_0] - k^2[K_0])k^2\begin{bmatrix}\phi \\ \varphi\end{bmatrix}_k - \frac{G\omega^4}{8}\int \frac{d^d k_1 d^d k_2 d^d k_3}{(2\pi)^{3d}}(k_1\cdot k_2) \times (k_3\cdot k_4)[\phi \quad \varphi]_{-k_1}\sigma^*\begin{bmatrix}\phi \\ \varphi\end{bmatrix}_{k_2}[\phi \quad \varphi]_{-k_3}\sigma^*\begin{bmatrix}\phi \\ \varphi\end{bmatrix}_{k_4}, \tag{32}$$

in which $k_4 = k_1 - k_2 + k_3$.

As in the acoustic problem, the only channel for cooperative semiclassical effects is diagonal in wave number, so it is convenient to transform to a symmetric basis $-k_1 \equiv k - k'$, $k_2 \equiv k + k'$, $-k_3 \equiv -k - k''$, and $k_4 \equiv -k + k''$, with the Jacobean of the measure simply carried along via notation

$$d^d k_1 d^d k_2 d^d k_3 \equiv \Delta d^d k d^d k' d^d k''. \tag{33}$$

The resulting directional inner products (density fluctuations are dipole scatterers) become $(k_1\cdot k_2) = (k'^2 - k^2)$, $(k_3\cdot k_4) = (k''^2 - k^2)$. The self-consistent ansatz of Ref. 10 was that the product of these contains the same weight as two positive delta-function terms, and a negative-semidefinite remainder:

$$(k^2 - k'^2)(k^2 - k''^2) = v_1(k^2)^2(2\pi)^{2d}\delta^d(k')\delta^d(k'') + v_2 k'^2 k''^2 (2\pi)^d \delta^d(k) - \mathcal{V}_{\rm rem}. \tag{34}$$

Because of the wrong sign of the interaction term (see Ref. 11 for this usage), the positive operator $\mathcal{V}_{\rm rem}$ simply contributes perturbative corrections, which are small in any weak-coupling, low-cutoff effective theory. The term v_2 in Eq. (34) renormalizes the dispersion at perturbative order if there are instantons, but because it only appears integrated, cannot create them. Only the v_1 term can lead to classical cooperative effects, so the others will be dropped or implicitly absorbed into weakly renormalized coefficients.

The rescaled coupling of the wave-number-diagonal interaction is

$$\gamma \equiv \Delta v_1 G \overline{\omega}^4. \tag{35}$$

Rather than make γ position dependent, nonuniformity will be introduced as in Ref. 10, by giving a parabolic profile to the density parameters,

$$[\rho_0] \rightarrow [\rho_0]\left(1-\frac{x^2}{D^2}\right) = [\rho_0]\left(1+\frac{\nabla_k^2}{D^2}\right). \tag{36}$$

Though not intended to represent particular physical parameters, this varies the wavelength, and thus regulates the strength of the Rayleigh cross section spatially. Because it contains the lowest order in derivatives in the dual basis, it should represent any effects not explicitly dependent on refraction in the simpler acoustic case, up to local rescaling of fields. In the Biot case it is less general, because the three components of ρ_0 can certainly vary independently. However, as a representation of nonuniformity of a single scattering coefficient γ, the form (36) is appropriate, and will therefore be kept. The resulting effective action, with appropriately symmetrized factors of k^i, is

$$L^{\mathrm{nonunif}} = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}(k^2)[\phi \quad \varphi]_{-k} k^i \left[\frac{\omega^2}{k^2}[\rho_0]\left(1+\frac{\nabla_k^2}{D^2}\right) - [K_0] - \frac{\gamma}{4}\sigma^* \begin{bmatrix}\phi \\ \varphi\end{bmatrix}_{-k} [\phi \quad \varphi]_k \sigma^*\right] k_i \begin{bmatrix}\phi \\ \varphi\end{bmatrix}_k. \tag{37}$$

The fields actually independent in the semiclassical expansion are not eigenvectors of k, but their symmetrized components with respect to the harmonic oscillator potential formed by the dispersion relation and the spatial regulator:

$$\begin{bmatrix}\phi \\ \varphi\end{bmatrix}_R (k) \equiv \frac{1}{2}\left(\begin{bmatrix}\phi \\ \varphi\end{bmatrix}_k + \begin{bmatrix}\phi \\ \varphi\end{bmatrix}_{-k}\right), \tag{38}$$

$$\begin{bmatrix}\phi \\ \varphi\end{bmatrix}_I (k) \equiv \frac{-i}{2}\left(\begin{bmatrix}\phi \\ \varphi\end{bmatrix}_k - \begin{bmatrix}\phi \\ \varphi\end{bmatrix}_{-k}\right). \tag{39}$$

(The explicit wave number argument will be suppressed when not needed.)

The R and I components at each k are then split into classical backgrounds and fluctuations,

$$\begin{bmatrix}\phi \\ \varphi\end{bmatrix}_{R,I} = \begin{bmatrix}\phi \\ \varphi\end{bmatrix}^{\mathrm{cl}}_{R,I} + \begin{bmatrix}\phi \\ \varphi\end{bmatrix}'_{R,I}, \tag{40}$$

so that the action has the expansion

$$L^{\mathrm{nonunif}} = L^{\mathrm{cl}} + \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\left\{[\phi \quad \varphi]'_R M_R \begin{bmatrix}\phi \\ \varphi\end{bmatrix}'_R + [\phi \quad \varphi]'_I M_I \begin{bmatrix}\phi \\ \varphi\end{bmatrix}'_I\right\}. \tag{41}$$

$L^{\mathrm{cl}} \equiv L^{\mathrm{nonunif}}[\phi^{\mathrm{cl}}, \varphi^{\mathrm{cl}}]$, and the notation for the second variation is introduced $L^{\mathrm{nonunif}} \equiv L^{\mathrm{cl}} + L''$.

Choosing classical solutions of the equations of motion ensures vanishing of terms linear in fluctuations:

$$\left\{\frac{\omega^2}{k^2}[\rho_0]\left(1+\frac{\nabla_k^2}{D^2}\right) - [K_0] - \frac{\gamma}{2}\left[\sigma^*\left([\phi \quad \varphi]^{\mathrm{cl}}_R \sigma^* \begin{bmatrix}\phi \\ \varphi\end{bmatrix}^{\mathrm{cl}}_R - [\phi \quad \varphi]^{\mathrm{cl}}_I \sigma^* \begin{bmatrix}\phi \\ \varphi\end{bmatrix}^{\mathrm{cl}}_I\right) + 2\sigma^* \begin{bmatrix}\phi \\ \varphi\end{bmatrix}^{\mathrm{cl}}_I [\phi \quad \varphi]^{\mathrm{cl}}_I \sigma^*\right]\right\} \begin{bmatrix}\phi \\ \varphi\end{bmatrix}^{\mathrm{cl}}_R = 0, \tag{42}$$

or likewise for $R \leftrightarrow I$. The solutions are further restricted to have the appropriate k-sign symmetry[10] implied by the splits (38) and (39).

With these conditions, the kernel for real fluctuations in L'' becomes

$$M_R = (k^2)^2 \left\{\frac{\omega^2}{k^2}[\rho_0]\left(1+\frac{\nabla_k^2}{D^2}\right) - [K_0] - \frac{\gamma}{2}\left[2\sigma^*\left(\begin{bmatrix}\phi \\ \varphi\end{bmatrix}^{\mathrm{cl}}_R [\phi \quad \varphi]^{\mathrm{cl}}_R + \begin{bmatrix}\phi \\ \varphi\end{bmatrix}^{\mathrm{cl}}_I [\phi \quad \varphi]^{\mathrm{cl}}_I\right)\sigma^* + \sigma^*\left([\phi \quad \varphi]^{\mathrm{cl}}_R \sigma^* \begin{bmatrix}\phi \\ \varphi\end{bmatrix}^{\mathrm{cl}}_R - [\phi \quad \varphi]^{\mathrm{cl}}_I \sigma^* \begin{bmatrix}\phi \\ \varphi\end{bmatrix}^{\mathrm{cl}}_I\right)\right]\right\} \tag{43}$$

with a symmetric form for $R \leftrightarrow I$. A single stationary point never has both real and imaginary backgrounds nonvanishing, because the apparent $\mathcal{O}(2)$ symmetry that would allow this arises only as an artifact of the harmonic-oscillator form of the regulator, and is broken correspondingly by nondegeneracy of the oscillator states.[10] As a result, the kernel for cross terms in R and I, though formally defined, always vanishes. Similarly, R fluctuations in the I background (or vice versa) do not have identically zero eigenvalue, and do not fall within the class of zero modes considered below.

The semiclassical Green's function is now expanded in a sum over stationary points. Unlike the case of uniform electronic systems,[11] where the only density of states comes from instantons, the potential in this problem preserves the free Green's function in the harmonic oscillator well as the term with no instantons. Single-instanton solutions are then parametrized by a center position in the forbidden region of the well. The sum over these stationary points is written formally as

$$\overline{\mathcal{G}(-k',k,\omega)}=\mathcal{G}^{\text{free}}(-k',k,\omega)+\frac{1}{\rho_f l_0^{d+2}}\lim_{n\to 0}\sum_{\tilde{k}^2>k_s^2} e^{-L^{\text{cl}}} \times\int \mathcal{D}\phi'\mathcal{D}\varphi' e^{-L''}\begin{bmatrix} e \\ \zeta \end{bmatrix}^{\text{cl},(1)}_{-k'}[e \quad \zeta]^{\text{cl},(1)}_{k}, \tag{44}$$

representing a dimensionless measure that will be evaluated in the next section.

