Gas of elastic quantum strings in 2+1 dimensions: Finite temperatures

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Motivated by the stripes of the high-T_c cuprates the problem was introduced recently of a system of free elastic quantum strings interacting via a hard-core condition embedded in 2+1 dimensions. At zero temperature this system is always a solid due to "quantum-entropic" (or "kinetic") interactions which dominate at long wavelengths. The high-temperature limit of this problem corresponds with thermally meandering elastic lines in two dimensions and this system is well known to be dominated by literal entropic interactions. Here we analyze in detail what happens in between zero and high temperature. We identify a "renormalized classical" intermediate regime where the on-string fluctuations have become predominantly quantum mechanical. Surprisingly, the entropic interactions keep their high-temperature nature in this regime. At a low, but finite temperature the quantum-mechanical kinetic interactions take over rather suddenly. Despite their origin in long-wavelength quantum fluctuations these are not affected by thermal fluctuations when temperature is low enough.

I. INTRODUCTION

The focus of quantum condensed matter has been traditionally on systems of particlelike excitations. Recently the evidence has been growing that at least in the strongly interacting electron systems as realized in correlated oxides interesting types of quantum-mechanical self-organizations are taking place on mesoscopic time and length scales. Although these are fluctuating textures, there exists substantial empirical evidence that they bear a direct relationship with the static stripe phases.

The stripe phase is a different form of electronic order which is found in doped Mott-insulating oxides.[1] Because of topological reasons,[2] the carriers bind to linelike textures in the 2+1 dimensional (D) cuprates and nickelates, and these "rivers of charge" are separated by Mott-insulating magnetic domains. The static stripes are reasonably well understood as generic instabilities of the doped Mott-insulating state.[3] However, experimental results suggest that on intermediate scales (nanometer length scales, $\sim$10-meV energy scales) the electron system in the cuprate superconductors tends to stripe order,[4] to flow away to the anomalous physics of high-T_c superconductivity at larger distances and longer times. It is often speculated that the high-T_c superconducting state is in one or the other way related to this quantum disordered stripe state (the "dynamical stripes").

On the theoretical side, the obvious problem is that little is known in general about a "complex" quantum state of matter of the kind suggested by the dynamical stripe phenomenology. This motivated us to look into the following elementary question: what can be said in general on the physics of a gas of quantum fluctuating elastic lines in two embedding space dimensions? These lines can be alternatively called strings[5−10] and to have a meaningful definition of a gas it is natural to let these strings merely interact via an excluded volume (or noncrossing) condition.[11]

One of us recently suggested an answer:[13] due to an order-out-of-disorder mechanism, the ultimate faith of this gas is that it solidifies always at zero temperature. This can be either viewed as a 2+1D extension of the mechanism responsible for the algebraic order in the 1+1D Luttinger liquid, or as a quantum version of the classical entropic repulsions familiar from the statistical physics of lines and membranes.[12] A central result of Ref. 13 is that this string solid should be characterized by a long-wavelength compression modulus which depends on the average string separation d in a stretch exponential fashion, $\sim \exp(-Ad^\alpha)$ with $\alpha \sim 2/3$. This argument was based on an elegant, but nonexact, self-consistent phonon method introduced quite some time ago by Helfrich[14] in the context of biological membranes. The stretched exponential turns out to reflect a highly untrivial and counterintuitive phenomenon. Usually entropic interactions emerge from short distance physics. The essence of the mechanism is that entropy is paid at collisions because of the hard-core condition, and collisions occur at short distances. For the case of elastic quantum strings this is qualitatively different. A single string shows algebraic order and for this reason its long-wavelength fluctuations are the most dangerous ones. According to the Helfrich method, these long-wavelength on-string fluctuations are responsible for the induced modulus.[13] Very recently, this result was confirmed in a numerical simulation by Nishiyama,[15] finding $\alpha \simeq 0.8$ instead of $\alpha = 2$ as would follow from the argument based on simple collisions.

The main focus of this paper is on the *finite*-temperature physics of this quantum string gas. This finite-temperature behavior is in itself rather untrivial. At zero temperature the rigidity in the system is driven by a gain of *kinetic* energy associated with long-wavelength on-string fluctuations. At the same time, the high-temperature, fully classical limit corresponds with the problem of elastic lines meandering over the 2D plane due to thermal fluctuations. This is a classic

statistical physics problem which is fully understood.[5,12,16] It is also characterized by a net coarse-grained modulus, now proportional to the square of temperature, originating in literal entropic interactions. The question is, how to connect the zero-temperature limit with the high-temperature limit? Since the order at zero temperature is driven by long-wavelength fluctuations one could be tempted to argue that at any finite temperature the entropic interactions should take over—at sufficiently long wavelength, thermal fluctuations usually overwhelm quantum fluctuations for any finite temperature. If this would be the correct answer there could be a potential problem with the argument of Ref. 13. To show that the string gas solidifies at zero temperatures it has to be demonstrated that dislocations cannot proliferate. It is well known that in the classical (high-temperature limit) dislocations always proliferate. The effective elastic constant $\sim T$ and under this condition the criterium for the Kosterlitz-Thouless (KT) transition (binding of dislocations) is never satisfied.[5,12,16] However, upon adding any tension of nonentropic origin the KT transition immediately shifts to a finite temperature and the zero-temperature state is protected against free dislocations. One of us[13] asserted that the induced modulus is nonzero at zero temperature due to the quantum fluctuations and this modulus can therefore serve the purpose of protecting the crystallinity of the ground state.

Here we address these matters by analyzing the finite-temperature regime explicitly using the Helfrich method. Starting from the high-temperature, classical regime one first enters a renormalized classical regime where the quantum-mechanical nature of the on-string fluctuations becomes noticeable because temperature is lower than the characteristic Debye temperature of the on-string modes. Naively one would expect either a qualitative change in the nature of the entropic interactions as compared to the high-temperature limit, or at the least a quantitative change in the sense that numerical factors in the classical result for the induced modulus become different. We were quite surprised to find that in fact the induced modulus is not changed at all. As we will explain, the reason turns out to be that the fluctuations driving the entropic interactions occur at frequencies which are low as compared to the quantum UV cutoff. Upon further lowering temperature, suddenly the quantum-kinetic interactions take over, at a temperature scale associated with the zero-temperature modulus. This quantum modulus protects itself: despite its origin in long-wavelength on-string fluctuations, it carries an associated energy scale and when temperature becomes lower than this scale the thermal fluctuations freeze out. As a result, in this low-temperature regime the thermal contributions appear as corrections to the zero-temperature modulus.

