Critical current density from magnetization hysteresis data using the critical-state model

A. A. Tulapurkar*

*Department of Condensed Matter Physics and Materials Science, Tata Institute of Fundamental Research,
Colaba, Mumbai 400005, India*

We present an exact method to extract the critical current density (J_c) from the irreversible magnetization data (M-H) using the Bean's critical-state model. The weakly pinned superconducting samples are known to display an anomalous peak in J_c with the increase in H, which is referred to as the peak effect phenomenon and it is in contemporary focus. We show that the shape of the hysteresis bubble identifying peak effect in $J_c(H)$ in the M-H data is asymmetric.

I. INTRODUCTION

The hysteresis in magnetization seen in hard type-II superconductors was first related to the critical current density (J_c) by Bean.[1,2] His model designated as the critical-state model (CSM) predicts the response of a superconductor with a critical current density J_c to an isothermal variation of the external magnetic field. According to this model, any change in the magnetic induction, howsoever small, felt by any region of the sample, induces current with the critical current density J_c to flow locally. The direction of the current depends, through Lenz's law, on the direction of the emf that accompanied the last local change in the magnetic induction. In his original work, C. P. Bean had assumed J_c to be independent of the magnetic field.[1] Later this model was generalized for the cases of arbitrary dependence of J_c on the magnetic field (see, for example, Ref. 3). Numerous studies have employed the CSM to extract J_c from the magnetization hysteresis data using the relation

$$J_c(H) = [M\!\downarrow\!(H) - M\!\uparrow\!(H)]/kd, \tag{1}$$

where $M\!\downarrow$ and $M\!\uparrow$ are the magnetization values in the field decreasing and increasing cases, respectively. $2d$ is the sample dimension perpendicular to the field direction and k is a numerical constant depending on the geometrical shape of the sample. Values of k are quoted for samples in the shape of an infinite slab, an infinite cylinder and, also, for a sphere.[4,5] The use of above equation was justified by Fietz and Webb,[6] who expanded the J_c inside the sample in a Taylor series around its value at the surface.

However, the above method to get J_c is not self-consistent and errors are inevitable. The higher-order corrections to Eq. (1) contain the second derivative of $J_c^2(H)$ with a pre-factor that depends on the sample geometry.[7] Thus at low fields, where the J_c is rapidly varying, the use of this equation leads to a spurious inference of field-independent J_c.[8] Even at high magnetic fields, if $J_c(H)$ shows a sharp anomalous increase (as it may happen in the peak effect phenomenon[9]), the above equation will not be a good approximation. The above equation can also give misleading results when applied to the anomalous variation in $J_c(H)$ at low fields, i.e., the fishtail effect.[10] An alternate procedure to Eq. (1) to get J_c is to choose a particular functional form $J_c(B)$, from

which one can compute the magnetization hysteresis loop and choose the model parameters to fit the experimental data in a consistent manner.[11,12]

We present in this paper an exact method to obtain the J_c from the M-H hysteresis loop. The sample is assumed to be an infinite slab with magnetic field applied parallel to the infinite dimension (zero demagnetization factor geometry).

II. CALCULATION OF THE MAGNETIZATION CURVES

The isothermal magnetization curves of a superconductor are calculated following the procedure given by Bean.[1] Macroscopic shielding currents flow in the sample to shield the flux in the sample from the changes in the external field. The direction and the magnitude of the shielding current, for any change in the external field, is assumed to be such so as to minimize the change of the total flux contained in the sample. These assumptions result in a field profile inside the sample which is governed by the equation

$$\frac{dB(x)}{dx} = \pm \mu_o J_c[B(x)]. \tag{2}$$

Here, $B(x)$ is the local field inside the sample, which is assumed to be an infinite slab with surfaces at $x=0$ and $x=2d$. The negative sign in the above equation is for the field increasing case and the positive sign for the field decreasing case. This equation is to be solved with the boundary condition $B(x=0)=B(x=2d)=\mu_o H_a$, where H_a is the externally applied magnetic field. Once the field profile inside the sample is known, the magnetization is obtained by

$$\mu_o M = \frac{1}{d}\int_0^d B(x)dx - \mu_o H_a. \tag{3}$$

Thus for a given functional form of $J_c(B)$, the magnetization hysteresis loop can be obtained from the above formalism. It may be noted that the influence of the lower critical field and surface effects are neglected[2] in the present treatment.

