Power-law localization in two and three dimensions with off-diagonal disorder

Shi-Jie Xiong[1] and S. N. Evangelou[2]
[1]*Department of Physics, Nanjing University, Nanjing 210093, China*
[2]*Department of Physics, University of Ioannina, Ioannina 45110, Greece*

(Received 5 January 2001; revised manuscript received 30 April 2001; published 31 August 2001)

We describe the nonconventional localization of the midband $E=0$ state in square and cubic finite-bipartite lattices with off-diagonal disorder by solving numerically the linear equations for the corresponding amplitudes. This state is shown to display multifractal fluctuations, having many sparse peaks, and by scaling the participation ratio we obtain its disorder-dependent fractal dimension D_2. A logarithmic average correlation function grows as $g(r)\sim\eta\ln r$ at distance r from the maximum amplitude and is consistent with a typical overall power-law decay $|\psi(r)|\sim r^{-\eta}$ where η is proportional to the strength of off-diagonal disorder.

DOI: 10.1103/PhysRevB.64.113107 PACS number(s): 71.23.An, 73.21.−b, 73.20.Jc

It is well-known that off-diagonal disorder in the nearest neighbors of one-dimensional (1D) lattice is responsible for anomalous localization of the corresponding state at the band center $E=0$.[1–4] The log-amplitude $\ln|\psi_i|$ at site i executes a random walk and the typical wave function decays exponentially at distance r from its maximum as $|\psi(r)|\sim\exp(-\alpha\sqrt{r})$, with α a disorder-dependent parameter. The $E=0$ is a special state arising from a sublattice symmetry, which can exist in the presence of off-diagonal disorder and causes the Hamiltonian to change sign under the transformation $\psi_i\rightarrow(-1)^i\psi_i$. This symmetry exists in bipartite lattices with two sublattices one connected to the other[1,5–8] and is also known as chiral symmetry. The corresponding eigenvalues appear in pairs with energies $(E,-E)$ and since the band center $E=0$ is a special energy the corresponding wave function can be easily constructed.

In 2D and 3D bipartite systems, such as square or cubic lattices, the question of localization at the band center with off-diagonal disorder is not resolved as in 1D and the decay of the wave-function amplitude from its maximum is not yet firmly established. It is believed[5–9] that nonexponential decay occurs in 2D and even absence of decay in 3D where extended states exist for weak disorder. The approach of Ref. 5 predicted $\exp(-\gamma\sqrt{\ln r})$ decay in every dimension higher than one, which is rather weaker than a power law. The theoretical interest in this problem has been recently revived. On one hand, due to the related problem of random Dirac fermions[10,11] that, however, starts from a different zero-disorder limit. In the case of off-diagonal disorder one has a Fermi surface with many points forming a square while in the problem of Dirac fermions only four points exist at zero energy without disorder. In the former case the pure density of states at $E=0$ has the well-known two-dimensional log-singularity while in the Dirac case the density of states approaches zero at the band center.[11] On the other hand, the study of localization with off-diagonal disorder could be important for understanding properties of several realistic 2D systems, such as states in semiconductor quantum wells and high-mobility silicon metal-oxide-semiconductor field-effect transistors,[12] current-currying states in quantum Hall effect,[13,14] etc.

In this report we present an exclusive study of the midband $E=0$ state in 2D (squared) and 3D (cubic) bipartite lattices with disorder in the nearest-neighbor hoppings. We compute the coefficients of the wave function for large finite systems and obtain the participation ratio. The amplitudes are shown to display multifractal fluctuations with many scattered peaks, characterized by a fractal dimension that depends on the strength of disorder. An appropriate correlation function is shown to behave as $g(r)\sim\eta\ln r$ where η also depends on the strength of disorder. It implies a power-law decay $|\psi(r)|\sim r^{-\eta}$ of the peak heights from the maximum peak, in contrast to the typical one-dimensional decay $|\psi(r)|\sim\exp(-\alpha\sqrt{r})$.

