Topological interpretation of subharmonic mode locking in coupled oscillators with inertia

M. Y. Choi
Department of Physics, Seoul National University, Seoul 151-742, Korea

D. J. Thouless
Department of Physics, University of Washington, Seattle, Washington 98195

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A topological argument is constructed and applied to explain subharmonic mode locking in a system of coupled oscillators with inertia. Via a series of transformations, the system is shown to be described by a classical *XY* model with periodic bond angles, which is in turn mapped onto a tight-binding particle in a periodic gauge field. It is then revealed that subharmonic quantization of the average phase velocity follows as a manifestation of topological invariance. Ubiquity of multistability and associated hysteresis are also pointed out.

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In a nonlinear oscillator system driven periodically, the competition between the natural frequency and the driving frequency in general leads to either an almost periodic motion or a periodic one, depending on the parameter range.[1] The latter, called mode locking, is characterized by the quantization at rational values of the average phase velocity. In particular subharmonic mode locking, appearing in the presence of the inertia term, results in the devil's staircase structure. One of the well-known examples is the Josephson junction driven by combined direct and alternating currents, with the capacitance playing the role of inertia.[2] Governed by the same equation of motion as a driven pendulum, it displays dc voltage plateaus in the current-voltage characteristics, known as Shapiro steps.[3] Similar voltage quantization has also been observed in arrays of Josephson junctions, yielding integer giant Shapiro steps[4,5] and subharmonic steps[6] according to the absence/presence of capacitive terms. Unfortunately, in spite of the deceptively simple equation of motion, even the *single*-junction problem has resisted complete analytical solutions, especially in the presence of the capacitive term, except for the results mainly based on the circle map[7] and on the approximate analysis by means of expansion and averaging.[8,9] Accordingly, such mode-locking phenomena in *arrays* have been demonstrated mostly by numerical simulations. On the other hand, the topological argument, proposed for the system without the capacitive term,[10] reveals topological invariance of the system as the physical origin of quantization.[11] As in the case of the quantum Hall effect,[12] the topological argument does not provide quantitative information, e.g., on the locking structure. Nevertheless it not only clarifies the nature of quantization but also provides a link between dynamics and statics by interpreting (dynamical) mode locking in terms of (static) topological invariance.

In this work, we construct a topological argument for the system with inertia and apply the idea to Josephson-junction arrays or systems of coupled oscillators, with attention to the resulting subharmonic locking. For this purpose, we consider an appropriate canonical transformation of the dynamic equations of motion and the corresponding Fokker-Planck equation, the stationary solution of which gives the effective Hamiltonian in the form of a classical *XY* model with periodic bond angles. Via mapping onto a tight-binding particle in a periodic gauge field, subharmonic quantization of the average (dc) phase velocity is revealed as a manifestation of the topological invariance. Also suggested is ubiquity of multistability, providing a natural explanation of the observed hysteresis due to the inertia.

We begin with the set of equations of motion for a system of N coupled oscillators

$$\sum_{j=1}^{N} [M_{ij}\ddot{\phi}_j+\gamma M_{ij}\dot{\phi}_j+J_{ij}\sin(\phi_i-\phi_j)]=I_i, \tag{1}$$

where ϕ_i represents the phase of the ith oscillator, M_{ij} the (rotational) inertia matrix, γ the damping parameter, and J_{ij} measures the coupling strength between the oscillators i and j. The right-hand side describes the periodic driving with frequency Ω: $I_i \equiv I_{i,d}+I_{i,a}\cos\Omega t$, where $I_{i,d}$ and $I_{i,a}$ are the amplitudes of the dc and ac components, respectively, of the driving on the ith oscillator. There are two cases depending on the detailed form of the inertia matrix. The simple case that $M_{ij}=M\delta_{ij}$ describes the system of coupled oscillators, each of which possesses inertia M and suffers from dissipation of strength γM under driving I_i. On the other hand, with ϕ_i denoting the phase of the superconducting order parameter at site i, Eq. (1) describes the dynamics of the array of resistively and capacitively shunted junctions (RCSJs), where the combined direct and alternating current I_i is fed into the grain at site i.[6,13] In this case, M_{ij} corresponds to the capacitance matrix and assumes the form $M_{ij}=C\Delta_{ij}$ with the junction capacitance C and the lattice Laplacian $\Delta_{ij}\equiv z\delta_{ij}-\delta_{ij'}$, where j' represents the neighboring sites of j and z represents the number of such neighbors. The damping parameter γ is inversely proportional to the shunt resistance of the junction. For simplicity, we henceforth concentrate on the case $M_{ij}=M\delta_{ij}$ since the generalization to the case of $M_{ij}=C\Delta_{ij}$ is straightforward.

