Coordinate representation of the two-spinon wave function and spinon interaction in the Haldane-Shastry model

B. A. Bernevig,[1] D. Giuliano,[1,2] and R. B. Laughlin[1]
[1]*Department of Physics, Stanford University, Stanford, California 94305*
[2]*Istituto Nazionale di Fisica della Materia (INFM), Unità di Napoli, Napoli, Italy*
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By deriving and studying the coordinate representation for the two-spinon wave function, we show that spinon excitations in the Haldane-Shastry model interact. The interaction is given by a short-range attraction and causes an enhancement in the two-spinon wave function at short separations between the spinons. We express the spin susceptibility for a finite lattice in terms of the enhancement, given by the two-spinon wave function at zero separation. In the thermodynamic limit, the spinon attraction turns into the square-root divergence in the dynamical spin susceptibility.

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I. INTRODUCTION

One of the most important issues in contemporary physics is spin fractionalization, which takes place in strongly interacting one-dimensional (1D) antiferromagnets. The first exact solution of a 1D antiferromagnet goes back to an original paper by Bethe[1] about the ''Bethe ansatz'' (BA) solution of the Heisenberg model (HM). From Bethe's solution, des Cloizeaux and Pearson[2] found the dispersion relation of the low-lying excitations of the HM. Subsequently, Fadeev and Takhtajan[3] discovered that, at odds with what one would expect, the elementary excitation of these systems is not a spin-1 spin wave, but a gapless ''spin-$\frac{1}{2}$ spin wave,'' later named a ''spinon'' by Haldane.[4,5]

Spin fractionalization is a general phenomenon in 1D spin-$\frac{1}{2}$ antiferromagnets.[6] The large-scale physics of such systems is given by ''spinon gas'' dynamics.[5] Therefore, in this paper we will focus on a particular model, where the excitations are easier to visualize, that is, the Haldane-Shastry model (HSM),[4,7] where spins-$\frac{1}{2}$ located at the sites of a circular lattice antiferromagnetically interact and the interaction is inversely proportional to the square of the chord between the two sites.

Fully polarized N-spinon eigenstates of the HSM are derived by means of a correspondence with the spinless Sutherland's continuum version of the model[8] and are parametrized in terms of BA-like ''pseudomomenta.''[5] Once the energy of the many-spinon state is expressed in terms of the pseudomomenta, in the thermodynamic limit it appears to be the sum of the energies of N noninteracting particles.[10] However, this does not imply the absence of an interaction between spinons, which is encoded in the BA-like equation defining the pseudomomenta.[5,9]

In this paper we carefully investigate the spinon interaction in an exact solution of the HSM, by employing a formalism based on analytic variables on the unit radius circle.[11] By using real-space coordinates, the spin-$\frac{1}{2}$ excitations become easier to construct and visualize than by making use of plane waves.[12,10] The formalism can be easily generalized to the study of other models. Such a formalism allows us to write a ''real-space'' representation of the two-spinon wave function.

By analyzing the real-space two-spinon wave function, we show that spinons scatter by means of a short-range attractive potential and analyze in detail the physical consequences of the existence of this potential. The short rangeness of the interaction makes spinons free when they are widely separated. However, from the exact solution of the Schrödinger equation for two spinons we find that the amplitude of the wave function is greatly enhanced when they are on top of each other, a phenomenon that we refer to as ''probability enhancement.'' While the density of states is uniform at low energy, probability enhancement causes the overlap between the wave function for the localized spin wave and that for the spinon pair to be significant, but not enough to create a two-spinon bound state. The corresponding matrix element is enhanced so as to make the spin-1 excitation unstable.

Physical consequences of the instability of the spin wave appear in the functional form of the dynamical spin susceptibility (DSS) $\chi_q(\omega)$. The DSS is the Fourier transform of the spin-spin correlation function. Its functional form can be experimentally tested by means, for instance, of neutron-scattering experiments, the probed quantity being the spectral density of states $1/\pi \, \mathrm{Im}\, \chi_q(\omega)$.[13] A system with a stable spin-1 excitation would show a sharp pole in $\mathrm{Im}\, \chi_q(\omega)$ at the corresponding dispersion relation $\omega=\omega(q)$. On the other hand, instability of the spin wave against decay into spinons will generate a branch cut in $\mathrm{Im}\, \chi(q,\omega)$ at the threshold energy for the creation of a spinon pair, which is a signal of the opening of a decay channel, corresponding to the lack of spin-wave integrity. Consequently, a sharp square-root singularity shows up at the threshold for the creation of a spinon pair, on top of the broadening in the spectral density of states. Experiments performed onto quasi-one-dimensional antiferromagnets provide clear evidence for broad spectra, while no sharp spin-1 resonance has been seen.[13]

An exact calculation of the DSS cannot, in general, be performed, even for models exactly solvable with the Bethe ansatz. However, the HSM has the remarkable property that the wave function for a spin-1 excitation is fully decomposed in the basis of the two-spinon eigenstates.[14] This allows us to write an exact expression for the DSS even for a finite lattice, thus letting us explicitly show the relationship between the probability enhancement (Fig. 4) and the DSS.

The paper is organized as follows: In Sec. II we shortly review the HS Hamiltonian and its symmetries; in Sec. III we introduce the ground state of the HSM and its representation as a function of analytic variables on the unit circle. In terms of the analytic variables, the ground state takes the same functional form as the fractional quantum Hall wave function that corresponds to a nondegenerate disordered spin singlet. We discuss at length several properties of the ground state, how to derive the corresponding energy, and the meaning of the disorder in the ground state. In Sec. IV we analyze the one-spinon solution and derive its relevant properties; in Sec. V we focus on the two-spinon solution. We derive the energy eigenvalues, the corresponding eigenvectors, and their norms. A discussion about spinon statistics is provided at the end of the section. The original derivation of the results in Secs. III, IV, and V is mainly due to Haldane and Shastry.[4,7,5] Our formalism allows for a simple derivation of those results, which we discuss at length in Secs. III, IV, and V; Secs. VI and VII contain the key results of our work. In Sec. VI we write the Schrödinger equation for the two-spinon wave function, whose solutions are hypergeometric polynomials. From the behavior of the two-spinon wave function, we infer the nature of the interaction between spinons: a short-range attraction. The physical consequences of such an interaction are discussed at length in Sec. VII, where we derive an exact closed-form expression for the dynamical spin susceptibility in terms of the two-spinon wave functions and rigorously prove that the DSS is fully determined by spinon interaction. In the thermodynamic limit spinon interaction turns into the square-root divergence in the DSS. In Sec. VIII we provide our main conclusions.

II. HALDANE-SHASTRY HAMILTONIAN

The Haldane-Shastry model[4,7] is defined on a lattice with periodic boundary conditions. Let N be the number of sites. Let z_α, with $z_\alpha^N=1$, be a complex number representing a lattice site on which a spin-$\frac{1}{2}$ electron resides, and let $\vec{S}_\alpha$ be a Heisenberg spin operator acting on that electron. The Haldane-Shastry Hamiltonian takes the form

$$\mathcal{H}_{HS}=J\left(\frac{2\pi}{N}\right)^2\sum_{\alpha<\beta}^{N}\frac{\vec{S}_\alpha\cdot\vec{S}_\beta}{|z_\alpha-z_\beta|^2}, \tag{1}$$

where J is the coupling strength. The interaction is an analytic function of the coordinates. This is related to the property of a complex variable z laying on the unit circle $z^*=z^{-1}$, which implies

$$\frac{1}{|z_\alpha-z_\beta|^2}=-\frac{z_\alpha z_\beta}{(z_\alpha-z_\beta)^2}.$$

The representation in terms of the analytic variables z_α, which we will use throughout the paper, proves to be very useful for describing the properties of spinons in real space. The Hamiltonian in Eq. (1) is clearly invariant under spin rotations generated by the total spin:

$$[\mathcal{H}_{HS},\vec{S}]=0,\quad \vec{S}=\sum_{\alpha}^{N}\vec{S}_\alpha. \tag{2}$$

It also possesses an additional symmetry generated by a vector operator independent of $\vec{S}$:

$$[\mathcal{H}_{HS},\vec{\Lambda}]=0,\quad \vec{\Lambda}=\frac{i}{2}\sum_{\alpha\neq\beta}\left(\frac{z_\alpha+z_\beta}{z_\alpha-z_\beta}\right)(\vec{S}_\alpha\times\vec{S}_\beta). \tag{3}$$

The extra symmetry of $\mathcal{H}_{HS}$ is the reason for the exceptional degeneracy of the energy eigenstates and does ultimately allow for the solution of the model,[4,7] as pointed out and discussed in Ref. 4. The algebra generated by the two vector symmetries of $\mathcal{H}_{HS}$ is referred to as Yangian and is discussed in Refs. 12 and 10. $\vec{\Lambda}$ can be physically interpreted as the spin-current operator for the HSM, as we show in Appendix C.

Starting from the next section we will review the properties of the ground state and of the one- and two-spinon excited states of the HSM. This will allow us to to define the formalism we will use in order to describe the relevant physical properties of the model.

III. GROUND STATE

In this section we review some of the most important results of the HS model, obtained by Haldane and Shastry.[4,7] Let N be even. We proceed by first giving the representation of the ground state $|\Psi_{GS}\rangle$ in terms of the z coordinates and then proving that it is the actual ground state of $\mathcal{H}_{HS}$. $|\Psi_{GS}\rangle$ is defined in terms of its projection onto the set of states with $M=N/2$ spins up and the remaining spins down. If $z_1,\dots,z_M$ are the coordinates of the up spins, one defines the state $|z_1,\dots,z_M\rangle$ as $|z_1,\dots,z_M\rangle=\Pi_{j=1}^M S_j^+\Pi_{\alpha=1}^N c_{\alpha\downarrow}^\dagger|0\rangle$, where $|0\rangle$ is the empty state. The projections are given by[4,7]

$$\Psi_{GS}(z_1,\cdots,z_M)=\prod_{j<k}^{M}(z_j-z_k)^2\prod_{j=1}^{M}z_j, \tag{4}$$

where $z_1,\dots,z_M$ denote the locations of the ↑ sites with all others being ↓. We can imagine the spin system as a 1D string of boxes populated by hard-core bosons, the ↓ spin state corresponding to an empty box and the ↑ spin state corresponding to an occupied one. The total number of bosons is conserved, as it is physically the same thing as the eigenvalue of S^z. Let us, now, review the main properties of $\Psi_{GS}(z_1,\dots,z_M)$.

A. The norm of Ψ_{GS}

$\Psi_{GS}(z_1,\dots,z_M)$ is a homogeneous polynomial of degree $N-1$ in the variables $z_1,\dots,z_M$. Its norm can be computed by using the following identity:

$$C_M=\sum_{z_1,\cdots,z_M}\prod_{i<j}^{M}|z_i-z_j|^4$$
$$=\left(\frac{N}{2\pi i}\right)^M\oint\frac{dz_1}{z_1}\ldots\oint\frac{dz_M}{z_M}\prod_{i\neq j}^{M}\left(1-\frac{z_i}{z_j}\right)^2, \quad (5)$$

where the integrals are calculated on the circle of radius 1. The integral in Eq. (5) has been evaluated by Wilson.[15] The result is

$$\left(\frac{1}{2\pi i}\right)^M\oint\frac{dz_1}{z_1}\ldots\oint\frac{dz_M}{z_M}\prod_{i\neq j}^{M}\left(1-\frac{z_i}{z_j}\right)^2=\frac{(2M)!}{2^M}, \quad (6)$$

therefore

$$C_M=\frac{(2M)!}{2^M}N^M. \quad (7)$$

B. Singlet sum rule

We shall prove that the ground state is a spin singlet by showing that $|\Psi_{GS}\rangle$ is annihilated by both S^z and S^-. $S^z|\Psi_{GS}\rangle=0$ because $|\Psi_{GS}\rangle$ has an equal number of ↑ and ↓ spins

$$[S^-\Psi_{GS}](z_2,\ldots,z_M)$$
$$=\sum_{\alpha=1}^{N}\langle z_2,\ldots,z_M|S_\alpha^-|\Psi_{GS}\rangle$$
$$=\lim_{z_1\to 0}\sum_{l=1}^{N-1}\frac{1}{l!}\left\{\sum_{\alpha=1}^{N}z_\alpha^l\right\}\frac{\partial^l}{\partial z_1^l}\Psi_{GS}(z_1,\ldots,z_M)=0 \quad (8)$$

since $\Sigma_{\alpha=1}^N z_\alpha^l=N\delta_{l0}$, (mod N).

