Stripe *ansätze* from exactly solved models

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Using the Boltzmann weights of classical statistical-mechanics vertex models we define a new class of tensor product *Ansätze* for two-dimensional quantum-lattice systems, characterized by a strong anisotropy, which gives rise to stripelike structures. In the case of the six-vertex model we compute exactly, in the thermodynamic limit, the norm of the *Ansatz*, and other observables. Employing this *Ansatz* we study the phase diagram of a Hamiltonian given by the sum of XXZ Hamiltonians along the legs coupled by an Ising term. Finally, we suggest a connection between the six- and eight-vertex anisotropic tensor-product *Ansätze*, and their associated Hamiltonians, with the smectic-stripe phases recently discussed in the literature.

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I. INTRODUCTION

Low-dimensional spin systems constitute one of the most active areas in condensed-matter physics due to the experimental findings and the associated theoretical activity. These systems are strongly correlated with very rich phase diagrams studied by means of a miscellanea of analytical and numerical techniques, among which the study of simplified-variational *Ansätze* for the ground state (GS) and excitations have played a significant role.

In one-dimension (1D), there is a plethora of variational *Ansätze*: the AKLT states,[1] the finitely correlated *Ansätze*,[2] the matrix-product *ansätze* (MPA),[3–5] the recurrent-variational *Ansatz* (RVA),[6] etc. All these *Ansätze* have a common structure for the GS wave function that is given by the sum, over some auxiliary variables, of products of amplitudes that also depend on the spin variables at the sites. The basic quantity here is the ''matrix-product amplitude'' $A_{\alpha,\beta}[m_i]$, where m_i is the spin at the ith site and α and β are auxiliary variables, which can be associated to the links meeting at the site. For a spin chain with N sites and periodic-boundary conditions the corresponding state can be written as[4]

$$|\psi_{\mathrm{MPA}}\rangle=\sum_{m_i's} \mathrm{Tr}(A[m_1]\ldots A[m_N])|m_1\rangle\ldots|m_N\rangle, \quad (1)$$

where the trace is over the auxiliary variables α. Some MPA states, such as the AKLT ones, are exact ground states of a Hamiltonian, which is given by the sum of projectors between nearest-neighbor sites.[1] In other cases, the MPA states are used as variational *Ansätze* for a given Hamiltonian, with the MPA amplitudes $A_{\alpha,\beta}[m]$, playing the role of variational parameters. Within the latter category fall, the DMRG states[7] (for a review on the DMRG see[8,9]), which are in fact MPA states with open-boundary conditions and position-dependent amplitudes (i.e., inhomogeneous MPA's).[4,10,11] In the DMRG the auxiliary variables label the states kept in the blocks.

The MPA states can be generalized in a natural way to 2D systems, replacing the matrix amplitudes $A_{\alpha,\beta}[m]$ by ''tensor-product'' amplitudes $A_{\alpha_1,\alpha_2,\ldots,\alpha_z}[m]$, where z is given by the coordination number of the lattice, i.e., $z=3$ for an hexagonal lattice, $z=4$ for a square lattice, so on and so forth.[12–20] By analogy with statistical mechanics (SM) these states can be called tensor-product vertex *ansätze* (TPVA) because the auxiliary variables α_i are associated to the links of the lattice, while the amplitudes are associated to the vertices.[21] Another class of 2D *Ansätze* is formed by the tensor-product face *Ansätze* (TPFA), where the amplitudes are associated to the faces of a square lattice as in the face or interaction around a face models in statistical mechanics.[15,22] The 2D generalizations of the AKLT states for spins 3/2, 2, and higher belong to the TPVA class. The recipe to construct TPA's is as in Eq. (1), where the contraction of the auxiliary variables follows the pattern of the underlying vertex or face models.

Most of the TPA's studied in the literature are isotropic, meaning that their properties are largely independent of the spatial direction. However in 2D and 3D there are physical systems, like some high-temperature superconductors,[23] quantum Hall systems,[24] or manganites,[25] which exhibit strongly anisotropic properties due to the existence of stripes. These objects are static or dynamic charge inhomogeneities, which are linear in 2D or planar in 3D. One may wonder whether these systems can be modeled with simple TPA's, just as Haldane spin chains can be easily described as valence-bond states. In this work we shall not address directly this question, but the results we have obtained suggest the possibility of a simple description of stripes in terms of TPA's. More precisely, in this paper we shall investigate a class of TPA's based on classical exactly solvable 2D-vertex models with strong anisotropic properties reminiscent to the stripe systems investigated in Refs. 26,27. Any classical SM 2D-vertex model, not necessarily integrable, defined by its Boltzmann weights, give rise to an anisotopic tensor product *Ansatz* (ATPA). If, in addition, the SM model is exactly solvable, then the corresponding ATPA becomes quasi-exactly-solvable. The latter term is borrowed from the theory of spectral problems associated to the Schrödinger Equation,[28] meaning, in our context, that some quantities, as the norm of the ATPA's and some expectation values, can be computed exactly in the thermodynamic limit.

To illustrate our proposal we have chosen the well-known six-vertex model, whose 1D quantum-mechanical counterpart is the XXZ model or the 1D spinless fermion.[21] We shall show that the corresponding ATPA has some similarities with the striped states of 2D spinless fermions studied in the literature.[26,27]

The organization of this paper is as follows. First of all we review briefly the basic ingredients of vertex models in statistical mechanics (Sec. II) and the tensor-product vertex *Ansätze* (Sec. III). In Sec. IV we introduce the ATPA's based on SM-vertex models and study their general properties. The ATPA associated to the six-vertex model is used in Sec. V as a trial ground state for an anisotropic Hamiltonian closely related to the XXZ spin-chain Hamiltonian, and derive the phase diagram. In Sec. VI we briefly comment on the eight-vertex ATPA model. The possible connections between the six- and eight-vertex ATPA is explored in Sec. VII and finally in Sec. VIII we state our conclusions. In Appendices A and B we collect some technical results.