The plan of the paper is as follows. In Sec. II the string gas model is specified, and the dimensionless parameters governing the crossovers between the different temperature regimes are estimated using the simple collision picture. In Sec. III the Helfrich method[14] is introduced and used to refine the estimates for the characteristic scales of Sec. II. In Sec. IV the behavior of the induced modulus in the various regimes are derived and discussed in detail. In the concluding section we put our findings in perspective.

II. CHARACTERISTIC SCALES: THE COLLISION PICTURE

One can acquire some insight regarding the origin of entropic repulsions in terms of a simple physical picture. The basic idea is that the world lines (particles) or world sheets (strings) once in a while collide when they meander in space-time. Entropy (high temperature) or kinetic energy (zero temperature) is paid at these collisions because of the non-crossing condition. This raises the (free) energy, and this energy increase translates into repulsive interactions at longer distances. Although this argument turns out to be not quite right for the zero-temperature string gas, it is qualitatively correct at high temperatures. In this section we will use this argument to obtain a crude picture of the physics at all temperatures, which will be refined in the next sections using Helfrich's self-consistent phonon method.

The string gas is defined as a system of nonintersecting elastic quantum strings embedded in $2+1$D space-time. In path-integral language this corresponds with the statistical physics of a system of nonintersecting, elastic world sheets. In addition, since it follows self-consistently from the theory that dislocations do not proliferate at zero temperature the strings can be considered to be directed along the y axis from the on set. The transversal displacements of the strings are parametrized in terms of a field $\phi_i(y,\tau)$ describing the motion of the ith string in the x direction (τ is imaginary time), and supplemented by the avoidance condition

$$\phi_1(y,\tau)<\phi_2(y,\tau)<\cdots<\phi_N(y,\tau). \qquad (1)$$

The partition function written in the form of a functional integral is

$$Z=\prod_{i=1}^{N}\prod_{y,\tau}\int d\phi_i(y,\tau)\exp\{-S/\hbar\},$$

$$S=\frac{1}{2}\int_0^{1/T}d\tau\int dy\sum_i\left[\rho_c(\partial_\tau\phi_i)^2+\Sigma_c(\partial_y\phi_i)^2\right], \qquad (2)$$

where the temperature T is expressed in energy units ($\hbar=k_B=1$). In Eq. (2) ρ_c is the linear mass density and Σ_c the string tension, such that $c=\sqrt{\Sigma_c/\rho_c}$ is the on-string sound velocity. As in Ref. 13, a lattice regularization with lattice constant a is chosen, such that the average interstring distance is $d=a/n$, where n is the density of strings. The ultraviolet (UV) momenta and frequency cutoffs on a single string (i.e., for the intrastring vibrations) are therefore $Q_s=\pi/a$ and $\omega_s=cQ_s$, respectively. Throughout this paper we use as a convention a subscript s when a quantity relates to a single string and a subscript g when it relates to the system of strings ("gas") of strings.

Let us now turn to the collision argument. Since the strings only interact via a nonintersection condition it is obvious that at sufficiently short times and distances the strings behave like a noninteracting system because collisions do not occur. A characteristic scale (collision length) can be identified where the probability of a collision becomes of order unity and at larger scales the physics is set by the collisions. At every collision an entropy $\sim k_B$ (at high T) or

a kinetic energy $\sim \hbar$ (at $T=0$) is paid and an estimate follows for the induced modulus by dividing this characteristic (free) energy cost by the characteristic distance (d). Hence all that has to be calculated is the collision length and this can be determined by evaluating the mean square of the single string meandering amplitude $\langle [\phi(y) - \phi(0)]^2 \rangle$ as a function of a distance y between two points on a string. Equating this quantity with the square of the average inter-string distance d one obtains a characteristic distance $y = l_c$ (and time $\tau_c = l_c/c$) where the probability for a collision to occur becomes of order unity.

After Fourier transforming the single string action we arrive at the standard result,

$$\langle [\phi(y) - \phi(0)]^2 \rangle$$

$$\sim \int \langle |\phi_k|^2 \rangle [1 - \cos(ky)] d k$$

$$= \int \frac{\hbar}{\rho_c c k} (1/2 + n_k)[1 - \cos(ky)] d k$$

$$\sim \int_{\pi/y}^{\pi/a} \frac{\hbar}{\rho_c c k} \{1/2 + (\exp[\hbar c k/T] - 1)^{-1}\} d k. \quad (3)$$

Consider first Eq. (3) in the high-temperature (classical) limit,

$$T \gg \frac{\pi \hbar c}{a}. \quad (4)$$

The mean-square meandering amplitude becomes

$$\langle [\phi(y) - \phi(0)]^2 \rangle$$

$$\sim \int_{\pi/y}^{\pi/a} \frac{\hbar}{\rho_c c k} \{1/2 + (\exp[\hbar c k/T] - 1)^{-1}\} d k$$

$$\approx \int_{\pi/y}^{\pi/a} \frac{\hbar}{\rho_c c k} [1/2 - 1/2 + T/(\hbar c k)] d k \sim \frac{Ty}{\Sigma_c}. \quad (5)$$

Equating this to d^2 yields for the collision length l_c the well-known result[12,5,13]

$$T l_c / \Sigma_c \approx d^2. \quad (6)$$

In the classical context, the induced modulus follows directly by dividing the free-energy cost associated with the collisions $\sim TS \sim T \times (1/l_c) \times k_B$ ($1/l_c$ is the collision density) by the characteristic length d (interstring separation). Thus Eqs. (5) and (6) lead to an estimate: $B \propto T^2/d^3$. This shows the correct dependence on temperature and density. Alternatively, this directed 2D string gas can also be interpreted as a system of 1+1D hard-core bosons which is in turn equivalent to noninteracting spinless fermion gas. In the latter, the collision length corresponds with the Fermi energy and using these trivial spinless-fermion results one finds that the estimate for the high-temperature limit of the string gas is qualitatively correct.[13]