C. Coordinate invariance

Spin rotational invariance implies that Ψ_{GS} is invariant under the interchange of ↑ and ↓ coordinates. More generally, the quantization axis can be taken to be an arbitrary direction in spin space. Denoting the sites complementary to $z_1,\ldots,z_M$ by $\eta_1,\ldots,\eta_M$, so that

$$\prod_{k}^{M}(z-z_k)(z-\eta_k)=z^N-1, \quad (9)$$

we have for fixed j

$$\prod_{k\neq j}^{M}(z_j-z_k)(z_j-\eta_k)=\lim_{z\to z_j}\frac{z^N-1}{z-z_j}=Nz_j^{N-1}, \quad (10)$$

and thus

$$\prod_{j<k}^{M}(z_j-z_k)^2\prod_{j}^{M}z_j=N\prod_{j,k}^{M}\frac{1}{z_j-\eta_k}$$
$$=(-1)^M\prod_{j<k}^{M}(\eta_j-\eta_k)^2\prod_{j}^{M}\eta_j. \quad (11)$$

D. Reality

The ground state is its own complex conjugate and therefore is a real number:

$$\Psi_{GS}^*(z_1,\ldots,z_M)=\prod_{j<k}^{M}(z_j^*-z_k^*)^2\prod_{j}^{M}z_j^*$$
$$=\prod_{j<k}^{M}(z_k-z_j)^2\prod_{j}^{M}z_j^{1-N}$$
$$=\Psi_{GS}(z_1,\cdots,z_M). \quad (12)$$

E. Translational invariance

The crystal momentum of the state q is defined (mod 2π) by the equation

$$\Psi_{GS}(z_1z,\ldots,z_Mz)=e^{iq}\Psi_{GS}(z_1,\ldots,z_M), \quad (13)$$

where $z=\exp(2\pi i/N)$. From Eq. (13) it comes out that q can be either 0 or π, according to whether N is divisible by 4 or not. Ψ_{GS} equals itself, up to an overall minus sign, when translated by one lattice constant.

F. Disordered state

$|\Psi_{GS}\rangle$ is a disordered state. The way spin-spin correlations fall off with the distance defines whether a state of a magnetic system takes order or not. The relevant quantity is the spin-spin correlation function $\chi(z_\alpha)=\langle\Psi_{GS}|S_0^+S_\alpha^-|\Psi_{GS}\rangle/\langle\Psi_{GS}|\Psi_{GS}\rangle$, which can be expressed in terms of two-spinon wave functions only, as we show in Sec. VII.

One-dimensional systems do not break continuous symmetries, so they are not alleged to order. However, there is a substantial difference between half-integer spin chains and

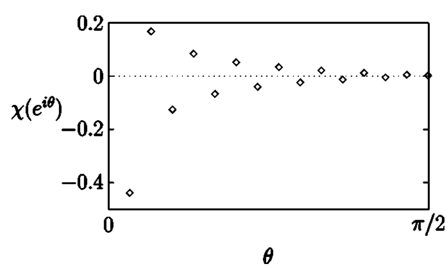

FIG. 1. Spin-spin correlation function decay for $N=60$.

integer spin ones.[6] Both have a disordered ground state, but the former have excitations above the ground state that are gapless in the thermodynamic limit while the latter have a gap that survives the thermodynamic limit and is given by $\Delta=\hbar v/\xi$, where v is the spin-wave velocity of a nearby ordered state and ξ is the correlation length. The consequence of this is that the falloff of the spin correlations in the ground state of half-odd spin chains is not as abrupt as for integer spin chains, where the correlations are suppressed within one or two lattice spacings (''Haldane's conjecture''). Figure 1 shows that the behavior of the HSM is the one expected for half-odd spin chains. Correlations decay as $(-1)^x/x$, according to Haldane's conjecture.

G. Ground-state energy

$|\Psi_{GS}\rangle$ is an eigenstate of $\mathcal{H}_{HS}$ with the eigenvalue[4,7]

$$\mathcal{H}_{HS}|\Psi_{GS}\rangle=-J\left(\frac{\pi^2}{24}\right)\left(N+\frac{5}{N}\right)|\Psi_{GS}\rangle. \tag{14}$$

We trade sums over spins on the lattice for derivative operators that are understood to act onto the analytic extension of $\Psi_{GS}(z_1,\ldots,z_M)$, in which the z_j's are allowed to take any value on the unit circle. After computing the derivatives, we constrain them again to lattice sites. We begin by observing that $[S^+_\alpha S^-_\beta\Psi_{GS}](z_1,\ldots,z_M)$ is identically zero unless one of the arguments $z_1,\ldots,z_M$ equals z_α. We have

$$\begin{aligned}\left[\left\{\sum_{\beta\neq\alpha}^{N}\frac{S^+_\alpha S^-_\beta}{|z_\alpha-z_\beta|^2}\right\}\Psi_{GS}\right](z_1,\ldots,z_M)&=\sum_{j=1}^{M}\sum_{\beta\neq j}^{N}\frac{1}{|z_j-z_\beta|^2}\Psi_{GS}(z_1,\ldots,z_{j-1},z_\beta,z_{j+1},\ldots,z_M)\\&=\sum_{j=1}^{M}\sum_{l=0}^{N-2}\left\{\sum_{\beta\neq j}^{N}\frac{z_\beta(z_\beta-z_j)^l}{l!|z_j-z_\beta|^2}\right\}\left(\frac{\partial}{\partial z_j}\right)^l\left\{\frac{\Psi_{GS}(z_1,\ldots,z_M)}{z_j}\right\}\\&=\sum_{l=0}^{N-2}\sum_{j=1}^{M}\frac{z_j^{l+1}}{l!}A_l\left(\frac{\partial^l}{\partial z_j^l}\right)\left\{\frac{\Psi_{GS}(z_1,\cdots,z_M)}{z_j}\right\}.\end{aligned} \tag{15}$$

The coefficients A_l are evaluated in Appendix B. Their remarkable property is that they are zero for $N>l>2$. Hence, Eq. (15) can be rewritten as

$$\begin{aligned}&\sum_{j=1}^{M}\left\{\frac{(N-1)(N-5)}{12}z_j-\frac{N-3}{2}z_j^2\frac{\partial}{\partial z_j}+\frac{1}{2}z_j^3\frac{\partial^2}{\partial z_j^2}\right\}\left\{\frac{\Psi_{GS}(z_1,\cdots,z_M)}{z_j}\right\}\\&=\left\{\frac{N(N-1)(N-5)}{24}-\frac{N-3}{2}\sum_{j\neq k}^{M}\frac{2z_j}{z_j-z_k}+\sum_{j\neq k\neq m}^{M}\frac{2z_j^2}{(z_j-z_k)(z_j-z_m)}+\sum_{j\neq k}^{M}\frac{z_j^2}{(z_j-z_k)^2}\right\}\Psi_{GS}(z_1,\ldots,z_M)\\&=\left\{-\frac{N}{8}-\sum_{j\neq k}^{M}\frac{1}{|z_j-z_k|^2}\right\}\Psi_{GS}(z_1,\ldots,z_M).\end{aligned} \tag{16}$$

In Eq. (16) we have made use of the rule

$$\frac{z_\alpha^2}{(z_\alpha-z_\beta)(z_\alpha-z_\gamma)}+\frac{z_\beta^2}{(z_\beta-z_\alpha)(z_\beta-z_\gamma)}+\frac{z_\gamma^2}{(z_\gamma-z_\alpha)(z_\gamma-z_\beta)}=1. \tag{17}$$

We also have

$$\begin{aligned}&\left[\left\{\sum_{\beta\neq\alpha}^{N}\frac{S^z_\alpha S^z_\beta}{|z_\alpha-z_\beta|^2}\right\}\Psi_{GS}\right](z_1,\ldots,z_M)\\&=\left\{-\frac{N(N^2-1)}{48}+\sum_{j\neq k}^{M}\frac{1}{|z_j-z_k|^2}\right\}\\&\quad\times\Psi_{GS}(z_1,\ldots,z_M).\end{aligned} \tag{18}$$

This completes the proof, since

$$\mathcal{H}_{HS}=\frac{J}{2}\left(\frac{2\pi}{N}\right)^2\left\{\sum_{\alpha\neq\beta}^{N}\frac{S^+_\alpha S^-_\beta}{|z_\alpha-z_\beta|^2}+\sum_{\alpha\neq\beta}^{N}\frac{S^z_\alpha S^z_\beta}{|z_\alpha-z_\beta|^2}\right\}. \tag{19}$$

The wave function $\Psi_{GS}(z_1,\ldots,z_M)$ was first introduced by Haldane and Shastry[4,7] in analogy to the exact Sutherland solution of the continuum limit of the problem.[8] The proof that this wave function is the actual ground state of $\mathcal{H}_{HS}$ is a consequence of the factorization of the HS Hamiltonian (first pointed out in Ref. 16, as we are going to discuss next).

H. Factorization of $\mathcal{H}_{HS}$

In Appendix D we prove that $\mathcal{H}_{HS}$ can be written as

$$\mathcal{H}_{HS}=J\left(\frac{2\pi}{N}\right)^2\left[\frac{2}{9}\sum_{\alpha}^{N}\vec{D}^\dagger_\alpha\cdot\vec{D}_\alpha-\frac{N(N^2+5)}{48}+\frac{N+1}{12}\vec{S}^2\right]. \tag{20}$$

The operators D_α are given by

$$\vec{D}_\alpha=\frac{1}{2}\sum_{\beta\neq\alpha}^{N}\frac{z_\alpha+z_\beta}{z_\alpha-z_\beta}[i(\vec{S}_\alpha\times\vec{S}_\beta)+\vec{S}_\beta] \tag{21}$$

and they annihilate $|\Psi_{GS}\rangle$ (see Appendix D). Equation (20) implies that $|\Psi_{GS}\rangle$ is the ground state of the HSM because $\mathcal{H}_{HS}$ can be written as a constant plus nonnegative definite operators, and the only state satisfying the requirement of minimum energy is $|\Psi_{GS}\rangle$.

I. Degeneracy

The HSM ground state is not degenerate, but is nearly so. We already pointed out that half-odd spin magnets have a gapless spectrum. In the next sections we will see that elementary excitations above the ground state are spinons and that their spectrum is relativistic. In particular, at the endpoints of the Brillouin zone, the energy becomes the same as the ground-state energy, modulo corrections that are subleading in the thermodynamic limit. This means that, in principle, one can have many states with the same energy as the ground state that are distinguished from one another by their number of spinons.

An example is provided by the singlet state of two spinons with total momentum π. It is given by

$$\Psi_S(z_1,\ldots,z_M)=\prod_{j<k}^{M}(z_j-z_k)^2\left[1-\prod_{j=1}^{M}z_j^2\right]. \tag{22}$$

Its energy is given by

$$\mathcal{H}_{HS}|\Psi_S\rangle=-J\left(\frac{\pi^2}{24}\right)\left(N-\frac{7}{N}\right)|\Psi_S\rangle \tag{23}$$

and is the energy of the ground state plus corrections that go to zero in the thermodynamic limit.

J. Spin current

We now show that $|\Psi_{GS}\rangle$ is an eigenstate of Λ^z belonging to the 0 eigenvalue. The action of Λ^z on the ground state gives

$$\Lambda^z|\Psi_{GS}\rangle=0. \tag{24}$$

Equation (24) can be proved as follows:

$$\begin{aligned}&[\Lambda^z\Psi_{GS}](z_1,\ldots,z_M)\\&\quad=\frac{1}{2}\sum_{j=1}^{M}\sum_{\beta\neq j}^{N}z_\beta\left(\frac{z_j+z_\beta}{z_j-z_\beta}\right)\\&\qquad\times\sum_{l=0}^{N-1}\frac{(z_j-z_\beta)^l}{l!}\frac{\partial}{\partial z_j^l}\left\{\frac{\Psi_{GS}(z_1,\ldots,z_M)}{z_j}\right\}\\&\quad=\left[-\frac{N}{4}(N-2)+\frac{N}{4}(N-2)\right]\Psi_{GS}(z_1,\ldots,z_M)=0,\end{aligned} \tag{25}$$

where we have made use of the results of Appendix B and of the technique described in detail in Sec. III F.

