Mesoscopic scattering in the half plane: Squeezing conductance through a small hole

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We model the two-probe conductance of a quantum-point contact (QPC), in linear response. If the QPC is highly nonadiabatic or near to scatterers in the open-reservoir regions, then the usual distinction between leads and reservoirs breaks down and a technique based on scattering theory in the full two-dimensional half plane is more appropriate. Therefore we relate conductance to the transmission *cross section* for incident plane waves. This is equivalent to Landauer's formula using a radial partial-wave basis. We derive the result that an arbitrarily small (tunneling) QPC can reach a *p*-wave channel conductance of $2e^2/h$ when coupled to a suitable reflector. If two or more resonances coincide the total conductance can even exceed this. This relates to recent mesoscopic experiments in open geometries. We also discuss reciprocity of conductance, and the possibility of its breakdown in a proposed QPC for atom waves.

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I. INTRODUCTION

The quantum-point contact[1,2] (QPC) has played a central role in the understanding of mesoscopic conductance. It is the simplest example of a two-dimensional electron gas 2DEG system where the quantum coherent nature of the electron controls the bulk transport properties. The Landauer-Büttiker (LB) formalism[3,4,2] reduces the calculation of quantum conductance in the linear response regime to the evaluation of single-particle wave-function transmission amplitudes. Traditionally, these amplitudes are measured between traveling wave basis states in the ''leads.'' Far from the scattering system the leads have constant profiles of finite width, and support a finite number of transverse modes (channels). Eventually it is assumed that the leads are impedence matched (that is, without reflection) into ''reservoirs'' that act as thermalized sources of electrons at their respective potentials; these potentials are taken to reflect the measured bias voltage. Such theoretical constructs have been remarkably successful at describing transport phenomena, for instance conductance quantization,[5,1,2] because the scattering systems involved have generally had good lead-to-reservoir matching.

We consider ''open'' two-terminal mesoscopic systems, namely, those where a QPC is *nonadiabatic* (possessing rapid longitudinal variation in transverse profile[6,1,7]) and has short or nonexistent leads (for instance, if it suddenly abutts onto the ''reservoir'' regions), or those where there can be scattering off nearby objects in the ''reservoir'' region. We call such systems ''open'' because the fully two-dimensional (2D) nature of the ''reservoirs'' (i.e., the surrounding semi-infinite regions of free space) is important, and therefore they cannot be modeled using the quasi-1D approach described above. This includes a variety of recent mesoscopic experiments, for example, the combination of QPC's with nearby resonator structures[8] or with a nearby depletion region caused by a moveable atomic force microscope (AFM) tip.[9] It also includes any QPC system where elastic backscattering from disorder in the reservoirs is significant,[10] or generally where the lead-reservoir matching is bad. In such systems, the conventional quasi-1D picture does not apply: the scattering system is not coupled to leads in the usual sense, indeed the distinction between leads and reservoirs is no longer clear.[11] The main aim of the present work is to introduce a 2D scattering theory approach that can handle such systems, and to apply it to the calculation of the maximum conductance of an open resonator structure of experimental relevance.

We imagine a geometry where a 2DEG exists in two semi-infinite half-plane regions, separated by an impenetrable potential barrier that we align with the y axis [see Fig. 1(a)]. Our general ''QPC scattering system'' is any gap in this barrier that allows coupling of the wave function on the left and right sides. This gap can be defined by an arbitrary form of the elastic potential, and may include other nearby scattering objects or disorder [which would all be placed