In Eq. (44), k_s is the maximal slow wave number allowed in the noninteracting well. Because by assumption $k_s > k_f$ (the maximal fast wave number) at any fixed frequency, both slow and fast waves are classically forbidden for $k^2 > k_s^2$. In this range only, both Biot components can form semiclassical solutions of finite action, independent of the value of D. For $k_f^2 < k^2 < k_s^2$, though the fast component might be expected to form bounded instantons, the slow component employs the propagating (extended) Green's function. The amplitude of instantons is set by the coupling, so the magnitude of a slow component coupled to fast waves by the classical equations of motion is fixed independent of well width. As the regulating well is widened (D increased), such solutions have diverging action and make zero contribution. Even at finite, large D (soft regulator or slow variation of coupling strength), possible solutions at $k^2 < k_s^2$ have large action, and so will be ignored.

This approximation will be justified by weak overlap of fast and slow waves in the effective theory, anywhere that slow wave localization can be calculated with a weak-coupling RG. As shown in Ref. 7 and recalled in the next section, the matrix form of the enhanced effective scattering vertex comes from that part of the compressibility matrix that couples principally to the slow wave combination of fluid and solid motions. Thus presumably, in systems with strong enough fast-wave scattering to localize the fast sector, slow wave coupling is already so strong that the Biot effective medium is of questionable validity, and conventional elastic models are as appropriate.

B. Particular solutions

The structure of instantons distinguishes between slow and fast waves, so extracting it requires identifying these components in terms of the background Biot parameters. The perturbative RG was controlled by the modulus matrix $[K_0]$, which was divided in terms of its eigenvalues and eigenvectors in Ref. 7 as

$$[K_0]\equiv K_+(v_+ v_+^{\text{T}})+K_-(v_- v_-^{\text{T}}). \tag{45}$$

($v_\pm$ is a unit-normalized column Biot polarization, and $v_\pm^T$ is its transpose.) With notation chosen so that $K_- < K_+$, the maximal stationary ray for fluctuations (18) lies along the projector

$$\sigma^* = (v_- v_-^{\text{T}}). \tag{46}$$

Thus $[K_0]$ is diagonalized as

$$\begin{bmatrix} v_+^{\text{T}} \\ v_-^{\text{T}} \end{bmatrix}[K_0][v_+ \quad v_-]=\begin{bmatrix} K_+ & \\ & K_- \end{bmatrix}, \tag{47}$$

and

$$\begin{bmatrix} v_+^{\text{T}} \\ v_-^{\text{T}} \end{bmatrix}\sigma^*[v_+ \quad v_-]=\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}. \tag{48}$$

Parameters for $[\rho_0]$ may be specified in the same basis by defining

$$\begin{bmatrix} v_+^{\text{T}} \\ v_-^{\text{T}} \end{bmatrix}[\rho_0][v_+ \quad v_-]\equiv\begin{bmatrix} \rho_+ & \rho_T \\ \rho_T & \rho_- \end{bmatrix}. \tag{49}$$

In this notation, it then follows that ρ^* in Eq. (22) is just the fluctuation $\delta\rho_-$, and that in the unrenormalized theory,

$$\frac{G_0/l_0^d}{\rho_-^2}=\overline{\left(\frac{\delta\rho_-}{\rho_-}\right)^2} \tag{50}$$

evaluated on uncorrelated patches at the natural scale. In Eq. (22) G (the only coupling actually used in calculations here), represents the renormalized effective coupling at the wavelength scale, denoted G^{eff} in Ref. 7.

The free Biot wave equation may be diagonalized by first rescaling the components multiplying $v_\pm$, and then introducing orthogonal matrices to diagonalize the combination

$$R_0\frac{\omega^2}{k^2}\begin{bmatrix} K_+^{-1/2} & \\ & K_-^{-1/2} \end{bmatrix}\begin{bmatrix} \rho_+ & \rho_T \\ \rho_T & \rho_- \end{bmatrix}\begin{bmatrix} K_+^{-1/2} & \\ & K_-^{-1/2} \end{bmatrix}R_0^T \equiv\begin{bmatrix} (\lambda_+ +1)^{-1} & \\ & (\lambda_- +1)^{-1} \end{bmatrix}, \tag{51}$$

where

$$R_0\equiv\begin{bmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{bmatrix}. \tag{52}$$

The degree of overlap between slow and fast wave polarizations is determined by the off-diagonal component ρ_T of $[\rho_0]$ in Eq. (49), and encapsulated in the dimensionless angle ξ, given by

$$\tan(2\xi)=\frac{2\rho_T\sqrt{K_+K_-}}{\rho_+K_- - \rho_-K_+}. \tag{53}$$

ξ will be assumed small in what follows, and checked quantitatively in Appendix A. In this notation, as $\rho_T \to 0$, the compressional sector of Biot theory degenerates to two decoupled acoustic waves, with speeds $K_+/\rho_+ \to c_f^2$ and $K_-/\rho_- \to c_s^2$. In the more general case, the eigenvalues produced under diagonalization (51) define $(\lambda_+ +1)\equiv k^2/k_f^2 = k^2c_f^2/\omega^2$ and $(\lambda_- +1)\equiv k^2/k_s^2 = k^2c_s^2/\omega^2$.

The harmonic oscillator potential created by finite D has its slowest speed at the spatial center, leading to branch points where k^2 attains the maximal values for either slow or fast waves. These correspond to the limits of the wave number potential, where instantons first appear. Specifically, the maximal allowed fast and slow wave numbers are defined by the vanishing conditions $\lambda_+|_{k_f}\equiv 0$ and $\lambda_-|_{k_s}\equiv 0$.

As in previous treatments,[11,10] the magnitude of instantons centered at each value of k^2 is determined by a dimensionless coupling strength, here given by

$$g_0\equiv\frac{\gamma}{2}\frac{D^4\cos^4\xi}{K_-^2k_s^4}(D\sqrt{\lambda_-})^{d-4}. \tag{54}$$

The magnitude of g_0 can be estimated from the definitions (35) and (22). In the bare theory with the Kronecker-δ regularization of fluctuations,[7] the bare coupling has approximate magnitude $G_{l_0}\sim l_0^d\langle(\delta\rho_-)^2\rangle_{l_0}$, where subscript l_0 indicates the mean square fluctuation on patches of correlation length l_0, assumed discrete and mutually independent. Using the fact that the approximate measure for the positive-definite central region of the vertex operator in Eq. (34) should scale as $v_1\sim k_s^{2d}$, as $\xi\rightarrow 0$, $2g_0/\Delta\rightarrow(v_1GD^d/\rho_-^2)(\sqrt{\lambda_-})^{d-4}(1+\mathcal{O}(\xi^2))$.

To conveniently express the Green's function in terms of nondimensionalized functional forms for the instantons, it is necessary to introduce one more transformation matrix, defined in terms of R_0:

$$\mathcal{R}_k\equiv\sqrt{k^2}\begin{bmatrix}1/k_f & \\ & 1/k_s\end{bmatrix}R_0\begin{bmatrix}K_+^{-1/2} & \\ & K_-^{-1/2}\end{bmatrix}\begin{bmatrix}v_+^{\mathrm{T}}\\ v_-^{\mathrm{T}}\end{bmatrix}. \tag{55}$$

Because the solutions of interest will all lie in a close neighborhood $k^2\gtrsim k_s^2$, it will often be acceptable to set $k^2\rightarrow k_s^2$, so that $\mathcal{R}_k$ becomes a constant.

The dilations e, ζ are then used to define two nondimensionalized, and as it will turn out, scale-free fields f_+, f_-:

$$\frac{1}{\sqrt{\rho_f}l_0^{(d/2+1)}}\begin{bmatrix}e\\ \zeta\end{bmatrix}\equiv i\frac{(D\sqrt{\lambda_-})^{d/2}}{\sqrt{\lambda_-}\sqrt{g_0}}\mathcal{R}_k^T\begin{bmatrix}f_+\\ f_-\end{bmatrix}. \tag{56}$$

Rescaling the coordinate differential $(D\sqrt{\lambda_-})dk\equiv dv$, and the corresponding gradient $(D\sqrt{\lambda_-})^{-1}\nabla_k\equiv\nabla_v$, it becomes possible to write the classical equations of motion (42) approximately, in the region where $\nabla_v(\ln\sqrt{\lambda_-})\ll 1$, as

$$\left(-\nabla_v{}^2+\begin{bmatrix}\lambda_+/\lambda_- & \\ & 1\end{bmatrix}\right)\begin{bmatrix}f_+\\ f_-\end{bmatrix}=\begin{bmatrix}\alpha\\ 1\end{bmatrix}(\alpha f_++f_-)^3. \tag{57}$$

The constant

$$\alpha\equiv\sqrt{\frac{\lambda_++1}{\lambda_-+1}}\tan\xi=\frac{c_f}{c_s}\tan\xi \tag{58}$$

is the small parameter that determines the coupling strength of slow and fast components of the instanton.