IV. ONE-SPINON WAVE FUNCTION

At odds with the naive idea that the elementary excitations for interacting magnets are integer spin states (spin flips), Fadeev and Takhtajan[3] first conjectured that one-dimensional half-odd spin chains exhibit excitations carrying half-odd spin, later called spinons.[4,5] For a chain with an even number of sites, the ground state is a disordered spin singlet but, if the number of sites is odd, the minimum possible value for the total spin is $\frac{1}{2}$. In the thermodynamic limit, it makes no difference whether one begins with an odd or an even number of sites. The short rangeness of the correlations in the ground state makes it insensitive to the boundary conditions, so in the thermodynamic limit, there is no way to distinguish between chains with an odd or even number of sites. States with half-odd spin are then alleged to appear as eigenstates of $\mathcal{H}_{HS}$ with an odd number of spinons. In this section we shall present the one-spinon wave function and discuss its properties. Following Haldane[4] we consider a wave function of the general form

$$\Psi(z_1,\ldots,z_M)=\Phi(z_1,\ldots,z_M)\prod_{j<k}^{M}(z_j-z_k)^2\prod_{j}^{M}z_j, \tag{26}$$

where $z_1,\ldots,z_M$ denote the position of the up spins. Here Φ is a homogeneous symmetric polynomial of degree less than $N-2M+2$ in each variable. This latter condition causes Ψ to be a polynomial of degree less than $N+1$ in each of its variables z_j, and thus allows the Taylor expansion technique used for the ground state to be applied. Doing so, we find that

$$\begin{aligned}\mathcal{H}_{HS}\Psi=\frac{J}{2}\left(\frac{2\pi}{N}\right)^2&\left\{\lambda+\frac{N}{48}(N^2-1)\right.\\&\left.+\frac{M}{6}(4M^2-1)-\frac{N}{2}M^2\right\}\Psi,\end{aligned} \tag{27}$$

provided that Φ satisfies the eigenvalue equation for λ,

$$\frac{1}{2}\left\{\sum_{j}^{M}z_j^2\frac{\partial^2\Phi}{\partial z_j^2}+\sum_{j\neq k}^{M}\frac{4z_j^2}{z_j-z_k}\frac{\partial\Phi}{\partial z_j}\right\}-\frac{N-3}{2}\sum_{j}^{M}z_j\frac{\partial\Phi}{\partial z_j}=\lambda\Phi. \tag{28}$$

A. One-spinon spin doublet

We look for one-spinon and two-spinon wave functions in the functional form given by Eq. (26). Here we analyze the one-spinon wave function. Let the number of sites N be odd and let

$$\Psi_\alpha(z_1,\ldots,z_M)=\prod_{j}^{M}(z_\alpha-z_j)\prod_{j<k}^{M}(z_j-z_k)^2\prod_{j}^{M}z_j, \tag{29}$$

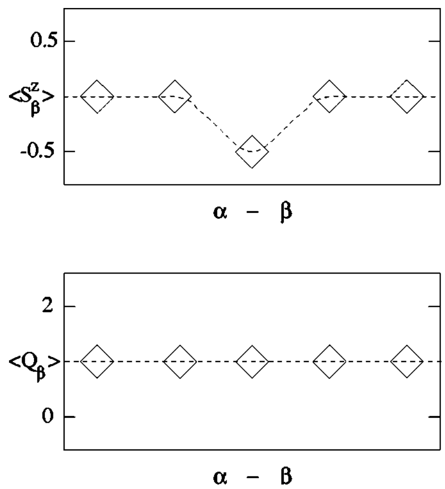

FIG. 2. Spin and charge profiles of the localized spinon $|\Psi_\alpha\rangle$ defined by Eq. (29). The dotted lines are a guide to the eye.

where $M=(N-1)/2$. This is a $\downarrow$ spin on site α surrounded by an otherwise featureless singlet sea. It is worth stressing that Eq. (29) makes perfect sense for any z_α on the unit circle. Nevertheless, as z_α coincides with a lattice site, it represents a spin $\downarrow$ localized at the corresponding site. The spin density of the corresponding state, plotted as a function of the spinon position, will be uniformly zero, as appropriate for the disordered spin singlet, except for an abrupt dip centered at $z=z_\alpha$ (see Fig. 2). Such a dip is what we refer to as a ''real-space representation'' of a spinon at z_α. Hence, a spinon can be visualized as a local defect in an otherwise featureless singlet sea. This defect behaves like a real quantum-mechanical particle, as we will show in the following.

By definition, Ψ_α is an eigenstate of S^z with eigenvalue $-\frac{1}{2}$. In order to prove that it is a spin-$\frac{1}{2}$ state, we need to show that S^- annihilates it. Indeed, per Eq. (8) we have

$$\sum_{\beta\neq\alpha}^{N} S_\beta^- \Psi_\alpha = 0, \tag{30}$$

which proves that Ψ_α is the spin-$\frac{1}{2}$ component of a spin doublet.

B. One-spinon energy

Equation (29) corresponds to a particular choice of Φ in Eq. (26) given by

$$\Phi(z_1,\ldots,z_M)=\Phi_\alpha(z_1,\ldots,z_M)=\prod_{j=1}^{M}(z_\alpha-z_j). \tag{31}$$

Equation (28), once written for the state Φ_α, takes the form

$$\left\{M(M-1)-z_\alpha^2\frac{\partial^2}{\partial z_\alpha^2}-\frac{N-3}{2}\left[M-z_\alpha\frac{\partial}{\partial z_\alpha}\right]\right\}\Phi_\alpha=\lambda\Phi_\alpha. \tag{32}$$

The eigenstate of $\mathcal{H}_{HS}$ is given by

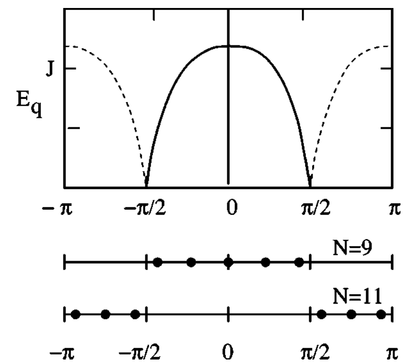

FIG. 3. Top: spinon dispersion given by Eq. (38). Bottom: allowed values of q for adjacent odd N.

$$\Psi_m(z_1,\ldots,z_M)=\frac{1}{N}\sum_{\alpha=1}^{N}(z_\alpha^*)^m\Psi_\alpha(z_1,\ldots,z_M) \tag{33}$$

and the energy eigenvalue is[4,7]

$$\mathcal{H}_{HS}|\Psi_m\rangle=\left\{-J\left(\frac{\pi^2}{24}\right)\left(N-\frac{1}{N}\right)+\frac{J}{2}\left(\frac{2\pi}{N}\right)^2 m\left(\frac{N-1}{2}-m\right)\right\}|\Psi_m\rangle, \tag{34}$$

with $0\leqslant m\leqslant(N-1)/2$ and $\lambda=m[(N-1)/2-m]$.

C. Crystal momentum

The state $|\Psi_m\rangle$ is a propagating $\downarrow$ spinon with crystal momentum

$$q=\frac{\pi}{2}N-\frac{2\pi}{N}\left(m+\frac{1}{4}\right)(\mathrm{mod}\ \ 2\pi), \tag{35}$$

per the definition

$$\Psi_m(z_1 z,\ldots,z_M z)=\exp(iq)\Psi_m(z_1,\ldots,z_M), \tag{36}$$

where $z=\exp(2\pi i/N)$. Rewriting the eigenvalue as

$$\mathcal{H}|\Psi_m\rangle=\left\{-J\left(\frac{\pi^2}{24}\right)\left(N+\frac{5}{N}-\frac{3}{N^2}\right)+E_q\right\}|\Psi_m\rangle, \tag{37}$$

we obtain the dispersion relation

$$E(q)=\frac{J}{2}\left[\left(\frac{\pi}{2}\right)^2-q^2\right](\mathrm{mod}\ \ \pi) \tag{38}$$

plotted in Fig. 3. Note that the momenta available to the spinon span only the inner or outer half of the Brillouin zone, depending on whether $N-1$ is divisible by 4 or not. The loss of half of the states available for a regular fermion is a peculiar property of the spinon spectrum. No negative energy states appear, i.e., there is nothing like an ''antispinon.'' One can picture a spinon as either an electron or a hole whose charge has been pulled out by the interaction. According to such a picture, a spinon can arise either from an electron with the same spin or from a hole with the opposite spin, which explains the halving of the Brillouin zone.

The spinon dispersion at low energies is linear in q with a velocity

$$v_{\mathrm{spinon}}=\frac{\pi}{2}J. \tag{39}$$

The half band of single elementary excitations for odd N are the only $S=\frac{1}{2}$ states without extra degeneracies. The ground state of the odd-N spin chain is four-fold degenerate and is given by $|\Psi_m\rangle$ for $m=0$ and $(N-1)/2$ and their $\uparrow$ counterparts. This corresponds physically to a ''leftover'' spinon with momentum $\pm\pi$.

D. Spin current

We now study the action of Λ^z on the state for one propagating spinon. Working as for the ground state one gets

$$\Lambda^z|\Psi_m\rangle=\left\{\frac{N-1}{4}-m\right\}|\Psi_m\rangle, \tag{40}$$

and the eigenvalue of Λ^z comes out to be proportional to the spinon velocity

$$\frac{dE(q)}{dq}=-\frac{2\pi J}{N}\left\{\frac{N-1}{4}-m\right\}. \tag{41}$$

Equation (40) is proven by first letting Λ^z act on the state Ψ_α defined in Eq. (29):

$$\begin{aligned}
[\Lambda^z\Psi_\alpha](z_1,\ldots,z_M)
&=\frac{1}{2}\sum_{j=1}^{M}\sum_{\beta\neq j}^{N} z_\beta\left(\frac{z_j+z_\beta}{z_j-z_\beta}\right)\sum_{l=0}^{N-1}\frac{(z_j-z_\beta)^l}{l!}\\
&\quad\times\frac{\partial}{\partial z_j^l}\left\{\frac{\Psi_\alpha(z_1,\ldots,z_M)}{z_j}\right\}\\
&=\frac{1}{2}\left\{-M(N-2)+2\sum_{j=1}^{M}\sum_{i\neq j}^{M}\frac{z_j}{z_j-z_i}+2\sum_{j=1}^{M}\frac{z_j}{z_j-z_\alpha}\right\}\\
&\quad\times\Psi_\alpha(z_1,\ldots,z_M).
\end{aligned} \tag{42}$$

After taking $M=(N-1)/2$ one gets

$$[\Lambda^z\Psi_\alpha](z_1,\ldots,z_M)=\left\{\frac{N-1}{4}-z_\alpha\frac{\partial}{\partial z_\alpha}\right\}\Psi_\alpha(z_1,\ldots,z_M), \tag{43}$$

which, on the basis of the states $|\Psi_m\rangle$, gives the result quoted in Eq. (40).

