Microscopic models of two-dimensional magnets with fractionalized excitations

Chetan Nayak and Kirill Shtengel*
Department of Physics and Astronomy, University of California at Los Angeles, Los Angeles, California 90095-1547

(Received 21 December 2000; published 24 July 2001)

We demonstrate that spin-charge separation can occur in two dimensions and note its confluence with superconductivity, topology, gauge theory, and fault-tolerant quantum computation. We construct a microscopic Ising-like model and, at a special coupling constant value, find its exact ground state as well as neutral spin-$\frac{1}{2}$ (spinon), spinless charge e (holon), and Z_2 vortex (vison) states and energies. The fractionalized excitations reflect the topological order of the ground state which is evinced by its fourfold degeneracy on the torus—a degeneracy which is unrelated to translational or rotational symmetry—and is described by a Z_2 gauge theory. A magnetic moment coexists with the topological order. Our model is a member of a family of topologically ordered models, one of which is integrable and realizes the toric quantum error correction code but does not conserve any component of the spin. We relate our model to a dimer model which could be a spin SU(2) symmetric realization of topological order and its concomitant quantum number fractionalization.

DOI: 10.1103/PhysRevB.64.064422 PACS number(s): 75.10.Jm

I. INTRODUCTION

The advent of polyacetylene[1] and the fractional quantum Hall effect[2,3] showed that quantum number fractionalization is a robust possibility in condensed-matter physics. The quantum numbers of the low-energy excitations of these systems are fractions of those of the microscopic degrees of freedom, the electrons. There are charge e, spin 0, and charge 0, spin-$\frac{1}{2}$ *spin-charge separated* excitations in polyacetylene and other one-dimensional (1D) systems. The fractional quantum Hall state at filling fraction $\nu=1/m$ has charge e/m, statistics π/m excitations; more exotic possibilities lurk at other ν. Despite a flurry of interest generated by the suggestion[4,5] that spin-charge separation is the mechanism for high-temperature superconductivity in the cuprates, it is, at present, unclear whether spin-charge separation can occur in a 2D magnet. There is a set of long-wavelength field theories[6–11] which describe the properties of putative fractionalized magnets, but their existence has been controversial for want of a concrete microscopic model of spin-$\frac{1}{2}$ moments coupled by short-ranged interactions in which fractionalization occurs. In this paper, we construct such a microscopic model of a 2D magnet. We find the exact ground state and neutral, spin-$\frac{1}{2}$ (spinon) and charge e, spinless (holon) excited eigenstates as well as a Z_2 vortex.[12–14,9] The fractionalized excitations reflect the *topological order*[15,16,14] of the ground state which is evinced by its fourfold degeneracy on the torus[17,18,13]—a degeneracy which is unrelated to translational or rotational symmetry—and is described by a Z_2 gauge theory.[19,20,9] Our construction implies that fractionalization is a reasonable possibility for magnets with short-ranged interactions. Our model is a member of a family of models, another of which is integrable and realizes the toric quantum error correction code.[21] The models are related to the quantum dimer model[22] and lie at the confluence between superconductivity, topology, gauge theory, and fault-tolerant quantum computation.

Our purpose here is to show that such microscopic models do exist, at least in principle, so we construct a model with the aim that it be deep within a phase supporting fractionalized excitations, not that it be a realistic description of any particular physical system. (As we discuss below, Kitaev,[21] in beautiful work, has constructed an exactly soluble model with many of the desired properties, but it does not have any conserved quantum numbers, so it is not 'fractionalized' in the sense of admitting fractional quantum numbers.) However, we insist that our model be expressed in terms of spin-$\frac{1}{2}$ electrons, so that it is truly microscopic. Consequently, our analysis differs in a number of key respects from earlier ones which dealt with models[10,11,6,9] which are not, strictly speaking, microscopic electronic models or else relied on various assumptions[22,8] in order to reduce the microscopic models to effective models which exhibit fractionalization. We avoid the need for such assumptions or modifications (however benign they may seem) by endowing our model with the following properties which distinguish it from other models which have been considered in this context: (i) Ising symmetry, (ii) translational symmetry which is broken by hand, and (iii) adiabatic continuability to an integrable model.[21]