We consider a tight-binding Hamiltonian in 2D and 3D lattices with random nearest-neighbor hoppings

$$H=\sum_{\langle ij\rangle}(t_{ij}c_i^{\dagger}c_j+\text{H.c.}),\tag{1}$$

where c_i is the annihilation operator of an electron on the ith site and the nearest-neighbor hopping integrals t_{ij} are real positive random variables satisfying the probability distribution

$$P(\ln t_{ij})=\frac{1}{W}\quad\text{for}\ -\frac{W}{2}\leqslant\ln t_{ij}\leqslant\frac{W}{2}.\tag{2}$$

This disorder is commonly used in 1D since it guarantees always positive hoppings $t_{i,j}$ with typical variation of the random elements W that is believed to be a good measure of the strength of off-diagonal disorder in any dimension.[15,16] If, instead, we choose to distribute directly t_{ij} between $-W/2$ to $+W/2$ the disorder can never be made strong enough since no energy scale remains in the Hamiltonian. In the following we consider off-diagonal disorder from Eq. (2) and ignore diagonal disorder by setting the site energies equal to zero.

A bipartite lattice consists of two interconnected sublattices A and B with the random hoppings t_{ij} connecting the sites of one sublattice to the sites of the other.[8] It was shown[1] that a finite system with n_A sites on sublattice A and n_B sites on sublattice $B(n_B>n_A)$ has at least n_B-n_A linearly independent eigenfunctions with eigenvalues exactly $E=0$. Moreover, the amplitudes of these states vanish on the sites of sublattice A. In the present study we consider finite square and cubic lattices with their L-site edges along the main directions in 2D and 3D. In the calculations L is odd and Dirichlet boundary conditions are applied with sublattice A having $L^2-1/2[(L^3-1)/2]$ sites and sublattice B has $L^2+1/2(L^3+1/2)$ sites for the square (cube). According to Ref. 1 only one $E=0$ state exists in this case with finite amplitude on sublattice B and zero amplitude on sublattice A. The corresponding amplitudes satisfy the linear system of equations

$$\sum_{\delta,(j+\delta\in B)} t_{j,j+\delta}\psi_{j+\delta}=0 \quad \text{for all} \quad j\in A, \tag{3}$$

where δ is summed over the nearest neighbors of site j so that $j+\delta$ belongs to the sublattice B where the amplitude is finite. The total number of equations $L^2-1/2[(L^3-1)/2]$ derived from the A sites allows to obtain the $L^2+1/2[(L^3+1)/2]$ amplitudes on the B sites. This procedure suffices to determine uniquely the $E=0$ state since the the normalization condition $\Sigma_i|\psi_i|^2=1$ accounts for the one missing equation.

We have solved numerically the linear system of equations (3) for different strengths W of off-diagonal disorder. In Fig. 1 typical pictures of the logarithmic amplitudes of the $E=0$ wave function in 2D are shown that display fractal characteristics. Similar fractal patterns are obtained for all examined disorder strengths in 2D and 3D. The logarithmic amplitude for very weak disorder (small W) is periodically distributed in Fig. 1(a), the fractal character of the state is obvious for intermediate disorder W in Fig. 1(b) and for higher values of disorder (large W) the area with significant amplitude becomes vanishingly small fraction of the total in Fig. 1(c). The latter case implies stronger decay of the $E=0$ state, similar to what one expects for ordinary-localized states with strong-diagonal disorder. However, the important difference to conventional localization is that the maximum is not concentrated in a small region of space, but many maxima of almost similar heights exist. These peaks with relatively large amplitude are randomly scattered over space and exist also for large disorder [Fig. 1(c)]. Therefore, we can conclude that the $E=0$ state in the presence of off-diagonal disorder shows unusual localization properties.