E. The norm

The squared norm of Ψ_m is defined as

$$\langle\Psi_m|\Psi_m\rangle=\sum_{z_1,\cdots,z_M}|\Psi_m(z_1,\ldots,z_M)|^2. \tag{44}$$

By means of a simple algebraic procedure, we generated a recursion relation between $\langle\Psi_m|\Psi_m\rangle$ and $\langle\Psi_{m-1}|\Psi_{m-1}\rangle$. Such a procedure can be straightforwardly extended to the norm of the multiple-spinon states. We discuss it at length in Appendix E. The induction relation is

$$\frac{\langle\Psi_m|\Psi_m\rangle}{\langle\Psi_{m-1}|\Psi_{m-1}\rangle}=\frac{\left(m-\frac{1}{2}\right)(M-m+1)}{m\left(M-m+\frac{1}{2}\right)}. \tag{45}$$

This recursively gives

$$\langle\Psi_m|\Psi_m\rangle=\frac{\Gamma[M+1]\Gamma\left[m+\frac{1}{2}\right]\Gamma\left[M-m+\frac{1}{2}\right]}{\Gamma\left[\frac{1}{2}\right]\Gamma\left[M+\frac{1}{2}\right]\Gamma[m+1]\Gamma[M-m+1]}C_M, \tag{46}$$

where C_M is the overall constant we have introduced in Eq. (5).

V. TWO-SPINON WAVE FUNCTION

Let us now focus on the two-spinon state. Spinons maintain their integrity when many of them are present. This does not mean that spinons are noninteracting. They can be separated at large distances, therefore being asymptotic states of the system. However, they also scatter strongly off each other by means of a short-range attractive potential. The interaction between spinons is not enough to create two-spinon bound states, but it generates a peculiar ''piling up'' of the relative wave function when the two spinons are on top of each other (Fig. 4). This is the reason for the huge decay amplitude for a spin wave into a pair of spinons, that is, the spin fractionalization.

In this section we derive the two-spinon eigenstates, their norm, and the corresponding value of the spin current. Moreover, we show that the appropriate statistics they obey is neither fermionic nor bosonic. They are semions, i.e., particles with $\frac{1}{2}$ fractional statistics.

A. Two-spinon energy

Two $\downarrow$ spinons can be pictured as two $\downarrow$ spins within an otherwise featureless disordered sea. The state with two $\downarrow$ spinons centered at z_α and z_β, respectively, is given by (N is even and $M=N/2-1$)[4,5,7]

$$\Psi_{\alpha\beta}(z_1,\ldots,z_M)=\prod_{j}^{M}(z_\alpha-z_j)(z_\beta-z_j)\times\prod_{j<k}^{M}(z_j-z_k)^2\prod_{j}^{M}z_j. \tag{47}$$

As for the one-spinon case, z_α and z_β are not necessarily lattice sites. If they are, Eq. (47) represents a pair of spinons at z_α, z_β. To derive the eigenvalue equation, we start from a wave function in the form of Eq. (26), where we take the function Φ to be equal to

$$\Phi_{\alpha\beta}=\prod_{j}^{M}(z_\alpha-z_j)(z_\beta-z_j). \tag{48}$$

Equation (28) can be rewritten for $\Phi_{\alpha\beta}$, yielding

$$\frac{1}{2}\left\{\sum_{j=1}^{M} z_j^2\frac{\partial^2\Phi_{\alpha\beta}}{\partial z_j^2}+\sum_{j\neq k}^{M}\frac{4z_j^2}{z_j-z_k}\frac{\partial\Phi_{\alpha\beta}}{\partial z_j}\right\}-\frac{N-3}{2}\sum_{j=1}^{M} z_j\frac{\partial\Phi_{\alpha\beta}}{\partial z_j}$$
$$=\left\{-\frac{z_\alpha^2}{z_\alpha-z_\beta}\frac{\partial}{\partial z_\alpha}-\frac{z_\beta^2}{z_\beta-z_\alpha}\frac{\partial}{\partial z_\beta}-z_\alpha^2\frac{\partial^2}{\partial z_\alpha^2}-z_\beta^2\frac{\partial^2}{\partial z_\beta^2}\right.$$
$$+\left(\frac{N-3}{2}\right)\left[z_\alpha\frac{\partial}{\partial z_\alpha}+z_\beta\frac{\partial}{\partial z_\beta}\right]$$
$$\left.+[2M^2-M(N-2)]\right\}\Phi_{\alpha\beta}=\lambda\Phi_{\alpha\beta}. \tag{49}$$

Let us now define the states Ψ_{mn} as follows:

$$\Psi_{mn}(z_1,\ldots,z_M)=\sum_{\alpha,\beta}^{N}\frac{(z_\alpha^*)^m}{N}\frac{(z_\beta^*)^n}{N}\Psi_{\alpha\beta}(z_1,\ldots,z_M). \tag{50}$$

A set of linearly independent states may be constructed by taking only the Ψ_{mn} with $M\geqslant m\geqslant n\geqslant 0$, which shows the overcompleteness of the set of states Ψ_{mn}. On such a set of states Eq. (49) becomes

$$\frac{1}{2}\left\{\sum_{j=1}^{M} z_j^2\frac{\partial^2}{\partial z_j^2}+\sum_{j\neq k}^{M}\frac{4z_j^2}{z_j-z_k}\frac{\partial}{\partial z_j}-(N-3)\sum_{j=1}^{M} z_j\frac{\partial}{\partial z_j}\right\}\Psi_{mn}$$
$$=\left\{-\frac{N^2}{48}\left(N-\frac{19}{N}+\frac{24}{N^2}\right)+m\left(\frac{N}{2}-1-m\right)\right.$$
$$\left.+n\left(\frac{N}{2}-1-n\right)+\frac{m-n}{2}\right\}\Psi_{mn}$$
$$-\sum_{l=0}^{l_M}(m-n+2l)\Psi_{m+l,n-l}, \tag{51}$$

where $l_M=n$ if $m+n<M$, $l_M=M-m$ if $m+n\geqslant M$, and, in deriving Eq. (51), we used the identity

$$\frac{x+y}{x-y}(x^my^n-x^ny^m)=2\sum_{l=0}^{m-n}x^{m-l}y^{n+l}-(x^my^n+x^ny^m). \tag{52}$$

We look for solutions to Eq. (51) that are linear combinations of the states $\Psi_{m+l,n-l}$:

$$\Phi_{mn}=\sum_{l=0}^{l_M}a_l^{mn}\Psi_{m+l,n-l}. \tag{53}$$

The coefficients a_l are found to be[8,4,7]

$$a_l^{mn}=\frac{-(m-n+2l)}{2l(l+m-n+1/2)}\sum_{k=1}^{l}a_{k-1}^{mn}\quad (a_0=1) \tag{54}$$

and the corresponding two-spinon energies are given by

$$E_{mn}=-J\frac{\pi^2}{24}\left(N-\frac{19}{N}+\frac{24}{N^2}\right)+\frac{J}{2}\left(\frac{2\pi}{N}\right)^2\left[m\left(\frac{N}{2}-1-m\right)\right.$$
$$\left.+n\left(\frac{N}{2}-1-n\right)-\frac{m-n}{2}\right]. \tag{55}$$

In terms of spinon momenta, the expression of the energy is

$$E_{mn}=-J\left(\frac{\pi^2}{24}\right)\left(N+\frac{5}{N}\right)$$
$$+\left[E(q_m)+E(q_n)-\frac{\pi J}{N}\frac{|q_m-q_n|}{2}\right](q_m\leqslant q_n). \tag{56}$$

E_{mn} is the sum of the ground-state contribution, $E_{GS}=-J(\pi^2/24)(N+5/N)$, and of the two-spinon energy above the ground state, $E(q_m,q_n)$. $E(q_m,q_n)$ is the sum of the energies of two isolated spinons plus a negative interaction contribution that becomes negligibly small in the thermodynamic limit.

Such a simple solution for the two-spinon problem is possible because the matrix to which Eq. (51) corresponds is lower triangular, i.e., takes the form

$$\text{matrix}=\begin{bmatrix} E_0 & 0 & 0 & 0 & \cdots \\ v_{10} & E_1 & 0 & 0 & \\ v_{20} & v_{21} & E_2 & 0 & \\ v_{30} & v_{31} & v_{32} & E_3 & \cdots \\ \vdots & & & \vdots & \end{bmatrix}, \tag{57}$$

where

$$E_j=\lambda_{m+j,n-j}\quad\text{and}\quad v_{pq}=-(m-n+2p+2q),$$

and

$$\lambda_{mn}=m\left(\frac{N}{2}-1-m\right)+n\left(\frac{N}{2}-1-n\right)-\frac{m-n}{2}. \tag{58}$$

The eigenvalues of such a matrix are its diagonal elements, and the corresponding eigenvectors are generated by recursion.

The transformation in Eq. (53) can be inverted and it takes the form

$$\Psi_{mn}=\sum_{l=0}^{l_M}b_l^{mn}\Phi_{m+l,n-l}. \tag{59}$$

The coefficients b_l^{mn} can be expressed in a closed-form formula in terms of the coefficients of the two-spinon wave functions. We provide their expression in Sec. VI.

B. The norm

The squared norm of the state Φ_{mn} is defined as

$$\langle \Phi_{mn}|\Phi_{mn}\rangle = \sum_{z_1,\ldots,z_M} |\Phi_{mn}(z_1,\ldots,z_M)|^2. \tag{60}$$

As for the one-spinon wave function, we calculate the norm of the two-spinon states by means of mathematical induction. The details of our calculation are discussed in Appendix E. The basic induction relations are given by

$$\frac{\langle \Phi_{mn}|\Phi_{mn}\rangle}{\langle \Phi_{m,n-1}|\Phi_{m,n-1}\rangle} = \frac{\left(n-\frac{1}{2}\right)\left(M-n+\frac{3}{2}\right)(m-n+1)^2}{n(M-n+1)\left(m-n+\frac{3}{2}\right)\left(m-n+\frac{1}{2}\right)}, \tag{61}$$

$$\frac{\langle \Phi_{mn}|\Phi_{mn}\rangle}{\langle \Phi_{m-1,n}|\Phi_{m-1,n}\rangle} = \frac{(M-m+1)\left(m-n+\frac{1}{2}\right)\left(m-n-\frac{1}{2}\right)m}{\left(M-m+\frac{1}{2}\right)(m-n)^2\left(m+\frac{1}{2}\right)}. \tag{62}$$

From Eqs. (61) and (62) one gets the formula for the squared norm:

$$\langle \Phi_{mn}|\Phi_{mn}\rangle = C_M \frac{M+\frac{1}{2}}{\pi} \frac{\Gamma\left[m-n+\frac{1}{2}\right]\Gamma\left[m-n+\frac{3}{2}\right]}{\Gamma^2[m-n+1]} \times \frac{\Gamma[m+1]\Gamma\left[M-m+\frac{1}{2}\right]}{\Gamma\left[m+\frac{3}{2}\right]\Gamma[M-m+1]} \times \frac{\Gamma\left[n+\frac{1}{2}\right]\Gamma[M-n+1]}{\Gamma[n+1]\Gamma\left[M-n+\frac{3}{2}\right]}. \tag{63}$$

Equation (63) basically agrees with the result quoted in Ref. 17, although we derived it by making direct use of the operator $\mathcal{H}_{HS}$ (see Appendix E).

C. Spin current

The Ψ_{mn} are eigenstates of Λ^z. Indeed, a manipulation similar to the one-spinon case yields

$$\Lambda^z|\Psi_{mn}\rangle = \left\{\frac{N-2}{2}-m-n\right\}|\Psi_{mn}\rangle, \tag{64}$$

with the eigenvalue given by the sum of the two-spinon velocities. We will skip the proof of Eq. (64), which works exactly like the proof of Eq. (40).