By inspection of Eq. (57), f_- is even, and f_+ odd, in α. Thus for small α and $\lambda_+>\lambda_->0$, through $\mathcal{O}(\alpha^1)$, f_- satisfies the conventional equation for instantons in a homogeneous medium:

$$\nabla_v{}^2f_-\approx f_--f_-^3. \tag{59}$$

Through $\mathcal{O}(\alpha^2)$, f_+ may be expressed in terms of the homogeneous f_-,

$$\left(-\nabla_v{}^2+\frac{\lambda_+}{\lambda_-}\right)f_+\approx\alpha f_-^3, \tag{60}$$

which may be solved in terms of the exponentially bounded + Green's function. Within the approximation of v-translation invariance of Eq. (57) (i.e., ignoring the slope of the harmonic oscillator potential), it is straightforward to show in $d=1$ that the continuation of this small-parameter expansion converges to bounded instantons for all values of α, proving existence of backgrounds of finite action. A similar result should be valid for spherically-symmetric solutions in higher d.[27]

In uniform media, the solutions of Eqs. (59) and (60) are formed by evaluating a single dimensionless + or − function offset from a continuous range of center points. Letting context indicate whether this function is being evaluated at its dimensional (k) or scale-free (v) argument, the classical solution will be variously denoted

$$f_\pm^{\mathrm{cl}}(k)\equiv f_{(\pm)}^{\mathrm{cl}}(D\sqrt{\lambda_-}(k-\tilde{k}))=f_{(\pm)}^{\mathrm{cl}}(v-\tilde{v})\equiv f_\pm^{\mathrm{cl}}(v). \tag{61}$$

$f_{(\pm)}^{\mathrm{cl}}$ is the canonical form, and the center point will always be implicitly $\tilde{k}$ or $\tilde{v}$.

The harmonic oscillator potential modifies the exact f_- from the approximate uniform solution to Eq. (59) exactly as in the acoustic case.[10] Coexistence of propagating solutions in the allowed region $k^2<k_s^2$ and instantons around $\tilde{k}^2>k_s^2$, makes the well of the instanton a leaky potential. Solutions to the wave equation in this well, including that of the instanton itself, continue through a tunneling barrier to allowed states of the free oscillator, and have eigenvalues with finite imaginary parts dependent on g_0. Such resonances do not actually appear in the spectrum of (appropriately symmetrized) eigenvectors at finite D. Rather, at large D, the dense sets of oscillator states approximate a branch cut looking through to unphysical sheets, and the unstable states appear as poles on these unphysical sheets, collectively represented in the residues of the actual spectrum.

For quite common parameters,[22,10] $\lambda_+/\lambda_-\gg 1$ over the whole range of k^2 contained in the effective field theory, and the Green's function solution to Eq. (60) becomes approximately $f_{(+)}^{\mathrm{cl}}\rightarrow(\lambda_-/\lambda_+)\alpha(f_{(-)}^{\mathrm{cl}})^3$. This result will be important below, because f_- is regular at small $(v-\tilde{v})$, and decays exponentially in large $(v-\tilde{v})$, toward the turning point to allowed propagation. Within the range of validity of the weak-coupling expansion, this exponential is already smaller than perturbative terms that have been dropped, but creates the leading nonzero terms in the localized density of states and resonant tunneling amplitude. However, direct coupling of f_+ through the turning points will be the cube of this small exponential, and much smaller than allowed fast waves perturbatively coupled to the classical slow-wave background. Finally, motivating the choice of dimensionless coupling g_0, it can be shown that the classical action for a single instanton reduces to $L^{\mathrm{cl}}=(a_d/g_0)(1+\mathcal{O}(\alpha))$, where a_d is a constant depending only on d.[11,10]

The formal sum over semiclassical backgrounds in Eq. (44) may now be defined. It is proportional to the integral $\int d^d\tilde{k}$ over the center positions of the instantons. The Jacobean from the measure $\mathcal{D}\phi\mathcal{D}\varphi$ is proportional to $d/2$ powers of the integrated square of the gradient of the classical solution, a zero mode of Eq. (43) (or its counterpart when $R\to I$). This translational Jacobean contributes a factor of $(1/g_0)^{d/2}$. $n-1$ rotational replica zero modes contribute a power of $g_0^{1/2}$ when $n\to 0$. The rotational zero modes, and a single negative eigenvalue proportional to $-\lambda_-$, contribute cancelling powers of $\sqrt{\lambda_-}$ and a single factor of i. All other scale factors are from the normalization (56). Combining these factors, the coherent Green's function becomes

$$\overline{\mathcal{G}(-k',k,\omega)}=\mathcal{G}^{\mathrm{free}}(-k',k,\omega)-iC_1\int_{\tilde{k}^2>k_s^2}\frac{d^d\tilde{k}}{(2\pi)^d}\frac{(D\sqrt{\lambda_-})^d}{\lambda_-}\left(\frac{1}{g_0}\right)^{(d+1)/2}e^{-a_d/g_0}\times\left(\mathcal{R}^T_{-k'}\begin{bmatrix} f_+ \\ f_- \end{bmatrix}^{\mathrm{cl}}_R(k')[f_+ \quad f_-]^{\mathrm{cl}}_R(k)\mathcal{R}_k\right)+R\to I. \tag{62}$$

The matrix-valued result (62) directly extends its acoustic counterpart.[10] The constant C_1 is proportional to the volume of the physical system $\int d^dx$, and is otherwise only formally defined at $n\to 0$. It can be specified physically in terms of the CPA attenuation,[10] as will be done below.

C. Resonant scattering

The ensemble average of the squared Green's function may now be computed without introducing additional machinery. Incorporating the parabolic regulator (36) and symmetrizing fields as in Eqs. (38) and (39), the free action (29) becomes

$$(L_0^++L_0^-)^{\mathrm{nonunif}}=\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}(k^2)^2[\Phi^{+T}\quad\Phi^{-T}]_R\left\{\frac{\overline{\omega}^2}{k^2}\begin{bmatrix}[\rho_0] & \\ & [\rho_0]\end{bmatrix}\left(1+\frac{\nabla_k^2}{D^2}\right)-\begin{bmatrix}[K_0] & \\ & [K_0]\end{bmatrix}+\frac{\overline{\omega}(\delta\omega+2i\eta)}{k^2}\begin{bmatrix}[\rho_0] & \\ & -[\rho_0]\end{bmatrix}\right\}\begin{bmatrix}\Phi^+ \\ \Phi^-\end{bmatrix}_R+R\to I. \tag{63}$$

Its important difference from the one-particle Lagrangian (37) is the term $\propto(\delta\omega+2i\eta)$. This term prevents the $O(n_+)$ symmetry of Φ^+ (a bounded group) from continuing to an $O(n_+,n_-)$ symmetry for (Φ^+,Φ^-) fluctuations. The unbounded generators of the latter are regulated by the opposite-sign blocks in $[\rho_0]$, and their finite value as $\delta\omega$ passes through zero is responsible for the pole term that creates resonancelike incoherent scattering. The corresponding interaction term couples Φ^+ and Φ^-, as well as $\phi_\pm$ and $\varphi_\pm$:

$$L^{\mathrm{int}}=-\frac{\gamma}{8}\int\frac{d^dk}{(2\pi)^d}(k^2)^2\left\{\left([\Phi^{+T}\quad\Phi^{-T}]_R\begin{bmatrix}\sigma^* & \\ & \sigma^*\end{bmatrix}\begin{bmatrix}\Phi^+ \\ \Phi^-\end{bmatrix}_R-[\Phi^{+T}\quad\Phi^{-T}]_I\begin{bmatrix}\sigma^* & \\ & \sigma^*\end{bmatrix}\begin{bmatrix}\Phi^+ \\ \Phi^-\end{bmatrix}_I\right)^2+4\left([\Phi^{+T}\quad\Phi^{-T}]_R\begin{bmatrix}\sigma^* & \\ & \sigma^*\end{bmatrix}\begin{bmatrix}\Phi^+ \\ \Phi^-\end{bmatrix}_I\right)^2\right\}. \tag{64}$$

Corresponding to the notation (25), it is convenient to define

$$\begin{bmatrix} f_+ \\ f_- \end{bmatrix}_{R,I}(k)\equiv\mathcal{F}_{R,I}(k). \tag{65}$$

For the two-particle function, it becomes necessary to distinguish the causal and anticausal solutions to the classical equations of motion. These have frequency eigenvalues differing by an imaginary part determined by the tunneling barrier, denoted $\mathrm{Im}(\omega_C)$, which remains finite at $\delta\omega+2i\eta\to 0$. The causal classical solution will be denoted

$$\mathcal{F}^{\mathrm{cl}}_{R,I}(k)\to\mathcal{F}^{+\,\mathrm{cl}}_{R,I}(k), \tag{66}$$

and the anticausal solution obtained by complex conjugation: $\mathcal{F}_k^{-\,\mathrm{cl}}\equiv(\mathcal{F}_k^{+\,\mathrm{cl}}*)$.