II. VERTEX MODELS IN STATISTICAL MECHANICS

Throughout this paper we shall follow closely Baxter's book.[21] Let us consider a rectangular lattice with N rows and L columns. Throughout these paper we shall also use the term ''legs'' for the rows and ''rungs'' for the columns. In a vertex model there is a local state variable α associated to every link and a Boltzmann weight associated to every vertex ●, which depends on the four link-variables meeting at it. We shall represent the Boltzmann weight as

$$W^{\beta,\eta}_{\alpha,\xi} = \begin{array}{ccccc} & & \eta & & \\ & & | & & \\ \alpha & - & \bullet & - & \beta. \\ & & | & & \\ & & \xi & & \end{array} \qquad (2)$$

The statistical weight of a global configuration is given by the product of the Boltzmann weights of all the vertices. The partition function Z is the sum of these weights over all the link configurations, which can also be expressed using transfer matrices. The row-to-row and column-to-column transfer matrices are defined as,

$$T^{\mathrm{row}}_{\boldsymbol{\eta},\boldsymbol{\xi}} = \sum_{\alpha' s} \prod_{i=1}^{L} W^{\alpha_{i+1},\eta_i}_{\alpha_i,\xi_i},$$

$$T^{\mathrm{col}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\xi' s} \prod_{i=1}^{N} W^{\beta_i,\xi_{i+1}}_{\alpha_i,\xi_i}, \qquad (3)$$

where $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_L)$, $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_N)$, etc., and the periodic-boundary conditions are assumed along both directions. Using Eq. (3) the partition function Z reads,

$$Z = \mathrm{Tr}\, T^{N}_{\mathrm{row}} = \mathrm{Tr}\, T^{L}_{\mathrm{col}}. \qquad (4)$$

As an example we display in Table I the Boltzmann weights for the allowed vertex configurations of the six-vertex model. The link variables take on two values, say 0 and 1, which in the standard notation correspond to the right and up pointing (for $\alpha=0$), and left and down pointing (for $\alpha=1$). The allowed configurations satisfy the ice rule $\alpha+\xi=\beta+\eta$, and the Boltzmann weights are invariant under the reversal of all arrows, which leaves three independent ones, called a, b, and c. The six-vertex model is integrable: there is a uniparametric family of transfer matrices $T(u)$ commuting among themselves. This is guaranteed by the Yang-Baxter equation satisfied by the Boltzmann weights.

TABLE I. Boltzmann weights of the six-vertex model.

Weight	a	b	c
η	0	0	0
α β	0 0	1 1	0 1
ξ	0	0	1
	1	1	1
	1 1	0 0	1 0
	1	1	0

III. TENSOR-PRODUCT VERTEX *ANSÄTZE*

As in the previous section we shall consider a lattice with N legs of length L. In the quantum-spin model there is a spin degree of freedom m at each vertex of the lattice. To construct a TPA we shall associate an auxiliary variable α to each link, as in the SM models. The TPA amplitudes will be denoted as

$$A^{\beta\eta}_{\alpha\xi}[m] = \begin{array}{ccccc} & & \eta & & \\ & & | & & \\ \alpha & - & m & - & \beta. \\ & & | & & \\ & & \xi & & \end{array} \qquad (5)$$

By analogy with SM we shall define the row-to-row and column-to-column transfer-matrix amplitudes

$$A^{\mathrm{row}}_{\boldsymbol{\eta},\boldsymbol{\xi}}[\mathbf{m}] = \sum_{\alpha' s} \prod_{i=1}^{L} A^{\alpha_{i+1},\eta_i}_{\alpha_i,\xi_i}[m_i],$$

$$A^{\mathrm{col}}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\mathbf{m}] = \sum_{\xi' s} \prod_{i=1}^{N} A^{\beta_i,\xi_{i+1}}_{\alpha_i,\xi_i}[m_i], \qquad (6)$$

where $\mathbf{m}=(m_1,\ldots,m_L)$ for A^{row} while $\mathbf{m}=(m_1,\ldots,m_N)$ for A^{col}. Using Eqs. (6) the TPA can be written in two alternative ways, i.e.,

$$|\psi\rangle_{\mathrm{row}} = \sum_{\mathbf{m}' s} \mathrm{Tr}(A^{\mathrm{row}}[\mathbf{m_1}]\ldots A^{\mathrm{row}}[\mathbf{m_N}])|\mathbf{m_1},\ldots,\mathbf{m_N}\rangle_{\mathrm{row}},$$

$$|\psi\rangle_{\mathrm{col}} = \sum_{\mathbf{m}' s} \mathrm{Tr}(A^{\mathrm{col}}[\mathbf{m_1}]\ldots A^{\mathrm{col}}[\mathbf{m_L}])|\mathbf{m_1},\ldots,\mathbf{m_L}\rangle_{\mathrm{col}}. \qquad (7)$$

TABLE II. Six-vertex TPA amplitudes.

Amplitude	a	b	c
η	0	0	0
$\alpha\ m\ \beta$	$0\ +\ 0$	$1\ +\ 1$	$0\ +\ 1$
ξ	0	0	1
	1	1	1
	$1\ -\ 1$	$0\ -\ 0$	$1\ -\ 0$
	1	1	0

These equations are identical to Eq. (1), which implies that the TPA can be regarded as a MPA through the legs or the rungs. The norm of the TPA is given by

$$\langle\psi|\psi\rangle = \mathrm{Tr}\,\mathcal{T}_{\mathrm{row}}^{N} = \mathrm{Tr}\,\mathcal{T}_{\mathrm{col}}^{L} \tag{8}$$

where

$$\mathcal{T}^{\mathrm{row}}_{\boldsymbol{\eta\eta'},\boldsymbol{\xi\xi'}} = \sum_{\mathbf{m}} A^{\mathrm{row}}_{\boldsymbol{\eta},\boldsymbol{\xi}}[\mathbf{m}] A^{\mathrm{row}}_{\boldsymbol{\eta'},\boldsymbol{\xi'}}[\mathbf{m}],$$

$$\mathcal{T}^{\mathrm{col}}_{\boldsymbol{\alpha\alpha'},\boldsymbol{\beta\beta'}} = \sum_{\mathbf{m}} A^{\mathrm{col}}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\mathbf{m}] A^{\mathrm{col}}_{\boldsymbol{\alpha'},\boldsymbol{\beta'}}[\mathbf{m}]. \tag{9}$$

Thus the computation of the norm (8) amounts to that of the partition function of a classical SM-vertex model where the link variables are twice those of the quantum-mechanical model.