III. OBTAINING J_c FROM MAGNETIZATION DATA

Figure 1 shows a typical field profile inside the slab when external magnetic fields of H_a and $H_a + \delta H_a$ are applied in the increasing field cycle. (The slab has a width of $2d$ with
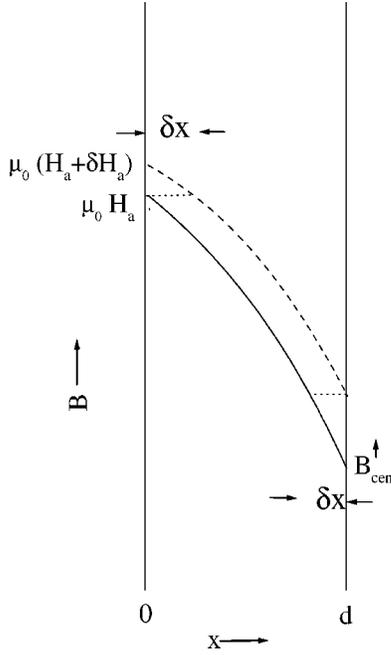
FIG. 1. The solid line shows the magnetic-field profile inside a slab of thickness $2d$ when an external magnetic field of H_a is applied in the forward cycle. The field profile is symmetric about the $x=d$ plane and only one-half of the field profile is shown. $B_{cen}^{\uparrow}$ is the field at the center of the slab. The dashed line shows the field profile when the applied field is $H_a + \delta H_a$.

its surfaces at $x=0$ and $x=2d$. The field profile is symmetric about the $x=d$ plane and Fig. 1 shows the field profile only in the first half.) At a given value of B, both the profiles have the same slope $[-\mu_o J_c(B)]$. Hence it can be seen from Fig. 1 that after a distance δx (marked in the figure), the field profiles are related by a rigid shift, i.e., they satisfy the relation

$$B(x, H_a + \delta H_a) = B(x - \delta x, H_a). \qquad (4)$$

The distance δx is given by

$$\delta x = \frac{\delta H_a}{J_c(B_a)}, \qquad (5)$$

where $B_a = \mu_o H_a$ is the magnetic field at the sample boundary. The difference in the value of B for the two profiles is given by

$$\delta B(x) = B(x, H_a + \delta H_a) - B(x, H_a)$$

$$= B(x - \delta x, H_a) - B(x, H_a)$$

$$= \frac{dB}{dx}(-\delta x). \qquad (6)$$

The difference in the magnetization values for the two cases can be obtained using Eqs. (3), (5), and (6) as

$$\mu_o \delta M(H_a) = \mu_o[M(H_a + \delta H_a) - M(H_a)]$$

$$= \frac{1}{d}\int_0^d \delta B \, dx - \mu_o \delta H_a$$

$$= \frac{-\delta H_a}{d \cdot J_c(B_a)}\int_0^d \frac{dB}{dx} \cdot dx - \mu_o \delta H_a$$

$$= \frac{-\delta H_a}{d \cdot J_c(B_a)}(B_{cen}^{\uparrow} - \mu_o H_a) - \mu_o \delta H_a,$$

where $B_{cen}^{\uparrow}$ is the magnetic field at the center of the slab, i.e., $B_{cen}^{\uparrow} = B(d, H_a)$. ($\uparrow$ denotes that the magnetic field at the center of the slab is obtained in the field increasing cycle.) The derivative of the magnetization in the field increasing case can be obtained from the above equation as

$$\frac{dM\uparrow}{dH_a} = \frac{(\mu_o H_a - B_{cen}^{\uparrow})}{d \cdot \mu_o \cdot J_c(B_a)} - 1. \qquad (7)$$

A similar equation can be written for the field decreasing case:

$$\frac{dM\downarrow}{dH_a} = \frac{(B_{cen}^{\downarrow} - \mu_o H_a)}{d. \mu_o . J_c(B_a)} - 1. \qquad (8)$$

Thus if the left-hand side of the above equations are obtained experimentally, $J_c(B_a)$ can be determined if one knows the magnetic fields at the center of the slab, for an applied magnetic field of B_a, in the field increasing and decreasing cycles, i.e., $B_{cen}^{\uparrow}$ and $B_{cen}^{\downarrow}$, respectively.

