Quantum tunneling of the order parameter in superconducting nanowires

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Quantum tunneling of the superconducting order parameter gives rise to the phase slippage process which controls the resistance of ultrathin superconducting wires at sufficiently low temperatures. If the quantum phase slip rate is high, superconductivity is completely destroyed by quantum fluctuations and the wire resistance never decreases below its normal state value. We present a detailed microscopic theory of quantum phase slips in homogeneous superconducting nanowires. Focusing our attention on relatively short wires we evaluate the quantum tunneling rate for phase slips, both the quasiclassical exponent and the pre-exponential factor. In very thin and dirty metallic wires the effect is shown to be clearly observable even at $T\to 0$. Our results are fully consistent with recent experimental findings [A. Bezryadin, C.N. Lau, and M. Tinkham, Nature (London) **404**, 971 (2000)] which provide direct evidence for the effect of quantum phase slips.

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I. INTRODUCTION

It is well established that superconducting fluctuations play a very important role in reduced dimension. Above the critical temperature T_C such fluctuations yield an enhanced conductivity.[1] Below T_C fluctuations are known to destroy the long-range order in low-dimensional superconductors.[2] Does the latter result mean that the resistance of such superconductors always remains finite (or even infinite), or can it drop to zero under certain conditions?

It was first pointed out by Little[3] that quasi-one-dimensional wires made of a superconducting material can acquire a finite resistance below T_C of a bulk material due to the mechanism of thermally activated phase slips (TAPS). This TAPS process corresponds to local destruction of superconductivity by thermal fluctuations. Superconducting phase $\varphi(t)$ can flip by 2π across those points of the wire where the order parameter is (temporarily) destroyed. According to the Josephson relation $V=\dot{\varphi}/2e$ (here and below we set $\hbar=1$) such phase slips cause a nonzero voltage drop and, hence, dissipative currents inside the wire. A theory of this TAPS phenomenon was developed in Refs. 4,5. This theory yields a natural result, that the TAPS probability and, hence, resistance of a superconducting wire R below T_C are determined by the activation exponent

$$R(T)\propto \exp(-U/T), \quad U\sim\frac{N_0\Delta^2(T)}{2}s\xi(T), \tag{1}$$

where $U(T)$ is the effective potential barrier for TAPS determined simply as the superconducting condensation energy [N_0 is the metallic density of states at the Fermi energy and $\Delta(T)$ is the BCS order parameter] for a part of the wire of a volume $s\xi$ where superconductivity is destroyed by thermal fluctuations [s is the wire cross section and $\xi(T)$ is the superconducting coherence length]. At temperatures very close to T_C Eq. (1) yields appreciable resistivity which was indeed detected experimentally.[6] Close to T_C the experimental results[6] fully confirm the activation behavior of $R(T)$ predicted in Eq. (1). However, as the temperature is lowered further below T_C the number of TAPS decreases exponentially and no measurable wire resistance is predicted by the theory[4,5] except in the immediate vicinity of the critical temperature.

Experiments[6] were done on small diameter whiskers and thin film samples of typical diameters ∼5000 Å. Recent progress in nanolithographic technique allowed to fabricate samples with much smaller diameters down to ∼10 nm. In such systems one can consider a possibility for phase slips to be created not only due to thermal but also due to *quantum* fluctuations of a superconducting order parameter. Mooij and co-workers[7] discussed this possibility and attempted to observe quantum phase slips (QPS) experimentally.

Later Giordano[8] performed experiments which clearly demonstrated a notable resistivity of ultrathin superconducting wires far below T_C. Their observations could not be adequately interpreted within the TAPS theory and were attributed to QPS. Other groups also reported noticeable deviations from the TAPS prediction in thin (quasi-)one-dimensional (1D) wires.[9,10]

First theoretical studies of the QPS effects[11–13] were performed within a simple approach based on the time-dependent Ginzburg-Landau (TDGL) equations. Later in Refs. 14,15 a microscopic theory of QPS processes was developed with the aid of the imaginary time effective action technique[16] which properly accounts for nonequilibrium, dissipative and electromagnetic effects during a QPS event. One of the main conclusions reached in Refs. 14,15 is that the QPS probability is considerably larger than it was predicted previously.[12] For ultrathin superconducting wires with sufficiently many impurities and with diameters in the 10 nm range this probability can already be large enough to yield experimentally observable phenomena. Also, further interesting effects including quantum phase transitions caused by interactions between quantum phase slips were discussed.[14,15]

In spite of all these developments an unambiguous interpretation of the results[8] in terms of QPS could still be questioned because of possible granularity of the samples used in these experiments. If that was indeed the case, QPS could

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easily be created inside weak links connecting neighboring grains. Also in this case superconducting fluctuations play a very important role,[17–19] however, in contrast to the QPS scenario,[14,15] the superconducting order parameter *needs not to be destroyed* during the QPS event.

Recently, Bezryadin, Lau, and Tinkham[20] developed a new technology which allowed them to fabricate essentially *uniform* superconducting wires with thicknesses down to 3–5 nm. According to our theory[14,15] the QPS effects should be sufficiently large in such systems to be observed in experiments. And indeed, the authors[20] observed that several wires showed no sign of superconductivity even at temperatures well below the bulk critical temperature. Moreover, at lower temperatures their resistance was found to *increase* with decreasing temperature, i.e., these samples could even turn insulating at $T\to 0$. The authors[20] also argued that their experimental data can be interpreted in terms of a quantum dissipative phase transition[21,22] which was predicted[15] also for ultrathin superconducting wires in a certain parameter range.

The results[20] are qualitatively consistent with previous experimental findings.[8] Both experimental works support our general understanding of the role of QPS processes in mesoscopic superconducting wires and call for more detailed theoretical studies of the QPS effects. In Refs. 14,15 an importance of collective modes[23] and QPS interaction effects was mainly emphasized. These are particularly important for long wires. On the other hand, for relatively short wires interaction between different phase slips—at least its spatially dependent part—should not play any significant role. Let us now recall that the wires studied in the experiments[20] are not only considerably thinner but also much *shorter* than those investigated by Giordano.[8] To give some numbers, the length of the wires[8] was typically 40–50 μm whereas the wires[20] were only 0.1–0.2 μm long. At the same time, the superconducting coherence length in the experiments[20] was even shorter, $\xi\sim 7-8$ nm, i.e., such samples can still be considered as quasi-1D superconductors.

Motivated by the experimental findings,[20] in this paper we will present a detailed microscopic investigation of single quantum phase slips. We will focus our attention on an accurate evaluation of the QPS tunneling rate rather than on the interaction effects between different phase slips.[14,15] We will go beyond the exponential accuracy and also evaluate a pre-exponential function in the expression for the QPS rate. We will then use our results for a direct quantitative comparison with the experimental results.[20]

The structure of the paper is as follows. In Sec. II we will formulate a simple derivation of the effective action for our problem with an emphasis put on the Ward identities. In Sec. III we will make use of our general results and derive the action for a special case of ultrathin superconducting wires. We also evaluate the QPS rate within the exponential accuracy. Section IV is devoted to an estimate of the pre-exponent for this rate. Comparison with experiments and brief conclusions are presented in Sec. V. Some further technical details are diverted to the Appendix.

II. THE MODEL AND EFFECTIVE ACTION

The starting point for our analysis is a model Hamiltonian that includes a short range attractive BCS and a long range repulsive Coulomb interaction. The idea is to integrate out the electronic degrees of freedom on the level of the partition function, so that we are left with an effective theory in terms of collective fields.[22,24,25] The partition function Z is conveniently expressed as a path integral over the anticommuting electronic fields $\bar\psi$, ψ and the commuting gauge fields V and $\mathbf{A}$, with Euclidean action

$$S=\int dx\left\{\bar\psi_\sigma\left[\partial_\tau-ieV+\xi\left(\boldsymbol{\nabla}-\frac{ie}{c}\mathbf{A}\right)\right]\psi_\sigma -\lambda\bar\psi_\uparrow\bar\psi_\downarrow\psi_\downarrow\psi_\uparrow+ienV+[\mathbf{E}^2+\mathbf{B}^2]/8\pi\right\}. \tag{2}$$

Here $\xi(\boldsymbol{\nabla})\equiv-\boldsymbol{\nabla}^2/2m-\mu+U(x)$ describes a single conduction band with quadratic dispersion and also includes an arbitrary impurity potential, λ is the BCS coupling constant, $\sigma=\uparrow,\downarrow$ is the spin index, and en denotes the background charge density of the ions. In our notation dx denotes $d^3\mathbf{x}d\tau$ and we use units in which $\hbar$ and k_B are set equal to unity. The field strengths are functions of the gauge fields through $\mathbf{E}=-\boldsymbol{\nabla}V+(1/c)\partial_\tau\mathbf{A}$ and $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ in the usual way for the imaginary time formulation.