A real magnet will have many additional complications, but these are unimportant so long as it shares the key feature of our model, namely *topological order*. In pioneering work, Wen[15,16,14] observed that phases of matter in two dimensions with fractionalized excitations are not characterized by a local order parameter, in contrast to more familiar phases such as crystals. Rather, their universal properties are encapsulated by topological quantum numbers, such as their ground-state degeneracy on a torus or higher genus surface, *over and above any degeneracy which is due to broken symmetry*. Degeneracy which is due to topological order persists in the presence of local perturbations such as impurities, which break translational and rotational symmetry. (This observation will prove important since it guides us to construct our model on lattices which penalize states which would break translational and rotational symmetry on a square lattice.) This is completely different from the twofold degeneracy associated with an Ising antiferromagnet, which is removed by the application of a small symmetry-breaking field at even one point. Topological order is well established theoretically in the fractional quantum Hall effect,[16] where it is manifested by the existence of excitations with nontrivial braiding statistics.[23]

Along with spinons and holons, a spin-charge separated state must have Z_2 vortices,[12–14] which have been recently dubbed ''visons.''[9] Topological order implies the existence of a gap in the vison spectrum. It is *not* necessary for all other excitations to be gapped (see, for instance, the construction of Ref. 6). This is analogous to the situation in a conventional ordered state such as a superconductor, which can have gapless quasiparticles if, for instance, it has *d*-wave symmetry or impurities. They do not preclude a stable sc state so long as there is a gap to the creation of vortices. Similarly, in our topologically ordered states the existence of a vison gap Δ_v is necessary to guarantee the existence of distinct topological sectors of the Hilbert space on the torus, as we will describe below. We compute the vison gap and present evidence that the rest of the spectrum is gapped (though, we reiterate, this is not a major issue). The integrable model in the family is fully gapped.

The concept of topological order is very attractive theoretically because it is precise, but it is sobering to note that it has not been possible, to date, to directly measure most of the topological quantum numbers—such as the braiding statistics—of a fractional quantum Hall state. On the other hand, this very feature has generated considerable interest in the use of topologically ordered states for quantum computation. The inaccessibility of topological degrees of freedom to local probes insulates them against many forms of decoherence, the *bête noir* of the quantum computation program.

This point was made by Kitaev[21] in a beautiful paper in which he constructed a concrete model exhibiting the requisite topological order and a fault-tolerant quantum error correcting code which could be implemented in it (see also Ref. 24). The integrable model in our family is equivalent to Kitaev's. For our purposes, the model of greater physical interest is the one which conserves S^z and exhibits quantum number fractionalization, which is of intrinsic interest and might be relevant to high-temperature superconductivity.[4] It could also prove useful for quantum computing since their spin and charge quantum numbers allows for the manipulation of spinons and holons. Harnessing the otherwise elusive visons also becomes a real possibility if the proposed experiment of Ref. 25 can be implemented. Finally and perhaps most importantly, the energy scale associated with topological order in a magnet is likely to be an exchange constant $J \sim 1000$ K. Thus a magnet with fractionalized excitations has many attractive features as a milieu for quantum computation (for another, see Ref. 26 and references therein).

II. MODEL

Our model has spin-$\frac{1}{2}$ degrees of freedom, $\mathbf{S}_\alpha$, living on the links of a lattice which we specify below. They are *not* gauge fields, but gauge-invariant, physical degrees of freedom which happen to be located on the links of the lattice (we return to this point later). The Hamiltonian is

$$H_0 = J_1 \sum_i g(S_i^z) - J_2 \sum_p F_p P_p + J_3 \sum_p P_p, \tag{1}$$

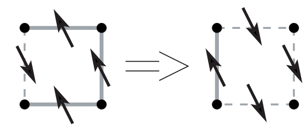

FIG. 1. The action of the flip operator F_p on a typical plaquette. Notice that the total z component of spin is generally not conserved under such operation. The links with the up spins are shown here as colored—this provides an alternative graphical representation which will be exploited later on.

where $S_i^z \equiv \sum_{\alpha \in \mathcal{N}(i)} S_\alpha^z$, and $\mathcal{N}(i)$ is the set of links emanating from site i. The definitions of F_p, P_p are

$$F_p \equiv \prod_{\alpha \in p} S_\alpha^x, \tag{2}$$

$$P_p \equiv f(S_{\alpha_1}^z + S_{\alpha_2}^z) \cdot f(S_{\alpha_2}^z + S_{\alpha_3}^z) \cdot f(S_{\alpha_3}^z + S_{\alpha_4}^z), \tag{3}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the links of plaquette p, enumerated clockwise. At a site with coordination number z, $g(x) = (2x + z - 2)^2/4$. $f(x) = 1 - x^2$. This model is closely related to the quantum dimer model[22] (please see below).