The high-temperature limit is defined as usual as the regime where temperature is larger than the highest phonon frequency which is in our case clearly corresponding with

the UV cutoff of the on-string modes (Q_s, ω_s). Hence upon lowering temperature one will reach an on-string Debye temperature below which the quantum-mechanical nature of the modes on the string will become noticeable. This cutoff frequency is $\omega_s = (\pi/a)\sqrt{\Sigma}/\sqrt{\rho}$, defining a dimensionless ratio ($\hbar = k_B = 1$),

$$\nu_s = \frac{\omega_s}{T} = \frac{\pi\sqrt{\Sigma}}{a\rho T}. \quad (7)$$

Therefore when $\nu_s > 1$ one enters a regime where the high-frequency on-string modes are freezing out, and we call this the renormalized classical regime. Since the on-string modes drive the meandering of the string one would expect that something happens with the entropic interactions around $\nu_s = 1$. As it turns out, however, the induced modulus continues to show its high-temperature behavior while even the prefactors are not affected. This counter-intuitive fact might be understood qualitatively on the basis of Eqs. (5) and (6). Let us estimate the characteristic energy of the on-string mode, which has a wavelength of order of the collision length l_c: $\omega_{coll.} = \hbar c/l_c \sim \hbar c T/d^2\Sigma_c$. This implies that this energy scales down linearly with the temperature T while it corresponds with the *lower bound* in energy on the modes which contribute to the collision length. Modes with lower energy have a wavelength larger than the collision length and do therefore not contribute. Hence, instead of having all modes contributing to the thermal fluctuations, for $\nu_s < 1$ only modes with frequencies $\omega_{coll} = \alpha_{coll} T < \omega < T$ contribute. At first sight one would expect this to give rise to a gross change. However, on closer inspection one finds it to be more subtle. A first requirement is that $\alpha_{coll} \ll 1$, otherwise the contribution of the thermal fluctuations would be completely quenched out. To estimate α_{coll} we recall that the importance of quantum fluctuations in the string gas is measured by the dimensionless ratio

$$\mu_g = \frac{\hbar}{\rho_c c d^2}, \quad (8)$$

corresponding with the ratio of $\hbar$ with the dimensionful quantities characterizing the problem: the coupling constant of the gas or the "de Boer parameter."[5] μ_g has to be small compared to unity because otherwise quantum fluctuations become important at the lattice scale, and the continuum description fails. Using Eq. (6) we find immediately that $\alpha_{coll} = \omega_{coll}/T \sim \mu_g \ll 1$! Hence the window in the mode spectrum contributing to the thermal meandering is in this sense large and it remains large regardless the fact that temperature becomes less than ω_s, at least initially.

It is still so that the short-wavelength modes with frequency $\omega > T$ are no longer available for the thermal meandering. However, a simple calculation shows that their contribution is exponentially small. Consider again Eq. (3), realizing that only the zero-point contribution matters for momenta in the interval $T/(\hbar c) < k < \pi/a$ while the thermal contribution dominates in the interval $\pi/l_c < k < T/(\hbar c)$. Carrying out the integrals,

$$\langle[\phi(y)-\phi(0)]^2\rangle \approx 1/2 \int_{T/\hbar c}^{\pi/a} \frac{\hbar}{\rho_c ck} dk + \int_{\pi/y}^{T/\hbar c} \frac{T}{\rho_c (ck)^2} dk$$

$$\approx \frac{\hbar}{2\rho_c c} \ln\frac{\pi\hbar c}{aT} + \frac{Ty}{\pi\Sigma_c} + O(\mu d^2). \quad (9)$$

The logarithmic term in Eq. (9) is due to the quantum zero-point string meandering coming from the modes with $\omega > T$ while the second term corresponds with the thermal meandering. Equating expression in Eq. (9) to d^2 we find a renormalized classical result (6):

$$Tl_c/\Sigma_c \approx d^2[1+O(\mu)], \quad (10)$$

provided that the logarithmic term is small. For this to be true, the temperature should be above a classical-to-quantum crossover temperature $T_0 \ll \omega_s$. At sufficiently low temperature eventually the logarithm will dominate and we can therefore estimate T_0 by neglecting the second term in Eq. (9) and equating the logarithm to d^2. It follows that

$$T_0 \sim \frac{\pi\hbar c}{a} \exp\left[-\frac{2}{\mu}\right]. \quad (11)$$

As we shall see in the following sections, this estimate for T_0 is actually flawed, while the correct answer is the stretched exponential following from the Helfrich self-consistent phonon method. However, for the purpose of crude estimations it suffices. T_0 is small compared to the on-string Debye temperature, because $\mu_g < 1$ and as long as $T \gg T_0$ the logarithm can be neglected. As long as this is the case nothing changes as compared to the high-temperature limit, because Eqs. (10) and (6) are just the same! Hence we arrive at the counterintuitive result that the collision density and thereby the induced modulus is insensitive to the freezing out of the on-string modes in the renormalized classical region in between T_0 and ω_s.

Finally, at temperatures $T < T_0$ quantum mechanics is expected to take over. The wavelength of the thermally excited modes are now large as compared to the collision length set by the quantum fluctuations. Therefore we can neglect the Bose factor in Eq. (3) completely,

$$\langle[\phi(y)-\phi(0)]^2\rangle \sim \int_{\pi/y}^{\pi/a} \frac{\hbar}{\rho_c ck} \frac{1}{2} dk = \frac{\hbar}{2\rho_c c} \ln\left(\frac{y}{a}\right). \quad (12)$$

Equating the latter estimate to d^2 we find an exponentially large, though finite, value of the collision length l_c:

$$l_c \sim a \exp\left[\frac{2}{\mu}\right]. \quad (13)$$

Thus an effective quantum-meandering induced repulsion is expected to be exponentially small but finite in the zero-temperature limit. We repeat, this last estimate is flawed as the comparison with numerical results shows. In any other regard we will find that the simple estimates presented in this section are consistent with the results of the more involved Helfrich method. This includes the basic observation that the zero-temperature induced modulus survives at small but finite temperatures $T < T_0$. This finding suffices to protect the solidification of the string gas at zero temperature because the dislocations can only proliferate at a Kosterlitz-Thouless transition at a temperature $T_{KT} \sim T_0$, according to the exact result of Pokrovsky and Talapov.[16]

III. CHARACTERISTIC SCALES ACCORDING TO HELFRICH'S METHOD

As explained in Ref. 13, within the Helfrich scheme[14] the avoidance condition Eq. (1) is dropped, and a self-consistently determined effective rigidity B is introduced instead. The parameter B governs the interstring interactions in the effective long-wavelength action S_{eff}, which substitutes for the exact action S as defined in Eq. (2),

$$S_{eff} = \frac{1}{2} \int d\tau\, dx\, dy [\rho(\partial_\tau\psi)^2 + \Sigma(\partial_y\psi)^2 + B(\partial_x\psi)^2]. \quad (14)$$