IV. ANISOTROPIC TENSOR PRODUCT *ANSÄTZE*

A. Generic case

Let us suppose we are given a vertex model with Boltzmann weights $W^{\beta,\eta}_{\alpha,\xi}$. Using them, we shall define a ATPA model by the equation:

$$A^{\beta,\eta}_{\alpha,\xi}[m] = \delta_{m,\eta} W^{\beta,\eta}_{\alpha,\xi} \tag{10}$$

where the spin variable at each site, i.e., m is identified with the link variable η. In the case of the six-vertex model we shall adopt the convention that $m=0$ corresponds to spin 1/2 and $m=1$ to spin $-1/2$. For the six-vertex model the corresponding TPA amplitudes are given in Table II.

The choice (10) is extremely anisotropic since it treats on a very different footing the vertical and horizontal directions of the lattice. This is the main reason to consider both the row-to-row and column-to-column transfer matrices, which give rise to complementary descriptions of the *Ansatz*. In SM models where the leg and rung variables run over different sets, one can obtain two inequivalent ATPA's, not related by a 90° rotation. In the rest of the paper we shall suppose that the link variables are of the same type in both directions.

Equation (10) implies a simple relationship between the row-to-row TPA amplitude (6) and the row-to-row transfer matrix (3) of the underlying SM model, namely,

$$A^{\mathrm{row}}_{\boldsymbol{\eta},\boldsymbol{\xi}}[\mathbf{m}] = \delta_{\mathbf{m},\boldsymbol{\eta}} T^{\mathrm{row}}_{\boldsymbol{\eta},\boldsymbol{\xi}}, \tag{11}$$

which leads to the following row representation of the ATPA state (7),

$$|\psi\rangle_{\mathrm{row}} = \sum_{\mathbf{m}'s} T^{\mathrm{row}}_{\mathbf{m_1},\mathbf{m_2}} T^{\mathrm{row}}_{\mathbf{m_2},\mathbf{m_3}} \ldots T^{\mathrm{row}}_{\mathbf{m_N},\mathbf{m_1}} |\mathbf{m_1}, \ldots, \mathbf{m_N}\rangle_{\mathrm{row}}, \tag{12}$$

where $T^{\mathrm{row}}_{\mathbf{m_1},\mathbf{m_2}}$ is the row-to-row transfer matrix (3) builtup from the SM Boltzmann weights $W^{\beta,\eta}_{\alpha,\xi}$ defining the ATPA [Eq. (10)]. The structure of this state is similar to the Kramers-Wannier variational state, first proposed for the GS of the Ising model,[13,29] where the analogue of T^{row} is played by 2×2 matrices. The ATPA built from the choice (10) can be seen as a superposition of leg states connected through the row-to-row transfer matrix of the SM model. For example, in the antiferroelectric phase of the six-vertex model, the state along the legs will be mostly of Neel type and correlated antiferromagnetically with their nearest-neighbor legs. In the spinless fermion picture the latter state is a Wigner crystal with charge-density wave (CDW) order.

The norm of Eq. (12) is given simply by

$$\langle\psi|\psi\rangle_{\mathrm{row}} = \sum_{\mathbf{m}'s} (T^{\mathrm{row}}_{\mathbf{m_1},\mathbf{m_2}})^2 (T^{\mathrm{row}}_{\mathbf{m_2},\mathbf{m_3}})^2 \ldots (T^{\mathrm{row}}_{\mathbf{m_N},\mathbf{m_1}})^2. \tag{13}$$

It is important to notice that Eq. (13) is not the partition function of the SM model defined with Boltzmann weights $W^{\beta,\eta}_{\alpha,\xi}$ or their square. The reason being that in general,

$$(T^{\mathrm{row}}_{\mathbf{m_1},\mathbf{m_2}})^2 \neq (T^{2}_{\mathrm{row}})_{\mathbf{m_1},\mathbf{m_2}}, \tag{14}$$

where the LHS of this equation is the square of the element $T^{\mathrm{row}}_{\mathbf{m_1},\mathbf{m_2}}$ of the row-to-row transfer matrix, while the RHS is the entry $(\mathbf{m_1},\mathbf{m_2})$ of the square of the row-to-row transfer matrix. In any case, the computation of Eq. (13) requires much less effort than Eq. (8) because the matrices involved contain half of the indices of those of the general case. In other words, the ATPA does not lead to a doubling of indices in the row representation.

The situation improves even further in the column representation. Using Eqs. (6) and (10) we see that the column ATPA amplitudes are given by the product of the Boltzmann weights on a column, i.e.,

$$A^{\mathrm{col}}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\mathbf{m}] = \prod_{i=1}^{N} W^{\beta_i,m_i}_{\alpha_i,m_{i-1}}, \tag{15}$$

where $m_0 = m_N$. Consequently the column-to-column ATPA transfer matrix (9) becomes

$$\mathcal{T}^{\mathrm{col}}_{\boldsymbol{\alpha\alpha'},\boldsymbol{\beta\beta'}} = \sum_{\mathbf{m}} \prod_{i=1}^{N} W^{\beta_i,m_i}_{\alpha_i,m_{i-1}} W^{\beta_i',m_i}_{\alpha_i',m_{i-1}}. \tag{16}$$

Let us suppose for a moment that we restrict ourselves to the ''diagonal'' sector of $\mathcal{T}^{\mathrm{col}}$, which is defined by the choices $\boldsymbol{\alpha}=\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}=\boldsymbol{\beta}'$. Then Eq. (16) becomes a column-to-column transfer matrix (3) with Boltzmann weights being the square of $W^{\beta,\eta}_{\alpha,\xi}$, namely,

$$\mathcal{T}^{\rm col}_{\boldsymbol{\alpha\alpha},\boldsymbol{\beta\beta}}=\sum_{\mathbf{m}}\prod_{i=1}^{N}(W^{\beta_i,m_i}_{\alpha_i,m_{i-1}})^2=T^{\rm col}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(W^2). \tag{17}$$

For a generic TPA this diagonal truncation may be a good approximation in certain regions of the parameter space, as has been shown by Niggemann *et al.* in a TPA for a spin-3/2 system on a hexagonal lattice.[12]

We shall show below that for a subclass of ATPA's, this diagonal truncation is in fact exact, which has important consequences.

B. *Ansätze* with conserved quantum numbers

Let us assume that the Boltzmann weights $W^{\beta,\eta}_{\alpha,\xi}$ satisfy a conservation law of the type,

$$W^{\beta,\eta}_{\alpha,\xi}=0 \quad \text{unless} \quad \alpha+\xi=\beta+\eta, \tag{18}$$

where the link-variables label the basis of an irreducible representation (irrep) of a Lie group $\mathcal{G}$. The six-vertex model corresponds to the spin 1/2 irrep of the group $\mathcal{G}=\mathrm{SU}(2)$, with the convention $\alpha=0$ (1) for the $s_z=1/2$ ($s_z=-1/2$). For a general Lie group the link variables will be given by the weights of the corresponding irrep.