The operator $g(S_i^z)$ annihilates states (and only those states) which have $S_i^z = -1$, i.e., which have one and only one neighboring spin. The operator F_p ''flips'' plaquette p by flipping the four spins around it; an example of such flip is shown in Fig. 1. The operator P_p is a projection operator which annihilates all states except those in which up and down spins alternate around p—see Fig. 2. Plaquettes are assumed to have four sides. however, they can be put together irregularly or can overlap, as the parallelogram-shaped plaquettes of the triangular lattice do.

We will take $J_1, J_2, J_3 > 0$.

III. LATTICE

Some care is required in the choice of lattice. As we will see below, the model (1) is tractable at $J_2 = J_3$. We would like to choose a lattice so that the ground-state exhibits the key feature from which all of the interesting physics follows: fourfold ground state degeneracy on the torus even in the presence of local translational symmetry-breaking fields.

This can be accomplished if (i) the lattice does not allow accidental symmetries which will increase the ground-state degeneracy, a requirement which can usually be satisfied by taking a nonbipartite lattice, and (ii) the lattice has a unit cell which includes several plaquettes, so as to frustrate states in

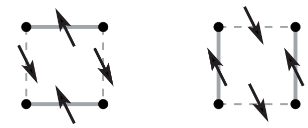

FIG. 2. The action of the projection operator P_p leaves plaquettes with the above shown spin configurations intact while annihilating *any* other type of plaquettes. A subsequent application of the flip operator F_p to these plaquettes simply transforms them into each other, therefore they will be referred to as ''flippable.''

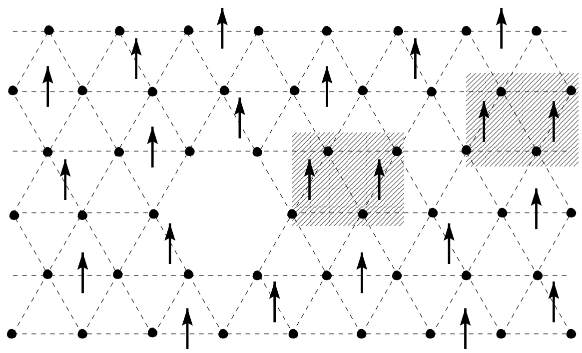

FIG. 3. The mutilated triangular lattice, T'. The spins (only up spins are shown) correspond to the maximally staggered configuration. In the presence of this type of lattice defects, there are strings of flippable plaquettes (shaded) which frustrate the true staggered state.

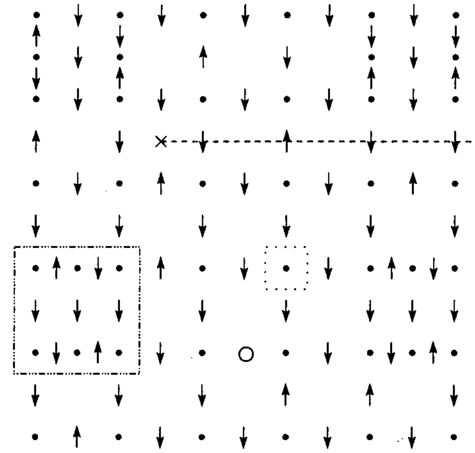

FIG. 4. A state of our model on the distorted square lattice with a spinon (centered about the site inside the dotted line), a holon (open circle), and a vison (a cross connected to a dashed line). The dotted-dashed line encloses one of the four defect plaquettes which has been split by additional sites.

which the up spins form an ordered crystal (i.e., a spin-density wave). There are many possible lattices which satisfy these requirements. Our basic strategy for constructing these lattices is to take a Bravais lattice and introduce a periodic array of ''defects.'' These defects pin a spin-density-wave state and make it nondegenerate, but they do not affect the fourfold degeneracy of the topologically ordered state. We arrange these defects with a spacing which is incommensurate with the likely spin-density-wave states so that these states are frustrated and lifted in energy. Certain types of defects will also make it easier to satisfy (i).