Equation (1) may be written in the form of Hamilton's canonical equations: $\dot{\phi}_i=\partial H/\partial p_i$ and $\dot{p}_i=-\partial H/\partial\phi_i$ with the Caldirola-Kanai Hamiltonian[14,15]

$$H_{CK}=\frac{1}{2M}\sum_i (p_i+Q_i)^2e^{-\gamma t}-\sum_{i<j} J_{ij}\cos(\phi_i-\phi_j)e^{\gamma t}, \tag{2}$$

where (ϕ_i, p_i) are conjugate variables and Q_i is the ''gauge charge'' given by $\dot{Q}_i e^{-\gamma t}=I_i$ or

$$Q_i=\frac{I_{i,d}}{\gamma}e^{\gamma t}+\frac{I_{i,a}}{\gamma^2+\Omega^2}e^{\gamma t}(\gamma\cos\Omega t+\Omega\sin\Omega t)+Q_i^0 \tag{3}$$

with arbitrary constant Q_i^0. In this sense the classical mechanical system with dissipation, governed by the equations of motion (1), can be described by the Hamiltonian in Eq. (2).[16] We then introduce new variables $(\theta_i, \tilde{p}_i)$ according to $p_i=\partial\Phi/\partial\phi_i$ and $\theta_i=\partial\Phi/\partial\tilde{p}_i$ with the generating function $\Phi(\{\tilde{p}_i\},\{\phi_i\})\equiv\Sigma_i\tilde{p}_i(\phi_i+a_i)$, where $M\dot{a}_i\equiv-e^{-\gamma t}Q_i$. Under this canonical transformation, Eq. (2) yields, apart from a constant term, the new Hamiltonian

$$\tilde{H}_{CK}\equiv H_{CK}+\frac{\partial\Phi}{\partial t}$$

$$=\frac{1}{2M}\sum_i \tilde{p}_i^2e^{-\gamma t}-\sum_{i<j} J_{ij}e^{\gamma t}\cos(\theta_i-\theta_j-a_{ij}), \tag{4}$$

where $\tilde{p}_i\,(=p_i)$ is conjugate to θ_i. In view of the RCSJ array, where a uniform current is usually fed into the sites along one edge and extracted from those along the opposite edge, we consider the case that some oscillators are driven by $I=I_d+I_a\cos\Omega t$ and some others by $-I$. Then the bond angle $a_{ij}\equiv a_i-a_j$, depending on (i,j), either vanishes or becomes

$$a_{ij}=\mp\frac{1}{M}\left[\frac{I_d}{\gamma}t+\frac{I_a}{\gamma^2+\Omega^2}\left(\frac{\gamma}{\Omega}\sin\Omega t-\cos\Omega t\right)\right] \tag{5}$$

apart from an arbitrary constant.

Note that the energy of the system is given by $e^{-\gamma t}\tilde{H}_{CK}$;[15] this also corresponds to the effective Hamiltonian describing the statistical mechanics of the system. To see this, we for the moment consider the system at finite temperatures and generalize the equations of motion (1) appropriately,

$$\sum_{j=1}^{N}[M_{ij}\ddot{\phi}_j+\gamma M_{ij}\dot{\phi}_j+J_{ij}\sin(\phi_i-\phi_j)]=I_i+\eta_i, \tag{6}$$

where η_i is the random (thermal) noise acting on the ith oscillator. In the system of oscillators with $M_{ij}=M\delta_{ij}$, the noise is characterized by the zero mean and the correlations

$$\langle\eta_i(t)\,\eta_j(t')\rangle=2\gamma Mk_BT\delta_{ij}\delta(t-t')$$

at temperature T. In the RCSJ array with $M_{ij}=C\Delta_{ij}$, η_i is given by the sum of the noise currents from neighboring sites, $\eta_i=\Sigma_j\eta_{ij}\delta_{ij'}$ with

$$\langle\eta_{ij}(t)\,\eta_{kl}(t')\rangle=2\gamma Ck_BT(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\delta(t-t').$$