We use a Hubbard-Stratonovich transformation to decouple the BCS interaction term and to introduce the superconducting order parameter field $\tilde\Delta=\Delta e^{i\varphi}$

$$\exp\left(\lambda\int dx\bar\psi_\uparrow\bar\psi_\downarrow\psi_\downarrow\psi_\uparrow\right) =\left[\int\mathcal{D}^2\tilde\Delta e^{-1/\lambda\int dx\Delta^2}\right]^{-1} \times\int\mathcal{D}^2\tilde\Delta e^{-\int dx[(1/\lambda)\Delta^2+\tilde\Delta\bar\psi_\uparrow\bar\psi_\downarrow+\tilde\Delta^*\psi_\downarrow\psi_\uparrow]}, \tag{3}$$

where the first factor is for normalization and will not be important in the following. As a result, the partition function now reads

$$Z=\int\mathcal{D}^2\tilde\Delta\int\mathcal{D}^3\mathbf{A}\int\mathcal{D}V\mathcal{D}^2\Psi e^{\left(-S_0-\int dx\bar\Psi\mathcal{G}^{-1}\Psi\right)},$$

$$S_0[V,\mathbf{A},\Delta]=\int dx\left(\frac{\mathbf{E}^2+\mathbf{B}^2}{8\pi}+ienV+\frac{\Delta^2}{\lambda}\right), \tag{4}$$

where the Nambu spinor notation for the electronic fields and the matrix Green function in Nambu space

$$\Psi=\begin{pmatrix}\psi_\uparrow\\ \bar\psi_\downarrow\end{pmatrix},\quad \bar\Psi=(\bar\psi_\uparrow\quad\psi_\downarrow),$$

$$\tilde{\mathcal{G}}^{-1}=\begin{pmatrix}\partial_\tau-ieV+\xi\left(\boldsymbol{\nabla}-\frac{ie}{c}\mathbf{A}\right) & \tilde\Delta\\ \tilde\Delta^* & \partial_\tau+ieV-\xi\left(\boldsymbol{\nabla}+\frac{ie}{c}\mathbf{A}\right)\end{pmatrix}. \tag{5}$$

has been introduced. After the Gaussian integral over the electronic degrees of freedom, we are left with the final effective action

$$S_{\rm eff} = -\,{\rm Tr}\ln \tilde{\mathcal{G}}^{-1} + S_0[V,\mathbf{A},\Delta]. \tag{6}$$

Here the trace Tr denotes both a matrix trace in Nambu space and a trace over internal coordinates or momenta and frequencies. In the following ''tr'' is used to denote a trace over internal coordinates only.

The gauge invariance of the theory enables us to rewrite the action (6) in a different form, which is more convenient for us,

$$S_{\rm eff} = -\,{\rm Tr}\ln \mathcal{G}^{-1} + S_0[V,\mathbf{A},\Delta], \tag{7}$$

where

$$\mathcal{G}^{-1} = \begin{pmatrix} \partial_\tau + \xi(\nabla) - ie\Phi + \dfrac{m\mathbf{v}_s^2}{2} - \dfrac{i}{2}\{\nabla,\mathbf{v}_s\} & \Delta \\ \Delta & \partial_\tau - \xi(\nabla) + ie\Phi - \dfrac{m\mathbf{v}_s^2}{2} - \dfrac{i}{2}\{\nabla,\mathbf{v}_s\} \end{pmatrix}, \tag{8}$$

and we have introduced the gauge invariant linear combinations of the electromagnetic potentials and the phase of the order parameter

$$\Phi = V - \frac{\dot{\varphi}}{2e}, \quad \mathbf{v}_s = \frac{1}{2m}\left(\nabla\varphi - \frac{2e}{c}\mathbf{A}\right). \tag{9}$$

The curly brackets $\{A,B\}$ denote an anticommutator.

A. Perturbation theory

The action (7) cannot be evaluated exactly. Here we will perform a perturbative expansion in Φ and $\mathbf{v}_s$. We will keep the terms up to the second order in these values. This perturbation theory is sufficient for nearly all practical purposes, because nonlinear electromagnetic effects (described by higher order terms) are known to be usually very small in the systems in question. Our general derivation holds for an arbitrary concentration and distribution of impurities as well as for arbitrary fluctuations of the order parameter field in space and time.

We split the inverse Green function (8) into two parts

$$\mathcal{G}_0^{-1} = \begin{pmatrix} \partial_\tau + \xi(\nabla) & \Delta \\ \Delta & \partial_\tau - \xi(\nabla) \end{pmatrix} \tag{10}$$

and

$$\mathcal{G}_1^{-1} = \begin{pmatrix} -ie\Phi + \dfrac{m\mathbf{v}_s^2}{2} - \dfrac{i}{2}\{\nabla,\mathbf{v}_s\} & 0 \\ 0 & ie\Phi - \dfrac{m\mathbf{v}_s^2}{2} - \dfrac{i}{2}\{\nabla,\mathbf{v}_s\} \end{pmatrix}. \tag{11}$$

The logarithm in Eq. (7) can now be expanded in powers of $\mathcal{G}_1^{-1}$ and we get

$${\rm Tr}\ln\mathcal{G}^{-1} = {\rm Tr}\ln\mathcal{G}_0^{-1} + {\rm Tr}(\mathcal{G}_0\mathcal{G}_1^{-1}) - \frac{1}{2}{\rm Tr}(\mathcal{G}_0\mathcal{G}_1^{-1})^2. \tag{12}$$

The Green function $\mathcal{G}_0$ has the form

$$\mathcal{G}_0 = \begin{pmatrix} G & F \\ F & \overline{G} \end{pmatrix}. \tag{13}$$

In Eq. (13) we used the fact that the non-diagonal component Δ in the matrix $\mathcal{G}_0^{-1}$ is real. As a result we have $\overline{F} = F$, $F(x_1,x_2) = F(x_2,x_1)$, and $\overline{G}(x_1,x_2) = -G(x_2,x_1)$.

B. Ward identities

The Green function $\mathcal{G}_0$ satisfies an important identity, which is easy to check:

$$\mathcal{G}_0^{-1}\chi - \chi\mathcal{G}_0^{-1} = \frac{\partial\chi}{\partial\tau} - \left\{\nabla, \frac{\nabla\chi}{2m}\right\}\sigma_3, \tag{14}$$

where χ is an arbitrary function of time and space, and σ_3 is one of the Pauli matrices. Multiplying this matrix identity by $\mathcal{G}$ from the left and from the right side and taking the diagonal components of the resulting matrix equation we get two identities

$$\chi G - G\chi = G\left(\dot{\chi} - \left\{\nabla, \frac{\nabla\chi}{2m}\right\}\right)G + F\left(\dot{\chi} + \left\{\nabla, \frac{\nabla\chi}{2m}\right\}\right)F,$$

$$\chi\overline{G} - \overline{G}\chi = F\left(\dot{\chi} - \left\{\nabla, \frac{\nabla\chi}{2m}\right\}\right)F + \overline{G}\left(\dot{\chi} + \left\{\nabla, \frac{\nabla\chi}{2m}\right\}\right)\overline{G}. \tag{15}$$

Below we will use these identities in order to decouple the effective action of the BCS superconductor and to reduce it to a transparent and convenient form. It is important to emphasize again that these identities are valid for any impurity distribution and for any time and spatial dependence of the order parameter field. It is also worth mentioning that the Ward identity (14) is *not* the result of the gauge invariance of the theory. It remains valid even for uncharged particles.

The Ward identity related to the gauge invariance of our theory has a different form

$$\mathcal{G}_0^{-1}\sigma_3\chi - \chi\sigma_3\mathcal{G}_0^{-1} = \frac{\partial\chi}{\partial\tau}\sigma_3 - \left\{\nabla, \frac{\nabla\chi}{2m}\right\} - 2i\sigma_2\Delta\chi. \tag{16}$$

We will use this identity to transform the first order correction to the action. It is interesting, that in the absence of superconductivity the identities (14) and (16) are equivalent because the inverse Green function commutes with σ_3 in this case. For superconductors, however, these two identities are different.