Here $\psi(x,y,\tau)$ is the coarse grained, long-wavelength displacement field taking over from the bare field $\phi_i(y,\tau)$, and $\Sigma = \Sigma_c/d$, $\rho = \rho_c/d$. The self-consistency equation from which the induced rigidity B can be determined is nothing else than the well-known relation between the free energy F of an elastic medium and its compression modulus,

$$B = d^2 \frac{\partial^2(\Delta F(B)/V)}{\partial d^2}. \quad (15)$$

Here V is the system's real-space volume, i.e., in the case of two-dimensional space the area $V \to L^2$, where L is the linear dimension of this space. The free-energy difference $\Delta F(B)$ is defined as

$$\Delta F(B) \equiv F(B) - F(B=0) = -T \ln\{Z(B)/Z(B=0)\}. \quad (16)$$

$\Delta F(B)$ is most conveniently derived using the standard procedure, representing the free field as a set of noninteracting quantum harmonic oscillators with frequencies

$$\omega_q = \sqrt{\Sigma q_y^2 + B q_x^2}, \quad (17)$$

defined for wave vectors $\vec{q}$ in momentum space bound by UV cutoffs: $|q_y| \leq \pi/a \equiv Q_s$; $|q_x| \leq \pi/d\eta \equiv Q_g$, where Q_s and Q_g are the cutoffs along and perpendicular to the string direction, respectively. $\eta \sim 1$ is a "fudge-factor," correcting for inaccuracies coming from short distance mode-coupling effects.[13] Using this representation for S_{eff}, one easily obtains familiar expressions for the partition function Z and the free energy F of an ideal gas of harmonic oscillators[17]

$$Z = \prod_q \exp(-\beta\hbar\omega_q/2)(1-\exp\{-\beta\hbar\omega_q\})^{-1},$$

$$F = -T \ln Z = \sum_q \frac{\hbar\omega_q}{2} + k_B T \sum_q \ln(1-\exp\{-\beta\hbar\omega_q\})$$

$$= F_0 + F_T, \quad (18)$$

where $\beta \equiv 1/T$. In the thermodynamic limit $LQ_s \gg 1$, $LQ_g \gg 1$, and the momentum summation in Eq. (18) can be substituted with an integration, leading to the following expression for $\Delta F(B)$ ($\hbar = 1$),

$$
\begin{aligned}
\Delta F(B)/L^2 &= \frac{1}{(2\pi)^2} \int_{-Q_s}^{Q_s} dq_y \int_{-Q_g}^{Q_g} dq_x \\
&\times \left\{ \frac{\sqrt{\Sigma q_y^2 + B q_x^2}}{2\sqrt{\rho}} - \frac{\sqrt{\Sigma q_y^2}}{2\sqrt{\rho}} \right\} + \frac{T^3 \rho}{\pi^2 \sqrt{\Sigma B}} I(\nu_s, \nu_g) \\
&= \frac{T^3 \rho}{\pi^2 \sqrt{\Sigma B}} \left[\int_0^{\nu_s} dy \int_0^{\nu_g} dx \right. \\
&\times \left. \left(\frac{\sqrt{x^2 + y^2}}{2} - \frac{y}{2} \right) + I(\nu_s, \nu_g) \right] \\
&\equiv \Delta_0 + \Delta_T .
\end{aligned}
\tag{19}
$$

The first term corresponds with the zero-temperature quantum contribution to the free energy (Δ_0), while the second term (Δ_T) contains the thermal contributions, proportional to

$$
I(\nu_s, \nu_g) = \int_0^{\nu_s} dy \int_0^{\nu_g} dx \ln\left(\frac{1 - \exp\{-\sqrt{x^2+y^2}\}}{1 - \exp\{-y\}} \right) .
\tag{20}
$$

Besides the prefactor in front of I, temperature enters only through the upper bounds in the integrals, Eq. (20). ν_s we encountered already in the previous section as the dimensionless on-string Debye temperature Eq. (7). The interesting point is that we have also to introduce a dimensionless ratio associated with the Debye temperature for the fluctuations perpendicular to the strings:

$$
\nu_g = (\pi/\eta d) \sqrt{B}/(T\sqrt{\rho}) = \omega_g / T ,
\tag{21}
$$

where ω_g plays the role of frequency cutoff (Debye temperature) for the interstring vibrations along the x axis. This cutoff has to be determined self-consistently because it obviously depends on the induced rigidity itself. This is a complicating factor. This ω_g is of crucial importance for balancing the relative importance of thermal and quantum fluctuations, while at the same time it is itself determined by this balance. We will demonstrate later that ω_g in fact acts according to the naive expectations. When T becomes less than ω_g the frequency window available for thermal fluctuations quickly diminishes and the quantum fluctuations take over completely already at a finite temperature. All what remains is to calculate self-consistently what ω_g is and this will turn out to be proportional to the zero temperature B for any temperature such that $\nu_g > 1$.

Another matter is the renormalized classical regime, defined by $\omega_g < T < \omega_s$. We will find that the self-consistent phonons do confirm the arguments of the previous section; the induced modulus is highly insensitive to the difference between the quantum-mechanical versus classical nature of the "high"-frequency phonons.

To complete the analysis of the dimensionless parameters characterizing this problem, we notice that there are actually two coupling constants. The first one arises already on the level of the bare strings, and it parametrizes the importance of quantum physics for an isolated string,

$$
\mu_s = \hbar/(\rho_c c a^2),
\tag{22}
$$

and we notice that this is related to the coupling constant of the string gas, Eq. (8), through,

$$
\mu_g \equiv \mu_s a^2/d^2 = \hbar/(\rho_c c d^2).
\tag{23}
$$

It now becomes directly clear why $\mu_g \ll 1$; the ratio $(a/d)^2 < 1$ while the meaning of $\mu_s = 1$ is that the quantum fluctuations become strong on the scale of the lattice constant thereby invalidating the notion of a single continuum string.

Interestingly, the single string parameter μ_s arises naturally in the Helfrich method when the self-consistency Eq. (15) for the induced rigidity $B(d)$ is written in dimensionless form:

$$
z = \mu_s \frac{\partial}{\partial s} \left(s^2 \frac{\partial \Delta(z, s, \zeta)}{\partial s} \right),
\tag{24}
$$

where the dimensionless variables are defined as follows:

$$
s = \frac{Q_g}{Q_s} = \frac{a}{\eta d}, \quad z = s\frac{B}{\Sigma}, \quad \Delta(z, s, \zeta) = \frac{a^2}{\hbar \omega_s} \frac{\Delta F}{L^2} .
\tag{25}
$$

This demonstrates that μ_s is the quantity setting the overall scale of the problem—everything else follows from the self-consistency condition. Summarizing, we have established that the string gas problem is characterized by four dimensionless quantities. Besides the coupling constants μ_s and μ_g, governing the zero-temperature physics, we also find the two Debye temperatures ν_s and ν_g governing the balance between quantum and thermal fluctuations. There are altogether three regimes: (a) $\nu_g > 1$, the low-temperature, quantum dominated regime, (b) $\nu_s < 1$, the high-temperature regime dominated by thermal fluctuations, (c) $\nu_g < 1 < \nu_s$, the renormalized classical regime where the dynamics is quantum mechanical at short distances while the system as a whole still behaves as if it is in the high-temperature limit.