D. Spinon statistics

Spinons are semions, i.e., particles obeying 1/2 fractional statistics. Since the two-spinon wave function $\Psi_{\alpha\beta}$ has the property

$$\Psi^*_{\alpha\beta}(z_1,\ldots,z_{N/2-1}) = (z_\alpha z_\beta)^{1-N/2}\Psi_{\alpha\beta}(z_1,\ldots,z_{N/2-1}),$$

the Berry phase vector potential for adiabatic motion of spinon α in the presence of β is

$$\frac{1}{2}\left[\frac{\left\langle \psi_{\alpha\beta}\middle| z_\alpha \frac{\partial}{\partial z_\alpha}\psi_{\alpha\beta}\right\rangle + \left\langle z_\alpha\frac{\partial}{\partial z_\alpha}\psi_{\alpha\beta}\middle|\psi_{\alpha\beta}\right\rangle}{\langle \psi_{\alpha\beta}|\psi_{\alpha\beta}\rangle}\right] = \frac{1}{2}\left(1-\frac{N}{2}\right). \tag{65}$$

The phase to ‘‘exchange’’ the spinons by moving α all the way around the loop is thus

$$\Delta\phi = \oint \frac{1}{2}\left(1-\frac{N}{2}\right)\frac{dz_\alpha}{z_\alpha} = \pm\frac{\pi}{2}i \quad (\mathrm{mod}\ 2\pi). \tag{66}$$

This number is 0 or π for bosons or fermions. The number of states available to $l\downarrow$ spinons, determined by counting the number of distinct symmetric polynomials of the form

$$\Phi_{z_{A_1},\ldots,z_{A_l}}(z_1,\ldots,z_{(N-l)/2}) = \prod_j^{(N-l)/2} (z_j-z_{A_1})\times\cdots\times(z_j-z_{A_l}) \tag{67}$$

is

$$\mathcal{N}_l^{\mathrm{semi}} = \binom{N/2+l/2}{l}. \tag{68}$$

This is just halfway between the numbers

$$\mathcal{N}_l^{\mathrm{fermi}} = \binom{N/2}{l} \quad \mathcal{N}_l^{\mathrm{bose}} = \binom{N/2+l}{l}, \tag{69}$$

likewise calculated assuming that the number of states available for one particle is $N/2$.

VI. PROBABILITY ENHANCEMENT

In this section we provide one of the key results of our study: the analysis of the interaction between two spinons. First, we properly define the real-space representation for the two-spinon relative wave function. Then, we study the behavior of the corresponding amplitude as a function of the spinon separation. Here we construct the real-space wave function for a spinon pair and we show that our results provide clear evidence for spinons being interacting particles.

The real-space representation for the two-spinon wave functions corresponding to the energy eigenstate $|\Phi_{mn}\rangle$, $z_\alpha^m z_\beta^n p_{mn}(z_\alpha/z_\beta)$, is defined by the decomposition of the state of two localized spinons at z_α and z_β, $|\Psi_{\alpha\beta}\rangle$, in the

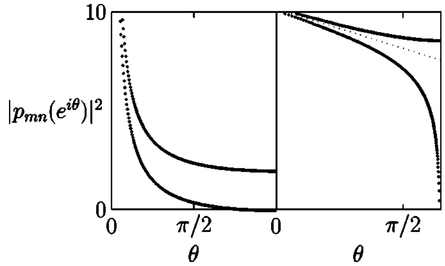

FIG. 4. Left panel: square modulus of the two-spinon wave function as a function of the separation between the two spinons for $N=1000$. Right panel: the same plot on a log-log scale. The dashed line is a guide to the eye. It is a plot of $1/x$ with an appropriate offset $[z=\exp(i\theta)]$.

basis of the fully polarized two-spinon eigenstates

$$\Psi_{\alpha\beta}=\sum_{m=0}^{M}\sum_{n=0}^{m}(-1)^{m+n}z_\alpha^m z_\beta^n p_{mn}\left(\frac{z_\alpha}{z_\beta}\right)\Phi_{mn}. \tag{70}$$

$|\Phi_{mn}\rangle$ is an eigenstate of $\mathcal{H}_{HS}$ with eigenvalue E_{mn}. This implies

$$\langle\Phi_{mn}|\mathcal{H}_{HS}|\Psi_{\alpha\beta}\rangle=E_{mn}\langle\Phi_{mn}|\Psi_{\alpha\beta}\rangle. \tag{71}$$

From Eq. (49) we see that $\langle\Phi_{mn}|\mathcal{H}_{HS}|\Psi_{\alpha\beta}\rangle$ can be written as a differential operator acting on $\langle\Phi_{mn}|\Psi_{\alpha\beta}\rangle$. $\Psi_{\alpha\beta}$ is perfectly defined for any z_α, z_β on the unit circle, so the differential operator acts on the analytic extension of $\langle\Phi_{mn}|\Psi_{\alpha\beta}\rangle$ as

$$\begin{aligned}\langle\Phi_{mn}|\mathcal{H}_{HS}|\Psi_{\alpha\beta}\rangle=&E_{GS}\langle\Phi_{mn}|\Psi_{\alpha\beta}\rangle+\frac{J}{2}\left(\frac{2\pi}{N}\right)^2\\&\times\left\{\left(M-z_\alpha\frac{\partial}{\partial z_\alpha}\right)z_\alpha\frac{\partial}{\partial z_\alpha}\right.\\&+\left(M-z_\beta\frac{\partial}{\partial z_\beta}\right)z_\beta\frac{\partial}{\partial z_\beta}\\&\left.-\frac{1}{2}\frac{z_\alpha+z_\beta}{z_\alpha-z_\beta}\left(z_\alpha\frac{\partial}{\partial z_\alpha}-z_\beta\frac{\partial}{\partial z_\beta}\right)\right\}\\&\times\langle\Phi_{mn}|\Psi_{\alpha\beta}\rangle.\end{aligned} \tag{72}$$

In the differential operator in Eq. (72) we recognize the sum of the energies of the two free spinons and a velocity-dependent interaction that diverges at small spinon separations. Equation (72) allows for determination of the exact expression of $p_{mn}(z_\alpha/z_\beta)$. Indeed, by using Eqs. (70)–(72), we find the following equation for $p_{mn}(z)$ $(z=z_\beta/z_\alpha)$:

$$z(1-z)\frac{d^2p_{mn}}{dz^2}+\left[\frac{1}{2}-m+n-\left(-m+n+\frac{3}{2}\right)z\right]\frac{dp_{mn}}{dz}+\frac{m-n}{2}p_{mn}=0. \tag{73}$$

This equation is a special case of the hypergeometric equation[18] where the parameters c,b,a are given by

$$c=\frac{1}{2}-m+n,\quad b=\frac{1}{2},\quad a=-m+n.$$

The solution is a hypergeometric series whose regular solution stops at a power of z given by z^{m-n}, thus becoming the ''hypergeometric polynomial''

$$p_{mn}(z)=\frac{\Gamma[m-n+1]}{\Gamma\left[\frac{1}{2}\right]\Gamma\left[m-n+\frac{1}{2}\right]}\times\sum_{k=0}^{m-n}\frac{\Gamma\left[k+\frac{1}{2}\right]\Gamma\left[m-n-k+\frac{1}{2}\right]}{\Gamma[k+1]\Gamma[m-n-k+1]}z^k. \tag{74}$$

The value of the spinon wave function at zero separation between spinons can be computed by means of general identities among hypergeometric series.[18] It is given by

$$p_{mn}(1)=\Gamma[1/2]\Gamma[m-n+1]/\Gamma[m-n+1/2]. \tag{75}$$

According to Eq. (74), $|p_{mn}(z)|^2$ is the density of probability for two spinons as a function of the distance between them.

The interpretation of our results is straightforward. Spinons do actually behave like real particles. Indeed, we have been able to determine a differential equation for p_{mn}, which for the two-spinon wave function is the same as the Schrödinger equation for a pair of ordinary particles. The interaction between spinons is clearly shown in Fig. 4, where we plot $|p_{mn}(z)|^2$. At large separations, the probability density oscillates and averages to 1, independently on the distance between the spinons. This is a typical feature of noninteracting particles. Indeed, at large separations spinons are in fact noninteracting. However, at small separations, $|p_{mn}|^2$ shows a probability enhancement that corresponds to a huge increase of the probability of configurations with the two spinons on top one of each other. The enhancement is highest at a relative spinon momentum of π. Such features are clear evidence for the interaction among spinons, which can be characterized as follows: (1) It is attractive: it favors configurations with spinons on top one of each other, and (2) it is short ranged: spinons are free at large distances. As N gets larger, the probability enhancement peaks up.[19] Hence, the enhancement safely survives the thermodynamic limit, even though in this limit, the energy for the two-spinon solution is the sum of the energies of the two isolated spinons. However, the attraction is not strong enough to create a two-spinon bound state, even in the thermodynamic limit. This corresponds to the absence of a low-energy stable spin-1 excitation, which is a typical feature of 1D spin-$\frac{1}{2}$ antiferromagnets.

The attractive force may also be inferred from the energy eigenvalue if we rewrite it as

$$\mathcal{H}_{HS}|\Phi_{mn}\rangle=\left\{-J\left(\frac{\pi^2}{24}\right)\left(N+\frac{5}{N}\right)+E_{q_m}+E_{q_n}+V_{q_m-q_n}\right\}|\Phi_{mn}\rangle=\left\{-J\left(\frac{\pi^2}{24}\right)\left(N+\frac{5}{N}\right)+E(q_m,q_n)\right\}|\Phi_{mn}\rangle, \tag{76}$$

where

$$V_q=-J\frac{\pi}{N}|q|. \tag{77}$$

Note that this potential vanishes as $N\to\infty$, as expected for particles that interact only when they are close together. However, as we already pointed out, the vanishing of the interaction potential in the thermodynamic limit does not mean that no residual effects survive such a limit. The probability enhancement when the two spinons are on the same site does survive the thermodynamic limit and it is the main reason for the instability of the spin-1 spin wave, as we discuss at length in the next section.

Before concluding this section, we provide the expression of the coefficients b_l^{mn} in Eq. (59). From Eq. (74) it is straightforward to prove that

$$b_l^{mn}=\frac{\Gamma[m-n+2l+1]}{\Gamma\left[\frac{1}{2}\right]\Gamma\left[m-n+2l+\frac{1}{2}\right]}\times\frac{\Gamma\left[l+\frac{1}{2}\right]}{\Gamma[l+1]}\frac{\Gamma\left[m-n+l+\frac{1}{2}\right]}{\Gamma[m-n+l+1]}. \tag{78}$$

VII. SPIN SUSCEPTIBILITY

In this section, we work out the dynamical spin susceptibility for the HSM. We show that the DSS depends only on the p_{mn}'s calculated at $z=1$, which allows us to obtain for any finite N a simple closed-form expression for the DSS and to relate it to the spinon interaction. By carefully taking the thermodynamic limit of our result, we obtain the Haldane-Zirnbauer formula for the DSS in the thermodynamic limit.[14] The Haldane-Zirnbauer formula shows that there is no low-energy spin-1 pole in the DSS, but the function takes a sharp square-root singularity at the two-spinon threshold on top of a branch cut, corresponding to the lack of integrity of the spin-1 excitation. Our analysis definitely proves that the square-root sharp edge on top of the broad spectrum is nothing but the interaction between spinons. The probability enhancement is the square-root singularity in the spin susceptibility. This result is of the utmost importance, since it represents a way to experimentally test the interaction among spinons in one dimension. We will come back to such a point in the concluding remarks.

Let us begin with the calculation of the spin susceptibility for a finite lattice. The DSS is the dynamical propagator for a spin-1 spin flip. A spin flip with momentum q is created by acting on $|\Psi_{GS}\rangle$ with S_q^-, defined as

$$S_q^-=\sum_\alpha (z_\alpha^*)^k(S_\alpha^x-iS_\alpha^y) \quad (q=2\pi k/N). \tag{79}$$

A peculiar property of the HSM is that a spin flip at z_α is the same as a spinon pair at the same site.[14] Therefore, we can fully decompose $S_q^-\Psi_{GS}$ in the basis of the two-spinon eigenstates:

$$S_q^-\Psi_{GS}=\sum_\alpha (z_\alpha^*)^k\Psi_{\alpha\alpha}=N\sum_{m=0}^{M}\sum_{n=0}^{m}(-1)^{m+n}p_{mn}(1)\times\delta(m+n-k)\Phi_{mn}. \tag{80}$$

The susceptibility is given by

$$\chi_q(\omega)=\sum_X\frac{|\langle X|S_q^-|\Psi_{GS}\rangle|^2}{\langle X|X\rangle\langle\Psi_{GS}|\Psi_{GS}\rangle}\frac{2(E_X-E_{GS})}{(\omega+i\eta)^2-(E_X-E_{GS})^2}, \tag{81}$$

($|X\rangle$ is an exact eigenstate of $\mathcal{H}_{HS}$ with energy E_X). Then, from Eqs. (80) and (81) we have that $\chi_q(\omega)$ takes a nonzero contribution only if $|X\rangle=|\Phi_{mn}\rangle$. Then, Eq. (81) becomes

$$\chi_q(\omega)=N^2\sum_{m=0}^{M}\sum_{n=0}^{m}\frac{\langle\Phi_{mn}|\Phi_{mn}\rangle}{\langle\Psi_{GS}|\Psi_{GS}\rangle}p_{mn}^2(1)\times\delta(m+n-k)\frac{2(E_{mn}-E_{GS})}{(\omega+i\eta)^2-(E_{mn}-E_{GS})^2}. \tag{82}$$

Equation (82) is another relevant result of our work. It shows that only the $p_{mn}(z)$'s at $z=1$ determine $\chi_q(\omega)$. Therefore, the spin susceptibility is completely determined by spinon interaction.