C. First order

The first order correction to the effective action is

$$S_1 = -\mathrm{Tr}(\mathcal{G}_0 \mathcal{G}_1^{-1})$$
$$= -\mathrm{tr}\left[\left(\frac{m\mathbf{v}_s^2}{2} - ie\Phi\right)(G-\overline{G}) - \frac{i}{2}\{\nabla,\mathbf{v}_s\}(G+\overline{G})\right]. \tag{17}$$

With the aid of the Ward identity (16) it is easy to show that the phase of the order parameter drops out from the first order terms in the electromagnetic fields. The action S_1 can therefore be rewritten as

$$S_1 = -\mathrm{tr}(m\mathbf{v}_s^2 G) - \int dx\left(ien_e[\Delta]V + \frac{1}{c}\mathbf{j}_e[\Delta]\mathbf{A}\right). \tag{18}$$

We note that in general the electron density $n_e[\Delta]$ and the current density $\mathbf{j}_e[\Delta]$ explicitly depend on the absolute value of the order parameter.

D. Second order

It is convenient to introduce the following notations:

$$\dot{\theta} = 2e\Phi, \quad \mathcal{L} = \{\nabla,\mathbf{v}_s\}. \tag{19}$$

In terms of these new variables the second order correction to the action reads

$$S_2 = \frac{1}{2}\mathrm{Tr}(\mathcal{G}_0\mathcal{G}_1^{-1})^2$$
$$= -\frac{1}{8}\mathrm{tr}[G\dot{\theta}G\dot{\theta} + \overline{G}\dot{\theta}\overline{G}\dot{\theta} - 2F\dot{\theta}F\dot{\theta} + G\mathcal{L}G\mathcal{L}$$
$$+ \overline{G}\mathcal{L}\overline{G}\mathcal{L} + 2F\mathcal{L}F\mathcal{L} + 2G\dot{\theta}G\mathcal{L} - 2\overline{G}\dot{\theta}\overline{G}\mathcal{L}]. \tag{20}$$

Here we used the properties of the Green function (13). The form (20) of the second order correction is not quite convenient, because it contains the $\dot{\theta}\mathcal{L}$ terms. In order to separate $\dot{\theta}$ and $\mathcal{L}$ we use the Ward identities (15). We write

$$G\dot{\theta}G = \theta G - G\theta + G\left\{\nabla,\frac{\nabla\theta}{2m}\right\}G - F\left(\dot{\theta} + \left\{\nabla,\frac{\nabla\theta}{2m}\right\}\right)F,$$

$$\overline{G}\dot{\theta}\overline{G} = \theta\overline{G} - \overline{G}\theta + \overline{G}\left\{\nabla,\frac{\nabla\theta}{2m}\right\}\overline{G} - F\left(\dot{\theta} - \left\{\nabla,\frac{\nabla\theta}{2m}\right\}\right)F.$$

Inserting these expressions into Eq. (20) after some simple transformations we rewrite the second order contribution as follows:

$$S_2 = -\mathrm{tr}(G(\mathbf{v}_s\nabla\theta)) - \frac{1}{8}[GKGK + \overline{G}K\overline{G}K$$
$$- GMGM + \overline{G}M\overline{G}M - G\dot{\theta}G\dot{\theta} - \overline{G}\dot{\theta}\overline{G}\dot{\theta}$$
$$- 2FKFK - 2F\dot{\theta}F\dot{\theta} + 2FMFM + 4F\mathcal{L}F\mathcal{L}]. \tag{21}$$

Here we have introduced

$$K = \{\nabla,\mathbf{u}\}, \quad M = \left\{\nabla,\frac{\nabla\theta}{2m}\right\},$$

$$\mathbf{u} = \frac{\nabla\theta}{2m} + \mathbf{v}_s = \frac{e}{m}\left(\int_{-\infty}^{\tau} d\tau'[\nabla V(\tau')] - \frac{1}{c}\mathbf{A}\right).$$

The values $\dot{\theta}$ and $\mathbf{v}_s$ are now almost decoupled. The terms containing both these values were transformed into the terms containing the linear combination of these values $\mathbf{u}$ which does not depend on the phase of the order parameter field. The action (21) can be simplified further. We rewrite the first term of Eq. (21) as follows:

$$-\mathrm{tr}[G(\mathbf{v}_s\nabla\theta)] = \mathrm{tr}\left[\left(m\mathbf{v}_s^2 + \frac{(\nabla\theta)^2}{4m} - m\mathbf{u}^2\right)G\right].$$

Again we decouple θ and $\mathbf{v_s}$. Making use of the identities (15) yet a couple of times we arrive at the final expression for the second order contribution to the effective action

$$S_2 = \mathrm{tr}(m\mathbf{v}_s^2 G) - \mathrm{tr}(m\mathbf{u}^2 G) - \frac{1}{4}\mathrm{tr}(G\{\nabla,\mathbf{u}\}G\{\nabla,\mathbf{u}\})$$
$$+ \frac{1}{4}\mathrm{tr}(F\{\nabla,\mathbf{u}\}F\{\nabla,\mathbf{u}\}) + \frac{1}{2}\mathrm{tr}(F\dot{\theta}F\dot{\theta})$$
$$- \frac{1}{2}\mathrm{tr}(F\{\nabla,\mathbf{v}_s\}F\{\nabla,\mathbf{v}_s\}). \tag{22}$$

E. Resulting action

Combining all contributions, we get the final result[16]

$$S = S_s[\Delta,\Phi,\mathbf{v}_s] + S_N[\Delta,V,\mathbf{A}] + S_{em}[\mathbf{E},\mathbf{B}], \tag{23}$$

where

$$S_s = \int dx\left(\frac{\Delta^2}{\lambda}\right) - \mathrm{Tr}\ln\mathcal{G}_0^{-1}[\Delta] + \mathrm{Tr}\ln\mathcal{G}_0^{-1}[\Delta=0]$$
$$+ \frac{1}{2}\mathrm{tr}(F\dot{\theta}F\dot{\theta}) - \frac{1}{2}\mathrm{tr}(F\{\nabla,\mathbf{v}_s\}F\{\nabla,\mathbf{v}_s\}), \tag{24}$$

$$S_N = \int dx\left(-ie(n_e[\Delta]-n)V - \frac{1}{c}\mathbf{j}_e[\Delta]\mathbf{A} + \frac{m\mathbf{u}^2}{2}n_e[\Delta]\right)$$
$$- \frac{1}{4}\mathrm{tr}(G\{\nabla,\mathbf{u}\}G\{\nabla,\mathbf{u}\}) + \frac{1}{4}\mathrm{tr}(F\{\nabla,\mathbf{u}\}F\{\nabla,\mathbf{u}\}), \tag{25}$$

$$S_{em} = \int dx\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}. \tag{26}$$

III. EFFECTIVE ACTION FOR ULTRA-THIN WIRES

A. Averaging over the electromagnetic field

The above expressions are complicated and in general can hardly be evaluated in a closed form. In this section we will focus our attention specifically on the case of quasi-one-dimensional superconducting wires and calculate the effective action performing several approximations. We will argue that our procedure allows us to evaluate the QPS action up to a numerical prefactor of order one.

If one assumes that deviations of the amplitude of the order parameter field from its equilibrium value are relatively small, the above effective action can be significantly simplified. We expand the general effective action (23) in powers of $\delta\Delta(x,\tau)=\Delta(x,\tau)-\Delta$ (here $\Delta\equiv\Delta_{\rm BCS}$) up to the second order terms. The next step is to average over the random potential of impurities.[16] After that the effective action becomes translationally invariant both in space and in time. Performing the Fourier transformation we obtain

$$S=\frac{s}{2}\int\frac{d\omega dq}{(2\pi)^2}\left\{\frac{|A|^2}{Ls}+\frac{C|V|^2}{s}+\chi_E\left|qV+\frac{\omega}{c}A\right|^2+\chi_J\left|V+\frac{i\omega}{2e}\varphi\right|^2+\frac{\chi_L}{4m^2}\left|iq\varphi-\frac{2e}{c}A\right|^2+\chi_A|\delta\Delta|^2\right\}. \tag{27}$$

Here L and C are, respectively, the inductance times unit length and the capacitance per unit length of the wire. The functions χ_E, χ_J, χ_L, and χ_A, which depend both on the frequencies and the wave vectors, are expressed in terms of the averaged products of the Green functions appearing in the Eqs. (24), (25) (see Ref. 16 for more details). These functions can be evaluated analytically for most limiting cases. For the sake of completeness some explicit expressions are presented in the Appendix.