We will give two examples, T' and S'. T' is based on the triangular lattice (which, without defects, was exploited in this context by Moessner and Sondhi,[27] see below); the defects are missing sites, as depicted in Fig. 3. Even a single such missing site frustrates the staggered state—a special type of crystalline state with no flippable plaquettes—as depicted in Fig. 3. (A flippable plaquette is a plaquette which is not annihilated by P_p—see Fig. 2. It has alternating up and down spins whose direction can be reversed by application of $F_pP_p \equiv S^+_{\alpha_1}S^-_{\alpha_2}S^+_{\alpha_3}S^-_{\alpha_4}+\text{H.c.}$ Left to its own devices, F_p will flip any plaquette, even the ones which are not flippable; P_p prevents this. On the triangular lattice, a plaquette is any primitive parallelogram.) In T', an array of sites is missing, so that the lattice is given by

$$\left\{\mathbf{R}\,\middle|\,\mathbf{R}=n_1(a\hat{\mathbf{x}})+n_2\left(\frac{a}{2}\hat{\mathbf{x}}+\frac{\sqrt{3}a}{2}\hat{\mathbf{y}}\right);\quad n_1,n_2\not\equiv 0 \bmod k\right\}, \tag{4}$$

where k is an arbitrary integer (this is just one such example, many other T'-type lattices can be constructed along these lines). Another possibility, S', is the square lattice in which some of the plaquettes are split in two, as in Fig. 4.[29] This must be done so as to split some plaquettes horizontally and others vertically, in order to frustrate staggered states aligned in both directions. These split plaquettes may be viewed as elementary dislocations in a perfect square lattice and thus they serve to model ''real-life'' defects.

IV. GROUND STATES AND TOPOLOGICAL ORDER

The first term in Eq. (1) requires each site to have one and only one neighboring up spin, so that $S^z_i=-1$, in which case it vanishes. On the square lattice, this leads to a magnetic moment which is half that of a fully polarized state. In our model, this magnetic moment does not result from spontaneous ordering; the magnetization is actually fixed at a nonzero value. However, it is possible to have a system which will spontaneously undergo an Ising-like transition into such phase.[28] We will call a state with both spin-charge separation and ferromagnetism F^*, following the nomenclature of Ref. 7. The coexistence of conventional long-range order and quantum number fractionalization is familiar in 1D and 2D: in polyacetylene, fractionalization coexists with charge-density wave (CDW) order; in easy-axis magnetic chains, with antiferromagnetism; at the $\nu=\frac{1}{3}$ quantum Hall plateau, charge $e/3$ quasiparticles can form a crystal and the topological order (and quantum Hall effect) will not be disrupted.

The ground state of this model on T' or S' can be found exactly for $J_2=J_3$. Every plaquette costs zero energy, so long as all flippable plaquettes are taken in the linear combination $|\psi\rangle+F_p|\psi\rangle$.[22] Since every spin configuration is obtainable from every other one by the repeated application of F_pP_p,[29,30] the ground state is the superposition with equal amplitudes of all possible configurations of spins satisfying $S^z_i=-1$. The ground state is annihilated by H_0.

Let us consider the crystalline states which compete with the topologically ordered state. On the square and triangular lattices,[27] the staggered state does not mix with other states under F_pP_p since it is annihilated by this operator. It is a zero-energy ground state which is degenerate with the topologically ordered state. These two distinct ground states have become degenerate at the first-order phase transition point $J_2=J_3$.

Fortunately, this is not the case on our lattices, as we now demonstrate. The staggered state has finite-energy density at $J_2=J_3$ on T' and S' since it is frustrated on these lattices.

On T', there is no perfect staggered state. Consider a single defect (missing site). It is clearly impossible to have a perfect staggered state in the presence of the defect. The closest that we can come to a perfect staggered state (which we will call a maximally staggered state) is either (i) a state which has a string of flippable plaquettes originating at the defect and extending to infinity or (ii) a state with one vertex which is frustrated by having $S_i^z = -2$. If we take $J_1 \to \infty$, then only (a) is possible. When we introduce an array of defects, the strings of flippable plaquettes will originate at one defect and terminate at another. Under the action of F_p, this state will mix with all of the others. Hence a maximally staggered state will have energy density proportional to $\sqrt{\rho} J_2$ in this limit (where ρ is the defect density). However, for $J_2 \to \infty$, only (ii) is possible and the energy density of the maximally staggered state is proportional to ρJ_1. More generally, for J_1 and J_2 finite, the energy density of the maximally staggered state will be $\rho J_1/2$ for small defect density ρ and will be proportional to $\sqrt{\rho} J_2$ at large ρ. One might wonder whether there is some other crystalline state (e.g., one with a large unit cell) which has zero energy. However, if such a state contains flippable plaquettes, it will mix under the action of F_p with all of the other states with flippable plaquettes.[30] Hence such a state will cost finite energy.