Let us analyze the thermodynamic limit of Eq. (82). In the thermodynamic limit the gamma functions can be approximated by using Stirling's formula:

$$\Gamma[z]\approx\sqrt{\pi}(z-1)^{(z-1/2)}e^{-(z-1)}, \tag{83}$$

From Eqs. (63), (75), and (83) we get in the thermodynamic limit

$$N^2\frac{\langle\Phi_{mn}|\Phi_{mn}\rangle}{\langle\Psi_{GS}|\Psi_{GS}\rangle}p_{mn}^2(1)=\frac{\pi N\left(m-n+\frac{1}{2}\right)}{\sqrt{n(M-n)\left(m+\frac{1}{2}\right)\left(M-m-\frac{1}{2}\right)}}. \tag{84}$$

Since the joint two-spinon density of states is flat, the sums over m and n become integrals over the (halved) one-spinon Brillouin zone

$$\sum_m \to -M\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}. \tag{85}$$

From Eqs. (83)–(85), we see that Eq. (82) turns, in the thermodynamic limit, into the Haldane-Zirnbauer formula for the DSS (Ref. 14):

$$\chi_q(\omega)=\frac{J}{2}\int_{-\pi/2}^{\pi/2}dq_1\int_{-\pi/2}^{q_1}dq_2\frac{|q_1-q_2|\,\delta(q_1+q_2-q)}{\sqrt{E(q_1)E(q_2)}}\times\frac{2E(q_1,q_2)}{(\omega+i\eta)^2-E^2(q_1,q_2)}, \tag{86}$$

where $E(q)$ and $E(q_1,q_2)$ are the one-spinon and the two-spinon energies, respectively. Integration over q_1, q_2 in Eq. (86) provides

$$\chi_q(\omega)=\frac{J}{4}\frac{\Theta[\omega_2(q)-\omega]\Theta[\omega-\omega_{-1}(q)]\Theta[\omega-\omega_{+1}(q)]}{\sqrt{\omega-\omega_{-1}(q)}\sqrt{\omega-\omega_{+1}(q)}}, \tag{87}$$

where $\omega_{-1}(q)$ and $\omega_{+1}(q)$ are the threshold energies for a spinon pair with momentum q, according to whether $0\leqslant q\leqslant\pi$ or $\pi\leqslant q\leqslant 2\pi$, respectively. They are given by $\omega_{-1}(q)=(J/2)q(\pi-q)$, $\omega_{+1}(q)=(J/2)(2\pi-q)(q-\pi)$. $\omega_2(q)=(J/2)q(2\pi-q)$ is the upper threshold for the spin-1 excitation. From Eq. (87) we see that the enhancement given by $p_{mn}^2(1)$ has turned into a square-root singularity in $\chi_q(\omega)$ vs ω at fixed q, with the branch cut originating either at $\omega_{-1}(q)$ or at $\omega_{+1}(q)$, depending on the value of q. Because the two-spinon joint density of states is uniform, the main conclusion we trace out from our calculation is that the branch cut in $\chi_q(\omega)$, i.e., the broadness of the spectral density of states, is the spinon interaction.

A measurement of $\chi_q(\omega)$ in 1D spin-$\frac{1}{2}$ antiferromagnets can be performed by means of neutron-scattering experiments.[13] The result of the measurements[13] does, in fact, show a sharp threshold followed by a broad spectrum, in good agreement with predictions of Eq. (87). In light of our present discussion, we conclude that what is actually seen in such an experiment is a direct consequence of the spinon interaction in 1D antiferromagnets. Hence, the experiments provide evidence that spinons do interact and that the spinon interaction is what determines the peculiar low-energy physics of spin-$\frac{1}{2}$ antiferromagnetic chains.

From Eq. (80) we also derive the formula for the spin-spin correlation function $\chi(z_\alpha)$ in terms of the two-spinon wave function at $z=1$:

$$\chi(z_\alpha)=\frac{\langle\Psi_{GS}|S_1^+S_{z_\alpha}^-|\Psi_{GS}\rangle}{\langle\Psi_{GS}|\Psi_{GS}\rangle}=\sum_{m=0}^{M}\sum_{n=0}^{m}(z_\alpha)^{m+n}\frac{\langle\Phi_{mn}|\Phi_{mn}\rangle}{\langle\Psi_{GS}|\Psi_{GS}\rangle}p_{mn}^2(1). \tag{88}$$

Equation (88) is the formula we have plotted in Fig. 1.

VIII. CONCLUSIONS

In this paper we developed a simple approach to the study of spinon excitations of the Haldane-Shastry model, based on the formalism of the analytic variables. Within our approach we picture spinons as local defects in the disordered sea. Our formalism allows for a consistent real-space representation of the wave function for two spinons. We construct the Schrödinger equation, whose solution is the two-spinon wave function, which shows that spinons behave as real quantum-mechanical particles. By means of a careful study of the real-space two-spinon wave function, we reveal the main result: the existence of spinon interaction and its survival in the thermodynamic limit. Spinon interaction is a short-range attraction that generates an enhancement of the probability for two spinons to be at the same site. Such an interaction determines the low-energy physics of 1D interacting antiferromagnets. Since the low-energy joint density of states is uniform, the broadness in the spectral density is exclusively caused by the enhancement, as we show from the finite-N expression for the spin susceptibility (Sec. VII). In the thermodynamic limit the probability enhancement develops a square-root singularity followed by a branch cut, which is the broadness in the spectral density of states. The branch cut reflects the instability of the spin wave towards decay into a spinon pair. Then, we show that even though in the thermodynamic limit the interaction is irrelevant, its main effect, the probability enhancement, peaks up. In conclusion, we analyzed spinon interaction in an exact solution of the Haldane-Shastry model and its consequences for the low-energy physics of 1D spin-$\frac{1}{2}$ antiferromagnets.

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APPENDIX A: FOURIER SUMS

In this appendix we will prove some of the formulas we used throughout the paper. Since the lattice sites z_α are roots of unity we have

$$\prod_\alpha^N (z-z_\alpha)=z^N-1. \tag{A1}$$

Then for $0\leqslant m\leqslant N$ we have

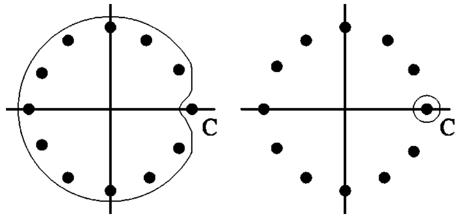

FIG. 5. Contours used in Eqs. (A2) and (A3).

$$\sum_{\alpha=1}^{N-1}\frac{z_\alpha^m}{z_\alpha-1}$$
$$=\frac{N}{2\pi i}\oint_C\frac{z^{m-1}dz}{(z-1)(z^N-1)}$$
$$=-\frac{N}{2\pi i}\oint_{C'}\frac{z^{m-1}dz}{(z-1)(z^N-1)}$$
$$=-\frac{N}{2\pi i}$$
$$\times\oint\left\{\frac{1+\binom{m-1}{1}x+\binom{m-1}{2}x^2+\cdots}{\binom{N}{1}+\binom{N}{2}x+\binom{N}{3}x^2+\cdots}\right\}\frac{dx}{x^2}$$
$$=\frac{N+1}{2}-m, \tag{A2}$$

and

$$\sum_{\alpha=1}^{N-1}\frac{z_\alpha^m}{|z_\alpha-1|^2}$$
$$=-\sum_{\alpha=1}^{N-1}\frac{z_\alpha^{m+1}}{(z_\alpha-1)^2}$$
$$=-\frac{N}{2\pi i}\oint_C\frac{z^m dz}{(z-1)^2(z^N-1)}$$
$$=\frac{N}{2\pi i}\oint_{C'}\frac{z^m dz}{(z-1)^2(z^N-1)}$$
$$=\frac{1}{2\pi i}$$
$$\times\oint\left\{\frac{1+\binom{m-1}{1}x+\binom{m-1}{2}x^2+\cdots}{\binom{N}{1}+\binom{N}{2}x+\binom{N}{3}x^2+\cdots}\right\}\frac{dx}{x^3}$$
$$=\frac{N^2-1}{12}-\frac{m(N-1)}{2}+\frac{m(m-1)}{2}. \tag{A3}$$

(Integration contours are shown in Fig. 5.)

APPENDIX B: CALCULATIONS OF THE COEFFICIENTS A_L

In this appendix we work out the coefficients A_l that appear in the eigenvalue equations for the eigenfunctions of the HSM are defined as

$$A_l=-\sum_{\alpha=1}^{N-1}\frac{z_\alpha^2}{(z_\alpha-1)^{2-l}}. \tag{B1}$$

They can be computed by using the equations from Appendix A. In particular we have

$$A_0=-\sum_{\alpha=1}^{N-1}\frac{z_\alpha^2}{(z_\alpha-1)^2},$$
$$\sum_{\alpha=1}^{N-1}\frac{z_\alpha}{|z_\alpha-1|^2}=\frac{(N-1)(N-5)}{12},$$
$$A_1=-\sum_{\alpha=1}^{N-1}\frac{z_\alpha^2}{z_\alpha-1}=-\frac{N-3}{2},$$
$$A_2=-\sum_{\alpha=1}^{N-1}z_\alpha^2=1,$$
$$A_l=-\sum_{\alpha=1}^{N-1}z_\alpha^2(z_\alpha-1)^{l-2}=0,\quad (l>2). \tag{B2}$$

APPENDIX C: THE SPIN CURRENT

In this appendix we provide the physical interpretation of the operator $\vec{\Lambda}$, as the spin-current operator. In order to do so, we first construct the continuous interpolation of the lattice spin field, given by the spin-density operator $\vec{\rho}(z)$. Then, we define a current density on the unit circle $\vec{j}(z)$. We prove that $\vec{\rho}$ and $\vec{j}$ obey an equation, which once restricted to the lattice, takes the form of the continuity equation for the spin density. The operator $\vec{\Lambda}$ comes out to be the global operator whose density is given by $\vec{j}(z)$.

The first step of such a construction is defining the field interpolating the spin operators into the interstices by means of the formula

$$\vec{\sigma}(z)=\left[\frac{z^{N/2}-z^{-N/2}}{2N}\right]\sum_{\beta}^{N}\left(\frac{z+z_\beta}{z-z_\beta}\right)\vec{S}_\beta. \tag{C1}$$

Then we can associate to $\vec{\sigma}(z)$ a "σ model-like" Hamiltonian given by

$$\frac{1}{2\pi i}\oint\left[z\frac{d\vec{\sigma}}{dz}\right]\cdot\left[z\frac{d\vec{\sigma}}{dz}\right]\frac{dz}{z}$$
$$=-\frac{2}{N}\sum_{\alpha\neq\beta}^{N}\frac{\vec{S}_\alpha\cdot\vec{S}_\beta}{|z_\alpha-z_\beta|^2}+\frac{3}{8}(N-1)+\frac{S^2}{8}, \tag{C2}$$

where the integral is performed over the unit circle. Eq. (C2) gives the Haldane-Shastry Hamiltonian plus an irrelevant constant and an operator that commutes with it.

