Wigner crystals in the lowest Landau level at low-filling factors

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We report on results of finite-size numerical studies of partially filled lowest Landau level, at low electron filling factors. We find convincing evidence suggesting that electrons form Wigner crystals at sufficiently low filling factors, and the critical filling factor is $\nu_c \approx 1/7$. At $\nu = 1/7$ we find the system undergoes a phase transition from the Wigner crystal to the incompressible Laughlin state when the short-range part of the Coulomb interaction is modified slightly. This transition is either continuous or very weakly first order.

Interacting electrons at sufficiently low densities and temperatures were shown by Wigner[1] to form a crystalline state. This ground state results from the dominance of the potential over the kinetic energy that occurs in the dilute limit. A Wigner crystal (WC) state has been observed in a low-density *two-dimensional* (2D) electron gas system trapped on the surface of helium liquid.[2] While unlikely to be observed at metallic densities in three dimensions this crystalline phase is in fact expected to occur in the 2D electron gas confined in semiconductor heterostructures or quantum wells, and subject to a strong perpendicular magnetic field [such that all electrons are in the lowest Landau level (LLL)]. Under these conditions the kinetic energy is completely quenched making it an ideal environment in which to observe Wigner crystallization; the WC state is expected to have the lowest Coulomb interaction energy when the Landau level filling factor (ν) is sufficiently low. Indeed, substantial experimental evidence, mostly from transport measurements, points to a pinned WC state in the insulating phases of low ν.[3] However, the situation is not yet definitive as direct experimental probe of the lattice structure of the WC is still lacking. In addition, the ubiquitous disorder in these systems could lead to other insulating phases[3] such as the Hall insulator, or just complicate the detection of the crystal itself. On the theoretical side, early work[4] suggested that a 2D electron gas is always unstable against formation of a WC when subject to a strong magnetic field, at zero temperature. However, the discovery of fractional quantum Hall effect (FQHE) has led to the realization[5] that electrons can also form incompressible fluid states, especially at the primary sequence $\nu = 1/m$, where $m = 3, 5, \ldots$. By comparing the energy of the Laughlin[5] state with that of a *correlated* WC state (whose energy is considerably lower than that of a simple Hartree-Fock estimate[6,7]), Lam and Girvin[8] concluded that the critical filling factor ν_c below which the WC forms is slightly above 1/7. This is in good agreement with transport experiments on clean samples. A more recent study[9] using a formalism based on composite fermion[10] wave functions calculated the collective excitation energy variationally and found that the ground state become unstable against proliferation of magnetorotons (which points to formation of WC) for $\nu \lesssim 1/9$, although such an instability was not found at $\nu = 1/7$. A similar trend was observed earlier in

the single mode approximation[11] where the magnetoroton gap while not zero was found to be diminished considerably at 1/7 and further at 1/9. This was interpreted as a precursor to magnetoroton mode softening and the transition to the WC phase.

In this paper we present finite-size exact diagonalization studies of systems with low ν that directly reveal the formation of the WC in the LLL. We study systems with torus geometry. Our previous work on charge density wave ground states in high Landau levels[12,13] has demonstrated that the torus geometry is advantageous as compared to, say, the spherical geometry,[14] in studies of states with broken translational symmetry. The reasons are as follows (i) On a torus one can adjust the geometry to better accommodate a lattice in a finite-size system. (ii) The presence of translational symmetry allows one to define a many-body momentum[15] that can be used to label the eigenstates. By inspecting the momenta of the low-lying states, as well as the wave-vector dependence of the density-density correlation functions, one can easily detect whether the translational symmetry is broken in the system, and determine lattice structure of the broken symmetry (crystal) state.

We find strong evidence that WC forms at filling factors $\nu \lesssim 1/7$, and that the critical filling factor ν_c is very close to 1/7. Furthermore, we have studied the competition between the WC state and the Laughlin state at $\nu = 1/7$ by varying the geometries of our finite-size systems, as well as the short-range part of the Coulomb interaction. We find that the Laughlin state is always *continuously* deformed into one of the low-lying states that correspond to the WC phase, when the geometry or the interaction is varied. This suggests that the quantum phase transition between the incompressible FQHE and compressible WC phases is either continuous or very weakly first order. The methods used here are similar to our previous studies and provide a reliable way of detecting broken symmetry in finite systems. We first investigate dependence of the spectrum on PBC geometry.