We can repeat the above analysis for S'. On S', we can frustrate one plaquette for each defect, with energy density ρJ_2. On the perfect square lattice, the F^* ground state is not fourfold degenerate. It is critical,[22] and unstable to a columnar state as J_3 is decreased.[31] This is not the case on T' or S', so we do not need to worry about the columnar state, either.

The ground state may be visualized in the following way. Consider some reference configuration of spins which is annihilated by the first term of Eq. (1) and color all of the links which have up spins. Now take any other configuration which is also annihilated by the first term of Eq. (1) and do the same. By placing one graph on top of the other, and erasing all links at which both graphs coincide, we obtain a collection of loops on the lattice. If we visualize states in terms of their associated loop graphs, then the ground state is given by a superposition of different loop configurations.

Since H_0 conserves modulo 2 the winding numbers of these loops about either of the generators of the torus, there are four degenerate ground states on the torus, $\psi_{(n_1,n_2)}$, $n_1, n_2 = 0,1$, with 0,1 corresponding to even or odd winding numbers. By straightforward extension, the degeneracy on a genus g surface is 4^g. Although we have computed this degeneracy only at the special coupling constant value $J_2 = J_3$, we believe that it is robust over some range of parameters because it is characterized by an integer, 4. This integer cannot change as a result of infinitesimal perturbations, but only as a result of a perturbation which is sufficiently strong that it moves the system across a phase transition at which this integer changes discontinuously.

On the perfect square lattice, the directed winding number (not merely the winding number modulo 2) is conserved because the lattice is bipartite. As a result, there are $L \times L$ sectors. S' is not bipartite, so it has only four topologically distinct ground states.[32]

V. SPINONS AND HOLONS

The fourfold ground-state degeneracy implies the existence of fractionalized excitations, so long as it is unrelated to translational and rotational symmetry, a condition which is satisfied as a result of our choice of lattice. To see this, imagine cutting open the torus along its second generator, thereby producing an annulus. The ground state $\psi_{(0,0)}$ has nonzero projection on the ground state of the annulus because it will have some amplitude to have zero loops encircling the torus. $\psi_{(1,0)}$ does not, but it does have finite projection on a finite-energy excited state of the annulus because it must have at least one loop circuiting the torus and it must have some amplitude to have only one. This finite-energy excitation has, by construction, a spinon at the inner and outer edges of the annulus, as in Laughlin's construction of charge $e/3$ quasiparticles in the fractional quantum Hall effect.[3] The inner edge of the annulus can be shrunk and filled in since our discussion depends only on the topology, but not the geometry of the lattice. This construction can be done on a torus of any size, so the spinon at the boundary can be taken arbitrarily far away from the one in the interior with finite energy cost.

This general argument can be substantiated in our model by a direct construction of spinon and holon excitations. Two spinons may be created by flipping a single up spin into a down spin. This changes S^z by -1 and creates 2 sites with $S_i^z = -2$. These sites can be moved apart; each one carries $S^z = -\frac{1}{2}$ and costs energy J_1. A holon may be constructed by simply removing a spin from one of these spinon sites. This removes charge e and spin $S^z = -\frac{1}{2}$ from a neutral $S^z = -\frac{1}{2}$ excitation, thereby producing a spinless charge e excitation. There will now be a site with $S_i^z = -\frac{3}{2}$, so the holon costs energy $J_1/4$. In the loop picture, spinons and holons reside at the endpoints of broken loops.

VI. VISONS

Consider now the operator

$$\Phi_p \equiv \prod_{\alpha \in c_p} 2 S_\alpha^z, \tag{5}$$

where c_p is any curve which starts at the center of plaquette p, connects it to the center of a neighboring plaquette p', and continues in this manner through the centers of a sequence of neighboring plaquettes, running to infinity (or the boundary of the system). The product in Eq. (5) is over all links α which intersect c_p. Under the action of Φ_p, each loop configuration receives a -1 if c_p has an odd number of intersections with colored links and 1 if it has an even number of intersections with colored links. When a holon or spinon follows a trajectory encircling p, the intersection number must change by one, so Φ_p creates a Z_2 vortex, or ''vison,''[9] at plaquette p.

The statistics of holons and spinons depend on the energetics of the model: by binding to a vison, they can switch their statistics between bosonic and fermionic.[12,13] Our Hamiltonian does not allow holons or spinons to move: they are infinitely heavy. However, a small perturbation will allow them to move and will give rise to the energetics which determines whether they bind with visons and thereby their statistics.