We also have spin-density and spin-current density operators

$$\vec{\rho}(z) = -i\vec{\sigma}(z)\times\vec{\sigma}(z),$$

$$\vec{j}(z) = \frac{1}{2i}\left\{\vec{\sigma}\times\left[z\frac{d\vec{\sigma}}{dz}\right]-\left[z\frac{d\vec{\sigma}}{dz}\right]\times\vec{\sigma}\right\}. \quad \text{(C3)}$$

That $\rho(z)$ is an appropriate definition of the spin density may be seen by taking the limit $z\to z_\alpha$, z_α being a site on the lattice. One gets

$$\lim_{z\to z_\alpha}\vec{\rho}(z) = \vec{S}_\alpha. \quad \text{(C4)}$$

That $\vec{j}(z)$ is a proper spin current can be inferred from the continuity equation

$$\lim_{z\to z_\alpha}\left\{z\frac{d\vec{j}}{dz}+\left[\sum_{\alpha\neq\beta}^{N}\frac{\vec{S}_\alpha\cdot\vec{S}_\beta}{|z_\alpha-z_\beta|^2},\vec{\rho}\right]\right\}=0. \quad \text{(C5)}$$

The zero-momentum component of this conserved current density is

$$\frac{1}{2\pi i}\oint\vec{j}\frac{dz}{z}=\vec{\Lambda}. \quad \text{(C6)}$$

The operator $\vec{\Lambda}$ is then a scaled spin current. Its action on the state with a fixed number of propagating spinons, Eqs. (25), (40), and (64), is definitely consistent with such an interpretation.

APPENDIX D: FACTORIZABILITY OF $\mathcal{H}_{\mathrm{HS}}$

In this appendix we will prove the factorization formula, Eq. (20). In order to do so, we split the proof in two steps. First, we will show that the operator $\vec{D}_\alpha$ annihilates $|\Psi_{GS}\rangle$, then we will prove the factorization equation. Let us begin with the first proof. The operator

$$\Omega_\alpha=\sum_{\beta\neq\alpha}^{N}\frac{z_\alpha}{z_\alpha-z_\beta}\left[S_\alpha^+S_\beta^- - \left(S_\alpha^z+\frac{1}{2}\right)\left(S_\beta^z+\frac{1}{2}\right) - \frac{N-1}{2}\left(S_\alpha^z+\frac{1}{2}\right)\right] \quad \text{(D1)}$$

annihilates $|\Psi_{GS}\rangle$. Indeed, by using the technique we developed in Sec. III, we find

$$[\Omega_\alpha\Psi_{GS}](z_1,\ldots,z_M)=\sum_{l=0}^{N-2}\left\{\frac{1}{l!}\sum_{\beta\neq\alpha}^{N}\frac{z_\alpha z_\beta(z_\beta-z_\alpha)^l}{z_\alpha-z_\beta}\right\}\times\frac{\partial^l}{\partial z_\alpha^l}\left\{\frac{\Psi_{GS}(z_\alpha,\ldots,z_{N/2})}{z_\alpha}\right\} - \sum_{\beta\neq\alpha}^{N}\frac{z_\alpha}{z_\alpha-z_\beta}\left[-2\left(\frac{1}{2}+S_\alpha^z\right)\times\left(\frac{1}{2}+S_\beta^z\right)+\frac{N-1}{2}\right]\times\Psi_{GS}(z_1,\ldots,z_M)=0 \quad \text{(D2)}$$

for all α. However since $|\Psi_{GS}\rangle$ is a spin singlet the irreducible representations of the rotation group present in this operator must destroy $|\Psi_{GS}\rangle$ separately. The scalar component is identically zero. The vector component is

$$\sum_{\beta\neq\alpha}^{N}\frac{z_\alpha}{z_\alpha-z_\beta}[i(\vec{S}_\alpha\times\vec{S}_\beta)+\vec{S}_\beta]|\Psi_{GS}\rangle=0. \quad \text{(D3)}$$

Since $|\Psi_{GS}\rangle$ is also its own time-reverse it must be destroyed by the time-reverse of the vector operator, i.e.,

$$\sum_{\beta\neq\alpha}^{N}\frac{z_\alpha^*}{z_\alpha^*-z_\beta^*}[i(\vec{S}_\alpha\times\vec{S}_\beta)+\vec{S}_\beta] = -\sum_{\beta\neq\alpha}^{N}\frac{z_\beta}{z_\alpha-z_\beta}[i(\vec{S}_\alpha\times\vec{S}_\beta)+\vec{S}_\beta]. \quad \text{(D4)}$$

The difference of these is the trivial operator $\vec{S}_\alpha\times\vec{S}$, and their sum is $2\vec{D}_\alpha$.

We prove, now, the factorizability of $\mathcal{H}_{HS}$. In order to do so, we need the following identities:

$$\sum_{\alpha}^{N}[i(\vec{S}\times\vec{S}_\alpha)+\vec{S}]\cdot\vec{D}_\alpha=\sum_{\alpha}^{N}[i\vec{S}\cdot(\vec{S}_\alpha\times\vec{D}_\alpha)+\vec{S}\cdot\vec{D}_\alpha] = \frac{3}{2}\vec{S}\cdot\vec{\Lambda}, \quad \text{(D5)}$$

$$\sum_{\alpha}^{N}[i(\vec{S}\times\vec{S}_\alpha)+\vec{S}]\cdot[i(\vec{S}_\alpha\times\vec{S})+\vec{S}]=\frac{3}{2}[N-1]S^2, \quad \text{(D6)}$$

$$\sum_{\beta\neq\gamma\neq\alpha}^{N}\frac{\vec{S}_\beta\cdot\vec{S}_\gamma}{(z_\alpha^*-z_\gamma^*)(z_\alpha-z_\beta)}=-\frac{1}{2}S^2+\frac{3}{8}N+2\sum_{\alpha\neq\beta}^{N}\frac{\vec{S}_\alpha\cdot\vec{S}_\beta}{|z_\alpha-z_\beta|^2}, \quad \text{(D7)}$$

$$
\begin{aligned}
i\sum_{\alpha\neq\beta\neq\gamma}^{N}&\frac{\vec{S}_\gamma\cdot(\vec{S}_\alpha\times\vec{S}_\beta)}{(z_\alpha^*-z_\gamma^*)(z_\alpha-z_\beta)}\\
&=i\sum_{\alpha\neq\beta\neq\gamma}^{N}\frac{z_\alpha z_\gamma}{(z_\alpha-z_\gamma)(z_\alpha-z_\beta)}\vec{S}_\alpha\cdot(\vec{S}_\gamma\times\vec{S}_\beta)\\
&=\frac{i}{2}\sum_{\alpha\neq\beta\neq\gamma}^{N}\left(\frac{z_\alpha+z_\gamma}{z_\alpha-z_\gamma}\right)(\vec{S}_\alpha\times\vec{S}_\gamma)\cdot\vec{S}_\beta\\
&=\vec{\Lambda}\cdot\vec{S}. \qquad \text{(D8)}
\end{aligned}
$$

By putting together the identity

$$
\begin{aligned}
\sum_{\alpha}^{N}\sum_{\beta\neq\alpha}^{N}\sum_{\gamma\neq\alpha}^{N}&\frac{[i(\vec{S}_\alpha\times\vec{S}_\gamma)+\vec{S}_\gamma]^\dagger\cdot[i(\vec{S}_\alpha\times\vec{S}_\beta)+\vec{S}_\beta]}{(z_\alpha^*-z_\gamma^*)(z_\alpha-z_\beta)}\\
&=\sum_{\alpha}^{N}\vec{D}_\alpha^\dagger\cdot\vec{D}_\alpha+\frac{3}{2}\vec{S}\cdot\vec{\Lambda}+\frac{3}{8}(N-1)S^2 \qquad \text{(D9)}
\end{aligned}
$$

and the identity

$$
\begin{aligned}
\sum_{\alpha}^{N}\sum_{\beta\neq\alpha}^{N}\sum_{\gamma\neq\alpha}^{N}&\frac{[-i(\vec{S}_\alpha\times\vec{S}_\gamma)+\vec{S}_\gamma]\cdot[i(\vec{S}_\alpha\times\vec{S}_\beta)+\vec{S}_\beta]}{(z_\alpha^*-z_\gamma^*)(z_\alpha-z_\beta)}\\
&=\frac{3}{2}\sum_{\alpha}^{N}\sum_{\beta\neq\alpha}^{N}\sum_{\gamma\neq\alpha}^{N}\frac{1}{(z_\alpha^*-z_\gamma^*)(z_\alpha-z_\beta)}\\
&\quad\times[\vec{S}_\gamma\cdot\vec{S}_\beta+i\vec{S}_\gamma\cdot(\vec{S}_\alpha\times\vec{S}_\beta)]\\
&=\frac{3}{2}\left[3\sum_{\alpha\neq\beta}\frac{\vec{S}_\alpha\cdot\vec{S}_\beta}{|z_\alpha-z_\beta|^2}+\frac{N(N^2+5)}{16}-\frac{S^2}{2}+\vec{S}\cdot\vec{\Lambda}\right], \qquad \text{(D10)}
\end{aligned}
$$

the proof is complete.

Since $\langle\Phi|\vec{D}_\alpha^\dagger\cdot\vec{D}_\alpha|\Phi\rangle$ is nonnegative for any wave function $|\Phi\rangle$, this provides an explicit demonstration that $|\Psi_{GS}\rangle$ is the true ground state. The annihilation operators and their equivalence to $\mathcal{H}_{HS}$ when squared and summed were originally discovered by Shastry.[16] They are lattice versions of the Knizhnik-Zamolodchikov operators known from studies of the Calogero-Sutherland model, the 1D Bose gas with inverse-square repulsions.[20,21,8]

APPENDIX E: THE NORM OF THE STATES

In this appendix we provide the proof of the formula for the norm of the one-spinon and of the two-spinon eigenstates. Throughout this section and the following one, we will make use of the scalar product between symmetric polynomials $f(z_1,\ldots,z_M)$ defined as

$$
\langle f|g\rangle=\sum_{z_1,\cdots,z_M}f^*(z_1,\ldots,z_M)g(z_1,\ldots,z_M). \qquad \text{(E1)}
$$

Let us begin with the one-spinon eigenstates. The state for one spinon in coordinate space $\Psi_\alpha(z_1,\ldots,z_M)$ has been defined in Eq. (29), in the odd-N case $[M=(N-1)/2]$, to be

$$
\Psi_\alpha(z_1,\ldots,z_M)=\prod_j^M(z_\alpha-z_j)\prod_{i<j}^M(z_i-z_j)^2\prod_j^M z_j. \qquad \text{(E2)}
$$

Ψ_α is of the form $\Phi_\alpha\times\Psi_{GS}$, where $\Phi_\alpha=\Pi_j^M(z_\alpha-z_j)$.