IV. INDUCED MODULI FROM LOW TO HIGH TEMPERATURE

In this section we will analyze in detail the behaviors of the induced moduli in the various temperature regimes as follow from the Helfrich method. In the first two subsections we will revisit the zero-temperature and high-temperature regimes which were already analyzed by one of us.[13] In the last two sections we approach the intermediate temperature renormalized classical regime both from the high-temperature and low-temperature side, demonstrating that (i) the induced modulus does not change upon entering the renormalized classical regime from the high-temperature side, while (ii) ν_g is actually finite, with the implication that

the zero-temperature quantum rigidity already takes over at a small but finite temperature.

A. Zero temperature: Quantum-entropic rigidity

Let us first evaluate the free energy Eq. (19) at zero temperature. It reduces to Δ_0 and this becomes with logarithmic accuracy:

$$
\begin{aligned}
\Delta_0 &= \frac{T^3 \rho}{\pi^2 \sqrt{\Sigma B}} \int_0^{\nu_s} dy \int_0^{\nu_g} dx \left(\frac{\sqrt{x^2+y^2}}{2} - \frac{y}{2} \right) \\
&\approx \frac{T^3 \rho}{\pi^2 \sqrt{\Sigma B}} \int_0^{\nu_g} dx \int_{\nu_g}^{\nu_s} dy \frac{x^2}{4y} = \frac{T^3 \rho}{\pi^2 \sqrt{\Sigma B}} \frac{\nu_g^3}{12} \ln \left\{ \frac{\nu_s}{\nu_g} \right\} \\
&\equiv \frac{Q_g^3 B}{12 \pi^2 \sqrt{\rho \Sigma}} \ln \left\{ \frac{Q_s \sqrt{\Sigma}}{Q_g \sqrt{B}} \right\}.
\end{aligned}
\tag{26}
$$

This result illustrates the crucial observation in Ref. 13 that the dominating contribution to Δ_0 originates in the *long-wavelength* on-string quantum fluctuations. The logarithm lives at the lower limit ν_g of the y integration, associated with the long-wavelength on-string fluctuations with momentum $q_y(y=\nu_g) \equiv \nu_g T \sqrt{\rho}/\sqrt{\Sigma} \sim Q_g \sqrt{B/\Sigma}$. At the same time, however, this free energy is dominated by large inter-string momenta, $q_x(x=\nu_g) \equiv \nu_g T \sqrt{\rho}/\sqrt{B} \sim Q_g$, and in this regard it is a short-wavelength physics, as in the high-temperature regime (next subsection). A more careful evaluation of the integral in Eq. (26) yields

$$
\Delta_0 = \frac{Q_g^3 B}{24 \pi^2 \sqrt{\rho \Sigma}} \left[2 \ln \left\{ \frac{2 Q_s \sqrt{\Sigma}}{Q_g \sqrt{B}} \right\} + \frac{5}{3} \right].
\tag{27}
$$

Substituting the estimate Eq. (26) in the self-consistency Eq. (15) we find the following equation for the induced rigidity B:

$$
\begin{aligned}
\frac{B}{d^2} &= \frac{\partial^2}{\partial d^2} \left[\frac{Q_g^3 B}{24 \pi^2 \sqrt{\rho \Sigma}} \ln \left\{ \frac{Q_s^2 \Sigma}{Q_g^2 B} \right\} \right] \\
&= -\frac{\partial^2}{\partial d^2} \left[\frac{\pi c}{24 \eta^3 \Sigma_c} \left(\frac{B}{d^2} \right) \ln \left\{ d \left(\frac{B}{d^2} \right) \frac{a^2}{\eta^2 \Sigma_c} \right\} \right].
\end{aligned}
\tag{28}
$$

Taking as an ansatz for $B(d)/d^2 \equiv \exp[-\Phi(d)]$, Eq. (28) can be solved with exponential accuracy (quasiclassical approximation), which is valid when μ_g is small. It follows that[13]

$$
\begin{aligned}
B &= A d^2 \exp \left\{ -\eta \left(\frac{54}{\pi} \right)^{1/3} \frac{1}{\mu_s^{1/3}} \left(\frac{d}{a} \right)^{2/3} \right\} \\
&= A d^2 \exp \left\{ -\eta \left(\frac{54}{\pi} \right)^{1/3} \frac{1}{\mu_g^{1/3}} \right\}.
\end{aligned}
\tag{29}
$$

Hence, instead of $B \sim \exp(-A/\mu_g)$ as followed from the "naive" collision argument of Sec. II (Eq. 13), the Helfrich method yields a stretched-exponential dependence on the coupling constant μ_g. This stretched exponential comes from the logarithm in Eq. (27); upon neglecting this log one

recovers the full exponential. This logarithm finds in turn its origin in the long-wavelength on-string fluctuations which are particularly dangerous for the zero-temperature elastic strings, reflecting the algebraic order of a single string. In the collision language there is no room for these on-string long-wavelength fluctuations and the Helfrich method suggests therefore a qualitatively different physics behind the induced quantum modulus. This seems now confirmed by a numerical simulation. According to the numerical work of Nishiyama,[15] $B \sim \exp(-A'd^\beta)$ with $\beta = 0.808(1)$, very close to our prediction $\beta = 2/3$ and very different from the naive expectation $\beta = 2$. We believe that the small difference between the numerical result and our result is due to "fluctuation" corrections; Helfrich's method has the structure of a mean-field theory and it should be possible to construct a perturbation expansion based on the difference between the exact action Eq. (2) and the "saddle-point" action Eq. (14).