$\mathcal{F}^{+\mathrm{cl}}$ will always be chosen for stationary points in the causal Green's function. Reduction of the determinant is then identical to that for the one-particle function (62). However, the lowest-eigenvalue fluctuation for Φ^- now has two solutions, corresponding to $\mathcal{F}^{+\text{cl}}$ and $\mathcal{F}^{-\text{cl}}$, separated by exponentially small imaginary eigenvalues. Normalizing these fluctuations in terms of the classical solution, and solving for a two-particle normalization constant

$$C_2 \equiv C_1 \Bigg/ \left(2 \int \frac{d^d \tilde{v}}{(2\pi)^d} \mathcal{F}_R^{\text{cl}T} \mathcal{F}_R^{\text{cl}} \right), \tag{67}$$

the squared Green's function (28) of wave number arguments evaluates as in the scalar acoustic problem:[10]

$$\begin{aligned}\overline{\mathcal{G}^+(-k',k,\omega_+)\mathcal{G}^-(-k',k,\omega_-)} = {} & \mathcal{G}^{\text{free}}(-k',k,\omega_+ + i\eta)\mathcal{G}^{\text{free}}(-k',k,\omega_- - i\eta) \\ & + iC_2\bar{\omega}\int_{\tilde{k}^2>k_s^2} \frac{d^d\tilde{k}}{(2\pi)^d} \frac{(D\sqrt{\lambda_-})^{2d}}{\lambda_-}\left(\frac{1}{g_0}\right)^{(d+1)/2} e^{-a_d/g_0}(\mathcal{R}_{-k'}^T,\mathcal{F}_R^{+\text{cl}}(k')\mathcal{F}_R^{+\text{cl}T}(k)\mathcal{R}_k) \\ & \times\left(\frac{\mathcal{R}_{-k'}^T,\mathcal{F}_R^{+\text{cl}}(k')\mathcal{F}_R^{+\text{cl}T}(k)\mathcal{R}_k}{\delta\omega+2i\eta} + \frac{\mathcal{R}_{-k'}^T,\mathcal{F}_R^{-\text{cl}}(k')\mathcal{F}_R^{-\text{cl}T}(k)\mathcal{R}_k}{\delta\omega-2\,\text{Im}(\omega_C)+2i\eta}\right) + R\to I.\end{aligned} \tag{68}$$

Both fluctuations are not relevant to evaluating the causal incoherent tail of impulse responses. That tail, which will appear again in calculations of the incoherent noise spectrum, is obtained by integrating over the frequency difference, where a contour integral selects only causal components:

$$\begin{aligned}\int \frac{d(\omega_+-\omega_-)}{2\pi} & e^{-i(\omega_+-\omega_-)t}\overline{\mathcal{G}^+(-k',k,\omega_+)\mathcal{G}^-(-k',k,\omega_-)} \\ & = C_2\bar{\omega}\int_{\tilde{k}^2>k_s^2} \frac{d^d\tilde{k}}{(2\pi)^d} \frac{(D\sqrt{\lambda_-})^{2d}}{\lambda_-}\left(\frac{1}{g_0}\right)^{(d+1)/2} e^{-a_d/g_0}(\mathcal{R}_{-k'}^T,\mathcal{F}_R^{+\text{cl}}(k')\mathcal{F}_R^{+\text{cl}T}(k)\mathcal{R}_k) \\ & \quad \times(\mathcal{R}_{-k'}^T,\mathcal{F}_R^{-\text{cl}}(k')\mathcal{F}_R^{-\text{cl}T}(k)\mathcal{R}_k)e^{2\,\text{Im}\,\omega_C t} + R\to I.\end{aligned} \tag{69}$$

Both $\text{Im}(\omega_C)$ and the magnitude of $\mathcal{F}_R^{\pm\text{cl}}$ are determined by the tunneling barrier between the well of the instanton and the allowed region $k^2<k_s^2$. At large $\bar{\omega}$, the correction from dimension-dependent lensing effects on the one-dimensional WKB approximation goes to a constant,[10] so that to $\mathcal{O}(\alpha^0)$,

$$-\frac{\text{Im}(\omega_C)}{\bar{\omega}} \propto \lambda_-\sqrt{S^{\text{WKB}}}e^{-S^{\text{WKB}}}, \tag{70}$$

where

$$S^{\text{WKB}} \propto (\lambda_-)^{3/2}\left(\frac{D\bar{\omega}}{c_s}\right). \tag{71}$$

In a stationary-point estimate of the maximum amplitude, the factors of λ_- in the exponential of $-S^{\text{WKB}}$ balance positive polynomials at $S^{\text{WKB}}\sim$const. Hence the scaling of the decay constant in Eq. (69), at the stationary point of maximum amplitude, satisfies

$$-\frac{\text{Im}(\omega_C)}{\bar{\omega}} \sim \left(\frac{c_s}{D\bar{\omega}}\right)^{2/3}, \tag{72}$$

as in the acoustic case.[10]

Scattering couples general k and k', because the solutions $\mathcal{F}^{\pm}$ describe tunneling from a nearly spherically symmetric instanton well through the hyper-planar wall of the oscillator potential to the allowed region. In general, the angular dependence of this tunneling solution is complicated, but could be found numerically.

Looking through the factors of $\mathcal{R}_k$ to the fast and slow eigenvectors $f_{\pm}$, the content of Eq. (69) is that slow waves, excited in the classically allowed region, couple to instanton fluctuations in the forbidden region with exponentially suppressed amplitude. These excitations then decay incoherently, with time constant $2\pi/\omega_C$, integrated over instantons at all $\tilde{k}$. The fluctuations are resonances in the instanton background, but the phenomenon differs from conventional resonance in that nonlinear coupling creates that background. Also unlike resonance from discrete features, the decay of Eq. (69) has a continuous spectrum of time constants.

The range of validity of the solutions (62), (68), and (69) requires comment. Use of the stationary form (18) for the interaction is only implied in an effective theory with a cutoff close above k_s, and a long scaling range between k_s and the grain scale. Thus the integral on $\tilde{k}$ cannot extend above $\lambda_-\approx 1$. On the other hand, while the weak coupling expansion for the instantons is valid for $g_0>1$, $g_0\gg 1$ give exponentially small contribution. Only the range $g_0\gtrsim 1$ contributes significantly. Using Eq. (54), $D\to\infty$ at fixed coupling G is inconsistent with the bound on $\tilde{k}$ at finite g_0. Therefore the explicit form of these results can only be used for a strong-scattering region of finite size, immersed in a larger weak-scattering background.

D. Spatial dependence

For finite *D*, the coherent Green's function (62) can be used to estimate the density of localized states. At $k'^2=k^2>k_s^2$, the only contribution comes from $(k-\tilde{k})^2$ small, so g_0 and λ_- may be written as functions of k in the weak-coupling regime (because the instantons are narrow). The remaining integral over $\tilde{k}$ can be nondimensionalized, to give

$$\overline{\mathcal{G}(-k,k,\omega)}\to-\frac{iC_1}{\lambda_-}\left(\frac{1}{g_0}\right)^{(d+1)/2}e^{-a_d/g_0}\mathcal{R}^T_{-k'}\times\left(\int\frac{d^d\tilde{v}}{(2\pi)^d}\begin{bmatrix}f_+\\f_-\end{bmatrix}^{\mathrm{cl}}_R(v)[f_+\quad f_-]^{\mathrm{cl}}_R(v)\right)\times\mathcal{R}_k+R\to I. \tag{73}$$

It is first interesting to check whether localized states do indeed arise in the center of the *spatial* potential, where scattering is strongest. To answer this, it is convenient to first define a characteristic length from the coupling,

$$x_0\equiv\left[\frac{\gamma}{2}\frac{D^4\cos^4\xi}{K_-^2k_s^4}\right]^{1/(4-d)}=g_0{}^{1/(4-d)}(D\sqrt{\lambda_-}), \tag{74}$$

and then to decompose the symmetrized components for $k^2>k_s^2$ into classical solutions with only one well around $\tilde{v}$,

$$\begin{bmatrix}f_+\\f_-\end{bmatrix}^{\mathrm{cl}}_{\substack{R\\I}}(v)\equiv\begin{bmatrix}f_+\\f_-\end{bmatrix}^1(v-\tilde{v})\pm\begin{bmatrix}f_+\\f_-\end{bmatrix}^1(v+\tilde{v}). \tag{75}$$

The dimensionless Fourier transforms of $f^1_\pm$ are denoted

$$\begin{bmatrix}f_+\\f_-\end{bmatrix}^1(v-\tilde{v})\equiv\int d^dz\,e^{-i(v-\tilde{v})\cdot z}\begin{bmatrix}\tilde{f}_+\\\tilde{f}_-\end{bmatrix}_z. \tag{76}$$

Using the definition (31) of the transform of $\mathcal{G}$, the Green's function may be expressed in terms of the wave number volume of the allowed region, k_s^d, x_0, the K components in $\mathcal{R}_k$, and dimensionless functions, as

$$\overline{\mathcal{G}(x,x,\omega)}\to-\frac{4iC_1k_s^d}{(4-d)}\int\frac{d^{d-1}\Omega}{(2\pi)^d}\int_0^\infty d\left(\frac{1}{g_0}\right)\times\left(\frac{1}{g_0}\right)^{(d-1)/2}e^{-a_d/g_0}\left(\frac{g_0{}^{1/(4-d)}}{x_0}\right)^d\mathcal{R}^T_{-k_s}\times\left(\begin{bmatrix}\tilde{f}_+\\\tilde{f}_-\end{bmatrix}[\tilde{f}_+\quad\tilde{f}|_-]\right)\Bigg|_{xg_0{}^{1/(4-d)}/x_0}\mathcal{R}_{k_s}. \tag{77}$$

Here $\int d^{d-1}\Omega$ is the area of the unit d sphere, from the integral over instanton centers $\tilde{k}$. Oscillatory interference terms arising from symmetrized $f_\pm$ have cancelled under $R\to I$, leaving a smooth, radially symmetric envelope in the harmonic well.