The immediate consequence of Eq. (18) is that the non vanishing terms of Eq. (16) must satisfy

$$\alpha_i+m_{i-1}=\beta_i+m_i, \quad i=1,\ldots,N,$$
$$\alpha'_i+m_{i-1}=\beta'_i+m_i, \tag{19}$$

which implies

$$\alpha'_i-\alpha_i=\beta'_i-\beta_i=Q_i, \tag{20}$$

where $Q_i=1,0,-1$ for the six-vertex model. In the general case Q_i, being the difference of two weights of irreps, is either zero or a root of the Lie group $\mathcal{G}$.

Defining the Boltzmann weights W^Q as

$$(W^Q)^{\beta,\eta}_{\alpha,\xi}=W^{\beta,\eta}_{\alpha,\xi}W^{\beta+Q,\eta}_{\alpha+Q,\xi}, \tag{21}$$

we see from Eqs. (16) and (20) that $\mathcal{T}^{\rm col}$ breaks into a block transfer matrices $T^{\mathbf{Q}}$ labeled by the vector $\mathbf{Q}=(Q_1,\ldots,Q_N)$, whose entries are given by

$$T^{\mathbf{Q}}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\equiv\mathcal{T}^{\rm col}_{\boldsymbol{\alpha\alpha}+\mathbf{Q},\boldsymbol{\beta\beta}+\mathbf{Q}}=\sum_{\mathbf{m}}\prod_{i=1}^{N}(W^{Q_i})^{\beta_i,m_i}_{\alpha_i,m_{i-1}}. \tag{22}$$

The case $\mathbf{Q}=\mathbf{0}$ corresponds to the matrix (17), and hence the truncation of the model to the "diagonal" sector is not an approximation, but an exact result. This fact greatly simplifies the computation of the norm of the ATPA in the thermodynamic limit $L\to\infty$, which is given by $\Lambda_{\max}$ where $\Lambda_{\max}$ is the biggest of all largest eigenvalues $\Lambda_0^{\mathbf{Q}}$ of the matrices $T^{\mathbf{Q}}$. In Appendix A we show that this eigenvalue belongs to the $\mathbf{Q}=0$ sector and thus,

$$\lim_{L\to\infty}\langle\psi|\psi\rangle_{\rm col}=\Lambda_{\max}, \quad \Lambda_{\max}=\Lambda_0^{\mathbf{Q}=\mathbf{0}}. \tag{23}$$

This is quite a useful result for it implies that if the SM model defined by the Boltzmann weights $W^0=(W)^2$ [Eq. (21)] is integrable then $\Lambda_{\max}$ can be computed exactly, at least in the limit $N\to\infty$. In the case of the ATPA based on the six-vertex model, the Boltzmann weights W^0 are simply the square of the original ones, i.e.,

$$W^0(a,b,c)=W(a^2,b^2,c^2), \tag{24}$$

and hence the norm of the ATPA can be computed exactly in the thermodynamic limit.

There is yet another important consequence of the conservation law (18). As it is well known in the theory of transfer matrices, Eq. (18) implies that the row-to-row transfer matrix preserves the sum of all quantum numbers of every row, i.e.,

$$T^{\rm row}_{\mathbf{m},\mathbf{m}'}=0 \quad \text{unless} \quad \sum_{i=1}^{L}m_i=\sum_{i=1}^{L}m'_i. \tag{25}$$

Hence all the terms appearing in the sum (12), giving $|\psi\rangle_{\rm row}$, must have the same value of "angular momenta" per leg. In the six-vertex model, this implies the vanishing of all correlators between raising and lowering spin operators among different rows/legs, i.e.,

$$\langle S^+_{i,a}S^-_{j,b}\rangle=0, \quad \text{if} \quad i\neq j, \tag{26}$$

where $S^{\pm}_{i,a}$ is the raising (lowering) spin operator on the ath site of the ith leg. In other words, the quantum fluctuations across the legs of the ATPA are strictly forbidden. In the six-vertex model the spins may only fluctuate along the legs. Using the spinless fermion terminology, the only allowed charge fluctuations occur inside the legs. As mentioned in the introduction, this lack of quantum fluctuations across the legs is reminiscent to that occurring in some models of high-T_c superconductors (see Sec. VII).

The previous considerations give us a hint on what sort of Hamiltonians, the ATPA's may become approximate ground states. After all, we want to use the ATPA as variational *Ansätze* for physically interesting systems. We postpone this question until next section after a discussion on correlators and density matrices for ATPA's.

C. Correlators and density matrices

Let $\mathcal{O}^{\rm d}_{i,i+1}$ be a diagonal operator acting between the legs i and $i+1$ that does not change their states and with matrix element $\hat{\mathcal{O}}^{\rm d}_{\mathbf{m_i},\mathbf{m_{i+1}}}$. A typical example in the six-vertex model is provided by the Ising term $\sigma^z_{i,a}\sigma^z_{i+1,a}$. The expectation value of $\mathcal{O}^{\rm d}_{i,i+1}$ in the ATPA is be given by,

$$\langle\psi|\mathcal{O}^{\rm d}_{i,i+1}|\psi\rangle_{\rm row}=\sum_{\mathbf{m}'s}(T^{\rm row}_{\mathbf{m_1},\mathbf{m_2}})^2(T^{\rm row}_{\mathbf{m_2},\mathbf{m_3}})^2\ldots(T^{\rm row}_{\mathbf{m_N},\mathbf{m_1}})^2\hat{\mathcal{O}}^{\rm d}_{\mathbf{m_i},\mathbf{m_{i+1}}}. \tag{27}$$

It can be shown that the square of the row-to-row transfer matrix can be written as [recall Eqs. (20) and (21)]

$$(T^{\rm row}_{\mathbf{m_1},\mathbf{m_2}})^2=\sum_Q T(W^Q)_{\mathbf{m_1},\mathbf{m_2}}, \tag{28}$$

and hence the sum (27) becomes

$$\langle\psi|\mathcal{O}^{\rm d}_{i,i+1}|\psi\rangle_{\rm row} = \sum_{\mathbf{m}'s}\sum_{Q_1,\ldots,Q_N} T(W^{Q_1})_{\mathbf{m_1},\mathbf{m_2}}\ldots T(W^{Q_N})_{\mathbf{m_N},\mathbf{m_1}} \times\hat{\mathcal{O}}^{\rm d}_{\mathbf{m_i},\mathbf{m_{i+1}}}. \tag{29}$$

In the thermodynamic limit this sum will be dominated by the term $Q_1=\cdots=Q_N=0$, just as in the computation of the norm of the state and hence the expectation value of $\mathcal{O}^{\rm d}_{i,i+1}$ reduces to an expectation value in the SM model with Boltzmann weights W^0. This is a property of all diagonal operators that allow their exact evaluation, provided they are known in the underlying exactly solved model.