B. High-temperature classical regime: $\nu_g \ll \nu_s \ll 1$

Consider now the high-temperature, fully classical regime $\nu_g \ll \nu_s \ll 1$, where all mode frequencies are small compared to temperature. In this limit the integral I in Eq. (20) becomes

$$
\begin{aligned}
I(\nu_s, \nu_g) &= \int_0^{\nu_s} dy \int_0^{\nu_g} dx \ln \left(\frac{1 - \exp\{-\sqrt{x^2+y^2}\}}{1 - \exp\{-y\}} \right) \\
&\approx -\int_0^{\nu_s} dy \int_0^{\nu_g} dx \left(\frac{\sqrt{x^2+y^2}}{2} - \frac{y}{2} \right) + \tilde{I}(\nu_s, \nu_g),
\end{aligned}
\tag{30}
$$

where

$$
\tilde{I}(\nu_s, \nu_g) \approx \int_0^{\nu_s} dy \int_0^{\nu_g} dx \ln \left(\frac{\sqrt{x^2+y^2}}{y} \right).
\tag{31}
$$

Thus from Eqs. (30) and (19) it is obvious that at high temperature, such that $\nu_g \ll \nu_s \ll 1$, the I term splits into two parts with the opposite signs. The *negative* part of the integral I exactly cancels the Δ_0 part in the free energy ΔF, in Eq. (19), while the positive part $\tilde{I}(\nu_s, \nu_g)$ plays the role of the high-temperature, "classical" limit of the free energy:

$$
\Delta F(B)/L^2 \approx \frac{T^3 \rho}{\pi^2 \sqrt{\Sigma B}} \tilde{I}(\nu_s, \nu_g).
\tag{32}
$$

The integration over y in the integral in Eq. (31) is dominated by the interval $y < \nu_g \ll \nu_s$. Therefore the upper integration limit over y could be made infinite with minor mistake: $\nu_s \to \infty$. After that, the integral is made dimensionless by a change of integration variables: $x \to \nu_g x'$ and $y \to \nu_g y'$. In this way, ν_g can be scaled out from the double integral and we find

$$
\tilde{I} = \text{const} \times \nu_g^2.
\tag{33}
$$

The const can be calculated exactly to be equal to $\pi/4$. Substituting Eq. (33) into Eq. (32) we find

$$\Delta F(B)/L^2 \approx \frac{\pi T \sqrt{B}}{4d^2 \eta^2 \sqrt{\Sigma}} = \frac{\pi T \sqrt{B}}{4d^{3/2}\eta^2 \sqrt{\Sigma_c}}. \quad (34)$$

Substituting this result into the self-consistency Eq. (15) yields

$$\frac{B}{d^2} = \frac{\partial^2}{\partial d^2}\left[\frac{\pi T \sqrt{B}}{4d^{3/2}\eta^2 \sqrt{\Sigma_c}}\right]. \quad (35)$$

Using as an ansatz for the unknown function $B(d) = Cd^{\alpha}$, one finds from Eq. (35)

$$B = \frac{9\pi^2 T^2}{\Sigma_c d^3}. \quad (36)$$

To be valid it should be checked if this result is consistent with the initial assumption $\nu_g \ll 1$. Substituting Eq. (36) in the definition of ν_g and demanding that $\nu_g \ll 1$ we find

$$\nu_g \equiv \omega_g/T = (\pi/\eta d)\sqrt{B}/(T\sqrt{\rho}) = \frac{3\pi^2}{\eta^3}\mu_g \sim \mu_g \ll 1 \quad (37)$$

which is indeed consistent with the general condition that $\mu_g \ll 1$ in Eq. (23).

The solution (36) can actually be directly checked using the fact that the high temperature limit of the directed string gas in $2+1$D is actually the same problem as the zero-temperature system of hard-core bosons in $1+1$D. As discussed in Ref. 13, the latter can be considered as the compactified version of the former, where temperature plays the role of wrapping up one of the dimensions. The $1+1$D hard core boson problem can be mapped on the trivial problem of noninteracting spinless fermions in $1+1$D. Rewriting Eq. (36) in terms of the dimensionful quantities characterizing the $1+1$D hard core boson gas (mass of the bosons M, interparticle distance d) one finds[13]

$$B_{1d}(d) = \frac{9\pi^2}{\eta^4}\frac{\hbar^2}{Md^3}. \quad (38)$$

Comparing this result with the exact results following from the spinless fermions one finds that[13] $\eta = \sqrt{6}$, indicating that the Helfrich self-consistent phonon method is remarkably accurate.

C. Approaching the renormalized classical regime from the high-temperature side

In the previous two subsections we demonstrated that the Helfrich method was completely tractable analytically both in the zero- and high-temperature limit. This is a lucky circumstance because the equations are in principle quite cumbersome and the calculations only become straightforward because we could exploit the smallness of μ_g. This becomes different in the intermediate temperature regime. Despite the expectation that also in this renormalized classical regime the answer is simple and actually the same as in the high-
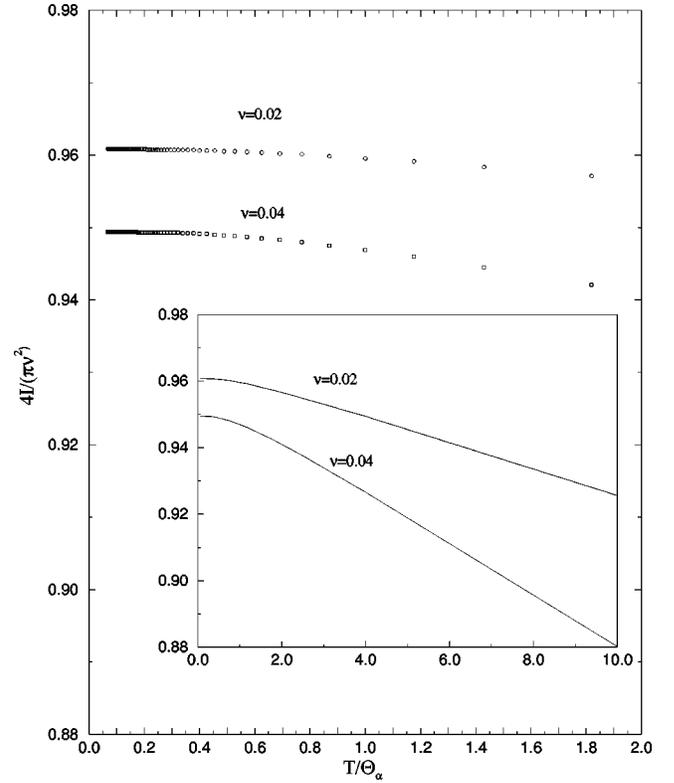

FIG. 1. Dependence of $I/(\pi \nu_g^2/4)$ on $\nu_s^{-1} = T/\omega_s$ for two different values of ν_g: $\nu_g = 0.02$ and $\nu_g = 0.04$.