The voltage V and the vector potential A enter the action in a quadratic form and, hence, can be integrated out exactly. After that the effective action will only depend on φ and $\delta\Delta$. We find

$$S=\frac{1}{2}\int\frac{d\omega dq}{(2\pi)^2}\{\mathcal{F}(\omega,q)|\varphi|^2+\chi_A|\delta\Delta|^2\}, \tag{28}$$

where

$$\mathcal{F}(\omega,q)=\frac{\left(\frac{\chi_J}{4e^2}\omega^2+\frac{\chi_L}{4m^2}q^2\right)\left(\frac{C}{sL}+\chi_E\left[C\omega^2+\frac{q^2}{L}\right]\right)+\frac{\chi_J\chi_L}{4m^2}\left[C\omega^2+\frac{q^2}{L}\right]}{\left(\frac{C}{s}+\chi_J+\chi_Eq^2\right)\left(\frac{1}{sL}+\chi_E\omega^2+\frac{e^2}{m^2}\chi_L\right)-\chi_E^2\omega^2q^2}. \tag{29}$$

The electromagnetic potentials are expressed as follows:

$$V=\frac{\chi_J\left(\frac{1}{sL}+\chi_E\omega^2+\frac{e^2}{m^2}\chi_L\right)+\frac{e^2}{m^2}\chi_E\chi_Lq^2}{\left(\frac{C}{s}+\chi_J+\chi_Eq^2\right)\left(\frac{1}{sL}+\chi_E\omega^2+\frac{e^2}{m^2}\chi_L\right)-\chi_E^2\omega^2q^2}\times\left(\frac{-i\omega}{2e}\varphi\right), \tag{30}$$

$$A=\frac{\frac{e^2}{m^2}\chi_L\left(\frac{C}{s}+\chi_J+\chi_Eq^2\right)+\chi_E\chi_J\omega^2}{\left(\frac{C}{s}+\chi_J+\chi_Eq^2\right)\left(\frac{1}{sL}+\chi_E\omega^2+\frac{e^2}{m^2}\chi_L\right)-\chi_E^2\omega^2q^2}\times\left(\frac{icq}{2e}\varphi\right). \tag{31}$$

In most of the situations the wire inductance is not important and can be neglected. Therefore here and below we put $L=0$. Then we get

$$S=\frac{s}{2}\int\frac{d\omega dq}{(2\pi)^2}\times\left\{\frac{\left(\frac{\chi_J}{4e^2}\omega^2+\frac{\chi_L}{4m^2}q^2\right)\left(\frac{C}{s}+\chi_Eq^2\right)+\frac{\chi_J\chi_L}{4m^2}q^2}{\frac{C}{s}+\chi_J+\chi_Eq^2}\times|\varphi|^2+\chi_A|\delta\Delta|^2\right\} \tag{32}$$

and

$$V=\frac{\chi_J}{\frac{C}{s}+\chi_J+\chi_Eq^2}\left(\frac{-i\omega}{2e}\varphi\right), \tag{33}$$

$$A=0. \tag{34}$$

Let us note that the Josephson relation $V=\dot{\varphi}/2e$ is in general not satisfied. According to Eq. (33) this relation may approximately hold only in the limit $\chi_J\gg C/s+\chi_Eq^2$. Making use of the results presented in the Appendix one easily observes that in a practically important limit of small elastic mean free paths l the latter condition is obeyed only at low frequencies and wave vectors $\omega/\Delta \ll 1$ and $Dq^2/\Delta \ll 1$, where $D = v_F l/3$ is the diffusion constant.

Let us now perform yet one more approximation and expand the action in powers of ω and q^2. Keeping the terms of the order q^4 and $\omega^2 q^2$ we find

$$S = \frac{s}{2}\int \frac{d\omega dq}{(2\pi)^2}\left\{\left(\frac{C}{s}\omega^2 + \pi\sigma\Delta q^2 + \frac{\pi^2}{8}\sigma D q^4 + \frac{\pi\sigma}{8\Delta}\omega^2 q^2\right)\right.$$
$$\left.\times\left|\frac{\varphi}{2e}\right|^2 + 2N_0\left(1 + \frac{\omega^2}{12\Delta^2} + \frac{\pi D q^2}{8\Delta}\right)|\delta\Delta|^2\right\}. \tag{35}$$

The term $\propto \omega^4$ turns out to be equal to zero. In Eq. (35) we introduced the normal state conductance of the wire $\sigma = 2e^2 N_0 D$. At even smaller wave vectors $Dq^2/2\Delta \ll 2C/\pi e^2 N_0 s \ll 1$, we get

$$S = \frac{1}{2}\int \frac{d\omega dq}{(2\pi)^2}\left\{(C\omega^2 + \pi\sigma\Delta s q^2)\left|\frac{\varphi}{2e}\right|^2 + s\chi_A|\delta\Delta|^2\right\}. \tag{36}$$

Here we have assumed $C/2e^2 N_0 s \ll 1$. This inequality is usually well satisfied for sufficiently good metals, perhaps except for the case of some specially chosen substrates. The form of the action suggests the existence of the plasma modes which can propagate along the wire. These are the so-called Mooij-Schön modes,[23] the velocity of which is given by the equation

$$c_0 \simeq \sqrt{\frac{\pi\sigma\Delta s}{C}}. \tag{37}$$

B. QPS action

One can show[14] that for very long wires the action (36) yields a QPS solution described by a simple formula $\varphi(x,\tau) = -\arctan(x/c_0\tau)$. The long time behavior of this solution results in the logarithmic interaction between two phase slips (x_1,τ_1) and (x_2,τ_2):

$$S_{\rm int} = \frac{\mu}{2}\ln\left[\frac{(x_1 - x_2)^2 + c_0^2(\tau_1 - \tau_2)^2}{\xi^2}\right], \tag{38}$$

where

$$\mu = \frac{\pi}{4\alpha}\sqrt{\frac{sC}{4\pi\lambda_L^2}}. \tag{39}$$

Here $\alpha \simeq 1/137$ is the fine structure constant. In short wires, however, the above logarithmic interaction (38) does not play an important role and can be essentially neglected.

Let us estimate the contribution of a single phase slip to the effective action. First we rewrite the action (35) in the space-time domain dropping the unimportant term $\propto (Dq^2)^2$:

$$S = \frac{s}{2}\int dx d\tau\left\{\frac{C}{4e^2 s}\left(\frac{\partial\varphi}{\partial\tau}\right)^2\right.$$
$$\left. + \frac{\pi\sigma\Delta}{4e^2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{\pi\sigma}{32e^2\Delta}\left(\frac{\partial^2\varphi}{\partial x\partial\tau}\right)^2\right\}$$
$$+ sN_0\int dx d\tau\left\{\delta\Delta^2 + \frac{1}{12\Delta^2}\left(\frac{\partial\delta\Delta}{\partial\tau}\right)^2 + \frac{\pi D}{8\Delta}\left(\frac{\partial\delta\Delta}{\partial x}\right)^2\right\}. \tag{40}$$

Then we assume that the absolute value of the order parameter is equal to zero at a time $\tau = 0$ and at a point $x = 0$. The size of the QPS core is denoted as x_0, and its time duration is τ_0. The amplitude of the fluctuating part of the order parameter field $|\delta\Delta(x,\tau)|$ can be approximately expressed as follows:

$$|\delta\Delta(x,\tau)| = \Delta\exp(-x^2/2x_0^2 - \tau^2/2\tau_0^2). \tag{41}$$

The QPS phase dependence on x and τ should satisfy several requirements. In a short wire and outside the QPS core the phase φ should not depend on the spatial coordinate in the zero current bias limit. On top of that, at $x = 0$ and $\tau = 0$ the phase should flip in a way to provide the change of the net phase difference across the wire by 2π. For concreteness, let us present two different trial functions which obey the above requirements. For instance, one may choose

$$\varphi(x,\tau) = -\frac{\pi}{2\cosh(\tau/\tau_0)}\tanh\left(\frac{x}{x_0\tanh(\tau/\tau_0)}\right) \tag{42}$$

or

$$\varphi(x,\tau) = -\frac{\pi}{2}\tanh\left(\frac{x\tau_0}{x_0\tau}\right). \tag{43}$$

Similar other trial functions can also be considered.