First, we investigate the fractal structure of the normalized wave function by computing the inverse participation ratio

$$p=\sum_i |\psi_i|^4 \sim L^{-D_2}, \tag{4}$$

where the scaling of p with size L defines the fractal dimension D_2. It is well known that extended states are space filling with $D_2=d$, the space dimension, and localized states are pointlike with $D_2=0$. The probability distributions of the $\ln p$ for various finite-size L systems over a sufficient number of disorder configurations are shown in Fig. 2 to be similar to the ones reported in Ref. 17. For example, in Fig. 2(b) we plot the distributions for the intermediate disorder strength $W=1$ that are roughly invariant in shape as the size L gets larger and a sudden drop occurs for small values of $\ln p$ with a tail for large $\ln p$. We focused on the peaks of such distributions by determining $p_{\max}$ from the maximum probability

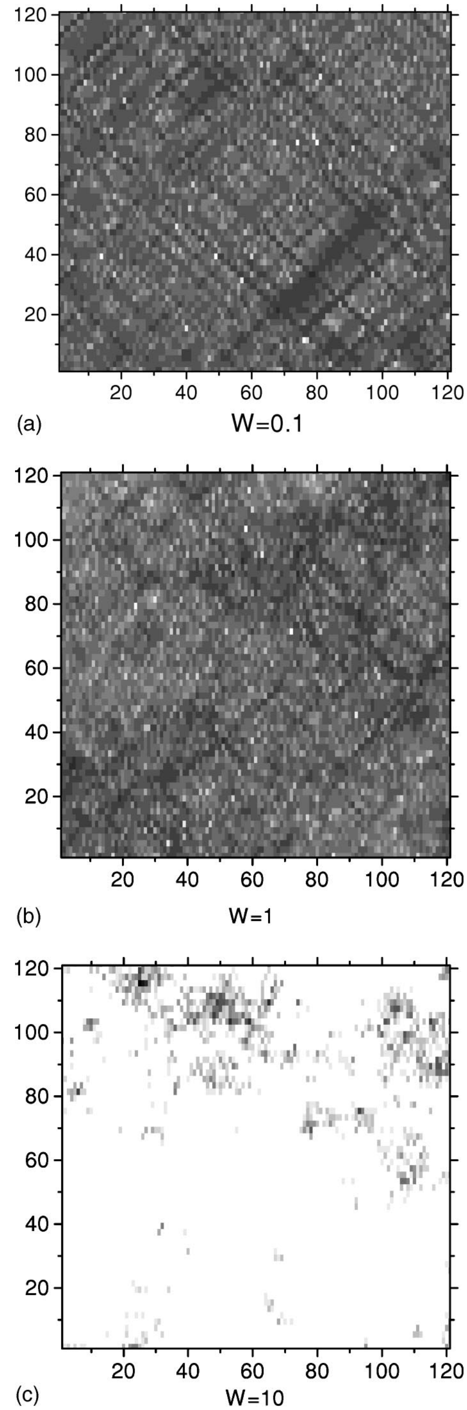

FIG. 1. The logarithmic amplitude $\ln|\psi|$ of the $E=0$ wave function in 2D-squared lattices with off-diagonal disorder strength (a) $W=0.1$, (b) $W=1$, and (c) $W=10$. The darkest regions cover areas with the higher $\ln|\psi|$, between 0 and -1, and the lightest regions areas with the lowest $\ln|\psi|$, between -12 and -13, while the various degrees of gray denote 13 linear scales.

of p as suggested in Ref. 17. The $p_{\max}$ shift towards lower values by increasing the size so that the variation of this typical value of p according to Eq. (4) is shown in the insets of Fig. 2 where we plot $\ln(p_{\max})$ vs $\ln L$. The obtained fractal dimension for $W=1$ is $D_2\approx1.55$.