temperature regime (Sec. II), we failed to find a closed analytic solution for the induced modulus at intermediate temperatures. We recall that the renormalized classical regime is defined as $\nu_g \ll 1 \ll \nu_s$ which means that the on-string Debye temperature is now large as compared to temperature. The largeness of ν_s makes it impossible to expand the integral I in Eq. (19) in a way it was done in Eq. (30), because now the integration variable y is not $\ll 1$ in the region $1 < y \ll \nu_s$. To put it another way, the difference between the two cases is due to the quantum zero-point contributions, because when $\nu_s \gg 1$ the on-string fluctuations have entered the quantum regime. Nevertheless, guided by the discussion in Sec. II (see text after Eq. (9)), we calculated numerically the ratio of the integral I to the high-temperature expression $\tilde{I}$ in Eq. (33) in the renormalized classical region of parameters $\nu_g \ll 1 \ll \nu_s$, see Fig. 1. As one might expect, the ratio remains close to 1 and nearly constant in the wide interval of the values of the parameter ν_s at the two fixed values of $\nu_g \ll 1$: $\nu_g = 0.02$; 0.04. Namely, as is seen in Fig. 1, the interval of ν_s's starts in the classical region: $\nu_g \ll \nu_s < 0.5$ and goes deep into the renormalized classical region: $\nu_g \ll 1 \ll \nu_s \approx 20$. Nevertheless, the ratio $I/\tilde{I}$ is ≈ 0.95 ($\nu_g = 0.04$) or ≈ 0.96 ($\nu_g = 0.02$) and changes by less than 2% of its value in the interval $0.5 \leq \nu_s \leq 20$. Having this numerical evidence, we conclude that, within a numerical factor which is very close to unity, the high-temperature solution for the induced rigidity B, Eq. (36), remains valid in the renormalized classical regime.

D. Approaching the renormalized classical regime from low temperatures

As we already emphasized, it is a most crucial issue if the zero-temperature quantum-kinetic interactions survive at a finite temperature. Using the collision arguments, we already argued in Sec. II that this should be the case. However, we also learned that these arguments fail qualitatively at zero temperature. At the same time, as we discussed in Sec. IV A the Helfrich method indicates that the on-string long-wavelength fluctuations are decisive for the zero-temperature rigidity—on general grounds one expects that any quantum phenomenon related to long wavelength should be quite vulnerable to the finiteness of temperature and it is therefore *a priori* unclear what to expect at a small but finite temperature. Fortunately, it turns out that the Helfrich method remains tractable in this regime of small but finite temperature and here we will derive a controlled solution for the induced modulus in this regime. It turns out that the zero-temperature modulus is robust at finite temperatures, up to a crossover temperature T_0 which is, however, different from the simple estimate presented in Sec. II.

Let us consider the low-temperature corrections to the quantum result Eq. (27). We assume that in this regime the induced modulus is finite so that a regime exists where temperature can become small as compared to the interstring Debye temperature. We will find that this assumption is self-consistently satisfied. Hence we are now dealing with the limit $1 \ll \nu_g \ll \nu_s$ and in this case the double integral in Eq. (20) can be rewritten as, in exponential accuracy,

$$I \equiv I(\nu_s, \nu_g) = \int_0^\infty dy \int_0^{\nu_g} dx \ln\left(\frac{1 - \exp\{-\sqrt{x^2 + y^2}\}}{1 - \exp\{-y\}}\right)$$

$$\approx \int_0^\infty dy \int_0^\infty dx \ln(1 - \exp\{-\sqrt{x^2 + y^2}\})$$

$$- \nu_g \int_0^\infty dy \ln(1 - \exp\{-y\}). \quad (39)$$

Recalling the definition Eq. (19), this leads to the following estimate ($\zeta(3) = 1.202$):

$$\Delta_T \approx \frac{T^2 Q_g \sqrt{\rho}}{\sqrt{\Sigma}} \frac{\pi}{12} - \frac{T^3 \rho}{\sqrt{\Sigma B}} \left(\frac{\zeta(3)}{2\pi}\right) \quad \text{if} \quad 1 \ll \nu_g \ll \nu_s. \quad (40)$$

Combining this estimate with Eq. (27) the free energy becomes in the low-temperature limit

$$\Delta F(B)/L^2 = \frac{Q_g^3 B}{24\pi^2 \sqrt{\rho\Sigma}} \left[2 \ln\left\{\frac{2Q_s \sqrt{\Sigma}}{Q_g \sqrt{B}}\right\} + \frac{5}{3}\right]$$

$$+ \frac{T^2 Q_g \sqrt{\rho}}{\sqrt{\Sigma}} \frac{\pi}{12} - \frac{T^3 \rho}{\sqrt{\Sigma B}} \left(\frac{\zeta(3)}{2\pi}\right). \quad (41)$$

It is directly clear that at sufficiently low temperatures the temperature-dependent corrections can be neglected in the self-consistency differential equation, which means that the quantum modulus Eq. (27) remains valid at the lowest tem-

peratures. Hence there has to be a temperature T_0 where the thermal fluctuations become of similar importance as the quantum fluctuations. This T_0 can be estimated from Eq. (41) by equating the leading temperature-dependent correction $\sim T^2$ to the zero-temperature term,

$$\frac{Q_g^3 B}{24\pi^2 \sqrt{\rho\Sigma}} \left[2 \ln\left\{\frac{2Q_s \sqrt{\Sigma}}{Q_g \sqrt{B}}\right\} + \frac{5}{3}\right] \sim \frac{T^2 Q_g \sqrt{\rho}}{\sqrt{\Sigma}} \frac{\pi}{12}. \quad (42)$$

Substituting in Eq. (42) the zero-temperature result for B, Eq. (29), one finds

$$T_0 \propto \sqrt{B} \sim \exp\left\{-\eta\left(\frac{54}{\pi}\right)^{1/3} \frac{1}{2\mu_s^{1/3}} \left(\frac{d}{a}\right)^{2/3}\right\}$$

$$= \exp\left\{-\eta\left(\frac{54}{\pi}\right)^{1/3} \frac{1}{2\mu_g^{1/3}}\right\}. \quad (43)$$