The fact that the v instantons are regular at $\tilde{v}$ (Ref. 27) implies exponential decay of $\tilde{f}_\pm$ at large argument. The integral (77) is a sum over this canonical function, with argument scaled by g_0 and a weight function of g_0. The density of localized states is largest at $x\to0$, as expected physically, and dies off exponentially for $x\gg x_0$. Unfortunately, the integral cannot be evaluated at $x\to0$ in $d>2$, because the dominant weight comes from small g_0. In this region, the weak coupling expansion becomes invalid, and the form of the instantons is not well-defined in the effective theory, because it depends on arbitrarily large values of k which lie above the running cutoff. The form for $x\gtrsim x_0$, however, should be valid.

A curious prediction of Eq. (77) is that the localized density of states has a spatially varying admixture of fast and slow polarizations, as diagonalized in the propagating wave equation. The approximation $f_+(k)\propto(f_-(k))^3$, with f_- regular at $k\to\tilde{k}$, implies an f_+ with three times the curvature at its center. Transforming to x, the long-range $+$ exponential decay must have $1/\sqrt{3}$ times the $-$ spatial decay constant. Thus, while the fast-wave component starts with a smaller coefficient ($\propto\alpha$), it decays more slowly, and the wings of the localized distribution are characterized by a relatively greater fast-wave polarization.

IV. THE UNIFORM LIMIT AND CPA RENORMALIZATION

The semiclassical Ginzburg-Landau expansion for nonuniform systems, and the self-consistent CPA ansatz for uniform systems, provide rather different-looking descriptions of the density of localized states. This is true in the electronic case, and even more so in the acoustic, because of the subtlety of wave-number-diagonal instantons. Yet it has been shown,[11] in a way that applies equally to electronic or acoustic problems, that the perturbation theory in the wrong-sign four-field interaction that gave the relevant RG (Ref. 7) cannot converge in the region of localized states, and semiclassical configurations are *necessary* to stabilize it. In the acoustic problem, the two descriptions are connected by relations between the attenuation length for the coherent Green's function, and the incoherent noise spectrum following impulse excitation,[10] both of which are recovered in the uniform limit.

The same relations all exist for Biot theory. However, unlike the acoustic problem, where the noise spectrum cannot be remotely probed unless the medium has an exposed interface (or equivalent), Biot media with localized slow waves imprint this spectrum perturbatively on fast wave impulses. The relation between coherent attenuation and the resulting spectrum of fast-wave coda is a signature by which to distinguish the presence of extra Biot degrees of freedom from purely dissipative attenuation.

A. The noise spectrum

The noise spectrum in the acoustic problem was originally defined for seismic displacements, with $d=1$ and a purely reflecting interface representing the exposed surface of the Earth.[28] In randomly-layered media, broadband energy delivered to the interface was trapped between perfect reflection at the surface and localization in the interior, leading to a stationary spectrum of incoherent noise at late times. To extend that definition to the Biot problem in general d, it is necessary to express the canonical Green's function $\mathcal{G}$ in terms of a displacement Green's function Γ^{ij} between spatial components i and j, as

$$\mathcal{G}(x',x,\omega) \equiv \nabla^i_{x'} \nabla^j_x \Gamma^{ij}(x',x,\omega). \tag{78}$$

If Biot polarization indices are explicitly represented, as on the matrix

$$\sigma^* \equiv [\sigma^*]_{\mu\nu}, \tag{79}$$

with the convention that repeated indices are summed, the ensemble-averaged coherent and incoherent Green's functions are related by a Schwinger-Dyson equation equivalent to the acoustic form,[10]

$$\begin{aligned}
&\overline{[\sigma^*\Gamma^{ij+}(x',x,\omega_+)\sigma^*]_{\lambda\mu}[\sigma^*\Gamma^{kl-}(x',x,\omega_-)\sigma^*]_{\nu\rho}} \\
&\quad = \overline{[\sigma^*\Gamma^{ij+}(x',x,\omega_+)\sigma^*]}_{\lambda\mu}\overline{[\sigma^*\Gamma^{kl-}(x',x,\omega_-)\sigma^*]}_{\nu\rho} + G\overline{\omega}^4 \int d^d\tilde{x}\overline{[\sigma^*\Gamma^{im+}(x',\tilde{x},\omega_+)\sigma^*]}_{\lambda\alpha} \\
&\quad\quad \times \overline{[\sigma^*\Gamma^{kn-}(x',\tilde{x},\omega_-)\sigma^*]}_{\nu\beta}\overline{[\sigma^*\Gamma^{mj+}(\tilde{x},x,\omega_+)\sigma^*]_{\alpha\mu}[\sigma^*\Gamma^{nl-}(\tilde{x},x,\omega_-)\sigma^*]_{\beta\rho}}.
\end{aligned} \tag{80}$$

Equation (80) admits an interpretation in $d=1$,[10] that the first term on the right-hand side propagates excitations directly from x to x', while the second propagates them to an arbitrary point x_0 near x, which may be viewed as a reflecting interface. Reflection at the interface is represented by the cumulative effect of the two-particle Green's function at $\tilde{x}$ in the integral, and the coupled coherent Green's functions propagate this reflected signal back to x'. This description may be generalized by considering an arbitrary point x_0 a source of internal "reflection," and defining the paired reflection coefficient

$$[\sigma^*]_{\lambda\mu}[\sigma^*]_{\nu\rho}\frac{\delta^{ik}\delta^{jl}}{d^2}\overline{R(\omega_+)R^*(\omega_-)} = G\overline{\omega}^4 \int d^d\tilde{x}\overline{[\sigma^*\Gamma^{ij+}(\tilde{x},x,\omega_+)\sigma^*]_{\lambda\mu}[\sigma^*\Gamma^{kl-}(\tilde{x},x,\omega_-)\sigma^*]_{\nu\rho}}. \tag{81}$$

In Eq. (81), three symmetries have been used. At late times, high-order scattering must be isotropic, so the paired Green's functions are uncorrelated between i and j, or k and l. However, due to replica symmetry, the two-particle function is a product of uncorrelated, coherent Green's functions, and hence vanishes, unless $i=k$ and $j=l$. Finally, because σ^* is a projector, sandwiching Green's functions between σ^* pairs must give a matrix proportional to σ^*. Because localized states at large D come from a neighborhood $k^2 \approx k_s^2$, contraction of Eq. (81) with δ^{ik} and δ^{jl} may be used to reduce

$$\overline{R(\omega_+)R^*(\omega_-)} = Gc_s^4 \int d^d\tilde{x}\,\overline{\mathrm{Tr}(\sigma^*\mathcal{G}^+(\tilde{x},x,\omega_+))\mathrm{Tr}(\sigma^*\mathcal{G}^-(\tilde{x},x,\omega_-))}. \tag{82}$$

The total energy at the "interface" (equivalently, at a probe located at x_0) is the integral of this reflection over ω_+ and ω_-. Therefore, the noise spectrum is defined as

$$N(t,\overline{\omega}) \equiv \int \frac{d(\omega_+ - \omega_-)}{2\pi} e^{-i(\omega_+ - \omega_-)t}\overline{R(\omega_+)R^*(\omega_-)}. \tag{83}$$

Omitting the model of the reflection coefficient, it becomes apparent that the noise spectrum may be defined directly in terms of the two-point function in general d as

$$[\sigma^*]_{\lambda\mu}[\sigma^*]_{\nu\rho}\frac{\delta^{ik}\delta^{jl}}{d^2}N(t,\overline{\omega}) = G\overline{\omega}^4 \int \frac{d(\omega_+ - \omega_-)}{2\pi} e^{-i(\omega_+ - \omega_-)t} \int d^d\tilde{x}\overline{[\sigma^*\Gamma^{ij+}(\tilde{x},x,\omega_+)\sigma^*]_{\lambda\mu}[\sigma^*\Gamma^{kl-}(\tilde{x},x,\omega_-)\sigma^*]_{\nu\rho}}. \tag{84}$$

(Physically, the picture of reflection from a small d-dimensional volume, or point at x_0, corresponds to the derivation of localization from coherent backscatter: Rays sent out from x_0 may be viewed as having been sent "through" this point from its far side. These are primarily reflected back to x_0 by coherent backscatter, and appear on the far side to have been reflected "through" x_0, as if it were a perfectly reflecting interface, but one having definite spectral characteristics.)