In the case of the operator $\sigma^z_{i,a}\sigma^z_{i+1,a}$, its correlator is equivalent to the SM expectation value

$$P\equiv\langle\sigma^z_{i,a}\sigma^z_{i+1,a}\rangle_{\rm ATPA}=\langle s_{i,a}s_{i+1,a}\rangle_{\text{six-vertex}}, \tag{30}$$

where $s=1,-1$ is related to the link variable $\alpha=0,1$ by the equation $s=1-2\alpha$. In Appendix B we shall compute this quantity using the exact solution of the six-vertex model.

Let us next consider an off-diagonal operator $\mathcal{O}^{\rm od}_i$ acting on the ith leg with matrix elements $\hat{\mathcal{O}}^{\rm od}_{\mathbf{m_i},\mathbf{m_i'}}=\langle\mathbf{m_i}|\mathcal{O}^{\rm od}_i|\mathbf{m_i'}\rangle$. Its expectation value will be given by,

$$\langle\mathcal{O}^{\rm od}_i\rangle=\mathrm{Tr}(\rho_i\hat{\mathcal{O}}^{\rm od}), \tag{31}$$

where ρ_i is the density matrix of the ith leg whose entries are,

$$\rho_{\mathbf{m_i},\mathbf{m_i'}}=\frac{1}{\langle\psi|\psi\rangle}\sum_{\mathbf{m}'s\neq\mathbf{m_i}\ \text{or}\ \mathbf{m_i'}}(T^{\rm row}_{\mathbf{m_1},\mathbf{m_2}})^2\ldots T^{\rm row}_{\mathbf{m_{i-1}},\mathbf{m_i}} \times T^{\rm row}_{\mathbf{m_{i-1}},\mathbf{m_i'}}T^{\rm row}_{\mathbf{m_i},\mathbf{m_{i+1}}}T^{\rm row}_{\mathbf{m_i'},\mathbf{m_{i+1}}}\ldots(T^{\rm row}_{\mathbf{m_N},\mathbf{m_1}})^2. \tag{32}$$

Using again Eq. (28) in the thermodynamic limit we can write ρ_i as

$$\rho_{\mathbf{m_i},\mathbf{m_i'}}=\frac{1}{\Lambda^2_{\max}}\sum_{\mathbf{m_{i-1}},\mathbf{m_{i+1}}} v^l_{\mathbf{m_{i-1}}}T^{\rm row}_{\mathbf{m_{i-1}},\mathbf{m_i}}T^{\rm row}_{\mathbf{m_{i-1}},\mathbf{m_i'}} \times T^{\rm row}_{\mathbf{m_i},\mathbf{m_{i+1}}}T^{\rm row}_{\mathbf{m_i'},\mathbf{m_{i+1}}}v^r_{\mathbf{m_{i+1}}} \tag{33}$$

where $v^{l/r}_{\mathbf{m}}$ are the left and right eigenvectors with highest eigenvalue $\Lambda_{\max}$ of the transfer matrix $T(W^0)$. Equation (33) shows that the regions located above or below of a given leg behave as if they were in a single coherent state, which in some cases can be identified with the ground state of the underlying quantum-mechanical model. This is indeed the case if we assume that $T(W^0)$ is a symmetric matrix, which is achieved in the six-vertex model if the Boltzmann weights a and b are equal. From now on we shall assume the latter condition, which implies that $v^l_{\mathbf{m}}=v^r_{\mathbf{m}}=v_{\mathbf{m}}$.

The computation of Eq. (31) is in general quite difficult depending on the operator in question. An approximation can however be made using the following result. If $\mathcal{O}^{\rm od}_i$ is a positive definite operator then

$$\langle\mathcal{O}^{\rm od}_i\rangle\leqslant\langle v|\hat{\mathcal{O}}^{\rm od}|v\rangle. \tag{34}$$

The proof of Eq. (34) uses the Perron-Frobenius theorem and the Schwartz inequality and it is similar to the one given in Appendix A to prove Eq. (23). For diagonal operators acting on a leg, Eq. (34) becomes an equality.

V. THE SIX-VERTEX ATPA

The considerations made above suggest that an ATPA based on the six-vertex model should be a reasonable approximation to the ground state of the following Hamiltonian:

$$H=H_{\rm leg}+H_{\rm rung},$$
$$H_{\rm leg}=-\frac{1}{2}\sum_{i=1}^{N}\sum_{a=1}^{L}(\sigma^x_{i,a}\sigma^x_{i,a+1}+\sigma^y_{i,a}\sigma^y_{i,a+1} +\Delta_0\sigma^z_{i,a}\sigma^z_{i,a+1}), \tag{35}$$
$$H_{\rm rung}=\frac{1}{2}J'\sum_{i=1}^{N}\sum_{a=1}^{L}\sigma^z_{i,a}\sigma^z_{i+1,a},$$

which is a combination of the XXZ Hamiltonian along the legs and an Ising one along the rungs. The latter choice is motivated by the absence of quantum fluctuations across the legs. This model has also been studied in Ref. 32 using bosonization techniques. Using the Hellman-Feynman theorem and Eqs. (30) and (34), one can find the following lower bound of the energy per site of the ATPA

$$E(\Delta,\Delta_0,J')=E_0(\Delta)+(\Delta_0-\Delta)\frac{\partial E_0}{\partial\Delta}+\frac{1}{2}J'\,P(\Delta), \tag{36}$$

where Δ is the anisotropy parameter associated to the Boltzmann weights W^0, i.e.,

$$\Delta=\frac{a_0^2+b_0^2-c_0^2}{2a_0b_0},$$
$$a_0=a^2,b_0=b^2,c_0=c^2. \tag{37}$$

$P(\Delta)$ is the expectation value defined in Eq. (30), and $E_0(\Delta)$ is the GS energy per site of the XXZ model with anisotropy Δ.