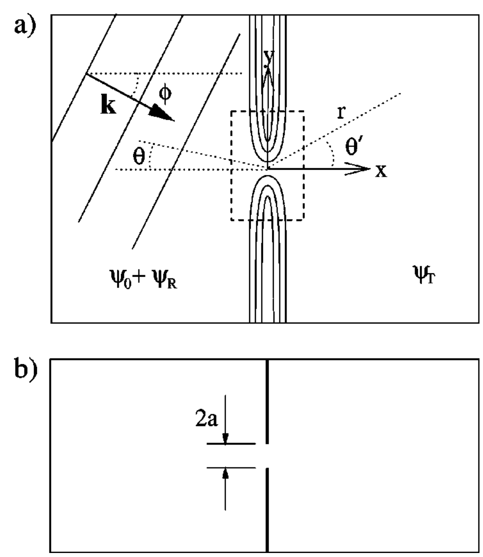

FIG. 1. Schematic QPC geometry in 2D: (a) general point-contact scatterer coupling two semi-infinite regions of free space. The solid curves are contours of an elastic scattering potential $V(\mathbf{r})$. The ''system'' size L (dashed box) we take to be the region where $V(\mathbf{r})$ has not yet reached its asympotic form (which is zero apart from a y-invariant profile around the y axis). Also shown are an incoming plane wave, and the coordinate system. (b) The idealized ''slit'' aperture in a thin, hard wall considered in Sec. IV.

within the box shown in Fig. 1(a)]. The only important limitation is that this coupling region (the ''system'') be of finite y extent, so that electrons that leave the system do so via a well-defined terminal: either the left $(x<0)$ or the right $(x>0)$. We also assume that the system size L is much smaller than both the dephasing length l_ϕ and the momentum relaxation (elastic scattering) length l_e. The former requirement allows treatment using a coherent wave function across the system; the latter allows free-space elastic scattering concepts to be applied. We will stay within the noninteracting quasiparticle picture, consider zero-applied magnetic field, and assume spin degeneracy of 2 throughout.

The conventional distinction between ''reservoir'' and ''lead'' is no longer applicable, however at short distances outside the system ($r>L$ but $r \ll l_\phi$ and $r \ll l_e$) the two semi-infinite free-space regions behave like leads, since they support scattering-free ''channels'' (see Sec. III). At large distances the *same* regions behave as reservoirs: for $r \gg l_e$ ergodicity ensures that the momentum distribution is uniform in angle, and for $r \gg l_\phi$ the energy is redistributed to ensure equilibrium at the relevant (experimentally measured) chemical potential of each terminal. In the intermediate region, there is a broad crossover from lead to reservoir.

In this work we first derive a general relation between transmission cross section (a concept we define using scattering in the half plane) and conductance for this open geometry, in Sec. II. In Sec. III we show that partial-wave-type states, defined in the half-plane regions, can take the place of transverse lead modes in the Landauer formula. In Sec. IV we discuss the maximum conductance through an idealized, highly nonadiabatic QPC (a hole in a thin hard wall) that is reached when a resonator is placed on one side of the QPC. We find a universal result, namely, a single conductance quantum, *regardless* how small the hole is. This illuminates the findings of a recent experiment[8] in such an open geometry. In Sec. V we discuss attempts to exceed this universal quantum of conductance through a single channel. A reciprocity relation for cross section is derived in Sec. VI, and the possibility of breaking this reciprocity, due to a nonthermal reservoir occupation, is described. We discuss an application to matter-wave ''conductance'' through a 3D QPC. We conclude in Sec. VII.

II. CONDUCTANCE IN TERMS OF CROSS SECTION

We consider scattering of a single-quasiparticle wave function from the general two-terminal system described in the Introduction [see Fig. 1(a)]. The Hamiltonian is $\mathcal{H} = -(\hbar^2/2m)\nabla^2 + V(\mathbf{r})$, for a quasiparticle mass m. The elastic scattering potential $V(\mathbf{r})$ completely defines the system. We imagine a monochromatic unit plane wave $\psi_\mathrm{I} = e^{i\mathbf{k}\cdot\mathbf{r}}$ incident from the free-space left-hand region.[12] The wave vector is $\mathbf{k} \equiv (k,\phi)$ in polar coordinates, ϕ being the angle of incidence. The free-space wave-vector magnitude is taken as $k = k_\mathrm{F}$ (corresponding to a total energy $E = \hbar^2 k^2/2m$ equal to the Fermi energy), unless stated otherwise.