Substituting the trial functions (41), (42) [or (41), (43)] into the action (40) one arrives at the expression

$$S(x_0,\tau_0) = \left[a_1\frac{C}{e^2} + a_2 s N_0\right]\frac{x_0}{\tau_0} + a_3 s N_0 D\Delta\frac{\tau_0}{x_0}$$
$$+ a_4\frac{sN_0 D}{\Delta}\frac{1}{x_0\tau_0} + a_5 s N_0\Delta^2 x_0\tau_0 + a_6\frac{\tilde{C}}{e^2\tau_0}, \tag{44}$$

where a_j are numerical factors of order one which depend on the precise form of the trial functions, $\tilde{C} = CX$ is the total capacitance of the wire, and X is the wire length. Note that fictitious divergences emerging from a singular behavior of the functions (42), (43) at $x = x_0$ and $\tau = \tau_0$ are eliminated since the order parameter vanishes inside the QPS core.

Let us first disregard capacitive effects neglecting the last term in Eq. (44). Minimizing the remaining action with respect to the core parameters x_0 and τ_0 and making use of the inequality $C/e^2 N_0 s \ll 1$, we obtain

$$x_0 = \left(\frac{a_3 a_4}{a_2 a_5}\right)^{1/4} \sqrt{\frac{D}{\Delta}}, \quad \tau_0 = \left(\frac{a_2 a_4}{a_3 a_5}\right)^{1/4} \frac{1}{\Delta}. \tag{45}$$

These values provide the minimum for the QPS action, and we find

$$S_{\rm QPS} = 2(\sqrt{a_2 a_3} + \sqrt{a_4 a_5}) N_0 s \sqrt{D\Delta}. \tag{46}$$

One can also express Eq. (46) in the form convenient for further comparison with experiments:

$$S_{\rm QPS} = A \frac{R_q}{R} \frac{X}{\xi}. \tag{47}$$

Here $A = 2(\sqrt{a_2 a_3} + \sqrt{a_4 a_5})/\pi$, R is the total wire resistance, $R_q = \pi\hbar/2e^2 = 6.453$ kΩ is the resistance quantum, and $\xi = \sqrt{D/\Delta}$ is the superconducting coherence length.

As it was already pointed out, the results (45) and (46) hold provided the capacitive effects are small. This is the case for relatively short wires

$$X \ll \xi \frac{e^2 N_0 s}{C}. \tag{48}$$

In the opposite limit the same minimization procedure of the action (44) yields

$$x_0 \sim \xi, \quad \Delta\tau_0 \sim \sqrt{XC/\xi e^2 N_0 s} \gg 1. \tag{49}$$

The QPS action again takes the form (47) with $A \sim \sqrt{XC/\xi e^2 N_0 s}$.

For the sake of clarity, let us summarize the approximations performed in this section. As a first step, we expanded the action derived in Sec. II up to the second order in $\delta\Delta(x,\tau) = \Delta(x,\tau) - \Delta$. Obviously this approximation is sufficient everywhere except inside the QPS core where $\Delta(x,\tau)$ is small. In these space- and time-restricted regions one can expand already in $\Delta(x,\tau)$ again arriving at Eq. (27) with $\delta\Delta(x,\tau) \to \Delta(x,\tau)$ and with all the χ functions defined in the Appendix with $\Delta \equiv 0$. Both expansions match smoothly at the scale of the core size $x_0 \sim \xi$, $\tau_0 \sim 1/\Delta$. Hence, the approximation (27) is sufficient to obtain the correct QPS action, perhaps up to a numerical prefactor of order 1.

In order to simplify our analysis further, in Eq. (35) we expanded Eq. (27) in powers of ω/Δ and Dq^2/Δ. Again this approximation is sufficient within the same accuracy. Indeed, one can—even without performing this expansion—substitute the trial functions (41), (42) [or (41), (43)] directly into the action (32). If the capacitive effects are neglected (48), the resulting QPS action can be represented as a function of the dimensionless parameters x_0/ξ and $\Delta\tau_0$ only. Making use of the general expressions for the χ functions collected in the Appendix and minimizing the QPS action with respect to x_0 and τ_0 one again arrives at the result (47) with $A \sim 1$. If the inequality (48) is violated, the accuracy of our expansion in powers of ω/Δ may only become better [see Eq. (49)].

Finally, the particular choice of the trial functions [e.g., Eqs. (41) and (42)] describing the QPS event also appears not to play any significant role as long as these trial functions obey the general requirements formulated above. In addition to Eqs. (41), (42), and (43) we have used several other trial QPS functions. In all cases we have obtained A within the interval $A \approx 0.8-2.5$. In a way our method can be regarded as a variational procedure. Therefore, even though the exact value of a numerical prefactor A in Eq. (47) cannot be established within our approach, we do not expect A to deviate substantially from the above values.

Our last remark in this section concerns the role of dissipation. From the form of the result (46) one could naively assume that the correct QPS action could be guessed, e.g., from a simple TDGL-based approach (or, alternatively, only from the ''condensation energy'' term proportional to χ_A) without taking into account dissipative and electromagnetic effects. Indeed, minimization of the contribution $\sim|\delta\Delta|^2$ [the last three terms in Eq. (40)] is formally sufficient to arrive at the correct estimate $S_{\rm QPS} \sim N_0 s \sqrt{D\Delta}$. It is obvious, on the other hand, that not only the amplitude but also the phase fluctuations of the order parameter are important during the QPS event. If the latter fluctuations are taken into account *without* including dissipative effects [this would correspond to formally setting $\sigma \to 0$ in Eq. (40)] the estimate for the QPS action $S_{\rm QPS} \sim \mu$ would follow immediately. This result would be parametrically different from Eq. (46). Note, however, that within our model the dissipative effects can be ignored only for $C/e^2 N_0 s \gtrsim 1$. Usually the latter condition cannot be satisfied for metallic systems, perhaps except for some specially chosen substrates. In the opposite—more realistic—limit $C/e^2 N_0 s \ll 1$ dissipation plays a dominant role during the phase slip event, and the correct QPS action *cannot* be obtained without an adequate microscopic description of dissipative currents flowing inside the wire.

IV. PRE-EXPONENT

The above results allow us to estimate the exponential suppression of QPS in ultrathin superconducting wires depending on thickness, impurity concentration, and other parameters. These results, however, are not yet sufficient to evaluate the whole QPS rate which has the form

$$\gamma_{\rm QPS} = B \exp(-S_{\rm QPS}). \tag{50}$$

The task at hand is to provide a reliable estimate for the pre-exponential factor B in Eq. (50). A general strategy to be used for this purpose is well known.[26] One can start, e.g., from the expression for the grand partition function of the wire

$$Z = \int \mathcal{D}\Delta \mathcal{D}\varphi \exp(-S) \tag{51}$$

and evaluate this path integral within the saddle point approximation. The least action paths

$$\delta S/|\delta\Delta| = 0, \quad \delta S/\delta\varphi = 0 \tag{52}$$

determine all possible QPS configurations. Integrating over small fluctuations around all QPS trajectories one represents the grand partition function in terms of infinite series (each term in such series corresponds to one particular QPS saddle point). Then—at least if interaction between different quantum phase slips is small and can be neglected—one can easily sum these series and represent the final result in the form of the exponent

$$Z=\exp(-F/T), \tag{53}$$

where a formal expression for the free energy F reads

$$F=F_0-T\frac{\int \mathcal{D}\delta Y \exp(-\delta^2 S_1[\delta Y])}{\int \mathcal{D}\delta Y \exp(-\delta^2 S_0[\delta Y])}\exp(-S_{\mathrm{QPS}})$$
$$\equiv F_0-\frac{\gamma_{\mathrm{QPS}}}{2}. \tag{54}$$

Here F_0 is the free energy without quantum phase slips, $\delta Y=(\delta\Delta,\delta\varphi)$ denote the fluctuations of relevant coordinates (fields), $\delta^2 S_{0,1}[\delta Y]$ are the quadratic in δY parts of the action, and the subscripts ''0'' and ''1'' denote the action, respectively, without and with one QPS.

The integrals over fluctuations in Eq. (54) can be evaluated exactly only in simple cases. Technically such a calculation can be quite complicated even if the saddle point trajectories can be determined explicitly. In our case an analytical expression for the QPS trajectory is not even known. Hence, an exact evaluation of the path integrals in Eq. (54) is not possible.