The problem is: fixing Δ_0 and J', find the value of Δ that minimizes the total energy (36), i.e.,

$$\Delta=\Delta(\Delta_0,J'). \tag{38}$$

It is easy to see that if $J'=0$ then $\Delta=\Delta_0$.

In the antiferromanetic region (AF) of the XXZ model, i.e., $\Delta<-1$, the parametrization of the Boltzmann weights is given by,[21]

$$\Delta=-\cosh\lambda, \quad \lambda>0,$$
$$a_0=\rho\sinh\frac{\lambda-v}{2},$$
$$b_0=\rho\sinh\frac{\lambda+v}{2}, \tag{39}$$
$$c_0=\rho\sinh\lambda,$$

where ρ is an overall factor and v is the spectral parameter that is set to zero in order to have a symmetric transfer matrix.

The total energy and its derivative in the AF region can be found from the Bethe *Ansätz* solution and they read,[21]

$$E_0=\frac{1}{2}\cosh\lambda-\sinh\lambda-4\sinh\lambda\sum_{m=1}^{\infty}\frac{1}{e^{2m\lambda}+1}, \tag{40}$$

$$\frac{\partial E_0}{\partial\Delta}=-\frac{1}{2}+\frac{1}{\tanh\lambda}+\frac{4}{\tanh\lambda}\times\sum_{m=1}^{\infty}\frac{1}{e^{2m\lambda}+1}-8\sum_{m=1}^{\infty}\frac{m\,e^{2m\lambda}}{(e^{2m\lambda}+1)^2}. \tag{41}$$

The matrix element P in this region is derived in Appendix B and it reads,

$$P=-1-\frac{2}{\sinh^2(\lambda/2)}+\frac{4}{\tanh(\lambda/2)}\sum_{m=1}^{\infty}\frac{\sinh m\lambda}{\cosh^2 m\lambda}. \tag{42}$$

In the critical region (C), i.e., $-1<\Delta<1$, the parametrization of the Boltzmann weights is given by,[21]

$$\Delta=-\cos\mu, \quad 0<\mu<\pi,$$
$$a_0=\rho\sin\frac{\mu-v}{2},$$
$$b_0=\rho\sin\frac{\mu+v}{2}, \tag{43}$$
$$c_0=\rho\sin\mu.$$

The GS energy per site and its derivative read[21]

$$E_0=\frac{1}{2}\cos\mu-\frac{\sin\mu}{\mu}\left[2\log 2-2\pi\int_0^{\infty}dx\,\frac{\log\cosh\mu x}{\sinh^2\pi x}\right], \tag{44}$$

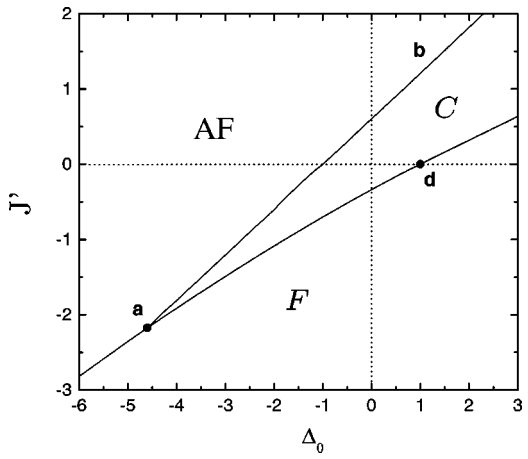

FIG. 1. Phase diagram of the Hamiltonian (35) obtained using the six-vertex ATPA. The point a corresponds to $\Delta_0=-4.6$ and $J'=-2.17$, b is a generic point on the line AF/C above the point a, and finally d corresponds to $\Delta_0=1$ and $J'=0$.

$$\frac{\partial E_0}{\partial\Delta}=-\frac{1}{2}+\frac{2\pi}{\mu}\int_0^{\infty}dx\,\frac{x\tanh\mu x}{\sinh^2\pi x}+\left(\frac{1}{\mu\tan\mu}-\frac{1}{\mu^2}\right)\left[-2\log 2+2\pi\int_0^{\infty}dx\frac{\log\cosh\mu x}{\sinh^2\pi x}\right] \tag{45}$$

and P is given by (see Appendix B),

$$P=1-\frac{4\pi}{\mu\tan(\mu/2)}\int_0^{\infty}dx\,\frac{\sinh^2\mu x}{\cosh\mu x\sinh^2\pi x}. \tag{46}$$

Figure 1 shows the phase diagram of the Hamiltonian (35) obtained by minimization of the energy (36). We recall that Eq. (36) is a lower bound of the energy of the ATPA, and hence does not yield an upper bound of the exact GS energy of Eq. (35).

The region denoted AF in Fig. 1 corresponds to the cases where $\Delta(\Delta_0,J')$ lies inside the antiferromagnetic regime $\Delta<-1$. The region C describes the critical regime, i.e., $-1<\Delta<1$, while the region F denotes the cases where $\Delta=1$. The phase boundaries between these regions have different properties. The AF/C boundary line **ab** corresponds to $\Delta=-1$, and hence the transition between the AF and C phases seems to be continuous. Below the point **a** the value of Δ, near the line AF/F but on the AF side, is smaller than -1, indicating that the AF/F boundary is discontinuous. The C/F line **ad** is also discontinuous, meaning that Δ jumps across it. Finally, there is no discontinuities across the C/F boundary above the point **d**.

In Ref. 32 the Hamiltonian (35) was studied using bosonization, mean-field and renormalization-group (RG) methods. Disregarding the interleg forward scattering and umklapp terms, that arise upon bosonization, the main conclusion of Ref. 32 is the existence of an AF region whenever $|J'|>2\Delta_0$ (mean-field result) or $|J'|>4\Delta_0$ (RG result). Furthermore, the interchain forward scattering terms can be taken into account[33] using the sliding-Luttinger liquid ap-

proach of Refs. 26,27. In[33] it was shown that the effect of the inter-chain forward scattering is to modify the phase boundaries separating the AF and C regions in an asymmetric way. Indeed the AF region appears when $J'>C_+\Delta_0$ (if $J'>0$) and $-J'>C_-\Delta_0$ (if $J'<0$), with $C_+\neq C_-$. Hence the results of Refs. 32,33, which should be valid in the weak coupling regime $|\Delta_0|,|J'|\ll 1$, suggest that the system should be in an AF phase whenever the legs are antiferromagnetic, i.e., $\Delta_0<0$. This is in contradiction with the ATPA result where there exist critical region with $\Delta_0<0$. On the other hand the ATPA agrees with the aforementioned works on the existence of large regions in the phase diagram where the system is critical, which we identify with the sliding or smectic Luttinger-liquid fixed points of Ref. 26,27.