The fourfold degeneracy of the ground state—or, in other words, the topological order—guarantees the existence of a vison energy gap. To see this, consider the degenerate states $\psi_{(0,0)} \pm \psi_{(1,0)}$ on the torus. Now imagine creating a vison pair at plaquette p, taking one vison around the second generator of the torus, and annihilating the pair at p. This is equivalent to acting on our state with an operator similar to Φ_p, but with the curve c_p in Eq. (5) replaced by a closed curve which passes through p and encircles the torus along its second generator. This operator exchanges $\psi_{(0,0)} \pm \psi_{(1,0)}$. The amplitude for such a process is essentially the exponential of the Euclidean action required for such a virtual process to occur, $\sim e^{-cL\Delta_v}$ where L is the length of the loop around the torus, Δ_v is the vison gap, and c is a constant. Hence the energy splitting between states $\psi_{(0,0)}$ and $\psi_{(1,0)}$ is $\sim e^{-cL\Delta_v}$. Since we know that this splitting vanishes in the thermodynamic limit, the vison gap Δ_v must be finite.

This conclusion is supported by a direct calculation. The creation of a vison at p takes the state $|\psi\rangle + F_p|\psi\rangle$ into $|\psi\rangle - F_p|\psi\rangle$, with an energy cost Δ_v. Since the vison creation operator Φ_p commutes with all of the terms in Eq. (1) except for the J_2 term at plaquette p, with which it anticommutes, a state with one vison, $|\Phi_p\rangle$, has excitation energy

$$\langle \Phi_p | H | \Phi_p \rangle = 2J_2 \langle 0 | P_p | 0 \rangle. \tag{6}$$

Hence the vison gap is equal to $2J_2$ multiplied by the density of flippable plaquettes. This may be computed at $J_2 = J_3$ by the Grassmann techniques discussed below. In an integrable model which we discuss below, exact vison eigenstates and energy eigenvalues may be found.

From Eq. (6), we see that the vison gap will be nonvanishing whenever $\langle 0|F_p P_p|0\rangle \neq 0$, i.e., whenever the spins fluctuate in the ground state, as they generically do in our model, even outside the topologically ordered phase. However, this is not particularly consequential. Consider the analogous situation in a superfluid: it is possible to define a vortex energy above the transition (e.g., the Kosterlitz-Thouless transition) which varies smoothly across the transition. However, this energy is only meaningful in the superfluid state (or, perhaps, near it). Similarly, the vison gap becomes meaningful in the topologically ordered phase. Outside this phase, Φ_p is merely an operator which creates some complicated gapped excitation.

A vison gap is necessary for topological order; a gap in the rest of the spectrum is not. However, the equal-time spin-spin correlation function in the ground state is exponentially decaying, as may be seen from an exact mapping between the ground state of our model and the field theory of free lattice fermions,[33] according to which it is a square root of the eight-fermion correlator. This decays exponentially with distance since these fermions, though massless on a regular square lattice, acquire mass in the presence of lattice distortions such as that shown in Fig. 4 (the details of this calculation will be published elsewhere). Hence it is natural to conclude that there is an energy gap to all excited states; this may be argued via the single-mode approximation. This computation has been carried out on the triangular lattice by Moessner and Sondhi,[27] who found that the single-mode approximation suggests that the system is indeed gapped.

VII. A FAMILY OF MODELS

We can gain further insight into the spectrum of our model by generalizing it to the following family of models:

$$H = H_0 + J_1' \sum_i y(S_i^z) - J_2' \sum_p [F_p(1-P_p) - (1-P_p)], \tag{7}$$

where $y(x) = \frac{2}{3}x(x^2-4)(x+1)$ (for $z=4$) and $J_2 = J_3$ in H_0. As J_1', J_2' are increased, two things occur: loop crossings are allowed[34] and plaquette flips of unflippable plaquettes are allowed. The topological order is preserved because the winding number modulo 2 is conserved. In the extreme limit $J_1' = J_1$, $J_2' = J_2$, there are eight equally likely configurations at each site (corresponding to those of the eight-vertex model) and the model is now integrable because $[g(S_i^z) + y(S_i^z)]$ and F_p commute among themselves and hence with the Hamiltonian, for all i, p. Hence we can simultaneously diagonalize all of these operators. They have eigenvalues 0 and 1, respectively, in the ground state; a $g(S_i^z) + y(S_i^z) = 1$ eigenvalue is a quasiparticle excitation at i and $F_p = -1$ is a vison at p. This model is equivalent to Kitaev's.[21] The ground state of this integrable model has the same topological order (fourfold degeneracy) as that of Eq. (1), but its quasiparticle excitations do not carry spin since it is not conserved. Since there is no projection operator P_p in H, the vison energy is exactly $2J_2$. Crystalline states have energy $J_2/2$ per plaquette above the ground state.