We are at liberty to choose our definition of the ''unscattered'' wave ψ_0. We take it to be the wave function which would result from reflection of the incident wave off a wall uniform in the y direction. We can imagine creating such a wall by replacing the system box shown in Fig. 1(a) by the surrounding y-invariant wall profile. Note that ψ_0 exists only on the left side. In the left free-space region it is

$$\psi_0 = e^{i(k_x x + k_y y)} - e^{\{i(-k_x x + k_y y + \gamma_\mathbf{k})\}}, \tag{1}$$

where the first term is ψ_I, and the angle-dependent reflection phase $\gamma_\mathbf{k}$ of the second term depends on both (k,ϕ) and the wall profile.[13] Upon introduction of our true system potential, the full wave function becomes

$$\psi \equiv \psi_0 + \psi_\mathrm{R} + \psi_\mathrm{T}, \tag{2}$$

where the change in reflected wave ψ_R exists only on the left side, and the new transmitted wave ψ_T exists only on the right. These scattered waves have the asymptotic ($r>L$ and $kr \gg 1$) forms of 2D scattering theory,[14]

$$\psi_\mathrm{R} = f_\mathrm{R}(\theta)\frac{e^{ikr}}{\sqrt{r}}, \quad \psi_\mathrm{T} = f_\mathrm{T}(\theta')\frac{e^{ikr}}{\sqrt{r}}. \tag{3}$$

See Fig. 1(a) for definitions of θ and θ'.

The transmission cross section $\sigma_\mathrm{T}(k,\phi)$ is the ratio of Γ_T, the transmitted particle flux (number per unit time), to j_I, the incident particle flux per unit length normal to the incident beam:

$$\sigma_\mathrm{T}(k,\phi) \equiv \frac{\Gamma_\mathrm{T}}{j_\mathrm{I}}. \tag{4}$$

Physically, $\sigma_\mathrm{T}(k,\phi)$ is the length required of an aperture-oriented normal to the incident beam in order to transmit an equivalent flux of classical particles. [Note that $\sigma_\mathrm{T}(k,\phi)$ is proportional to the *injection distribution*[1] that can be measured in mesoscopic systems.[15]] It depends on the incident angle because $V(\mathbf{r})$ has no radial symmetry. j_I is the magnitude of the incoming probability flux density vector $\mathbf{j} \equiv (\hbar/m)\mathrm{Im}[\psi_\mathrm{I}^* \boldsymbol{\nabla} \psi_\mathrm{I}]$, which for a unit wave gives $j_\mathrm{I} = v$, the particle speed. The transmitted flux is defined as

$$\Gamma_\mathrm{T} \equiv \int dl\ \hat{\mathbf{n}}\cdot\mathbf{j} = \frac{\hbar}{m}\int dl\ \hat{\mathbf{n}}\cdot \mathrm{Im}[\psi_\mathrm{T}^* \boldsymbol{\nabla} \psi_\mathrm{T}], \tag{5}$$

where the line integral encloses the entire transmitted wave, and the (rightwards pointing) surface normal is $\hat{\mathbf{n}}$. Applying this and Eq. (4) to the asymptotic form gives

$$\sigma_\mathrm{T}(k,\phi) = \int_{-\pi/2}^{\pi/2} d\theta' |f_\mathrm{T}(\theta')|^2, \tag{6}$$

familiar from scattering theory apart from the restriction to the right half plane. There is a corresponding form

$$\sigma_\mathrm{R}(k,\phi) = \int_{-\pi/2}^{\pi/2} d\theta |f_\mathrm{R}(\theta)|^2, \tag{7}$$

for the reflective cross section (removal from the unscattered wave without being transmitted).

We will calculate the conductance by assuming the chemical potential is slightly higher on the left side than the right, and as is usual[1,4] consider only the left-to-right transport of the states in this narrow energy range. We take the left region to be a large ($\gg l_\phi$) closed region of area A containing single-particle states, and find their decay rate through the QPC into the right side. Semiclassically each single-particle state occupies a phase-space volume h^d, where we have $d=2$. Therefore the phase-space density in the 2DEG Fermi sea is $2/h^2$ where the factor of 2 comes from the spin degeneracy. We can project this density onto momentum space in order to find the effective number of plane-wave states impinging on the wall:[16] this corresponds to a uniform density of states in **k** space given by

$$\rho(k,\phi)k\ dk\,d\phi=\frac{A}{2\pi^2}k\ dk\,d\phi. \tag{8}$$

Each state has an amplitude $A^{-1/2}$ due to the requirement of unity area normalization in the left region, so has incoming flux density $j_{\mathrm{I}}=v/A$. Substituting this into Eq. (4) gives the decay rate of a state i as

$$\Gamma_{\mathrm{T}}^{(i)}=\frac{v}{A}\sigma_{\mathrm{T}}(k_i,\phi_i). \tag{9}$$