Motivated by the canonical transformation in the absence of noise, we write Eq. (6) in the form

$$\dot{\theta}_i=M^{-1}p_ie^{-\gamma t},$$

$$\dot{p}_i=-\sum_j J_{ij}e^{\gamma t}\sin(\theta_i-\theta_j-a_{ij})+\eta_ie^{\gamma t}, \tag{7}$$

where $\theta_i\equiv\phi_i+a_i$. The set of Langevin equations (7) may be transformed into the Fokker-Planck equation[17]

$$\frac{\partial P}{\partial t}=-\sum_i\frac{p_i}{M}e^{-\gamma t}\frac{\partial P}{\partial\theta_i}+\sum_{ij} J_{ij}e^{\gamma t}\sin(\theta_i-\theta_j-a_{ij})\frac{\partial P}{\partial p_i}$$

$$+\gamma Mk_BT\sum_i e^{2\gamma t}\frac{\partial^2P}{\partial p_i^2}, \tag{8}$$

which describes the time evolution of the probability distribution $P(\{\theta_i\},\{p_i\},t)$ of phases and momenta at time t. Equation (8) yields the stationary solution valid in the limit $t\rightarrow\infty$,

$$P(\{\theta_i\},\{p_i\})\propto e^{-H_{eff}/k_BT},$$

where the *effective* Hamiltonian is given by

$$H_{eff}=\frac{1}{2\tilde{M}}\sum_i p_i^2-\sum_{i<j} J_{ij}\cos(\theta_i-\theta_j-a_{ij}) \tag{9}$$

with $\tilde{M}\equiv Me^{2\gamma t}$. It is thus concluded that the stationary distribution has the form of a Gibbs measure, with the effective Hamiltonian indeed corresponding to the energy of the system ($H_{eff}=e^{-\gamma t}\tilde{H}_{CK}$).

The first term in Eq. (9) becomes vanishingly small in the stationary state ($t\rightarrow\infty$); it is further obvious that the kinetic energy in the above classical system decouples from the interaction energy. We thus obtain the classical *XY* Hamiltonian[18]

$$H_{XY}=-J\sum_{\langle i,j\rangle}\cos(\theta_i-\theta_j-a_{ij}), \tag{10}$$

where the nearest-neighbor coupling ($J_{ij}=J\delta_{ij'}$) has been assumed for convenience. At zero temperature, which is our concern, the system described by the Hamiltonian (10) is equivalent to a tight-binding particle (of charge e), with $2k_BT/J$ taking the role of the energy eigenvalue[19]. The Hamiltonian describing such a tight-binding system has the position representation

$$\langle i|H|j\rangle=e^{-ia_{ij}}\delta_{ij'}, \tag{11}$$

where $|i\rangle$ is the position eigenket and a_{ij} may be viewed as the line integral of the appropriate gauge potential $\mathbf{a}$: $a_{ij}=(e/\hbar c)\int_i^j\mathbf{a}\cdot d\mathbf{l}$. Equation (5) shows that a_{ij}, defined modulo 2π, is periodic in time with period $\tau=2\pi m/\Omega$ only if

$$I_d=\frac{s}{m}\gamma M\Omega \tag{12}$$

where m and s are integers.

When Eq. (12) is satisfied, the Hamiltonian (11) as well as the gauge field a_{ij} has the periodicity τ and the Floquet theorem is applicable to the corresponding Schrödinger equation for the wave function $\Psi_i \equiv \langle i|\Psi\rangle$, giving the relation[20]

$$\Psi_i(t+\tau) = e^{-i\tilde{E}\tau}\Psi_i(t) \tag{13}$$

with the quasienergy $\tilde{E}$. This imposes that, apart from the dynamical contribution $\tilde{E}\tau$, the corresponding change in the phase of the wave function Ψ_i should be an integer multiple of 2π, $\theta_i(t+\tau)-\theta_i(t) = 2n_i\pi - \tilde{E}\tau$, where n_i is an integer depending on the form of a_{ij}, i.e., of the driving I_i and $\tilde{E}$ has been assumed to be real (see below). We thus have the average change rate of the phase

$$\langle \dot{\theta}_i \rangle \equiv \frac{1}{\tau}\int_0^\tau dt\ \dot{\theta}_i = \frac{n_i}{m}\Omega - \tilde{E} \tag{14}$$

or in terms of the original phase ϕ_i,

$$\langle \dot{\phi}_i \rangle = \langle \dot{\theta}_i \rangle + \frac{I_{i,d}}{\gamma M} = \frac{\tilde{n}_i}{m}\Omega - \tilde{E}, \tag{15}$$

where $\tilde{n}_i = n_i \pm s$ or n_i for driven ($I_{i,d} = \pm I_d$) or undriven ($I_{i,d}=0$) oscillators, respectively. Accordingly, the average change rate of the phase difference or the average (dc) phase velocity, which usually gives the appropriate physical quantity, e.g., the voltage in the case of an RCSJ array, indeed displays subharmonic mode locking,

$$\langle V_{ij} \rangle \equiv \langle \dot{\phi}_i \rangle - \langle \dot{\phi}_j \rangle = \frac{n}{m}\Omega \tag{16}$$

with $n \equiv \tilde{n}_i - \tilde{n}_j$. Note that for a given configuration of driving, the integer n in Eq. (16) is determined by the winding number n_i, manifesting the topological nature of the mode locking.