VIII. FIELD-THEORETIC DESCRIPTION

The configurations allowed in the ground state of the integrable model are described by closed loops—in other words, by the configurations of the Ising model on the dual lattice (equivalent to the eight-vertex model). The dynamics of the plaquette flip operator is the same as that of a transverse field in the Ising model. Hence the integrable model is equivalent at low energies to the transverse field Ising model which, in turn, is dual to a Z_2 gauge theory. Since the topological order associated with H_0 is the same as that of the integrable model, it, too, is described by a Z_2 gauge theory,[19] as proposed in Refs. 20 and 9.

Note that S_i^z is conserved for all i in our model (1). Hence there is an independent U(1) symmetry at each site of the lattice. However, only time-independent transformations leave the Lagrangian invariant. Hence this is an ordinary (but large) symmetry group; it is *not* a gauge symmetry, which must allow time-dependent transformations. All of the degrees of freedom in Eq. (1) are physical. This is similar to the symmetry of a set of noninteracting spins in a magnetic field, $H=-\Sigma_i \mathbf{B}\cdot\mathbf{S}_i$, which also has an independent U(1) at each site of the lattice.

IX. QUANTUM DIMER MODELS

Our model (1) can be mapped to the quantum dimer model,[5,22,12,35] in which it is assumed that there are spins located at the sites of a lattice and that each spin forms a singlet dimer with one of its nearest neighbors. In Eq. (1), an up-spin link corresponds to a dimer; a down-spin link to the absence of a dimer; spinons, to empty sites (which are holons in the dimer model). Then the first term in Eq. (1) requires each spin to form a dimer with exactly one of its neighbors. The J_2 and J_3 terms are precisely the dimer kinetic and potential energies of Ref. 22. Our F^* state of Eq. (1) is simply the resonating valence bond (RVB) (Refs. 4 and 22 ground state of the quantum dimer model on the same lattice. Recently, Moessner and Sondhi[27] gave compelling evidence that the triangular lattice quantum dimer model has an RVB ground state over a substantial range of parameters terminating at a first-order phase transition at $J_2=J_3$ into the staggered state. According to our arguments, the RVB state is the unique, exact ground state at $J_2=J_3$ on T'.

X. SU(2) SYMMETRIC MODELS

There is no reason to believe that an SU(2) symmetric magnet cannot be topologically ordered. If we wish to apply the preceding results, then the quantum dimer model can be the starting point for the discussion of an SU(2) symmetric topologically ordered magnet.[36,37,39] There is some numerical evidence[38] that a model of spins on the triangular lattice with strong four-spin exchanges—reminiscent of the plaquette flip operator—has a topologically ordered ground state with fourfold degeneracy.

XI. SUMMARY

We have demonstrated that quantum number fractionalization and topological order are an eminently reasonable possibility for two-dimensional magnets by constructing a microscopic model which exhibits these phenomena. Our result links two problems of great interest: quantum number fractionalization in 2D quantum magnets and the investigation of physical systems which are suitable platforms for fault-tolerant quantum computation (this link was also noted in Ref. 40). A possible nexus with ideas about superconductivity—either in the cuprates or elsewhere—leads to potentially fruitful avenues for further research in both areas.

ACKNOWLEDGMENTS

We would like to express our gratitude to S. Chakravarty, L. Chayes, M. Freedman, A. Kitaev, R. B. Laughlin, R. Shankar, R. Rajaraman, T. Senthil, and, especially, S. Kivelson for valuable conversations. C.N. and K.S. were both supported by the National Science Foundation under Grant No. DMR-9983544. C.N. is also supported by the A. P. Sloan Foundation.

---

*Present address: Department of Physics and Astronomy, University of California at Irvine, Irvine, CA 92697-4575.