The behavior of this special state for off-diagonal disorder

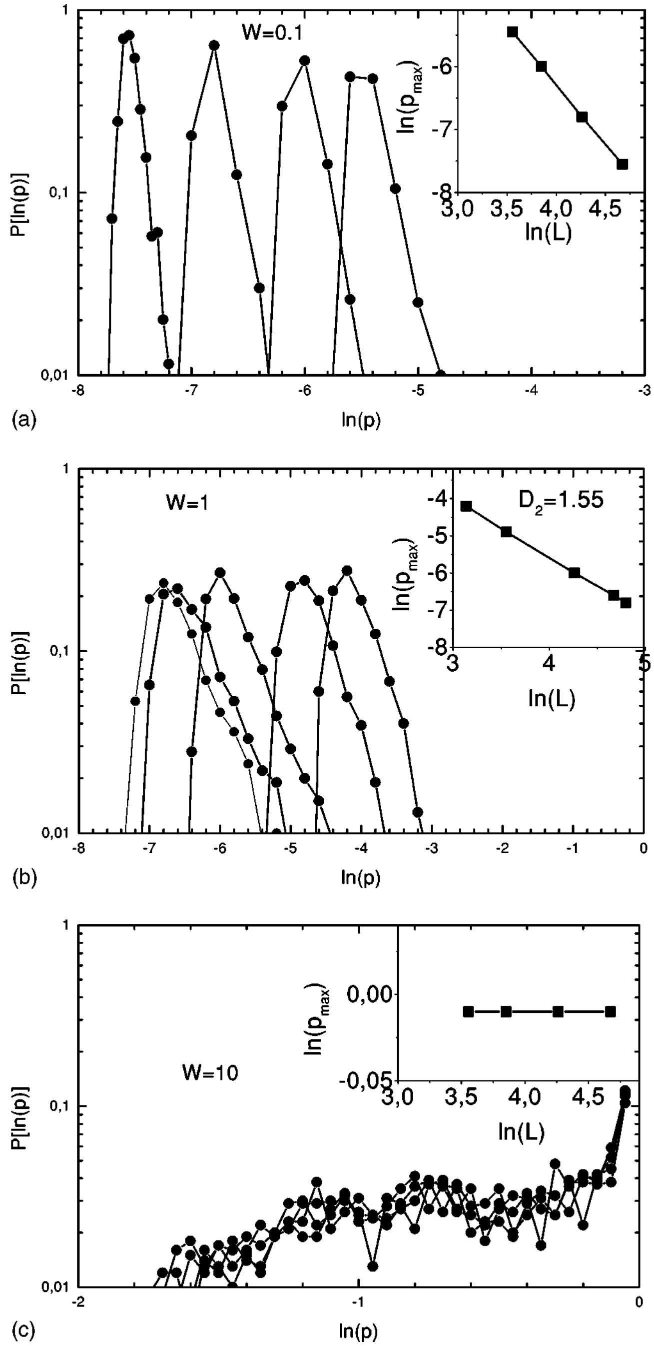

FIG. 2. The probability distribution for the logarithm of the inverse participation ratio for the $E=0$ wave function in square (2D) lattices with sizes $L=23$, 35, 71, 107, and 121. The inset shows $\ln(p_{\max})$ vs $\ln L$ for (a) $W=0.1$, (b) $W=1$, and (c) $W=10$.

is, somehow, reminiscent of the conclusions reached in Ref. 18 where the 2D wave functions are multifractal in the presence of disorder for length scales shorter than the localization length. For off-diagonal disorder a similar situation is encountered since the usual (exponential) localization length is guaranteed to diverge at $E=0$. Extensive numerical studies of the off-diagonal disorder in squared lattices that display the divergence of the localization length at $E=0$ and other properties of the chiral $E=0$ state can be found in Refs. 19 and 20. In Fig. 2 we plot the probability distribution