Below we will present a simple approach which allows to establish the correct expression for the pre-exponent B up to an unimportant numerical prefactor. Within our present analysis any attempt to find an explicit value for such a prefactor would make little sense simply because the numerical value of A in Eq. (47) is not known exactly. Also for other problems numerical prefactors in the pre-exponent are usually of little interest. Therefore we believe that our approach may be useful for various other situations because it allows to establish the correct functional form of the pre-exponent practically without any calculation. If needed, with a little extra effort our method may also allow to approximately evaluate a numerical coefficient in the pre-exponent.

In order to calculate the ratio of the path integrals in Eq. (54) let us introduce the basis in the functional space $\Psi_k(z)$ in which the second variation of the action around the instanton $\delta^2 S_1[\delta Y]$ is diagonal. Here the basis functions depend on a general vector coordinate z which is simply $z=(\tau,x)$ in our case. The first N functions Ψ_k are the so-called ''zero modes'' related to the invariance of the instanton action under arbitrary shifts in certain directions in the functional space (in our case, shifts of the QPS position along the wire and in imaginary time, i.e., $N=2$). Let us denote an instanton solution as $\tilde{Y}(z)$. Then the zero mode eigenfunctions are expressed as follows: $\Psi_k(X)=\partial\tilde{Y}/\partial z_k$, where $k\leqslant N$ and the number of zero modes N coincides with the dimension of the vector z. An arbitrary fluctuation $\delta Y(z)$ can be represented in terms of the Fourier expansion

$$\delta Y(z)=\sum_{k=1}^{N}\delta z_k\frac{\partial\tilde{Y}(z)}{\partial z_k}+\sum_{k=N+1}^{\infty}u_k\Psi_k(z). \tag{55}$$

Then we get

$$\delta^2 S_0[\delta Y]=\frac{1}{2}\sum_{k,n=1}^{\infty}A_{kn}u_k u_n^*,$$
$$\delta^2 S_1[\delta Y]=\frac{1}{2}\sum_{k=N+1}^{\infty}\lambda_k|u_k|^2, \tag{56}$$

where for $k\leqslant N$ the Fourier coefficients $u_k\equiv\delta z_k$ are just the shifts of the instanton position along the kth axis and λ_k are the eigenvalues of $\delta^2 S_1[\delta Y]$. Integrating over the Fourier coefficients one obtains

$$\frac{\int \mathcal{D}\delta Y \exp(-\delta^2 S_1[\delta Y])}{\int \mathcal{D}\delta Y \exp(-\delta^2 S_0[\delta Y])}=\int_0^{L_1}d\,\delta x_1\cdots\int_0^{L_N}d\,\delta x_N\sqrt{\frac{\det A_{kn}}{(2\pi)^N\prod_{k=N+1}^{\infty}\lambda_k}}, \tag{57}$$

where L_k is the system size in the kth dimension. The formula (57) is, of course, not at all new. It just represents the standard ratio of determinants with excluded zero modes.[26] We will argue now, that with a sufficient accuracy in the latter formula one can keep the contribution of only first N eigenvalues. Indeed, the contribution of the ''fast''eigenmodes (corresponding to frequencies and wave vectors much larger than the inverse instanton size in the corresponding dimension) is insensitive to the presence of an instanton. Hence, the corresponding eigenvalues are the same for both $\delta^2 S_0$ and $\delta^2 S_1$ and just cancel out from Eq. (57). In addition to the fast modes there are several eigenmodes with frequencies (wave vectors) of order of the inverse instanton size. The ratio between the product of all such modes for $\delta^2 S_1$ and the product of eigenvalues for $\delta^2 S_0$ with the same numbers is dimensionless and may only affect a numerical prefactor which is not interesting for us here. Dropping the contribution of all such eigenvalues one gets

$$\frac{\int \mathcal{D}\delta Y \exp(-\delta^2 S_1[\delta Y])}{\int \mathcal{D}\delta Y \exp(-\delta^2 S_0[\delta Y])}\approx\int_0^{L_1}d\,\delta x_1\cdots\int_0^{L_N}d\,\delta x_N\sqrt{\frac{\det A_{kn}|_{k,n\leqslant N}}{(2\pi)^N}}. \tag{58}$$

What remains is to estimate the parameters A_{kk} for $k\leqslant N$. For this purpose let us observe that the second variation of the action becomes approximately equal to the instanton action, $\delta^2 S_1=\frac{1}{2}A_{kk}z_{0k}^2\approx S_{\mathrm{QPS}}$, when the shift in the kth direction becomes equal to the instanton size in the same direction $\delta z_k=z_{0k}$. Then we find $A_{kk}\approx 2S_{\mathrm{QPS}}/z_{0k}^2$ and

$$\det A_{kn}|_{k,n<N} \approx \prod_{k=1}^{N} A_{kk} \approx \frac{(2S_{\mathrm{QPS}})^N}{\prod_{k=1}^{N} z_{0k}^2}. \tag{59}$$

Finally, combining Eqs. (50), (54), (58), and (59) we obtain

$$B = bT\left(\prod_{k=1}^{N} \frac{L_k}{z_{0k}}\right)\left(\frac{S_{\mathrm{QPS}}}{\pi}\right)^{N/2}. \tag{60}$$

Here b is an unimportant numerical prefactor. This result demonstrates that the functional dependence of the pre-exponent can be determined practically without any calculation. It is sufficient to know just the instanton action, the number of the zero modes N and the instanton effective size z_{0k} for each of these modes.

Let us also note that a similar observation has already been made[26] for some local Lagrangians equal to the sum of kinetic and potential energies. Here we have shown that the result (60) holds for arbitrary effective actions, including nonlocal ones. Hence, this result can be directly applied to our problem of quantum phase slips in thin superconducting wires. In this case we have $L_1 \equiv 1/T$, $L_2 \equiv X$ and Eq. (60) yields

$$B \approx \frac{S_{\mathrm{QPS}} X}{\tau_0 x_0}. \tag{61}$$

This equation provides an accurate expression for the pre-exponent B up to a numerical factor of order 1. As we have already discussed, such an accuracy is sufficient for our purposes. We also note that the result (61) is parametrically different from previous results obtained within a TDGL type of analysis[13] or suggested phenomenologically in Ref. 8.

Finally, it is worth mentioning that we have also compared our Eq. (60) with the exact results previously obtained for various other problems by means of different approaches. Here we will briefly discuss three different examples for the sake of illustration. The first example is the problem of a quantum particle in a cosine periodic potential. In this case an explicit expression for the interwell tunneling rate is well known. If we apply our method and evaluate only A_{11}, we will obtain the tunneling rate which is $\sim 40\%$ smaller than the exact result. If we also evaluate A_{22} and λ_2 and include their ratio into our formula, the result for the tunneling rate will be only 10% smaller as compared to the exact one. This example demonstrates that also a sufficient numerical accuracy in the pre-exponent can be achieved without a complicated calculation of the ratio of the determinants.

Two other examples concern the systems with nonlocal in time Lagrangians. Consider, e.g., the problem of quantum decay of a particle in the presence of dissipation.[29] In the limit of strong dissipation this problem was treated by Larkin and Ovchinnikov[30] who found the exact eigenvalues and, evaluating the ratio of the determinants, obtained the prefactor in expression for the decay rate $B \propto \eta^{7/2}/m^2$, where η is an effective friction constant and m is the particle mass. This result would imply that the pre-exponential factor in the decay rate[30] should be very large and may even diverge if one formally sets $m \to 0$.

Later it was realized[31] that this divergence is artificial. Performing a simple one-loop perturbative calculation[31] one arrives exactly at the same high frequency divergence as in the result.[30] It implies that this divergence has nothing to do with tunneling, and it is regularized by means of a proper renormalization of the bare parameters in the effective action. After that the high frequency contribution to the pre-exponent is eliminated and one finds[31] $B \propto 1/\sqrt{\eta}$. [Note a misprint in the power of η in Eq. (8) of Ref. 31.] This expression does not contain the particle mass m at all. It also allowed to fully resolve a discrepancy with the experiments.[32] Note that the result[31] can also be expressed in the form $B \sim \sqrt{S_b}/\tau_0$, where $S_b \propto \eta$ is the instanton (bounce) action and $\tau_0 \propto \eta$ is its typical size. As we have already argued, the result in this form can be guessed from Eq. (60) without any calculation. [For this particular problem our approach allows one to even reproduce an exact numerical prefactor.]