Figure 1 shows the energy spectra of systems with $N = 6$ electrons, at filling factors $\nu = 1/6$, 1/7, and 1/8, with rectangular geometry, for pure Coulomb interaction. For $\nu = 1/6$ and 1/7 we plot energy levels for a series of aspect ratio a_0, while for $\nu = 1/8$ we plot the energy level versus the magnitude of many-body momentum **K**, for $a_0 = 0.35$. One can see

FIG. 2. The allowed momenta **K** for the $\nu = 1/7$ system with $N = 6$ electrons with aspect ration $a_0 = 3/8$, and the **K**'s of the ground-state manifold. These low-lying **K**'s form a nearly triangular array, which represents the reciprocal lattice of the Wigner crystal that the electrons form.
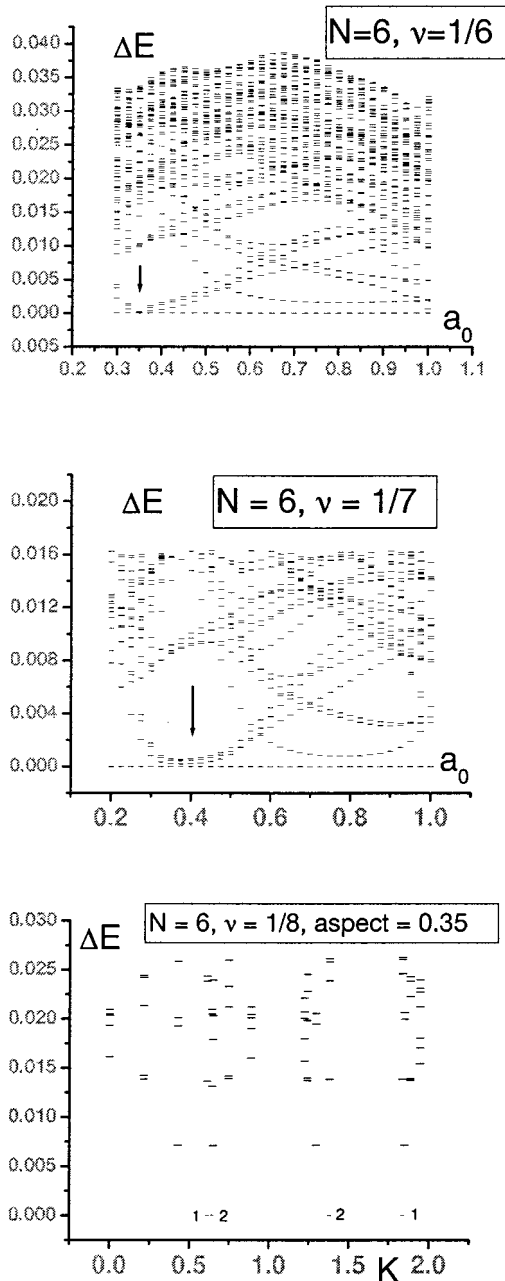
FIG. 1. The spectra of rectangular-shaped finite-size systems with $N = 6$ electrons, at filling factors 1/6, 1/7, and 1/8. The energy levels are plotted as functions of the aspect ratio for $\nu = 1/6$ and 1/7, and plotted as a function of the magnitude of momentum **K** for $\nu = 1/8$, at aspect ratio 0.35. The degeneracy of each low-lying level is indicated by the integer beside it for $\nu = 1/8$.

that for the $\nu = 1/8$ case, as well as $\nu = 1/7$, for a sizable range of a_0 ($0.3 < a_0 < 0.5$), there are $N_D = 6$ nearly degenerate low-lying states (including the ground state) that form what we call the ground-state manifold. The **K**'s of the states in this manifold form a 2D array, as we show in Fig. 2 for $\nu = 1/7$ at $a_0 = 3/8$, indicating that the translational symmetry is broken in both directions, and the system has 2D crystalline order. These **K**'s determine the lattice structure of the crystal. The number of unit cells included in the finite-size
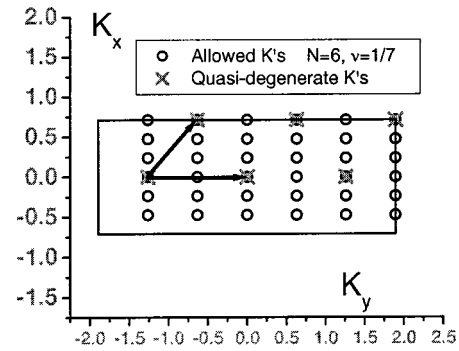
system can be determined easily:[13] $N_c = \bar{N}^2/N_D$, where $\bar{N}$ is the highest common divisor of N and the number of flux quanta N_ϕ. Here we have $\bar{N} = 6$ in both cases, thus $N_c = 6 = N$, indicating there is one electron per unit cell. This is precisely what one expects for a WC. This suggests that electrons form a WC at these filling factors, at least when a WC can be easily accommodated by the geometry of the finite-size system.