We can now sum the decay rates of all the left-hand states in a given wave-vector range k_{F} to $k_{\mathrm{F}}+\delta k$, to get the current

$$\delta I=e\sum_i \Gamma_{\mathrm{T}}^{(i)}=\frac{ev}{A}\int_{-\pi/2}^{\pi/2}d\phi\int_{k_{\mathrm{F}}}^{k_{\mathrm{F}}+\delta k}k\ dk\,\rho(k,\phi)\sigma_{\mathrm{T}}(k,\phi)$$

$$=\frac{ev\ k_{\mathrm{F}}\delta k}{2\pi^2}\int_{-\pi/2}^{\pi/2}d\phi\ \sigma_{\mathrm{T}}(k_{\mathrm{F}},\phi), \tag{10}$$

where the last step incorporated the linear-response assumption that σ_{T} is constant over the range δk.

When a potential difference δV is applied across the QPC, the energy range carrying current is $\delta E=e\,\delta V$, which we can equate with $\hbar v\,\delta k$ using the dispersion relation. This can be used with Eq. (10) to write the conductance

$$G\equiv\frac{\delta I}{\delta V}=\frac{2e^2}{h}\frac{1}{\lambda_{\mathrm{F}}}\int_{-\pi/2}^{\pi/2}d\phi\ \sigma_{\mathrm{T}}(k_{\mathrm{F}},\phi) \tag{11a}$$

$$=\frac{2e^2}{h}\frac{k_{\mathrm{F}}}{2}\langle\sigma_{\mathrm{T}}\rangle_\phi, \tag{11b}$$

where the particle wavelength is $\lambda_{\mathrm{F}}\equiv 2\pi/k_{\mathrm{F}}$. The latter form is written in terms of the angle-averaged cross section at the Fermi energy. The weighting of this average is uniform because of the ergodic assumption that incoming states are uniformly distributed in angle.

Equation (11) is a key result of this paper (an independent derivation is given by Barnett[17]). Like the Landauer formula, it directly connects conductance and scattering. In a scattering measurement from the left side, σ_{T} appears to be the QPC's inelastic cross section (since the transmitted waves never return to this side). In a current measurement the corresponding conductance is given by Eq. (11). Our derivation was for temperature $T=0$, but it applies at a finite T as long as σ_{T} does not change significantly over the energy range $k_{\mathrm{B}}T$. This can be seen by generalizing the above to include integration over the Fermi distribution.

In the limit where a QPC is adiabatic, its conductance is known to be quantized:[5,1,2] $G=(2e^2/h)N$, where N is the integer number of open channels at the Fermi energy. Looking at Eq. (11a), this corresponds to quantization of the angular integral of the cross section in units of λ_{F}.

III. PARTIAL-WAVE CHANNEL MODES FOR A TWO-TERMINAL SYSTEM

In free-space scattering theory, partial waves form a basis in which to decompose the asymptotic ($r\to\infty$) form of the full wave function ψ into incoming and outgoing states of definite angular momentum l. In 2D the basis functions are the cylindrical solutions to the free-space wave equation; the S matrix that takes incoming to outgoing waves can then be written in this basis.[14] Because there is only a single set of incoming channels and a single set of outgoing channels, this is equivalent to a scattering system (a stub) connected to a single lead, with an infinite number of open-channel modes. This contrasts the open two-terminal geometry we study, where we need to account for two new related facts: (1) in the $r\to\infty$ limit the potential V no longer preserves angular momentum, and (2) there are now distinct ways the particle can enter and exit the system, via different leads.