The last example is the problem of Coulomb blockade in normal tunnel junctions in the strong tunneling limit. This problem was treated within the instanton technique in Ref. 33. In this problem each instanton has two zero modes which correspond to its shifts in time and fluctuations of its frequency Ω. The value of the instanton action is well known and the parameters z_{01} and z_{02} for both zero modes can be evaluated directly by means of the approach presented above. Each of these parameters is found to depend on one of the zero modes Ω. However, the product $z_{01}z_{02}$ turns out to be an Ω-independent constant. Making use of this fact and integrating over the zero mode coordinates in Eq. (60), we arrive at the functional form of the pre-exponent derived in Eq. (10) of Ref. 33 by means of an explicit calculation of the fluctuation determinants, see also Ref. 22. The above examples demonstrate that our approach allows to easily derive the functional form of the pre-exponent in a variety of problems, including those where technically involved calculations appear to be inevitable otherwise.

V. COMPARISON WITH EXPERIMENT

Now let us compare our results with experimental findings. Recently Bezryadin, Lau, and Tinkham[20] reported clear experimental evidence for the existence of quantum phase slips in ultrathin (with diameters down to 3 nm) and uniform in thickness superconducting wires. Three out of eight samples studied in the experiments[20] showed no sign of superconductivity even well below the bulk critical temperature T_C. Furthermore, in the low temperature limit the resistance of these samples was found to show a slight upturn with decreasing T. In view of that one can conjecture that these samples may actually become insulating at $T \to 0$. The resistance of other five samples[20] decreased with decreasing T. Also for these five samples no clear superconducting phase transition was observed.

All three nonsuperconducting wires ($i1$, $i2$, and $i3$) had the normal state resistance below the quantum unit R_q, while the normal state resistance of the remaining five ''superconducting'' samples was larger than R_q. This observation allowed the authors[20] to suggest that a dramatic difference in

TABLE I.

Sample	R/d, kΩ/nm	S_0	$t_0\|_{A=1}$, sec	$t_0\|_{A=2}$, sec
$i1$	0.122	7.8	10^{-11}	10^{-8}
$i2$	0.110	8.7	10^{-11}	10^{-6}
$i3$	0.079	12.7	10^{-9}	10^{-4}
$s1$	0.038	25.1	10^{-4}	10^{6}
$s2$	0.028	33.7	1	10^{14}
$s3$	0.039	22.6	10^{-5}	10^{5}
$ss1$	0.054	15.4	10^{-7}	10^{-1}
$ss2$	0.044	19.6	10^{-6}	10^{2}

the behavior of these two group of samples (otherwise having similar parameters) can be due to the dissipative phase transition (DPT) (Refs. 15,21,22) analogous to that observed earlier in Josephson junctions.[27]

Without going into details here, let us just point out that DPT can be observed only provided quantum phase slips are easily created inside the wire. The results for $\gamma_{\rm QPS}$ derived in the present paper allow us to estimate a typical average time within which one QPS event occurs in the sample. Making use of Eqs. (47), (50), and (61) we performed an estimate of such a time $t_0=1/\gamma_{\rm QPS}$ for all eight samples studied in Ref. 20. In this experiment the samples were fabricated from $\mathrm{Mo_{79}Ge_{21}}$ alloy. For our estimates we will use the value of the density of states $N_0=1.86\times 10^{13}$ sec/m^3 for clean Mo, which can be extracted from the specific heat data. The resistivity of the material was measured to be $\rho=1.8$ $\mu\Omega$/m, the superconducting critical temperature is $T_C\simeq 5.5$ K. With these numbers we obtain the coherence length $\xi\simeq 7$ nm in agreement with the estimate.[20] The results for t_0 are summarized in Table I. The action S_0 is defined by Eq. (47) with $A=1$. The typical QPS time t_0

$$t_0=\frac{\xi}{XAS_0\Delta}\exp(AS_0).$$

is very sensitive to the particular value of the factor A, therefore here we present two estimates corresponding to $A=1$ and $A=2$.

In spite of remaining uncertainty in the prefactors some important conclusions can be drawn already from the above estimates. For instance, we observe that for both $A=1$ and $A=2$ the QPS rate $\gamma_{\rm QPS}=1/t_0$ is very high (as compared, e.g., to the typical experimental time scale ~ 1 sec) in the ''insulating'' wires $i1$, $i2$, and $i3$. This fact is fully consistent with the observations[20]: numerous quantum phase slips occurring in these wires completely destroy the phase coherence and, hence, superconductivity is washed out. Thus, nonsuperconducting behavior of these three samples should be due to quantum phase slips.

On the other hand, the QPS rate is notably lower for all the ''superconducting'' wires.[20] Possible interpretation of the experimental results for the samples $s1-ss2$ depends strongly on the value of A. For example, for $A=1$ the QPS rate is high enough practically in all samples. In this case quantum phase slips should in principle be important also for ''superconducting'' wires.[20] Then one can indeed relate the behavior of these samples to DPT[15], as a result of which quantum phase slips are bound in pairs and, hence, quantum fluctuations are strongly suppressed.

If, however, one chooses $A=2$ the QPS time for the samples s1-ss2 turns out to be very long, much longer than the experimental time. Then the QPS effects should be irrelevant, and one would expect these samples to show a superconducting behavior, perhaps with the renormalized critical temperature.[28] This conclusion would also be consistent with the experimental observations.[20]

Finally, let us note that all the above estimates are performed in the limit $T=0$. This is correct if temperature is considerably below T_C. Otherwise the expression for the QPS action $S_{\rm QPS}$ needs to be modified.

In conclusion, we have developed a detailed microscopic theory of quantum phase slips in ultrathin homogeneous superconducting wires. We have derived the effective QPS rate for such wires and evaluated this rate for the systems studied in recent experiments.[20] Our results are fully consistent with the experimental findings[20] which provide perhaps the first unambiguous evidence for QPS in mesoscopic metallic wires.

APPENDIX

Let us collect some rigorous expressions for the ''susceptibilities'' χ_E, χ_J, χ_L, and χ_A. In Ref. 16 these quantities have been related to the so-called polarization bubbles f_0, g_0 and h_0. In the interesting for us diffusive limit $\Delta l/v_F\ll 1$ these polarization bubbles are defined by the equations[16]

$$f_0=T\sum_{\omega_\nu}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\langle F(\omega+\omega_\nu,\mathbf{q}+\mathbf{k})F(\omega_\nu,\mathbf{k})\rangle_{\rm dis}$$

$$=\pi N_0T\sum_{\omega_\nu}\frac{\Delta^2}{WW'(W+W'+Dq^2)}, \tag{A1}$$

$$g_0=T\sum_{\omega_\nu}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\langle G(\omega+\omega_\nu,\mathbf{q}+\mathbf{k})G(\omega_\nu,\mathbf{k})\rangle_{\rm dis}$$

$$=-N_0+\pi N_0T\sum_{\omega_\nu}\frac{WW'-\omega(\omega+\omega_\nu)}{WW'(W+W'+Dq^2)}, \tag{A2}$$

$$h_0=T\sum_{\omega_\nu}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\langle G(\omega+\omega_\nu,\mathbf{q}+\mathbf{k})\bar{G}(\omega_\nu,\mathbf{k})\rangle_{\rm dis}$$

$$=-\pi N_0T\sum_{\omega_\nu}\frac{WW'+\omega(\omega+\omega_\nu)}{WW'(W+W'+Dq^2)}. \tag{A3}$$

In these equations we use the notations

$$W=\sqrt{\omega_\nu^2+\Delta^2},\quad W'=\sqrt{(\omega_\nu+\omega)^2+\Delta^2}, \tag{A4}$$

$\omega_\nu=\pi T(2\nu+1)$ and $\omega=2\pi Tn$, where ν and n are arbitrary integers, and $\langle\cdots\rangle_{\mathrm{dis}}$ implies disorder averaging.