For $\nu = 1/6$, on the other hand, such a (near) degeneracy is absent for almost all geometries, except for perhaps at $a_0 \approx 0.35$. This suggests that the translational symmetry is not broken at $\nu = 1/6$, although the tendency toward WC formation is present, especially when the geometry of the finite-size system is favorable. We thus conclude that the critical filling factor ν_c below which the Wigner crystal forms is close to and probably slightly above 1/7, which is in excellent agreement with the estimate of Lam and Girvin.[8] From the superlattice primitive basis vectors in the reciprocal space[13] we obtain the parameters of the WC unit cell $a_1 \approx 8.842$, $a_2 \approx 6.655$, and $\theta \approx 131.6°$. While this is not exactly a triangular lattice, it is close to it. We believe the discrepancy is caused by the finite size of the system, and in the thermodynamic limit we expect to recover a triangular lattice. We emphasize the fact that the WC state is stabilized near $a_0 = 0.4$ is *not* an artifact of the geometry of the finite-size system; as we will see below at filling factors $\nu = 1/3$ and 1/5 where the system is an incompressible fluid and well described by the Laughlin state, no crystalline order is developed and the ground state has high overlap with the Laughlin state for all reasonable geometries, including the ones that are favorable for WC formation.

At $\nu = 1/m$ with m being an odd integer, it is known[16] that for certain short-range repulsive interactions, the Laughlin state is the exact ground state that describes an incompressible fractional quantum Hall fluid. In Fig. 3 we show the square of the overlap between the Laughlin state and the exact ground state of the system for $m = 3$, 5, and 7, at different aspect ratios and system sizes. It is seen that for $\nu = 1/3$ and 1/5, the overlap is rather insensitive to the geometry of the unit cell and is close to 1, while for $\nu = 1/7$ the square of the overlap varies appreciably with the geometry,
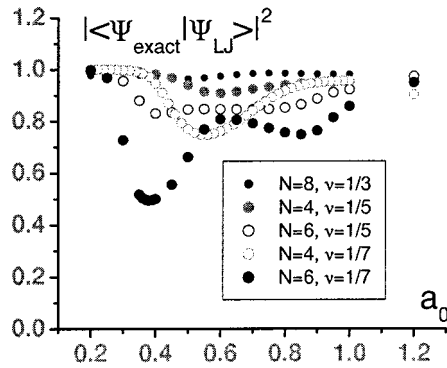
FIG. 3. The square of the overlap between the Laughlin state and the exact ground state of systems with rectangular geometry and a variety of filling factors and number of electrons, as a function of aspect ratio. The points at 1.2 are for hexagonal unit cells.

and dips below 0.75 and 0.5 in certain range of a_0 for $N=4$ and $N=6$, respectively. The sensitivity of the energy spectrum as well as the overlaps on the system geometry is a strong indication that the ground state is *compressible*. It reflects the fact that the system wants to form a WC, but is frustrated when the aspect ratio a_0 is far from the optimal one for forming a crystal, $a_0 \approx 0.4$ for $N=6$. It should be noted that the Laughlin state has $\mathbf{K}=0$, which is one of the $\mathbf{K}$'s in the ground-state manifold, and in the case of a pure Coulomb interaction the ground state turns out to be in the $\mathbf{K}=0$ sector for all the aspect ratios and sizes that we explored; thus the square of the overlap evolves continuously from being close to 1 near $a_0=1$, where the WC state is frustrated, to be less than 0.5 near $a_0=0.4$ where there is little geometric frustration for WC formation. Physically the reason that the overlap is large when the WC state is frustrated is that the Laughlin state at $\nu=1/7$, while describing an incompressible fluid state, already has substantial short-range crystal order built into it; it is thus energetically still competitive.

At $\nu=1/7$, the Laughlin state can be stabilized by slightly modifying the short-range part of the Coulomb potential. In Fig. 4 we plot the square of the overlap between the Laughlin state and the exact ground state of the system at $a_0=3/8$, as a function of δV_5, the change of the $m=5$ pseudopotential[16] from its Coulomb value. It is seen that the overlap is substantially improved to be above 0.8 by adding a small pseudopotential $\delta V_5 = 0.025 e^2/\epsilon l$ (which represents less than 10% increase of the V_5 pseudopotential), even though the system geometry is most favorable for WC formation. This is further indication that $\nu=1/7$ is very close to the phase boundary separating the crystal and liquid phases. The smooth change of the overlap as a function of δV_5 also suggests the transition is either continuous or very weakly first order in the thermodynamic limit. Adding a negative δV_5, on the other hand, further suppresses the overlap; in particular, at $\delta V_5 \approx -0.011 e^2/\epsilon l$ there is a level crossing for the ground state, and for $\delta V_5 < -0.011 e^2/\epsilon l$ the *absolute* ground state has a *different* wave vector from the Laughlin state; thus the overlap between the Laughlin state and the absolute ground state is, strictly speaking, zero here. However, the quasidegeneracies are well developed and the
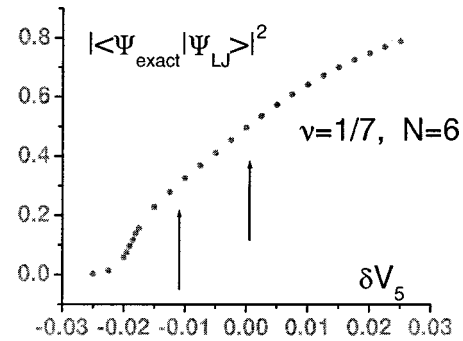