The subharmonic mode locking given by Eq. (16) can persist even for the (dc) driving slightly off the condition in Eq. (12). To see this, we take

$$\frac{I_d}{\gamma M \Omega} = \frac{s}{m} + \epsilon \tag{17}$$

for small ϵ, which leads to the bond angle either zero or $a_{ij} = a_{ij}^0 \mp \epsilon\Omega t$ with a_{ij}^0 representing the periodic part for $\epsilon = 0$. In this case the Hamiltonian of the system as well as a_{ij} is in general not periodic. However, in terms of the shifted phase $\chi_i \equiv \theta_i + \epsilon_i \Omega t$, where $\epsilon_i = \pm\epsilon$ or 0 for driven or undriven oscillators, the Hamiltonian in Eq. (10) reads

$$H_{XY} = -J\sum_{\langle i,j\rangle} \cos(\chi_i - \chi_j - a_{ij}^0), \tag{18}$$

where periodicity has been restored. Accordingly, the argument leading to Eq. (14) is applicable, giving $\langle \dot{\chi}_i \rangle = (n_i/m)\Omega - \tilde{E}$. This in turn leads to

$$\langle \dot{\phi}_i \rangle = \langle \dot{\theta}_i \rangle + \frac{I_{i,d}}{\gamma M} = \langle \dot{\chi}_i \rangle - \epsilon_i\Omega + \frac{I_{i,d}}{\gamma M} = \frac{\tilde{n}_i}{m}\Omega - \tilde{E}, \tag{19}$$

which is precisely Eq. (15). It is thus obvious that Eq. (16) remains unchanged: Given subharmonic quantization can persist in the finite interval of the dc driving strength $\delta I_d \propto \epsilon$, thus generating the step structure in the appropriate response characteristics. Since rational numbers form a dense set, this also indicates that there can exist multiple quantization states with different values of n/m in Eq. (16) for given driving strength I_d. However, some of those, corresponding to complex values of the quasienergy $\tilde{E}$ with nonzero imaginary parts are unstable,[8,20] and only stable states (with real values of $\tilde{E}$) among those determine the step structure in the actual response characteristics. Although such information as stability cannot be obtained by the topological argument, it is certainly plausible to have two or more states stable in some intervals of the driving strength. Such multistability in general gives rise to hysteresis behavior in the response characteristics, which has widely been observed in the oscillator systems with inertia terms.[6,9,17,21] Note also that the topological argument is in essence a zero-temperature analysis like in Refs. 10,12: At finite temperatures thermal fluctuations make the mapping onto Eq. (11) and the following argument inexact. Nevertheless, at sufficiently low temperatures, the phase slippage induced by fluctuations should have an exponentially low rate, hardly affecting the quantization itself determined by the winding number. On the other hand, it is expected that fluctuations tend to destabilize various quantization states, reducing the stability interval and the emergence of multistability. This is consistent with smoothing out of the step structure[4] and suppression of hysteresis,[22] observed in the presence of noise.

In summary, we have constructed a topological argument for subharmonic mode locking in a driven system of coupled oscillators with inertia. Starting from the dynamic equations of motion, we have derived the effective Hamiltonian for the system in the form of a classical XY model with periodic bond angles, which in turn has been mapped onto a tight-binding particle in a periodic gauge field. It has then been shown that subharmonic quantization of the average (dc) phase velocity follows as a manifestation of topological invariance, revealing the topological nature of the dynamical mode locking. Also revealed is the possibility of multistability, providing a natural explanation of the ubiquity of hysteresis due to the inertia. In view of the fact that the set of equations of motion (1) describes a prototype of oscillatory system, we believe the result of this paper to be rather general and applicable to a variety of oscillatory systems displaying mode locking phenomena.

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[1] M.H. Jensen, P. Bak, and T. Bohr, Phys. Rev. Lett. **50**, 1637 (1983); Phys. Rev. A **30**, 1960 (1984).

[2] For a list of references, see, e.g., *Proceedings of the 2nd CTP Workshop on Statistical Physics: KT Transition and Superconducting Arrays*, edited by D. Kim, J.S. Chung, and M.Y. Choi (Min-Eum Sa, Seoul, 1993); *Macroscopic Quantum Phenomena and Coherence in Superconducting Networks*, edited by C. Giovannella and M. Tinkham (World Scientific, Singapore, 1995); *Proceedings of the ICTP Workshop on Josephson Junction Arrays*, edited by H.A. Cerdeira and S.R. Shenoy [Physica B **222**, 253 (1996)].