VI. THE EIGHT-VERTEX ATPA

An ATPA closely related to the six-vertex model one that can be built from the Baxter's eight-vertex model, whose Boltzmann weights are those of the six-vertex model plus two new weights $W_{00}^{11}=W_{11}^{00}=d$.[21] The conservation law (18) now becomes,

$$W_{\alpha,\xi}^{\beta,\eta}=0 \quad \text{unless} \quad \alpha+\xi=\beta+\eta, \quad (\text{mod } 2). \tag{47}$$

The transfer matrix $\mathcal{T}^{\text{col}}$ also breaks into block matrices $T^{\mathbf{Q}}$ with the difference that Q_i only takes two values 0 and 1, since $Q_i=-1=1(\text{mod } 2)$.

The eight-vertex ATPA can be taken as the an *Ansatz* for the GS of the following Hamiltonian:

$$H=H_{\text{leg}}+H_{\text{rung}},$$

$$H_{\text{leg}}=-\frac{1}{2}\sum_{i,a}\left(J_x\sigma_{i,a}^x\sigma_{i,a+1}^x+J_y\sigma_{i,a}^y\sigma_{i,a+1}^y+J_z\sigma_{i,a}^z\sigma_{i,a+1}^z\right), \tag{48}$$

$$H_{\text{rung}}=\frac{1}{2}J'\sum_{i,a}\sigma_{i,a}^z\sigma_{i+1,a}^z.$$

The phase diagram of this model can be worked out using the Baxter's exact solution of the eight-vertex model, as we did for the six vertex one in the previous section. The results will be presented elsewhere.

VII. THE ATPA AND STRIPES

An interesting feature of the six- and the eight-vertex ATPA's is their possible connection with the stripes in high-T_c superconductors, specially when regarded as electronic liquid crystals.[26,27,30] In this section we shall briefly explore this issue that deserves a more detailed study in the future.

The first observation is that the eight-vertex Hamiltonian (48) [and similarly the six vertex one (35)] can be Jordan-Wigner transformed onto the following spinless fermion Hamiltonian,

$$H=-\frac{1}{2}\sum_{i,a}\Big[(J_x+J_y)(\psi_{i,a}^\dagger\psi_{i+1,a}+h.c.)+(J_y-J_x)\times(\psi_{i,a}\psi_{i+1,a}+h.c.)+2J_z\left(n_{i,a}-\frac{1}{2}\right)\left(n_{i+1,a}-\frac{1}{2}\right)-2J'\left(n_{i,a}-\frac{1}{2}\right)\left(n_{i,a+1}-\frac{1}{2}\right)\Big], \tag{49}$$

which describes the motion of holons along the legs of a 2D lattice [term (J_x+J_y)], which are coupled by density-density interactions (term J'), together with pair tunneling between the legs and the environment [term (J_x-J_y)].[31] Upon bosonization Eq. (49) has a structure similar, but not identical, to the ''smectic'' Hamiltonian in the spin-gap case considered in Ref. 26 and the spinless sliding-Luttinger model of Ref. 27. Indeed, the smectic symmetry $\phi_a\rightarrow\phi_a+\alpha_a$,[26] where ϕ_a is the boson field of the ath leg, is the dual version of the standard $U(1)$ symmetry of the six-vertex model, which corresponds to $\theta_a\rightarrow\theta_a+\alpha_a$, where θ_a is the dual boson.[34] The CDW coupling among the stripes in[26,27] corresponds to the term J', while the Josephson tunneling is somehow reflected by the pair creation and annihilation terms. Assuming these correspondences, it is quite natural to conjecture a relationship between the smectic phases of Refs. 26,27 and the corresponding phases of the eight-vertex model. The stripe-crystal phase should correspond to the antiferromagnetic phase, the smectic-superconducting phase should correspond to the disordered phase and finally, the smectic metal should be associated to the critical phase, which is the one of the six-vertex model when $-1<\Delta<1$.

VIII. CONCLUSIONS

In this paper we have proposed a new class of ATPA using the Boltzmann weights of classical statistical-mechanics vertex models.

We have shown that the computation of the norm and some observables simplifies enormously, becoming exact whenever the underlying SM model is exactly solvable.

The strong anisotropy of the ATPA's is reflected in the absence of quantum fluctuations across the legs of the 2D lattice, a property that suggests a possible connection with some current models of stripes.

We have studied the ATPA based on the six-vertex model, as a trial state for the ground state of a Hamiltonian given by the sum of XXZ Hamiltonians along the legs of a 2D lattice, which are coupled by an Ising term. Using the exact solution of the six-vertex model we have proposed the phase diagram of this model and compared it with the one obtain with other methods.[32,33]

We have suggested a connection between the six-vertex and eight-vertex ATPA's, and their associated 2D Hamiltonians, with the smectic-stripe phases considered in Refs. 26 and 27.

Let us finally comment on the relation between the ATPA and the DMRG. As we explained in Sec. IV, the link variables along the legs and the rungs of the SM-vertex model can be of different type. For example we can choose the rung

variables ξ, η to take only two values, say 0 and 1, as in the six-vertex model, while the legs variables α, β can take a large number of values, say $1,2,\ldots,m$, as in the DMRG. The ATPA so constructed would have a spin 1/2 at each site with strong correlations along the legs. This state would be a sort of anisotropic DMRG state with a stripelike structure builtin. The problem is to device an algorithm to update the local weights.