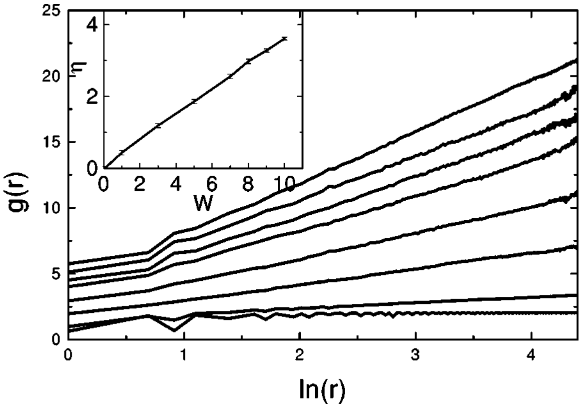

FIG. 3. The correlation function $g(r)$ vs $\ln r$ in 2D for $W=0.1,1,3,5,7,8,9,10$ from bottom to top and system-size $L=121$. All curves can be fitted by straight lines $g(r)\sim\eta\ln r+\beta$ and the power-law exponent η is plotted as a function of W in the inset.

of the inverse participation ratio for $W=0.1$, $W=1.0$, and $W=10$ where D_2 is well defined from the slope of the almost-linear curve $\ln p_{\max}$ vs $\ln L$ in the investigated sizes. For small disorder $W=0.1$ we find $D_2\approx 2$, which implies an almost-extended state, on the contrary, for strong disorder $W=10$ the logarithm of the typical inverse participation ratio is close to zero, which implies strong localization with $D_2\approx 0$.

Finally, we examine the typical decay of the $E=0$ state by introducing an appropriate correlation function. Following the definition of Ref. 1 we have studied

$$g(r)=\left\langle \ln\frac{|\psi_{i_{\max}}|}{|\psi_j|}\right\rangle,\quad r=|\mathbf{r}_j-\mathbf{r}_{i_{\max}}|, \tag{5}$$

where $i_{\max}$ is the lattice site where the maximum amplitude is located and the average $\langle ...\rangle$ is taken over a sufficient number of random configurations and various spatial directions.

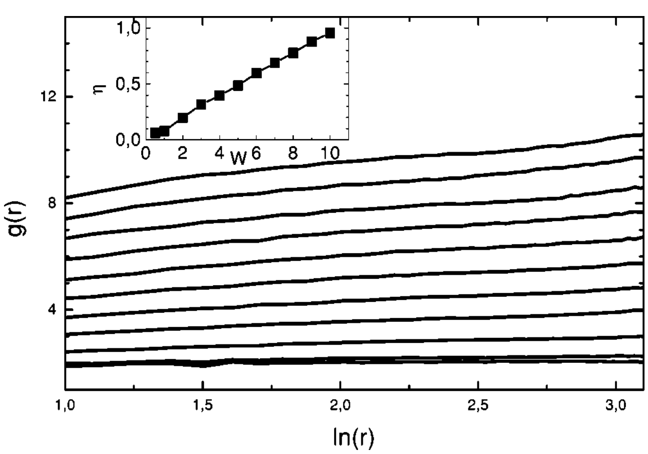

FIG. 4. The correlation function $g(r)$ vs $\ln r$ for 3D with $W=0.1,1,2,3,4,5,6,7,8,9,10$ from bottom to top and system-size $L=21$. The curves can be fitted by straight lines $g(r)\sim\eta\ln r+\beta$ and the obtained η is plotted in the inset as a function of W.

In 1D bipartite lattice, where the log amplitude of the $E=0$ state with off-diagonal disorder exhibits random-walk behavior, the correlation function is $g(r)\sim\sqrt{r}$ for large r.[1] In Fig. 3 we plot $g(r)$ vs $\ln r$ in 2D by increasing W. These curves for not too small r can be reasonably well fitted to the relation $g(r)=\eta\ln r+\beta$, with parameters η and β. For example, for $W=1$ we find $\eta=0.44\pm0.01$, $\beta=1.51\pm0.02$ and the average amplitude asymptotically decays from its maximum as a power law $|\psi(r)|\sim r^{-\eta}$. We conclude that the multifractal amplitude pattern shows many random peaks while the power-law decay reflects the long-range decay of the peak heights from the highest peak. Instead, the amplitude for short distances from each peak decays very sharply.