[3] S. Shapiro, Phys. Rev. Lett. **11**, 83 (1963); For the subharmonic steps in the junction with finite capacitance, see V.N. Belykh, N.F. Pedersen, and O.H. Sorensen, Phys. Rev. B **16**, 4860 (1977).

[4] S.P. Benz, M. Rzchowski, M. Tinkham, and C.J. Lobb, Phys. Rev. Lett. **64**, 693 (1990); K.H. Lee, D. Stroud, and J.S. Chung, *ibid.* **64**, 962 (1990).

[5] In an applied magnetic field, corresponding to k/q flux quanta per plaquette (k and q being relatively prime integers), the system in Ref. 4 also displays fractional steps with the (fixed) denominator q. Note the difference from the subharmonic steps, the denominators of which are arbitrary integers. To avoid possible confusion, we confine our investigation here to the system in the absence of a magnetic field.

[6] P.H.E. Tiesinga, T.J. Hagenaars, J.E. van Himbergen, and J.V. José, J. Phys.: Condens. Matter **9**, 1813 (1997); M. Lee and M.Y. Choi (unpublished).

[7] V.I. Arnol'd, Izv. Akad. Ser. Mat. **25**, 21 (1961) [Am Math. Soc. **46**, 213 (1965)]; Usp. Mat. Nauk **38**, 189 (1983) [Russ. Math. Surveys **38**, 189 (1983)].

[8] N. Minorsky, *Nonlinear Oscillations* (Van Nostrand, Princeton, 1962); C. Hayashi, *Nonlinear Oscillations in Physical Systems* (McGraw-Hill, New York, 1964).

[9] See also H. Hong, M.Y. Choi, J. Yi, and K.-S. Soh, Phys. Rev. E **59**, 353 (1999).

[10] M.Y. Choi, Phys. Rev. B **46**, 564 (1992); **50**, 13 875 (1994).

[11] D.J. Thouless, *Topological Quantum Numbers in Nonrelativistic Physics* (World Scientific, Singapore, 1998).

[12] D.J. Thouless, M. Kohmoto, M.P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982); Q. Niu, D.J. Thouless, and Y.S. Wu, Phys. Rev. B **31**, 3372 (1985).

[13] M.-S. Choi, M.Y. Choi, and S.-I. Lee, Europhys. Lett. **43**, 439 (1998).

[14] H. Bateman, Phys. Rev. **38**, 815 (1931); P. Caldirola, Nuovo Cimento **18**, 393 (1941); E. Kanai, Prog. Theor. Phys. **3**, 440 (1948).

[15] H.H. Denman and L.H. Buch, J. Math. Phys. **14**, 326 (1973); See also M.Y. Choi, K.-C. Lee, and D.I. Choi, Phys. Lett. **89A**, 1 (1983); D.H. Kobe, G. Reali, and S. Sienlutyez, Am. J. Phys. **54**, 997 (1986).

[16] There exists subtlety in the quantization of Eq. (2), giving rise to controversy as to the quantum-mechanical formulation. See, e.g., J.M. Cervero and J. Villarroel, J. Phys. A **17**, 2963 (1984); C.I. Um, K.H. Yeon, and W.H. Kahng, *ibid.* **20**, 611 (1987); S. Srivastava, Vishwamittar, and I.S. Minhas, J. Math. Phys. **32**, 1510 (1991). Here our system is classical and no such complication arises.

[17] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer-Verlag, Berlin, 1989).

[18] Alternatively, one may regard Eq. (9) as a quantum Hamiltonian, which at zero temperature maps onto the effective action of the form in Eq. (10) with the coupling given by $\sqrt{\widetilde{M}J}$. Since the stationary state corresponds to the strong coupling limit ($\sqrt{\widetilde{M}J} \to \infty$), there do not exist quantum fluctuations and the classical limit is automatically achieved. Note also that the extra (imaginary-time) dimension plays a trivial role in this limit.

[19] W.Y. Shih and D. Stroud, Phys. Rev. B **28**, 6575 (1983); M.Y. Choi, in *Progress in Statistical Mechanics*, edited by C.-K. Hu (World Scientific, Singapore, 1988), p. 385 and references therein.

[20] See, e.g., E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1952); C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).

[21] H. Tanaka, A.J. Lichtenberg, and S. Oishi, Phys. Rev. Lett. **78**, 2104 (1997).

[22] H. Hong, M.Y. Choi, B.-G. Yoon, K. Park, and K.-S. Soh, J. Phys. A **32**, L9 (1999).