We define a ''half-plane partial-wave basis'' as the subset of the cylindrical free-space solutions that go to zero on the entire y axis. This gives independent basis functions existing on either the left or right side of the y axis. The basis is expressed in terms of Hankel functions[18] on either side

$$\phi_l^{-L}(kr)\equiv H_l^{(2)}(kr)\sin\left[l\left(\frac{\pi}{2}-\theta\right)\right]$$

$$\phi_l^{+L}(kr)\equiv H_l^{(1)}(kr)\sin\left[l\left(\frac{\pi}{2}-\theta\right)\right]$$

$$\phi_l^{-R}(kr)\equiv H_l^{(2)}(kr)\sin\left[l\left(\frac{\pi}{2}-\theta'\right)\right]$$

$$\phi_l^{+R}(kr)\equiv H_l^{(1)}(kr)\sin\left[l\left(\frac{\pi}{2}-\theta'\right)\right], \tag{12}$$

where on the left (L) side θ is the angle from the negative x axis and on the right (R) side θ' is the angle from the positive x axis [see Fig. 1(a)]. The channel index is $l=1,2,\cdots\infty$, and $+(-)$ refers to outgoing (incoming) traveling waves. We note that the s wave $l=0$ is excluded because of the y-axis barrier, leaving the first channel as the p wave $H_1(kr)\cos(\theta)$. Assuming the width of the barrier is finite and constant as $|y|\to\infty$ [see Fig. 1(a)], then any wave function in the $r\to\infty$ limit can be written as a sum of the above basis functions. The separability of this basis in (r,θ) is directly analogous to the separability of conventional (constant-width) lead basis states[3] into a product of transverse modes and longitudinal traveling waves.

Our basis (12) is chosen such that unit-amplitude coefficients carry equal fluxes in all incoming and outgoing chan-

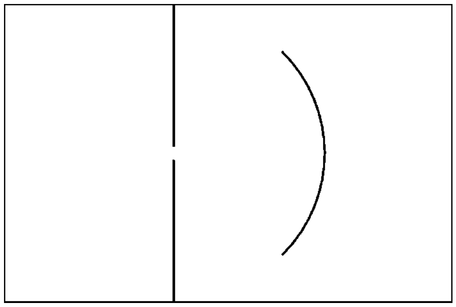

FIG. 2. A tunneling-regime QPC combined with a nearby circular reflector, forming a stable resonant cavity open at the sides.

nels, so flux conservation implies the unitarity of the S matrix when written in this basis. As with a conventional transverse lead mode basis, the familiar Landauer formula

$$G=\frac{2e^2}{h}\mathrm{Tr}(t^\dagger t),\tag{13}$$

holds.[19,20,4,17] The transmission matrix t is defined by $q_l^+ =\Sigma_m t_{lm}p_m^-$, where the outgoing (incoming) amplitude coefficients are p_l^+ (p_l^-) on the left and q_l^+ (q_l^-) on the right. Note that it is possible to ''mix and match'' different basis set types (for instance define a transmission matrix between transverse lead modes on the left side and partial-wave modes on the right), as long as equal-flux normalization, and transverse orthogonality, are preserved.

IV. POINT CONTACT COUPLED TO A RESONATOR

Figure 2 illustrates a QPC-plus-reflector system whose conductance has been experimentally measured.[8] The circular arc reflector and the vertical wall together form a cavity that can support long-lived resonances; the energy of these resonances can be swept by sweeping the reflector gate voltage. The classical condition[8] for stability of the cavity modes is that the arc center must lie at, or to the left of the wall $(x=0)$. The cavity modes are coupled to the left terminal via the QPC, and to the right terminal via leakage of the modes out through the cavity top and bottom. The system is interesting because it is ''open'' in the sense that it has no Coulomb blockade,[1] but ''closed'' in the sense that the dwell time is much greater than the ballistic time (the resonances are long lived). It has also been studied recently in our laboratory using microwave measurements.[21]

The actual potential in a mesoscopic experiment differs from the illustration: it has soft walls (on the scale $1/k_\mathrm{F}$), it may have deviations from the circle due to lithographic error, and it has modulations of the background potential due to elastic disorder.[8] However, we will not be interested in details of the resonator on the right-hand side. Rather, we will adopt the view of a 2D scattering-theorist ''looking'' from the left-hand side. In this section we discuss the maximum conductance of this system, when the ''bare'' QPC (i.e., without the reflector) is in the tunneling regime (conductance $\ll 2e^2/h$).