Following the results of the previous sections, traces of $\mathcal{G}$ with σ^* are proportional to contractions of $\mathcal{F}^{\mathrm{cl}}_{R,I}$ with the vector $[\alpha 1]$. Because the fast-wave component is itself $\mathcal{O}(\alpha)$, these contractions can be ignored at $\mathcal{O}(\alpha^1)$, and to that order Eq. (83) evaluates to

$$N(t,\overline{\omega})=\frac{2Gc_s^4k_s^d}{K_-^2}\frac{C_2}{\int d^dx}\left(\int\frac{d^d\tilde{v}}{(2\pi)^d}\mathcal{F}_R^{\mathrm{cl}T}\mathcal{F}_R^{\mathrm{cl}}\right)^2\frac{\overline{\omega}}{(4-d)}\int\frac{d^{d-1}\Omega}{(2\pi)^d}\int_0^\infty d\left(\frac{1}{g_0}\right)\left(\frac{1}{g_0}\right)^{(d-1)/2}e^{-a_d/g_0}e^{2\,\mathrm{Im}\,\omega_C t}. \tag{85}$$

The noise spectrum at early times, like the Rayleigh cross section, scales as ω^{d+1}. Further, using the fact that $C_2\propto\int d^dx$ (as the functional determinant must be) it is finite. At $t\to 0$, the integral over the coupling g_0 becomes independent of both G and D, and reduces to the constant $\Gamma(d+1)/a_d^{d+1}$. Thus the only dependence on coupling is through the direct perturbative factor of G. The finite time constants $1/\mathrm{Im}\,\omega_C$, however, cause the noise spectrum to decay in a way that is sensitive to g_0. To identify the behavior in the uniform limit, the back reaction of fast-wave scattering on slow waves must be taken into account, which is most easily done through the CPA itself.

B. Coherent attenuation

Normally, the coherent-potential approximation is solved self-consistently in a bare theory and then renormalized with a nonlinear sigma model.[2] The coherent attenuation that defines the initial coupling in the sigma-model renormalization group flow is due to the density of scattering states, but in the CPA these are never seen as explicitly localized. Rather, localization is only inferred from the absence of renormalized diffusion.

In nonuniform systems, where there is effectively a position-dependent band edge, this seems paradoxical; the Ginzburg-Landau treatment gives an explicitly localized description of the density of states sufficiently deep within the forbidden region, where the instanton expansion is valid,[11] and such a description is equivalent in electronic or acoustic systems. The paradox is resolved by considering uniform systems as the limits of nonuniform ones and using the instanton density of states to form an effective CPA action directly from the replica-field generating functional.[10]

The generating functional is the functional integral responsible for ensemble-averaged Green's functions, like Eq. (28), without the field insertions. Splitting fields into classical and fluctuating parts, and summing over instanton backgrounds as before, the leading quadratic action for coherent propagation can be formed as in Ref. 10:

$$L^{\mathrm{CPA}}=\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}k^2[\Phi^{+T}\quad\Phi^{-T}]_{-k}\left\{\begin{bmatrix}\omega_+{}^2[\rho_0] & \\ & \omega_-{}^2[\rho_0]\end{bmatrix}-k^2\begin{bmatrix}[K_0] & \\ & [K_0]\end{bmatrix}\right.$$
$$\left.-\frac{G\overline{\omega}^4}{\int d^dx}\int\frac{d^dk'}{(2\pi)^d}\frac{(k\cdot k')^2}{k^2(k'^2)^2}\begin{bmatrix}\sigma^*\overline{\mathcal{G}^+(-k',k',\omega_+)}\sigma^* & \\ & \sigma^*\overline{\mathcal{G}^-(-k',k',\omega_-)}\sigma^*\end{bmatrix}\right\}\begin{bmatrix}\Phi^+\\ \Phi^-\end{bmatrix}_k. \tag{86}$$

It has been assumed in Eq. (86) that a uniform limit exists, which leads to the correct normalization of the free-fluctuation Green's function,[10] and the factor of $\int d^dx$.

It is very important that only the tensor component of the second variation has been kept explicitly in this semiclassical expansion. The corresponding scalar component contributes a sum over $\mathcal{G}^+$ and $\mathcal{G}^-$, which has no imaginary part, and at most perturbatively renormalizes the dispersion relation to orders that are already implicitly absorbed in the effective couplings.

Using the contraction of the projection operator with $\mathcal{G}$

$$\sigma^*\overline{\mathcal{G}(-k',k',\omega)}\sigma^*=\sigma^*\mathrm{Tr}(\sigma^*\overline{\mathcal{G}(-k',k',\omega)}), \tag{87}$$

Eq. (86) reduces to a simple matrix form for Biot polarizations, multiplying a Green's function trace proportional to the density of states:

$$L^{\mathrm{CPA}}=\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}k^2[\Phi^{+T}\quad\Phi^{-T}]_{-k}\left\{\begin{bmatrix}\omega_+^2[\rho_0] & \\ & \omega_-^2[\rho_0]\end{bmatrix}-k^2\begin{bmatrix}[K_0] & \\ & [K_0]\end{bmatrix}-\frac{G\overline{\omega}^2c_s^2}{\int d^dx}\begin{bmatrix}\sigma^* & \\ & \sigma^*\end{bmatrix}\right.$$
$$\left.\times\left(\frac{\int d^{d-1}\Omega\cos^2\theta}{\int d^{d-1}\Omega}\right)\int\frac{d^dk'}{(2\pi)^d}\begin{bmatrix}\mathrm{Tr}(\sigma^*\overline{\mathcal{G}^+(-k',k',\omega_+)}) & \\ & \mathrm{Tr}(\sigma^*\overline{\mathcal{G}^-(-k',k',\omega_-)})\end{bmatrix}\right\}\begin{bmatrix}\Phi^+\\ \Phi^-\end{bmatrix}_k. \tag{88}$$

This CPA form transforms easily to the position basis, which will then be generalizable to nonuniform systems:

$$L^{\mathrm{CPA}}=\frac{1}{2}\int d^dx\,\nabla^i[\Phi^{+T}\quad\Phi^{-T}]\left\{\begin{bmatrix}\omega_+{}^2[\rho_0] & \\ & \omega_-{}^2[\rho_0]\end{bmatrix}+\nabla^2\begin{bmatrix}[K_0] & \\ & [K_0]\end{bmatrix}-G\overline{\omega}^2c_s^2\begin{bmatrix}\sigma^* & \\ & \sigma^*\end{bmatrix}\right.$$
$$\left.\times\left(\frac{\int d^{d-1}\Omega\cos^2\theta}{\int d^{d-1}\Omega}\right)\frac{1}{\int d^dx'}\int d^dx'\begin{bmatrix}\mathrm{Tr}(\sigma^*\overline{\mathcal{G}^+(x',x',\omega_+)}) & \\ & \mathrm{Tr}(\sigma^*\overline{\mathcal{G}^-(x',x',\omega_-)})\end{bmatrix}\right\}\nabla^i\begin{bmatrix}\Phi^+\\ \Phi^-\end{bmatrix}. \tag{89}$$

This rather roundabout way of arriving at the position-basis correction for the CPA was used to isolate the angular integral of $\cos^2\theta\equiv(k\cdot k')^2/(k^2k'^2)$, expressing the fact that density perturbations are dipole scatterers. This symmetry factor appears only as an overall weight function. It is significant, though, that spatially δ-correlated perturbations couple all wave numbers, so fast waves are attenuated by scattering into localized slow waves, with D-invariant weight, at leading perturbative order in G.

The CPA correction in Eq. (88) may be understood as an imaginary addition to the matrix blocks of density components for $\Phi^{\pm}$, which remains finite at $\eta\to 0$,

$$[\rho_0]^{\pm}\to[\rho_0]^{\pm}-\sigma^*Gc_s^2\left(\frac{\int d^{d-1}\Omega\cos^2\theta}{\int d^{d-1}\Omega}\right)\times\frac{1}{\int d^dx}\int d^dx\,\mathrm{Tr}(\sigma^*\overline{\mathcal{G}^{\pm}(x,x,\omega)}). \tag{90}$$

The explicit evaluation of the averaged Green's function trace, again correct to $\mathcal{O}(\alpha^1)$, is

$$\frac{1}{\int d^dx}\int d^dx\,\mathrm{Tr}(\sigma^*\overline{\mathcal{G}^+(x,x,\omega_+)})=-\frac{2ik_s^d}{K_-}\frac{C_1}{\int d^dx}\left(\int\frac{d^d\tilde{v}}{(2\pi)^d}\mathcal{F}_R^{\mathrm{cl}T}\mathcal{F}_R^{\mathrm{cl}}\right)\times\frac{1}{(4-d)}\int\frac{d^{d-1}\Omega}{(2\pi)^d}\int_0^\infty d\left(\frac{1}{g_0}\right)\left(\frac{1}{g_0}\right)^{(d-1)/2}e^{-a_d/g_0}. \tag{91}$$

Recalling the relation (67) of C_1 to C_2, and the expression for the noise spectrum (91), the CPA density blocks become

$$[\rho_0]^{\pm}\to[\rho_0]^{\pm}\pm i\sigma^*\frac{2K_-}{c_s^2}\left(\frac{\int d^{d-1}\Omega\cos^2\theta}{\int d^{d-1}\Omega}\right)\frac{N(t\to 0,\overline{\omega})}{\overline{\omega}}. \tag{92}$$

To $\mathcal{O}(\alpha^0)$, $[\rho_0]$ and σ^* can be simultaneously diagonalized and used to understand the correction to the on-shell slow wave number,

$$k_s\to k_s\left[1+\frac{iK_-}{c_s^2\rho_-}\left(\frac{\int d^{d-1}\Omega\cos^2\theta}{\int d^{d-1}\Omega}\right)\times\frac{N(t\to 0,\overline{\omega})}{\overline{\omega}}(1+\mathcal{O}(\alpha^2))\right]. \tag{93}$$