We have also obtained similar relations $g(r)=\eta\ln r+\beta$ for the cubic (3D) case by plotting $g(r)$ vs $\ln(r)$ for different values of disorder where $g(r)$ still increases linearly with $\ln r$. This means that the corresponding amplitudes show a power-law decay from the maximum, for any finite W, with the exponent η proportional to W. The unusual power-law localization of the midband state is rather surprising in 3D, since nearby states might be extended if the disorder is weak. In fact, η for both 2D and 3D is found to increase almost linearly with W, shown in the insets of Figs. 3 and 4, where we establish the approximate relations $\eta\sim W/3$ in 2D and $\eta\sim W/9$ in 3D.

In summary, we have investigated the $E=0$ wave function for square and cubic lattices with off-diagonal disorder. This zero-energy wave function in the adopted geometry exists for any disorder configuration and can be easily studied numerically. The wave-function amplitude is shown to display many sharp peaks randomly scattered in space and is characterized by the fractal dimension D_2 that strongly depends on the strength of disorder. The amplitude from each peak falls off very rapidly for short ranges but the heights of the peaks decay slowly from the main maximum via the power law $r^{-\eta}$, where η is almost linearly proportional to the strength of disorder W. We have shown that the special $E=0$ chiral state in bipartite lattices, usually refered as ''extended state,'' has a critical nature and can also be described as power law localized.

[1] M. Inui, S. A. Trugman, and E. Abrahams, Phys. Rev. B **49**, 3190 (1994).
[2] L. Fleishman and D. C. Licciardello, J. Phys. C **10**, L125 (1977).
[3] T. P. Eggarter and R. Riedinger, Phys. Rev. B **18**, 569 (1978).
[4] M. Steiner, Yang Chen, M. Fabricio, and A. O. Gogolin, Phys. Rev. B **59**, 14 848 (1999).
[5] T. A. L. Ziman, Phys. Rev. B **26**, 7066 (1981).
[6] R. Gade and F. Wegner, Nucl. Phys. B **360**, 213 (1991).
[7] R. Gade, Nucl. Phys. B **398**, 499 (1993).
[8] M. Fabrizio and C. Castellani, Nucl. Phys. B **583**, 542 (2000) studied localization on a generic bipartite lattice close to the band center by deriving an effective nonlinear σ model. In two dimensions they predict finite conductance and in three dimensions extended states close to the band center independently on the strength of disorder.
[9] C. M. Soukoulis, I. Webman, G. S. Grest, and E. N. Economou, Phys. Rev. B **26**, 1838 (1982).
[10] Y. Morita and Y. Hatsugai, Phys. Rev. Lett. **79**, 3728 (1997).
[11] A. Atland and B. D. Simons, J. Phys. A **32**, L353 (1999).
[12] S. V. Kravchenko, D. Simonian, M. P. Sarachik, W. Mason, and J. E. Fourneaux, Phys. Rev. Lett. **77**, 4938 (1996).
[13] B. Huckenstein, Rev. Mod. Phys. **67**, 357 (1995).
[14] M. Janssen, Phys. Rep. **295**, 1 (1998).
[15] S. N. Evangelou, J. Phys. C **19**, 4291 (1986); J. Phys.: Condens. Matter **2**, 2953 (1990).
[16] C. M. Soukoulis and E. N. Economou, Phys. Rev. B **24**, 5698 (1981).
[17] F. Evers and A. D. Mirlin, Phys. Rev. Lett. **84**, 3690 (2000).
[18] V. I. Falko and K. B. Efetov, Phys. Rev. B **52**, 17 413 (1995).
[19] A. Eilmes, R. A. Romer, and M. Schreiber, Eur. Phys. J B **1**, 29 (1998) and references therein. We have confirmed their value $D_2\approx1$ for energies close to the band center when t is randomly distributed within [0, 1].
[20] V. Z. Cerovski, Phys. Rev. B **62**, 12 775 (2000), and references therein.