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APPENDIX A: HIGHEST EIGENVALUE OF T^Q

In this appendix we shall give a proof of Eq. (23) under the condition that all the TPA amplitudes are non negative. Let us call $\Lambda_0^{\mathbf{Q}}$ the largest eigenvalue of the transfer matrix $T^{\mathbf{Q}}$ defined in Eq. (22). The statement is that

$$\Lambda_0^{\mathbf{Q}} \leqslant \Lambda_0^{\mathbf{Q}=0}, \quad \forall \mathbf{Q}. \tag{A1}$$

Choosing two vectors χ and ϕ, with positive entries and scalar product equal to one,

$$\langle \chi | \phi \rangle = 1,$$

$$\chi_{\boldsymbol{\alpha}} = u_{\boldsymbol{\alpha}}^2, \quad \phi_{\boldsymbol{\alpha}} = v_{\boldsymbol{\alpha}}^2, \quad u_{\boldsymbol{\alpha}}, v_{\boldsymbol{\alpha}} > 0, \tag{A2}$$

one has by the definition of $\Lambda_0^{\mathbf{Q}}$,

$$\langle \chi | T^{\mathbf{Q}} | \phi \rangle \leqslant \Lambda_0^{\mathbf{Q}} \tag{A3}$$

where the equality holds whenever χ and ϕ are the left and right eigenvectors of $T^{\mathbf{Q}}$ respectively (recall that, by the Perron-Frobenius theorem, the eigenvector of $T^{\mathbf{Q}}$, with highest eigenvalue, has all its entries positive).

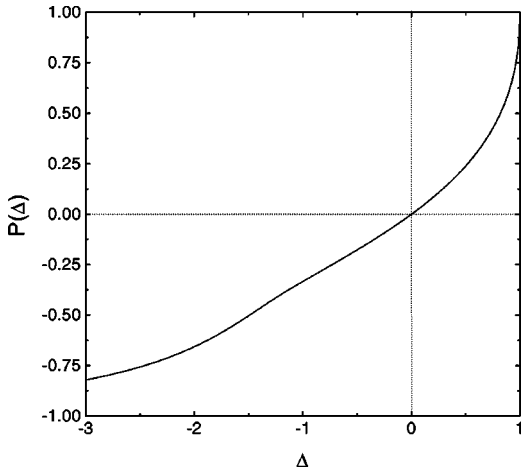

FIG. 2. Plot of $P(\Delta) = \langle s_{i,a} s_{i+1,a} \rangle_{\text{six-vertex}}$ in the AF region [Eq. (42)] and critical region [Eq. (46)].

Using Eqs. (9) and (22) we can write the LHS of Eq. (A3) as

$$\langle \chi | T^{\mathbf{Q}} | \phi \rangle = \sum_I x_I^{\mathbf{0}} x_I^{\mathbf{Q}} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{m}} u_{\boldsymbol{\alpha}}^2 v_{\boldsymbol{\beta}}^2 A^{\text{col}}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[\mathbf{m}] A^{\text{col}}_{\boldsymbol{\alpha}+\mathbf{Q},\boldsymbol{\beta}+\mathbf{Q}}[\mathbf{m}], \tag{A4}$$

where I denotes the triple $(\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{m})$ and $x_I^{\mathbf{Q}}$ stands for

$$x_I^{\mathbf{Q}} = u_{\boldsymbol{\alpha}} v_{\boldsymbol{\beta}} A^{\text{col}}_{\boldsymbol{\alpha}+\mathbf{Q},\boldsymbol{\beta}+\mathbf{Q}}[\mathbf{m}], \tag{A5}$$

it turns out that

$$\sum_I (x_I^{\mathbf{Q}})^2 \leqslant \Lambda_0^{\mathbf{0}}, \quad \forall \mathbf{Q}. \tag{A6}$$

On the other hand, the Schwartz inequality

$$\left| \sum_I x_I^{\mathbf{0}} x_I^{\mathbf{Q}} \right| \leqslant \sqrt{\left(\sum_I x_I^{\mathbf{0}} x_I^{\mathbf{0}} \right) \left(\sum_J x_J^{\mathbf{Q}} x_J^{\mathbf{Q}} \right)} \tag{A7}$$

implies

$$|\langle \chi | T^{\mathbf{Q}} | \phi \rangle| \leqslant \Lambda_0^{\mathbf{0}}. \tag{A8}$$

Hence, choosing χ and ϕ the left and right eigenvectors of $T^{\mathbf{Q}}$ one derives the desired result (A1).

APPENDIX B: THE TWO-POINT CORRELATOR $P(\Delta)$

In this appendix we indicate how to compute the expectation value

$$P = \langle s_{i,a} s_{i+1,a} \rangle_{\text{six-vertex}}, \tag{B1}$$

in the six-vertex model with Boltzmann weights a, b, and c when $a = b$, which is the case under study. This quantity is similar, but not identical, to the polarizability $P_0 = \langle \alpha_1 \rangle$ defined by Baxter.[21]

The partition function of the six-vertex model can be expanded as

$$Z = \sum a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6}, \tag{B2}$$

where n_1 and n_2 are the number of vertices with Boltzmann weight a, etc. The weights a and b contribute to P with $+1$ while c does it with -1, hence P is given by the formula,

$$P = \lim_{N,L\to\infty} \frac{1}{NL} \frac{1}{Z} \left(a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} - c \frac{\partial}{\partial c} \right) Z$$

$$= - \left(a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} - c \frac{\partial}{\partial c} \right) f \tag{B3}$$

where f is the free energy per site in the units $k_B T = 1$. It is important to realize that the derivatives in Eq. (B3) are performed keeping the remaining ones unchanged. Equation (B3) assumes that a, b, and c are independent quantities, however if $a = b$ the formula for P becomes,

$$P_{a=b} = -\left(a\frac{\partial}{\partial a} - c\frac{\partial}{\partial c} \right) f. \tag{B4}$$

In the AF region the free energy f is given, in the parametrization (39), by,[21]

$$-f = \log a + \frac{\lambda + v}{2} + \sum_{m=1}^{\infty} \frac{e^{-m\lambda} \sinh m(\lambda + v)}{m \cosh m\lambda} \tag{B5}$$

while in the critical region one has, in the parametrization (43)[21]

$$-f = \log a + \int_{-\infty}^{\infty} \frac{dx}{x} \frac{\sinh(\lambda + v)x \sinh(\pi - \mu)x}{\sinh \pi x \cosh \mu x}. \tag{B6}$$

Using Eqs. (B4), (B5), (B6), (39), and (43), with v set equal to zero, one can derive Eqs. (42) and (46) yielding $P(\Delta)$.

In Fig. 2 we plot $P(\Delta)$ in the AF and C regions. In the AF region one has $-1 < P < -1/3$, while in the C region $-1/3 < P < 1$. At the isotropic point $\Delta = -1$ one finds $P = -1/3$, while in the XY model, i.e., $\Delta = 0$, the result is $P = 0$. In all the ferromagnetic (F) region, i.e., $\Delta > 1$, one has $P = 1$.

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