Recognizing that $K_-/(c_s^2\rho_-)=1+\mathcal{O}(\alpha^2)$, and defining the coherent attenuation length l_{coh} in terms of the imaginary part of k_s, gives the desired relation between coherent attenuation and the noise spectrum to this order in α:

$$\mathrm{Im}(k_s)\equiv\frac{1}{l_{\mathrm{coh}}}\approx\left(\frac{\int d^{d-1}\Omega\cos^2\theta}{\int d^{d-1}\Omega}\right)\frac{N(t\to 0,\overline{\omega})}{c_s}\sim\overline{\omega}^{d+1}. \tag{94}$$

One difference of the true definition of the intrinsic noise spectrum from the model of interface reflection now becomes apparent. Interface reflection causes coherent displacement doubling, and so multiplies noise very near the interface by a factor of four relative to Eq. (94).[10,28] The fact that the intrinsic noise spectrum is due to weak localization is emphasized by the lack of coherence between the input and reflected waves through an apparent reflection point x_0, causing the extra factor of four not to appear. Up to this coherence factor, Eq. (94) agrees in $d=1$ (where $\cos^2\theta\equiv 1$) with the localization result obtained from stochastic differential equations,[28] and it agrees identically with the homogeneous acoustic result.[10]

Because waves localized in a sufficiently weakly nonuniform medium are still compact with respect to the dimensions of the nonuniformity, the uniform CPA must have a limited generalization to the case of spatially varying scattering strength, obtained by replacing

$$\frac{1}{\int d^dx}\int d^dx\overline{\mathcal{G}(x,x,\omega)}\to\mathcal{G}(x,x,\omega) \tag{95}$$

in Eq. (90). In the regions of the potential where the instanton expansion can be used, the form (77) then predicts the form of anomalous spatial attenuation associated with the onset of spatially limited localization.

C. Relating instantons to the free-field CPA

The result (92) made explicit use of the instanton expansion through the solution (91), and so distinguished between slow and fast wave contributions only through the form and wave number dependence of the instantons. Due to abstract constants like C_1, though, it is difficult to apply to the usual perturbative construction of the CPA, in which the entire density of states *at the propagating wave number* defines the initial diffusivity to be renormalized.[2,3] It will be argued here that, if there is a large ratio c_f/c_s, and small overlap angle ξ, the instanton expansion indicates how to carry the acoustic CPA construction over into the Biot problem.

The essential semiclassical result was that there are no instantons in the narrow shell around the allowed (propagating) value of the fast wave number, while there are around the allowed value of the slow wave number. Further, in the large-D limit, only states within a vanishingly small distance of the allowed wave numbers make nonvanishing contributions (these have the small values of g_0).

Now, the CPA creates an imaginary correction to the matrix $[\rho_0]$, from the Green's function in Eq. (90), and in the uniform limit, the spatial average is simply replaced with the uniform Green's function as in Eq. (95), whose imaginary part comes from the on-shell densities of states. The different instanton treatment of fast and slow wave numbers suggests that the fields be split into polarization components

$$\Phi^{\pm} \equiv \Phi_f^{\pm} + \Phi_s^{\pm}, \tag{96}$$

where $\Phi_f^{\pm}$ is the component corresponding to f_+ in the frequency-dependent diagonalization (56), and $\Phi_s^{\pm}$ to f_-. Separate integrations over the fast and slow components then give different CPA field theories, whose physical interpretations are given below.

D. The fast coherence length

The first interesting CPA is obtained by integrating out $\Phi_s^{\pm}$, and leaving $\Phi_f^{\pm}$ explicit. Because σ^* couples at order ξ^0 to the slow wave and ξ^2 to the fast wave, the result (90) holds, with $\mathcal{G}^{\pm}$ replaced at leading order in ξ^2 by the uniform, free-field, $\Phi_s^{\pm}\Phi_s^{\pm}$ Green's function. This is, of course, exactly the same correction as determines the slow wave coherence length, so that the two are proportional.

Working through the diagonalizations (51) and (52), correcting $[\rho_0]$ with the fixed term of Eq. (90), and using the fact that to leading order in ξ^2,

$$\sin^2\xi \approx \left(\frac{\rho_T^2}{\rho_+\rho_-}\right)\left(\frac{c_s}{c_f}\right)^2, \tag{97}$$

it follows that

$$\mathrm{Im}(k_f) = \frac{c_f}{c_s}\tan^2\xi\,\mathrm{Im}(k_s) \approx \left(\frac{\rho_T^2}{\rho_+\rho_-}\right)\frac{c_s}{c_f}\mathrm{Im}(k_s). \tag{98}$$

Equation (98), with Eq. (94), is the advertised relation between the fast and slow coherence lengths, and the slow-wave noise spectrum. Since fast-wave attenuation through conversion to slow waves is a local effect, the relation between the fast and slow coherence lengths does not depend on whether the slow waves are localized. The relation between the slow-wave coherence length and the noise spectrum relies essentially on the approximation that fast-wave coupling is small, which can now be quantified.

E. Corrections to the slow-wave CPA

The acoustic CPA (Ref. 2) essentially assumes that fluctuations are not severe enough to invalidate the effective medium obtained by integrating the Green's function in the scalar equivalent to Eq. (90). Thus, the fields left dynamical, and those contributing the averaged density of states are the same. In particular, all states associated with the dynamical fields are either localized or not, and renormalization of the diffusivity tells which.

When there is coherence loss due to both fast and slow scattering in a Biot CPA, there arises the possibility that one should contribute the initial value to the renormalized diffusivity, and the other appear merely dissipative. The lack of an instanton contribution at $k^2 \sim k_f^2$ suggests that if $\Phi_f^{\pm}$ is integrated out to form an effective slow wave theory, its free field density of states should contribute to the latter. Physically this makes perfect sense: it is coherent backscatter from strongly coupled slow waves that leads to their localization; more weakly coupled fast waves should simply carry away coherence information, when viewed on the same scale.

To obtain the effective slow-wave theory, the interaction Lagrangian is again split as for semiclassical backgrounds, except that the binomial expansion is performed in fast and slow components. Slow components are kept explicit, the interaction exponential expanded as a power series, and fast-wave products are averaged with the free Green's function. The first nonvanishing term in this series is

$$L^{\mathrm{int}} \ni -\frac{G\overline{\omega}^4}{2}\int \frac{d^dk_1 d^dk_2 d^dk_3}{(2\pi)^{3d}}\frac{(k_1\cdot k_2)}{k_2^2}\frac{(k_3\cdot k_4)}{k_3^2} \times [\Phi_s^{+T} \quad \Phi_s^{-T}]_{-k_1}\begin{bmatrix}\sigma^*\mathrm{Tr}(\sigma^*\mathcal{G}_f^{+}(k_2,-k_3,\omega_+)) & \\ & \sigma^*\mathrm{Tr}(\sigma^*\mathcal{G}_f^{-}(k_2,-k_3,\omega_-))\end{bmatrix}\begin{bmatrix}\Phi_s^{+}\\ \Phi_s^{-}\end{bmatrix}_{k_4}. \tag{99}$$

The fast-wave Green's function is denoted $\mathcal{G}_f^{\pm}$, and only the tensor component has been kept, as in Eq. (86), because only the imaginary part is of interest.

At large D, the perturbing kernel in Eq. (99) has imaginary part

$$\mathrm{Im}\left(\int \frac{d^dk_2 d^dk_3}{(2\pi)^{2d}}\frac{(k_1\cdot k_2)}{k_2^2}\frac{(k_3\cdot k_4)}{k_3^2}\mathrm{Tr}(\sigma^*\mathcal{G}_f^{+}(k_2,-k_3,\omega_+))\right) = -\pi\frac{\alpha^2\cos^2\xi}{K_-}k^2k_f^{d-2}\left(\frac{\int d^{d-1}\Omega\,\cos^2\theta}{\int d^{d-1}\Omega}\right)C_{\mathrm{SHO}}, \tag{100}$$

where C_{SHO} is a constant of order unity defined by the density of states of the nondimensionalized, simple harmonic oscillator. In the uniform limit, C_{SHO} is replaced by

$$C_{\mathrm{SHO}} \rightarrow \int \frac{d^{d-1}\Omega}{(2\pi)^d}. \tag{101}$$

Recalling the definition (58) of α, the density parameters for slow waves are modified by a term interpreted as dissipation:

$$[\rho_0]_s^{\pm} \rightarrow [\rho_0]_s^{\pm} \pm i\pi\sigma^* \frac{G c_s^2 k_s^d}{K_-} \sin^2\xi \left(\frac{c_f}{c_s}\right)^{4-d} \times \left(\frac{\int d^{d-1}\Omega \cos^2\theta}{\int d^{d-1}\Omega}\right) C_{\mathrm{SHO}}. \tag{102}$$

Making use of the density parametrization introduced in Eq. (49) and the definition (50) (and dropping the subscript s), Eq. (102) simply shifts the scalar ρ_-, as

$$\rho_-^{\pm} \rightarrow \rho_-^{\pm} \left\{ 1 \pm i\pi \overline{\left(\frac{\delta\rho_-}{\rho_-}\right)^2} \left(\frac{k_s l_0}{2\pi}\right)^d \left(\frac{G^{\mathrm{eff}}}{G_0}\right) \times \left(\int d^{d-1}\Omega \cos^2\theta \right) \times \sin^2\xi \left(\frac{c_f}{c_s}\right)^{4-d} \right\}. \tag{103}$$

The effective slow wave theory that remains is essentially just the scalar acoustic theory of Ref. 3, with ρ_- a complex constant given by Eq. (103). In particular, the scattering terms quartic in $\Phi_s^{\pm}$ have still been left explicit during the above integrations. If they are now approximated by the diffusive CPA, the on-shell density of free slow states is used for the Green's function in Eq. (91), and slow-slow scattering makes an additional coherence-loss correction

$$\rho_-^{\pm} \rightarrow \rho_-^{\pm} \left\{ 1 \pm i\pi \overline{\left(\frac{\delta\rho_-}{\rho_-}\right)^2} \left(\frac{k_s l_0}{2\pi}\right)^d \left(\frac{G^{\mathrm{eff}}}{G_0}\right) \left(\int d^{d-1}\Omega \cos^2\theta \right) \right\}. \tag{104}$$

There is thus a simple relation between the apparent dissipation of slow waves induced by fast wave scattering, and the coherent attenuation from self-scattering.