A. Obtaining $B_{cen}^{\uparrow}$ and $B_{cen}^{\downarrow}$ from the *M-H* hysteresis loop

The solid line in Fig. 2 shows a typical field profile inside the sample with an externally applied field of H_a (in the increasing cycle) and magnetic field at the center $B_{cen}^{\uparrow}$. The dashed line in the same figure shows the field profile in the decreasing field cycle, when the external field is $B_{cen}^{\uparrow}/\mu_o$. At a given value of B, these two profiles have the opposite slope $[\pm\mu_o J_c(B)]$. Hence it can be seen from Fig. 2 that they are related by the following equation:

$$B\uparrow(x, H_a) = B\downarrow(d - x, B_{cen}^{\uparrow}/\mu_o), \qquad (9)$$

where $\uparrow$ and $\downarrow$ indicate increasing and decreasing field cycles, respectively. These two curves have the same area $[\int_0^d B(x)dx]$ under them, i.e., both of them have the same value of magnetic induction ($\bar{B}$). Thus the magnetic fields on the forward and the reverse cycles, at which the sample has the same value of magnetic induction, correspond to the pair of $\mu_o H_a$ and magnetic field at the center. Figure 3(a) shows a part of a typical *M-H* hysteresis loop. The bottom panel shows the corresponding $\bar{B}$-H curve, which can be obtained from Eq. (3). As indicated in the bottom panel, $J_c(B_a)$ can be obtained in two ways: using Eq. (7) with $B_{cen}^{\uparrow}$ as the magnetic field at the center of the sample or using Eq. (8) with $B_{cen}^{\downarrow}$ as the magnetic field at the center of the sample. If the *M-H* loop can be produced by the CSM, the two values of J_c ought to match.
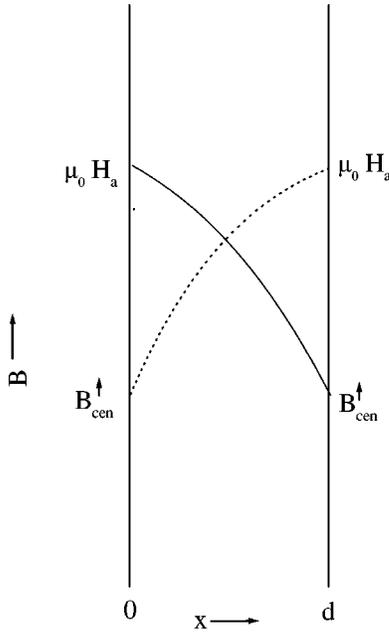
FIG. 2. The solid line shows the magnetic-field profile in the increasing field cycle. H_a is the external magnetic field and $B^{\uparrow}_{cen}$ is the magnetic field at the center of the slab. The dashed line shows the field profile in the decreasing field cycle.

The magnetization hysteresis data are usually obtained as a five quadrant loop, i.e., the zero-field-cooled sample is first subjected to an increasing magnetic field up to the upper critical field H_{c2} (the first leg). Then the field is decreased from H_{c2} to $-H_{c2}$ through zero field (i.e., the second and third legs). After that the field is again increased from $-H_{c2}$ to H_{c2} through zero field (i.e., the fourth and fifth legs). In the initial part of the first leg (virgin curve), the magnetic field does not penetrate the sample fully and it differs from the fifth leg of the magnetization. After the magnetic field exceeds the field of full penetration (H^*), the curves along the first leg and the fifth leg match. Equation (7) is valid for the virgin curve with $B^{\uparrow}_{cen}=0$ until the field of full penetration (H^*) and it can be used to get J_c. Alternatively, the full five quadrant loop can be used to get J_c in this region.