Let us start from the function χ_E defined as

$$\chi_E=-\frac{2e^2}{q^2}(f_0+g_0). \tag{A5}$$

At $T=0$ we find

$$\chi_E=\begin{cases}\dfrac{\sigma}{\Delta}\dfrac{1}{2\sqrt{1+x^2}}\displaystyle\int_0^{+\infty}dt\,\frac{1-t^2+\sqrt{(t^2-1)^2+4t^2/(1+x^2)}}{\sqrt{(t^2-1)^2+4t^2/(1+x^2)}\,(1+t^2+\sqrt{(t^2-1)^2+4t^2/(1+x^2)})}, & y=0,\\[2ex] \dfrac{\sigma}{\Delta}\left(\dfrac{1}{2y}-\dfrac{\pi}{4y}+\dfrac{\ln(y+\sqrt{y^2-1})}{2y\sqrt{y^2-1}}\right), & x=0,\quad y>1,\\[2ex] \dfrac{\sigma}{\Delta}\left(\dfrac{1}{2y}-\dfrac{\pi}{4y}+\dfrac{1}{y\sqrt{1-y^2}}\arctan\left(\sqrt{\dfrac{1-y}{1+y}}\right)\right), & x=0,\quad y<1.\end{cases} \tag{A6}$$

Here and below we set $x=\omega/2\Delta$ and $y=Dq^2/2\Delta$. In the limit $|\omega|\gg 2\Delta$ we obtain

$$\chi_E\simeq\frac{\sigma}{|\omega|+Dq^2}. \tag{A7}$$

At small ω and q we get

$$\chi_E=\frac{\pi\sigma}{8\Delta}\left[1-\frac{3}{8}\left(\frac{\omega}{2\Delta}\right)^2-\frac{8}{3\pi}\frac{Dq^2}{2\Delta}\right]. \tag{A8}$$

It is also possible to evaluate χ_E at $\omega=0$, $q=0$, and arbitrary T:

$$\chi_E(0,0)=\frac{\pi\sigma}{8\Delta}\left(\tanh\frac{\Delta}{2T}-\frac{\Delta}{2T\left(\cosh\frac{\Delta}{2T}\right)^2}\right). \tag{A9}$$

For $\Delta\ll T$ we find

$$\chi_E(0,0)=\frac{\pi\sigma\Delta^2}{96T^3}. \tag{A10}$$

In this limit also a more general expression for arbitrary ω and q can be obtained:

$$\chi_E=\frac{\sigma}{|\omega|+Dq^2}+\frac{4\sigma\Delta^2Dq^2}{(\omega^2-D^2q^4)^2}\times\left[\Psi\left(\frac{1}{2}+\frac{|\omega|+Dq^2}{4\pi T}\right)-\Psi\left(\frac{1}{2}\right)\right]-\frac{2\sigma\Delta^2(\omega^2+D^2q^4)}{|\omega|(\omega^2-D^2q^4)^2}\left[\Psi\left(\frac{1}{2}+\frac{|\omega|}{2\pi T}\right)-\Psi\left(\frac{1}{2}\right)\right]+\frac{\sigma\Delta^2}{2\pi T(\omega^2-D^2q^4)}\Psi'\left(\frac{1}{2}+\frac{|\omega|}{2\pi T}\right), \tag{A11}$$

where $\Psi(x)$ is the digamma function.

The remaining χ functions are evaluated analogously. Consider the function

$$\chi_J=4e^2f_0. \tag{A12}$$

After straightforward algebra we obtain

$$\chi_J=\begin{cases} 4e^2N_0\dfrac{\ln(|x|+\sqrt{1+x^2})}{2|x|\sqrt{1+x^2}}, \quad y=0,\\ 4e^2N_0\left(\dfrac{\pi}{4y}-\dfrac{\ln(y+\sqrt{y^2-1})}{2y\sqrt{y^2-1}}\right), \quad x=0, \quad y>1,\\ 4e^2N_0\left(\dfrac{\pi}{4y}-\dfrac{1}{y\sqrt{1-y^2}}\arctan\left(\sqrt{\dfrac{1-y}{1+y}}\right)\right), \quad x=0, \quad y<1. \end{cases} \tag{A13}$$

In the limit of low frequencies and wave vectors one has

$$\chi_J=2e^2N_0\left[1-\frac{2}{3}\left(\frac{\omega}{2\Delta}\right)^2-\frac{\pi}{4}\frac{Dq^2}{2\Delta}\right]. \tag{A14}$$

At $\omega=0,\ q=0$ and for arbitrary T we get

$$\chi_J=2\pi e^2N_0\Delta^2T\sum_{\omega_\nu}\frac{1}{(\omega_\nu^2+\Delta^2)^{3/2}}=\begin{cases} 2e^2N_0, & T\ll\Delta,\\ \dfrac{7\zeta(3)}{2\pi^2}\dfrac{e^2N_0\Delta^2}{T^2}, & T\gg\Delta, \end{cases} \tag{A15}$$

where $\zeta(3)\simeq 1.202$. In the limit $T\gg\Delta$ and $\omega\neq 0$ we find

$$\chi_J=\frac{8e^2N_0\Delta^2}{|\omega|(\omega^2-D^2q^4)}\left\{|\omega|\left[\Psi\left(\frac{1}{2}+\frac{|\omega|+Dq^2}{4\pi T}\right)-\Psi\left(\frac{1}{2}\right)\right]-Dq^2\left[\Psi\left(\frac{1}{2}+\frac{|\omega|}{2\pi T}\right)-\Psi\left(\frac{1}{2}\right)\right]\right\}. \tag{A16}$$

We proceed further with the function

$$\chi_L=\frac{8m^2\Delta^2}{q^2}\left[\frac{1}{\lambda}+h_0-\left(1+\frac{\omega^2}{2\Delta^2}\right)f_0\right]. \tag{A17}$$

First we consider the limit $T=0$ and find

$$\chi_L=\begin{cases} 4m^2\Delta N_0D\dfrac{1}{\sqrt{1+x^2}}\displaystyle\int_0^{+\infty}dt\,\frac{1}{\sqrt{(t^2-1)^2+4t^2/(1+x^2)}}, \quad y=0,\\ 4m^2\Delta N_0D\dfrac{1}{\sqrt{y^2-1}}\ln(y+\sqrt{y^2-1}), \quad x=0, \quad y>1,\\ 4m^2\Delta N_0D\dfrac{2}{\sqrt{1-y^2}}\arctan\left(\sqrt{\dfrac{1-y}{1+y}}\right), \quad x=0, \quad y<1. \end{cases} \tag{A18}$$

At low frequencies and wave vectors the above expressions yield

$$\chi_L=2\pi N_0Dm^2\Delta\left[1-\frac{1}{4}\left(\frac{\omega}{2\Delta}\right)^2-\frac{2}{\pi}\frac{Dq^2}{2\Delta}\right]. \tag{A19}$$

For high temperatures $T\gg\Delta$ we obtain

$$\chi_L=\frac{4m^2\sigma\Delta^2}{e^2(\omega^2-D^2q^4)}\left\{|\omega|\left[\Psi\left(\frac{1}{2}+\frac{|\omega|}{2\pi T}\right)-\Psi\left(\frac{1}{2}\right)\right]-Dq^2\left[\Psi\left(\frac{1}{2}+\frac{|\omega|+Dq^2}{4\pi T}\right)-\Psi\left(\frac{1}{2}\right)\right]\right\}. \tag{A20}$$

In the limit of zero frequency and wave vectors χ_L reduces to a very simple form

$$\chi_L = \frac{\pi m^2 \sigma \Delta^2}{2e^2 T}. \tag{A21}$$

Finally, let us evaluate the function

$$\chi_A = 2\left(\frac{1}{\lambda} + h_0 + f_0\right). \tag{A22}$$

We obtain

$$\chi_A = \begin{cases} 2N_0 \dfrac{\sqrt{1+x^2}}{|x|} \ln(|x| + \sqrt{1+x^2}), & y=0, \\ 2N_0\left(\dfrac{\pi}{2y} + \dfrac{\sqrt{y^2-1}}{y} \ln(y+\sqrt{y^2-1})\right), & x=0, \quad y>1, \\ 2N_0\left(\dfrac{\pi}{2y} - \dfrac{2\sqrt{1-y^2}}{y} \arctan\left(\sqrt{\dfrac{1-y}{1+y}}\right)\right), & x=0, \quad y<1. \end{cases} \tag{A23}$$

In the limit of high frequencies $|\omega| \gg \Delta$ one finds

$$\chi_A \simeq 2N_0 \ln \frac{|\omega| + Dq^2}{\Delta}. \tag{A24}$$

At low frequencies and wave vectors we derive

$$\chi_A \simeq 2N_0\left[1 + \frac{1}{3}\left(\frac{\omega}{2\Delta}\right)^2 + \frac{\pi}{4}\frac{Dq^2}{2\Delta}\right]. \tag{A25}$$

In the high-temperature limit $T \gg \Delta$ we find

$$\chi_A = 2N_0\left[\ln\frac{T}{T_C} + \Psi\left(\frac{1}{2} + \frac{|\omega| + Dq^2}{4\pi T}\right) - \Psi\left(\frac{1}{2}\right)\right]. \tag{A26}$$

The above expressions are sufficient to evaluate the QPS action practically in all interesting limiting cases.

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