Two relations can now be obtained directly from the scalar CPA of Ref. 3. The first is the classic Ioffe-Regel condition[29] in the nondissipative limit, from Eq. (23) of that work:

$$2\pi(d-2) \lesssim \left(\frac{c_s}{\omega l_{\mathrm{coh}}}\right)^{d-1}, \tag{105}$$

from the condition that the initial coupling strength be sufficient to overcome the classical scaling dimension. The second relation is a weakness condition on dissipation obtained from Ref. 3, Eq. (22), necessary to allow flow to strong coupling from any initial value:

$$\frac{\mathrm{Im}\,\omega_{(s)}}{\omega} \lesssim \frac{1}{2\pi(d-2)} \left(\frac{c_s}{\omega l_{\mathrm{coh}}}\right)^d. \tag{106}$$

The coherence length in both equations (105) and (106) is obtained from

$$\frac{c_s}{\omega l_{\mathrm{coh}}} = \frac{1}{2}\left(\frac{\mathrm{Im}\,\rho_-}{\rho_-}\right), \tag{107}$$

using the shift in Eq. (104), while the apparent dissipation in Eq. (106) is given by

$$\frac{\mathrm{Im}\,\omega_{(s)}}{\omega} = \frac{1}{2}\left(\frac{\mathrm{Im}\,\rho_-}{\rho_-}\right), \tag{108}$$

using the shift in Eq. (103).

The more general relation, that strong coupling be attainable from any combination of initial parameters, defines quantitatively the meaning of weak overlap. Making use of Ref. 3, Eq. (22), the definitions (107) and (108), and their relations implied by Eqs. (103) and (104), gives

$$\sin^2\xi \left(\frac{c_s}{c_f}\right)^{d-4} \approx \left(\frac{\rho_T^2}{\rho_+\rho_-}\right)\left(\frac{c_s}{c_f}\right)^{d-2} \lesssim \frac{1}{2\pi(d-2)}\left(\frac{c_s}{\omega l_{\mathrm{coh}}}\right)^{d-1} - 1. \tag{109}$$

As a final comment, it is now clear why a mobility edge in $d=3$ suggests localized slow and extended fast waves. Using the slow-slow scattering strength of Eq. (104) as a reference, the dissipative term from scattering to fast waves is smaller by ξ^2, due to the factor of σ^* in the trace of $\mathcal{G}$. For even slow waves to localize near the Ioffe-Regel criterion (which is rather strict), ξ^2 must be small as in Eq. (109). The slow-wave correction to fast-wave coherence is of comparable magnitude to the fast-slow term, this time due to the σ^* outside the trace, while the fast-fast scattering correction is order ξ^4 from both σ^* factors. Thus, if there is any separation of scales $c_s/c_f \ll 1$, an Ioffe-Regel criterion can never be satisfied for the fast wave until slow wave scattering is already so strong that the analysis presented here must be replaced by something else, in which case the medium probably no longer looks much like a Biot system.

F. Quantitative estimates

The Ioffe-Regel criterion can be translated into a required absolute heterogeneity, for the simple example like superfluid helium scattering by tortuosity fluctuations considered in Ref. 7. In $d=3$, Eqs. (104), (105), and (107) can be combined to read

$$\overline{\left(\frac{\delta\rho_-}{\rho_-}\right)^2}\left(\frac{k_s l_0}{2\pi}\right)^3\left(\frac{G^{\mathrm{eff}}}{G_0}\right) \geqslant \frac{3}{\sqrt{2\pi^3}} \approx 0.38. \tag{110}$$

Perhaps not surprisingly, this combination of parameters turns out not to lie within the weak-coupling regime of perturbation theory in Ref. 7, for any initial variance $\overline{(\delta\rho_-/\rho_-)^2}$, so the ratio G^{eff}/G_0 cannot be obtained from that calculation. However, one may simply ask what strength of variation *at* the natural scale is sufficient to localize, since there is then no renormalization, and $k_s l_0/2\pi$ and G^{eff}/G_0 both equal unity, in which case the answer is that

$\overline{(\delta\rho_-/\rho_-)^2} \gtrsim 0.38$. This rather severe condition for proper localization is the counterpart of that encountered in the other classical wave problems.[3] For the superfluid helium example of Ref. 7, ρ_- becomes simply the Biot parameter m, which is proportional to the inviscid tortuosity α_∞.

V. CONCLUSIONS

The localizing transition in a heterogeneous Biot medium has been found to be somewhat different qualitatively from the similar transition in linear acoustics or elasticity. The separation of scales between slow and fast wave speeds leads to an instanton density of states at the slow wave speed, but none at the fast wave speed if the two waves are coupled by scattering. Since the instanton density of states was interpreted as the localized part in the acoustic problem, this seems to argue that the fast-wave states remain extended.

This semiclassical result, used as physical input in forming a CPA description, leads to very different roles for the fast and slow wave densities in the nonlinear sigma model of Ref. 3. Whereas in the acoustic case, the coherence loss from all states is subsumed into the initial value for the renormalized diffusivity, here the fast-wave states contribute a coherence loss equivalent to absorption, and only the slow-wave states are considered diffusive.

Localization, to occur at all, requires both an Ioffe-Regel-like strong scattering condition for slow waves, and a condition of sufficiently weak overlap for slow-fast interconversion. When these are both satisfied, there is a strong hierarchy of scattering strengths, in which fast-fast scattering is weakest, slow-fast scattering intermediate, and slow-slow scattering strongest. This result argues from a second direction that fast-wave scattering is too weak to localize within any range where perturbation theory correctly describes slow waves. Thus, if fast waves localize at all, it must be as a secondary transition in an effective theory that takes into account strong slow wave scattering.

Finally, it may be observed that the slow-localizing transition derived here is the first connection between the Biot and elastic effective media by something akin to a phase transition.[30] When Biot theory was first derived, Anderson localization had not yet been discovered, and the difference between regular and random small-scale heterogeneity was not yet appreciated. (The small-scale models of pore flow[5] were regular lattices, which could have led to Bloch waves at strong coupling, but not to qualitative changes in the propagating degrees of freedom.) Biot clearly understood the universality of the effective medium he derived, but did not consider when it might become equivalent to simple elasticity, except in trivial cases like vanishing or total porosity, or when extreme heterogeneity caused both models to break down. This work identifies slow-wave localization as one transformation in a series potentially leading from the consolidation transition ultimately to elasticity.

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APPENDIX A: AN ESTIMATE OF α

At high frequencies, viscosity becomes a less severe perturbation than scattering in heterogeneous Biot theory. Therefore, it may be appropriate to use the parameters commonly accepted for water-saturated sand in the inviscid sector alone, to estimate polarization overlap of fast and slow waves. The parameter set used here will be the same as that used to define the most-relevant effective scattering vertex in Ref. 7.

Density parameters are $\rho_f = 1000$ kg/m^3, $\rho = 1875$ kg/m^3, $m = 2660$ kg/m^3, and compressibilities are $H = 4.073\times 10^9$ Pa, $C = 4.000\times 10^9$ Pa, and $M = 4.005\times 10^9$ Pa. These give eigenvalues for $[K_0]$ of $K_+ = 8.039\times 10^9$ Pa, $K_- = 3.9\times 10^7$ Pa. The corresponding parameters for $[\rho_0]$ in Eq. (49) are $\rho_+ = 3264$ kg/m^3, $\rho_- = 1271$ kg/m^3, $\rho_T = -793$ kg/m^3.

These parameters give speeds of $c_f = 1706$ m/s, $c_s = 175$ m/s, and a rotation angle $\xi = 0.022$ in Eq. (53). The resulting estimate for the instanton overlap parameter in Eq. (58) is $\alpha \approx 0.21$. Since α^2 is the coupling between fast and slow waves that determines the effective dissipation in Eq. (100), it is clear that these parameters allow only $\sim 4\%$ of the fast wave density of states to actually contribute to apparent slow wave dissipation.

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