The virgin curve cannot be used to get the $J_c(0)$, as Eqs. (7) and (8) just predict that $dM/dH=-1$, when $H_a=0$. However, there is a particularly simple way to get $J_c(0)$ from the five quadrant loop. It is easy to see that the magnetic field at the center of the slab for $H_a=0$ in the field decreasing cycle is the field of full penetration (H^*), which corresponds to the point at which the virgin curve merges with the fifth leg of the magnetization. Thus Eq. (8) with $H_a=0$ and $B^{\downarrow}_{cen}=H^*$ can be used to get $J_c(0)$. [This is the same as using Eq. (7) with $H_a=0$ and $B^{\uparrow}_{cen}=-H^*$ to get $J_c(0)$ from the forward cycle.]

One more value of the magnetic field below H^* at which it is easy to get the magnetic field at the center of the sample is the point at which $\bar{B}=0$, i.e., $M=-H_a$. It is easy to see that at this point $B^{\uparrow}_{cen}=-\mu_oH_a$. (We are considering the fifth leg of the five quadrant loop here.) Thus Eq. (7) can be
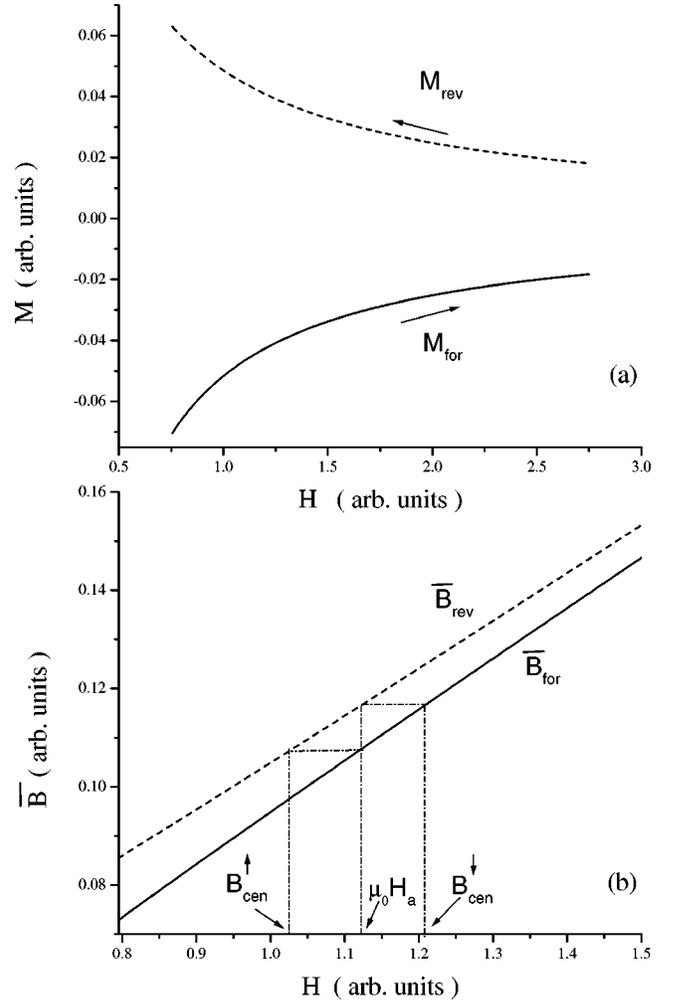
FIG. 3. The top panel shows a part of typical M-H hysteresis loop. The corresponding $\bar{B}$-H curve is shown in the bottom panel. The construction by dotted lines shows the two methods to obtain the pairs of μ_oH_a and B_{cen}. The pair μ_oH_a and $B^{\uparrow}_{cen}$ is to be used with Eq. (7) and the pair μ_oH_a and $B^{\uparrow}_{cen}$ is to be used with Eq. (8).

used to get J_c at this point or alternatively, as mentioned above, one can use the virgin curve with $B^{\uparrow}_{cen}=0$ to get the J_c at this point.