Hence we find that the elementary consideration in Sec. II leading to the estimate Eq. (11) is in essence correct, except that the stretched exponential result has to be used for the zero-temperature modulus. Of course, T_0 has the same status as ω_g and a regime of small but finite temperatures exists where $\nu_g \gg 1$ due to the zero-temperature quantum fluctuations. The physics behind this result can be deduced from the low-temperature expansion Eq. (41). In contrast to the renormalized classical regime entered at $T > T_0$, the string gas Debye temperature ω_g acts according to the expectations. For $T < \omega_g$ the fraction of the modes which are thermally occupied diminishes quickly, making it possible to arrive at the simple expansion Eq. (41). At the same time, it follows from the form of this expansion that the long-wavelength aspect of the zero-temperature modulus is not as simple as discussed in the beginning of this subsection. Equation (41) still starts out with the unmodified zero-temperature result despite the fact that temperature is finite and this term should be destroyed immediately if the argument that the long-wavelength on-string fluctuations immediately lose their quantum character for any finite temperature would be correct. The resolution of this apparent paradox is that the free energy is that of the effective 2+1D medium and the zero-temperature logarithm appears in the final answer, after integrating both interstring and intrastring momenta. This implies a quantum rigidity for the system as a whole, and this in turn leads to a diminishing of all thermal fluctuations, including those acting along the strings. The implication is that there is still a length scale where the quantum-induced modulus appears and when temperature becomes low enough the wavelength of the typical thermal fluctuations becomes large as compared to this length scale and thereby inconsequential for the induced modulus. In this sense the basic argument of Sec. II survives in this self-consistent phonon language. The T_0 vs d dependence as expressed in Eq. (43) is plotted in Fig. 2. Based on the observation made in Sec. II that the dislocations can only proliferate at a Kosterlitz-Thouless temperature $T_{KT} \sim T_0$, we consider Fig. 2 as the phase diagram of the directed hard-core string gas. $T_0(d)$ plays a role of the solidification line, which separates the
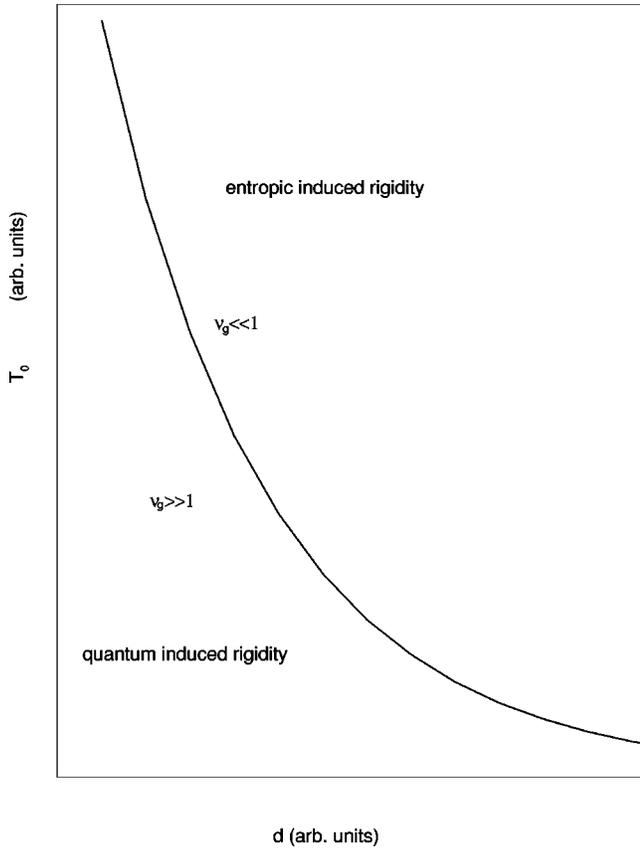
FIG. 2. Phase diagram of the directed hard-core string gas: a solidification temperature T_0 as function of the average interstring distance d.

low-temperature long-range ordered state of the string-system at $T<T_0(d)$ from the disordered string-gas state at the high temperatures: $T>T_0(d)$.

V. CONCLUSIONS

In this paper we have presented results of a detailed study of a gas of elastic quantum strings in 2+1 dimensions, interacting via a hard-core condition. The model mimics some essential features of a dynamical stripe state, arising in the underdoped cuprate superconductors. From a more general perspective, it relates to the theme of entropic interactions at finite temperatures and quantum-kinetic interactions at zero temperature.

Although in detail quite different, the physics at zero temperature falls in the same category as, for instance, superexchange which is for good reasons also called kinetic exchange. Physics which is associated with kinetic energy (electron hops) at short distances becomes physics associated with order and potential energy at long distances (antiferromagnetism). In the string gas, the short distance physics corresponds with the string fluctuations and the long distance physics is that of long-range order, breaking translational symmetry. As was emphasized in Ref. 13, the elastic string gas in 2+1D is a close relative of the Luttinger liquid in 1+1D. The same order-out-of-disorder mechanism which renders the algebraic order of the hard-core bose gas in

1+1D to be generic causes the true long-range order in the string gas at zero temperature.

Here we focused mainly on the finite-temperature physics of the quantum string gas. We found this to be a rather nontrivial affair. The same basic mechanism which is responsible for the zero-temperature kinetic interactions is responsible for the entropic interactions at high temperature. We focused on the question, how do these two regimes connect?

Against our initial expectations we found out that quantum mechanics takes over rather suddenly. We pointed out that there is a temperature scale below which the quantum-mechanical nature of the string fluctuations becomes important. However, we found that these quantum fluctuations are initially completely hidden: the entropic interactions continue to behave as if the strings are fully classical. The reason is that the typical string fluctuations responsible for the induced modulus live at frequencies which also decrease with temperature, in such a way that the quantum cutoff stays effectively at infinity. As a result, the entropic interactions stay in this renormalized classical regime in fact unrenormalized.

One could now have the impression that thermal fluctuations would continue to overrule the quantum fluctuations down to the lowest temperatures. An additional motive could be that the zero-temperature "stretched exponential modulus" which has been confirmed by numerical simulations is due to long-wavelength on-string quantum fluctuations. One would anticipate that these long-wavelength quantum fluctuations would be extremely vulnerable to finite temperature. However, we found that the zero-temperature kinetic interactions are self-protecting in the regime of small but finite temperatures. Below the scale T_0, which is set by the zero-temperature modulus, quantum mechanics starts to play a conventional role. Below T_0 the phase space for thermal fluctuations shrinks rapidly and thereby also the influence of the thermal fluctuations on the induced modulus. Hence, while this conventional intuition failed badly at intermediate temperatures, it is correct at temperatures less than T_0.

Of course, the above picture rests entirely on the self-consistent phonon method invented a long time ago by Helfrich. This approximate method is put here by us to the test in an interesting way. However, we have confidence that the above conclusions are trustworthy. After all, the hardest part is zero temperature where according to the Helfrich method a truly different mechanism is at work, giving rise to the stretched exponential. Except for some small correction, likely due to "fluctuations around the mean-field," this stretched exponential turns out to be correct. Given that the high-temperature limit is also described rather accurately, it has to be that the intermediate temperature regime is also described accurately.

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