We will prove the formula for the norm of the state $|\Phi_m\rangle$, Eq. (46), by mathematical induction. In order to do so, let us define the symmetric operator

$$
e_1(z_1,\ldots,z_M)=z_1+\cdots+z_M. \qquad \text{(E3)}
$$

For any wave function of the form $\Phi\times\Psi_{GS}$, where Φ is a symmetric polynomial with degrees less than $N-2M+2$, we have

$$
\begin{aligned}
\mathcal{H}_{HS}\Phi\Psi_{GS}=E_{GS}\Phi\Psi_{GS}&+\frac{J}{2}\left(\frac{2\pi}{N}\right)^2\Psi_{GS}\\
&\times\left\{\frac{1}{2}\left[\sum_j z_j^2\frac{\partial^2}{\partial z_j^2}+4\sum_{j\neq k}\frac{z_j^2}{z_j-z_k}\frac{\partial}{\partial z_j}\right]\right.\\
&\left.-\frac{N-3}{2}\sum_j z_j\frac{\partial}{\partial z_j}\right\}\Phi, \qquad \text{(E4)}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathcal{H}_{HS}&e_1\Phi\Psi_{GS}-e_1\mathcal{H}_{HS}\Phi\Psi_{GS}\\
&=\frac{J}{2}\left(\frac{2\pi}{N}\right)^2\Psi_{GS}\left[\sum_j z_j^2\frac{\partial}{\partial z_j}+\frac{N-3}{2}e_1\right]\Phi. \qquad \text{(E5)}
\end{aligned}
$$

From the definition of Φ_α one then obtains

$$
\sum_j^M z_j^2\frac{\partial}{\partial z_j}\Phi_\alpha=\left[e_1+Mz_\alpha-z_\alpha^2\frac{\partial}{\partial z_\alpha}\right]\Phi_\alpha. \qquad \text{(E6)}
$$

Equation (E6) implies the following identity for Ψ_m:

$$
\begin{aligned}
\left[\mathcal{H}-\frac{E_{GS}}{(J2\pi^2)/N^2}\right][e_1\Psi_m]&=[m(M-m)+M][e_1\Psi_m]\\
&\quad+[M-(m-1)]\Psi_{m-1}. \qquad \text{(E7)}
\end{aligned}
$$

[$\mathcal{H}$ is the scaled Hamiltonian: $\mathcal{H}=\mathcal{H}_{HS}/(J2\pi^2/N^2)$]. Then, the following identity chain is proven:

$$
\begin{aligned}
(m-1)&(M-m+1)\langle\Psi_{m-1}|e_1|\Psi_m\rangle\\
&=\left\langle\Psi_{m-1}\left|\left[\mathcal{H}-\frac{E_{GS}}{(J2\pi^2)/N^2}\right]\right|[e_1\Psi_m]\right\rangle\\
&=[M+m(M-m)]\langle\Psi_{m-1}|e_1|\Psi_m\rangle\\
&\quad+(M-m+1)\langle\Psi_{m-1}|\Psi_{m-1}\rangle, \qquad \text{(E8)}
\end{aligned}
$$

thus, Eq. (E8) implies the identity

$$\langle\Psi_{m-1}|e_1|\Psi_m\rangle=-\frac{M-m+1}{2\left(M-m+\frac{1}{2}\right)}\langle\Psi_{m-1}|\Psi_{m-1}\rangle. \tag{E9}$$

In order to determine a suitable induction relation, let us introduce the operator

$$e_M(z_1,\ldots,z_M)=z_1\cdot\cdots\cdot z_M.$$

Clearly

$$e_M(z_1,\ldots,z_M)e_M\left(\frac{1}{z_1},\ldots,\frac{1}{z_M}\right)=1. \tag{E10}$$

Since all the Ψ_m's are products of the ground-state factor Ψ_{GS} times a symmetric polynomial of degrees less than 2 in each variable, we have

$$\begin{aligned}\langle\Psi_{m-1}|e_1|\Psi_m\rangle&=\langle[e_M^*\Psi_m]|e_1|[e_M^*\Psi_{m-1}]\rangle\\&=\langle\Psi_{M-m}|e_1|\Psi_{M-m+1}\rangle.\end{aligned} \tag{E11}$$

At this point, we use again Eq. (E9) in order to write Eq. (E11) as

$$\begin{aligned}\langle\Psi_{M-m}|e_1|\Psi_{M-m+1}\rangle&=-\frac{m}{2\left(m-\frac{1}{2}\right)}\langle\Psi_{M-m}|\Psi_{M-m}\rangle\\&=-\frac{m}{2\left(m-\frac{1}{2}\right)}\langle\Psi_m|\Psi_m\rangle.\end{aligned} \tag{E12}$$

Equation (E12) closes the induction relation

$$\frac{\langle\Psi_m|\Psi_m\rangle}{\langle\Psi_{m-1}|\Psi_{m-1}\rangle}=\frac{\left(m-\frac{1}{2}\right)(M-m+1)}{m\left(M-m+\frac{1}{2}\right)}. \tag{E13}$$

The formula generated by recursion is

$$\begin{aligned}\langle\Psi_m|\Psi_m\rangle&=\prod_{j=1,\ldots,m}\frac{\left(j-\frac{1}{2}\right)(M-j+1)}{j\left(M-m+j+\frac{1}{2}\right)}C_M\\&=\frac{\Gamma[M+1]\Gamma\left[m+\frac{1}{2}\right]\Gamma\left[M-m+\frac{1}{2}\right]}{\Gamma\left[M+\frac{1}{2}\right]\Gamma[m+1]\Gamma[M-m+1]}C_M.\end{aligned} \tag{E14}$$

The constant C_M is expressed in terms of Wilson's integral as

$$\begin{aligned}C_M&=\left(\frac{N}{2\pi i}\right)^M\oint\frac{dz_1}{z_1}\ldots\oint\frac{dz_M}{z_M}\prod_{i\neq j}^{M}\left(1-\frac{z_i}{z_j}\right)^2\\&=N^M\frac{(2M)!}{2^M}.\end{aligned} \tag{E15}$$

Equations (E14) and (E15) complete the proof.

Now we work out the formula for the two-spinon eigenstates. In this case $M=N/2-1$ and Eq. (E5) becomes

$$\begin{aligned}&\mathcal{H}e_1\Phi\Psi_{GS}-e_1\mathcal{H}\Phi\Psi_{GS}\\&\quad=\Psi_{GS}\left[\sum_{j=1}^{M}z_j^2\frac{\partial}{\partial z_j}+\left(M-\frac{1}{2}\right)e_1\right]\Phi,\end{aligned} \tag{E16}$$

where again we work with the Hamiltonian in scaled units. Equation (E6) now becomes

$$\begin{aligned}\sum_{j=1}^{M}z_j^2\frac{\partial}{\partial z_j}\Phi_{\alpha\beta}&=2[e_1\Phi_{\alpha\beta}]\\&\quad+\left[Mz_\alpha-z_\alpha^2\frac{\partial}{\partial z_\alpha}+Mz_\beta-z_\beta^2\frac{\partial}{\partial z_\beta}\right]\Phi_{\alpha\beta},\end{aligned} \tag{E17}$$

where the state $\Phi_{\alpha\beta}$ has been defined in Eq. (48). Hence, by letting e_1 act onto the two-spinon eigenstate Φ_{mn}, we obtain

$$\begin{aligned}&\left[\mathcal{H}-\frac{E_{GS}}{(J2\pi^2)/N^2}\right][e_1\Phi_{mn}]-\left(\lambda_{mn}+M-\frac{3}{2}\right)[e_1\Phi_{mn}]\\&\quad=\sum_{j=1}^{M}z_j^2\frac{\partial}{\partial z_j}\Phi_{mn}\\&\quad=\sum_k a_k^{mn}\sum_{\alpha,\beta=1}^{N}\frac{(z_\alpha^*)^m}{N}\frac{(z_\beta^*)^n}{N}\\&\qquad\times\left[2[e_1\Psi_{\alpha\beta}]+\left(Mz_\alpha-z_\alpha^2\frac{\partial}{\partial z_\alpha}+Mz_\beta-z_\beta^2\frac{\partial}{\partial z_\beta}\right)\right.\\&\qquad\left.\times\Psi_{\alpha\beta}\right],\end{aligned} \tag{E18}$$

which implies

$$\begin{aligned}&\left[\mathcal{H}-\frac{E_{GS}}{(J2\pi^2)/N^2}-\left(\lambda_{mn}+M+\frac{1}{2}\right)\right]e_1\Phi_{mn}\\&\quad=\sum_k a_k^{mn}\Big\{[M-m-k+1]\\&\qquad\times\sum_j b_j^{m+k-1,n-k}\Phi_{m+k-1+j,n-k-j}\\&\qquad+[M-n+k+1]\\&\qquad\times\sum_j b_j^{m+k,n-k-1}\Phi_{m+k+j,n-k-1-j}\Big\}.\end{aligned} \tag{E19}$$

[See Eqs. (54) and (78) for the definition of the coefficient a_k^{mn},b_k^{mn} and Eq. (58) for the definition of λ_{mn}.]

Let us now take the scalar product of both sides of Eq. (E19) with $\Phi_{m-1,n}$. The result can be recast in the form

$$\frac{\langle\Phi_{m-1,n}|e_1|\Phi_{mn}\rangle}{\langle\Phi_{m-1,n}|\Phi_{m-1,n}\rangle}=-\frac{M-m+1}{2\left(M-m+\frac{1}{2}\right)}. \tag{E20}$$

On the other hand, by taking the scalar product of $\Phi_{m,n-1}$ with both sides of Eq. (E19), we obtain:

$$\begin{aligned}\left(\lambda_{m,n-1}-\lambda_{mn}-M-\frac{1}{2}\right)\langle\Phi_{m,n-1}|e_1|\Phi_{mn}\rangle &=\sum_k a_k^{mn}\Big\{[M-m-k+1]\\ &\times\sum_j b_j^{m+k-1,n-k}\langle\Phi_{m,n-1}|\Phi_{m+k-1+j,n-k-j}\rangle\\ &+[M-n+k+1]\sum_j b_j^{m+k,n-k-1}\\ &\times\langle\Phi_{m,n-1}|\Phi_{m+k+j,n-k-j-1}\rangle\Big\},\end{aligned} \tag{E21}$$

which implies the relation

$$\frac{\langle\Phi_{m,n-1}|e_1|\Phi_{mn}\rangle}{\langle\Phi_{m,n-1}|\Phi_{m,n-1}\rangle}=-\frac{\left(M-n+\frac{3}{2}\right)(m-n+1)^2}{2\left(m-n+\frac{3}{2}\right)\left(m-n+\frac{1}{2}\right)(M-n+1)}. \tag{E22}$$

In order to complete the proof, we need two more identities that can be proven in the same way as we did for Eq. (E13):

$$\begin{aligned}\frac{\langle\Phi_{m-1,n}|e_1|\Phi_{mn}\rangle}{\langle\Phi_{mn}|\Phi_{mn}\rangle}&=\langle\Phi_{M-n,M-m}|e_1|\Phi_{M-n,M-m+1}\rangle\\ &=-\frac{\left(m+\frac{1}{2}\right)(m-n)^2}{2\left(m-n+\frac{1}{2}\right)m\left(m-n-\frac{1}{2}\right)},\end{aligned} \tag{E23}$$

$$\begin{aligned}\frac{\langle\Phi_{m,n-1}|e_1|\Phi_{mn}\rangle}{\langle\Phi_{mn}|\Phi_{mn}\rangle}&=\frac{\langle\Phi_{M-n,M-m}|e_1|\Phi_{M-n+1,M-m}\rangle}{\langle\Phi_{mn}|\Phi_{mn}\rangle}\\ &=-\frac{n}{2\left(n-\frac{1}{2}\right)}.\end{aligned} \tag{E24}$$

Hence, the proof is given by the following induction relations:

$$\frac{\langle\Phi_{mn}|\Phi_{mn}\rangle}{\langle\Phi_{m,n-1}|\Phi_{m,n-1}\rangle}=\frac{\left(n-\frac{1}{2}\right)\left(M-n+\frac{3}{2}\right)(m-n+1)^2}{n(M-n+1)\left(m-n+\frac{3}{2}\right)\left(m-n+\frac{1}{2}\right)}, \tag{E25}$$

$$\frac{\langle\Phi_{mn}|\Phi_{mn}\rangle}{\langle\Phi_{m-1,n}|\Phi_{m-1,n}\rangle}=\frac{(M-m+1)\left(m-n+\frac{1}{2}\right)\left(m-n-\frac{1}{2}\right)m}{\left(M-m+\frac{1}{2}\right)(m-n)^2\left(m+\frac{1}{2}\right)}. \tag{E26}$$

Equations (E25) and (E26) imply

$$\begin{aligned}\langle\Phi_{mn}|\Phi_{mn}\rangle=C'_M&\frac{\Gamma\left[m-n+\frac{1}{2}\right]\Gamma\left[m-n+\frac{3}{2}\right]}{\Gamma^2[m-n+1]}\\ &\times\frac{\Gamma[m+1]\Gamma\left[M-m+\frac{1}{2}\right]}{\Gamma\left[m+\frac{3}{2}\right]\Gamma[M-m+1]}\\ &\times\frac{\Gamma\left[n+\frac{1}{2}\right]\Gamma[M-n+1]}{\Gamma[n+1]\Gamma\left[M-n+\frac{3}{2}\right]},\end{aligned} \tag{E27}$$

and the constant C'_M is now given by

$$C'_M=N^M\frac{(2M)!}{2^M}\frac{M+\frac{1}{2}}{\pi}. \tag{E28}$$

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