[1] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. B **22**, 2099 (1980).
[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[3] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[4] P. W. Anderson, Science **235**, 1196 (1987).
[5] S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865 (1987).
[6] L. Balents, M. P. A. Fisher, and C. Nayak, Int. J. Mod. Phys. B **12**, 1033 (1998).
[7] L. Balents, M. P. A. Fisher, and C. Nayak, Phys. Rev. B **60**, 1654 (1999).
[8] L. Balents, M. P. A. Fisher, and C. Nayak, Phys. Rev. B **61**, 6307 (2000).
[9] T. Senthil and M. P. A. Fisher, cond-mat/9910224 (unpublished).
[10] N. Read and S. Sachdev, Phys. Rev. Lett. **66**, 1773 (1991).
[11] N. Read and S. Sachdev, Int. J. Mod. Phys. B **5**, 219 (1991).
[12] S. A. Kivelson, Phys. Rev. B **39**, 259 (1989).
[13] N. Read and B. Chakraborty, Phys. Rev. B **40**, 7133 (1989).
[14] X. G. Wen, Phys. Rev. B **44**, 2664 (1991).
[15] X. G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990).
[16] X. G. Wen, Phys. Rev. B **41**, 9377 (1990).
[17] F. D. M. Haldane, Phys. Rev. Lett. **61**, 1029 (1988).
[18] N. E. Bonesteel, Phys. Rev. B **40**, 8954 (1989).
[19] E. Fradkin and S. H. Shenker, Phys. Rev. D **19**, 3682 (1975).
[20] R. Moessner, S. L. Sondhi, and P. Chandra, Phys. Rev. Lett. **84**, 4457 (1999).
[21] A. Kitaev, quant-ph/9707021 (unpublished).
[22] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).
[23] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).
[24] J. Preskill, quant-ph/9712048 (unpublished).
[25] T. Senthil and M. P. A. Fisher, cond-mat/0006481 (unpublished).
[26] M. H. Freedman, quant-ph/0003128 (unpublished), and references therein.
[27] R. Moessner and S. L. Sondhi, cond-mat/0007378 (unpublished).
[28] C. Nayak and K. Shtengel (unpublished).

[29] Notice that the introduction of certain types of lattice defects, such as those depicted in Fig. 4, generates a few plaquettes (namely two for every split) that contain an odd number of spins. These plaquettes should be considered ''unflippable'' since flipping them would break S_z conservation. Nevertheless, every spin belonging to such plaquettes is also a member of another, flippable plaquette and therefore a low density of such defects will not result in ''frozen'' spins so long as some extra constraints are added.

[30] This can be proven inductively by assuming that this is true on all 2,4, . . . ,$2N$ site lattices with free boundary conditions if every site has at least two neighbors. If we add a pair of sites satisfying

the same condition, then we can show by direct construction that the resulting $2N+2$ site lattices will also have the property that all spin configurations are reachable by repeated application of F_pP_p. This inductive step only fails when we add a pair of sites, each of which is part of the same two plaquettes and no others. This necessarily occurs if we try to impose periodic boundary conditions, and it signals the existence of winding number sectors.

[31] S. Sachdev, Phys. Rev. B **40**, 5204 (1989).

[32] A careful reader may have noticed at this point that even though the S' lattice is not bipartite, it consists of the subsets that are. Therefore within one such subset (consider, e.g., a ‘‘belt’’ made only of perfect squares that winds the torus) one can still assign directions to the bonds (along the lines of Ref. 22) and conclude that the ‘‘directed’’ winding number across such a belt is conserved under any dynamics local to this belt. This appears to restore the extensive ground-state degeneracy that plagued the perfect square lattice.[22] However, we argue that this apparent degeneracy is due to localized modes and does not affect the overall topological order. This can be supported by the following observations: First, the spin-spin correlations are exponentially decaying with distance, and second, the ‘‘directed’’ winding number is not conserved across different parallel ‘‘belts’’ separated by the nonbipartite regions. Both these features are in sharp contrast with the case of a perfect square lattice. We thank A. Kitaev for drawing our attention to this issue.

[33] S. Samuel, J. Math. Phys. **21**, 2806 (1980).

[34] As J_2' increases, some $S_i^z=1$ configurations will occur in the ground state. Let us color up-spin links as before and superimpose the resulting picture on that of a reference state which only allows $S_i^z=-1$. The ground state is again a sum over all loops, but with crossings allowed.

[35] E. Fradkin and S. A. Kivelson, Mod. Phys. Lett. B **4**, 225 (1990).
[36] J. T. Chayes, L. Chayes, and S. A. Kivelson, Commun. Math. Phys. **123**, 53 (1989).
[37] D. J. Klein, J. Phys. A **15**, 661 (1982).
[38] G. Misguich, C. Lhuillier, B. Bernu, and C. Waldtmann, Phys. Rev. B **60**, 1064 (1999).
[39] S. A. Kivelson (private communication).
[40] N. E. Bonesteel, quant-ph/0006092 (unpublished).