It should be noted that, since this method uses the derivative of the magnetization, the experimental data points at small intervals are required to get the reliable value of the derivative.

B. Shape of the magnetization hysteresis bubble pertaining to the peak effect

We have seen in the previous sections how Eqs. (7) and (8) can be used to get the critical current density from the magnetization hysteresis data. These equations also predict some interesting features about the locations of the maxima and minima in the magnetization hysteresis loop (i.e., points where $dM/dH=0$) and the shape of the magnetization around these points. This is particularly relevant to the case of the peak effect phenomenon. The critical current density is usually a monotonically decreasing function of the mag-
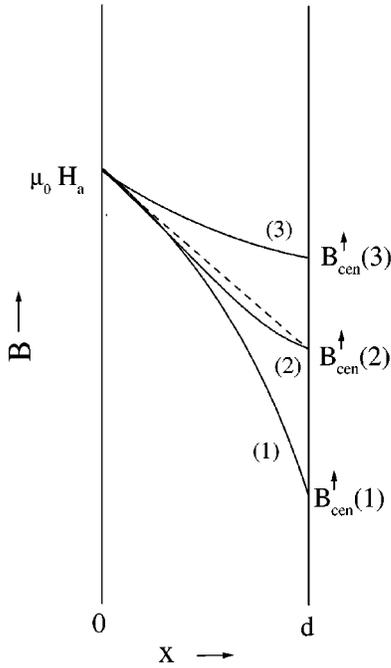
FIG. 4. Solid lines show the field profiles in the increasing field cycle for different functional forms of $J_c(B)$. $J_c(B_a)$ is the same for all three curves, i.e., they have the same slope at $x=0$. The dashed curve is a straight line with slope $-\mu_o J_c(B_a)$.

FIG. 5. The bottom panel shows an example of J_c with a peak. The top panel shows the qualitative nature of the M-H hysteresis loop obtained from Eqs. (7) and (8). Note that the positions of the peaks in the M-H loop is different from the peak position in J_c and the M-H loop is asymmetric.

netic field. However, in weakly pinned superconductors, J_c shows an anomalous peak before reaching the superconducting-normal phase boundary, which is termed as the peak effect.[9,13,14] We will show below that whenever the J_c shows a peak, the corresponding peaks in the magnetization in the field increasing and decreasing cycles occur at different values of the external fields and, moreover, the shape of the magnetization loop around the peaks is asymmetric.

Figure 4 shows three field profiles in the increasing field cycle with different functional forms of $J_c(B)$. $J_c(B_a)$ is the same for all three cases, i.e., they have the same slope, $-\mu_o J_c(B_a)$, at $x=0$. The dotted curve in the figure is a straight line drawn with slope $-\mu_o J_c(B_a)$. If the field profile curve meets the dotted line at the center of the slab, the first term in Eq. (7) is 1, which means that dM/dH is 0. This situation is shown in curve 2 which is an example of the field profile that is obtained when $J_c(B)$ is an increasing function of B over some part and a decreasing function over some part of B.

Curve 1 in Fig. 4 is a typical example of the field profile obtained when $J_c(B)$ is a monotonically decreasing function of B. Thus, as shown in the figure, the magnetic field at the center $[B_{cen}^{\uparrow}(1)]$ is less than the value of the magnetic field at which the dotted line meets the center $[B_{cen}^{\uparrow}(2)]$. As a result of this, the first term in Eq. (7) is greater than 1 and the dM/dH is positive.

Curve 3 in Fig. 4 is a typical example of the field profile obtained when the $J_c(B)$ is monotonically increasing function of B. Thus as shown in the figure, the magnetic field at

the center $[B_{cen}^{\uparrow}(3)]$ is more than $B_{cen}^{\uparrow}(2)$. As a result of this, the first term in Eq. (7) is less than 1 and the dM/dH is negative.