FIG. 4. The square of the overlap between the Laughlin state and the exact ground state, in a system with aspect ratio $a_0=3/8$. It is plotted as a function of δV_5 (see text). The arrow on the right-hand side indicates the position at which the near degeneracy of the low-lying states starts to develop. The arrow on the left-hand side indicates the position of δV_5 below which the absolute ground state is not in the $K=0$ sector. The overlap shown is between the Laughlin state and the lowest energy $K=0$ state.

ground state is not unique. We have tracked the overlap with the $K=0$ "ground state." In real systems, the short-range part of the Coulomb interaction is *softened* due to the finite extent of the electron wave function along the $\hat{z}$ direction (or finite layer thickness); thus the WC state is expected to be further stabilized; we have studied potentials of typical layer thickness and find it is indeed the case at $\nu=1/7$.

As in our previous studies we now turn to the density response functions. In Fig. 5 we show the LLL projected (or guiding center) density-density correlation function, in real space. This is the Fourier transform of the guiding center static structure factor.[13] The central peak represents the usual $\delta(\mathbf{r})$ singularity projected to the LLL, which is at the position of the electron where the correlation is measured from. The crystal structure can be clearly seen; the number of peaks (including the central one) equals the number of electrons $N=6$, indicating each unit cell includes one electron, which is what one expects for a Wigner crystal.

The tendency toward WC formation, as well as the structure of the resultant lattice, can also be seen in the density
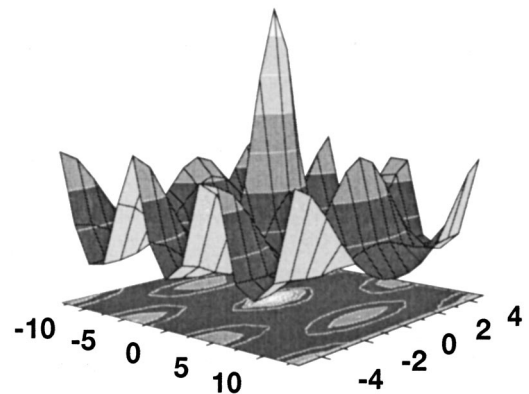

FIG. 5. The real-space lowest Landau level projected (or guiding center) density-density correlation function, for $N=6$ electrons at filling factor 1/7, in a rectangular system with aspect ratio 3/8. A 2D contour plot is also shown.
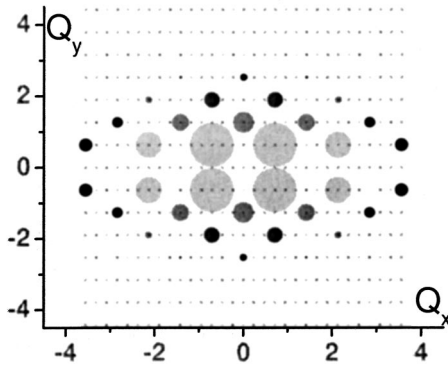
FIG. 6. A 2D plot of the LLL projected density response function $\chi(\mathbf{q})$, with the size of the dot representing the strength. The system is the same as that in Fig. 5. The four large dots represent a peak value of about 21 000. The nearby smaller dots along the Q_y axis represent a value of 4700, etc.

response function in momentum space, $\chi(\mathbf{q})$. Figure 6 is a plot of $\chi(\mathbf{q})$ for the same system as in Fig. 5. Here we see strong response at an array of momenta, which form a lattice that is close to being triangular. This is essentially the reciprocal lattice of the crystal that forms in real space. The origin of the large response at these momenta is the near degeneracy of the low-lying states that are connected by these momenta discussed earlier; small energy denominators lead to large response at these momenta.

In summary, we have shown that FQHE incompressible fluids become unstable around $\nu = 1/7$ in agreement with previous predictions. We find the Wigner crystal becomes the ground state as the system geometry is adjusted to accommodate the crystal. It is more difficult to determine the order of the transition by the small sizes that we have studied. However, a strong level crossing first-order transition seems to be ruled out by our study. We observe only a continuous evolution of the required quasidegenerate manifold that always includes the state at $K = 0$. This state for geometries that frustrate the crystal shows large overlap with the Laughlin state. This suggest the transition may be second order but a weakly first-order transition predicted by a mean-field approach[11] also remains a possibility.

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