Figure 5 shows the qualitative nature of the M-H loop in the case where J_c shows a symmetric peak. When the applied field in the increasing field cycle equals the field at which J_c shows a maximum, the field profile inside the sample corresponds to the curve 3 of Fig. 4 (J_c is an increasing function of B over the range of B covered by the field profile), which implies that dM/dH is negative. Thus the peak in the magnetization in the forward cycle does not coincide with the peak in J_c, but occurs at a higher value of the applied field such that the field profile inside the sample corresponds to curve 2 of Fig. 4. Since the magnetization drops to zero when J_c becomes zero, the peak in magnetization has an asymmetric shape as shown in Fig. 5. A similar analysis can be applied to the magnetization in the reverse cycle, and the results are shown in the same figure. Thus the CSM predicts that the peaks in the magnetization in the forward and reverse cycles and the peak in J_c do not coincide with each other. Moreover, the shape of the peak in magnetization is asymmetric. The asymmetry of course depends on the sample thickness, such that the larger the thickness, the more is the asymmetry. However, it should be noted that, in the experimental hysteresis data on the peak effect, the slope of the magnetization in the reverse cycle before the peak is less (i.e., opposite to that shown in Fig. 5) due to the supercooling of the disordered vortex phase.[15] The mismatch in

the peak positions in the field increasing and decreasing cycles is also shown in Ref. 10. There the authors have explicitly calculated the magnetization loop in the case of the fishtail effect by representing it by term in the current density having Lorenzian form. The present work shows the qualitative features of the peaks in magnetization without any explicit calculation.

IV. CONCLUSIONS

We have presented in this paper a method to extract J_c from the *M-H* hysteresis data using the critical-state model.

It is also shown that in the case of the peak effect phenomenon, the shape of the *M-H* hysteresis loop is asymmetric and the peak in J_c and the peaks in magnetization in the forward and reverse cycles occur at different values of the applied magnetic fields.

*Email address: ashwin@tifr.res.in

[1] C.P. Bean, Phys. Rev. Lett. **8**, 250 (1962).

[2] C.P. Bean, Rev. Mod. Phys. **36**, 31 (1964).

[3] Y.B. Kim, C.F. Hempstead, and A.R. Strand Phys. Rev. **129**, 528 (1963).

[4] J.R. Clem and V.G. Kogan, Jpn. J. Appl. Phys., Part 1 **26**, 1161 (1987).

[5] S. Senoussi, M. Ousséna, G. Collin, and I.A. Campbell, Phys. Rev. B **37**, 9792 (1988).

[6] W.A. Fietz and W.W. Webb, Phys. Rev. **178**, 657 (1969).

[7] T.H. Johansen and H. Bratsberg, J. Appl. Phys. **77**, 3945 (1995).

[8] P. Chaddah, K.V. Bhagwat, and G. Ravikumar, Physica C **159**, 570 (1989).

[9] Y. Paltiel, E. Zeldov, Y. Myasoedov, M.L. Rappaport, G. Jung, S. Bhattacharya, M.J. Higgins, Z.L. Xiao, E.Y. Andrei, P.L. Gammel, and D.J. Bishop, Phys. Rev. Lett. **85**, 3712 (2000).

[10] T.H. Johansen, M.R. Koblischka, H. Bratsberg, and P.O. Hotland, Phys. Rev. B **56**, 11 273 (1997).

[11] T.H. Johansen and H. Bratsberg, J. Appl. Phys. **76**, 8001 (1994).

[12] H. Ishii, T. Hara, S. Hirano, A.M. Figueredo, and M.J. Cima, Physica C **225**, 91 (1994).

[13] M.J. Higgins and S. Bhattacharya, Physica C **257**, 232 (1996).

[14] P. W. Anderson, *Basic Notions in Condensed Matter Physics*, Frontiers in Physics Vol. 55 (Addison-Wesley, Reading, MA, 1984).

[15] G. Ravikumar, P.K. Mishra, V.C. Sahani, S.S. Banerjee, A.K. Grover, S. Ramakrishnan, P.L. Gammel, D.J. Bishop, E. Bucher, M.J. Higgins, and S. Bhattacharya, Phys. Rev. B **61**, 12 490 (2000).