We use an idealized slit QPC model [see Fig. 1(b)] in which the potential V is zero everywhere except along a hard, thin wall where it is taken as infinite. The QPC is a gap in the wall of size $2a$. This model is highly nonadiabatic (see Ref. 17 for a review of its transmission properties). The hard wall simplifies the treatment of the left-hand side scattering problem, and we do not believe it alters our basic conclusion. We consider the ''unscattered'' wave to be the incident plus reflected wave Eq. (1) when the QPC is closed $(a=0)$. This we expand in Bessel functions,

$$\begin{aligned}\psi_0(\mathbf{r})&=e^{[i(k_x x+k_y y)]}-e^{[i(-k_x x+k_y y)]}\\&=-4iJ_1(kr)\cos(\theta)\cos(\phi)+\text{higher order terms.}\end{aligned}\tag{14}$$

The first term in the expansion is the incoming plus outgoing p wave, which in the tunneling limit will dominate in our consideration of the absorption.[17]

Now we open the slit, and replace $2J_1(kr)$ in the above by $H_1^{(2)}(kr)+e^{2i\delta}H_1^{(1)}(kr)$, where δ follows the usual definition of partial-wave phase shift.[14] The closed slit corresponds to $\delta=0$. An open slit leading into a closed resonator (imagine extending the arc in Fig. 2 to seal off the entire right side), in the case of infinite dephasing length, corresponds to $\delta=$ real, and would appear from the left side as an elastic dipole scatterer. An open slit with an open resonator corresponds to complex δ with positive imaginary part, and would appear as a general inelastic dipole scatterer. Therefore transmission though the QPC appears, to an observer on the left side, to be *absorption* of incident waves. σ_T is interpreted as an ''inelastic'' cross section (since exiting the right-hand terminal is equivalent to leaving in a new channel), and σ_R as an ''elastic'' one. $\sigma_\mathrm{T}(k,\phi)$ can be found from integrating the net incoming flux [as in Eq. (5)] of the total wave function on the *left* side. Substitution into Eq. (4) then gives $\sigma_\mathrm{T}(k,\phi)=4/k(1-|e^{2i\delta}|^2)\cos^2(\phi)$. For $\delta\to i\infty$ the maximal cross section is reached,

$$\sigma_{\mathrm{T,max}}(k,\phi)=\frac{4}{k}\cos^2(\phi).\tag{15}$$

This corresponds to an effective classical ''area'' (size) $a_\mathrm{eff} =\lambda_\mathrm{F}/2$. This is analogous to the fact[22] that in 3D the effective area of an arbitrarily small electromagnetic dipole aerial can be of order λ^2. To an observer on the left side who was able to ''see'' the electron waves living in the energy range $e\delta V$ responsible for conductance, the QPC would stand out as a ''black dot'' of size $\sim\lambda_\mathrm{F}$ against the surrounding uniform ''gray'' thermal luminosity reflected in the vertical wall mirror.

The associated maximum conductance is found easily using Eqs. (15) and (11) to be

$$G_{\max}=\frac{2e^2}{\hbar},\tag{16}$$

the universal quantum of conductance (for 2 spin channels), independent of the size of the QPC hole, even for an *arbitrarily small* hole $(ka\to 0)$. This universal resonant-tunneling maximum conductance was first found numerically;[23,24,1] however our system differs from those of

Xue and Lee[23] and Kalmeyer and Laughlin[24] because the resonance does not involve transmission though an *isolated* (zero-dimensional) quantum dot. The dramatic increase over the conductance of the bare QPC [which vanishes as $(ka)^4$, see Ref. 17] runs counter to the naive classical expectation, namely, that the reflector would *decrease* the left-to-right flow of electrons because it sends back into the QPC particles that would otherwise exit to the right.

How do we know that it is possible to build a resonant geometry that corresponds to $\delta \to i\infty$? The reflector can be described by r, the amplitude with which it returns an outgoing p wave back to the QPC as an incoming p wave. If $|r|^2 = 1 - |t_{11}|^2$, where the p-wave transmission of the QPC is t_{11} as defined in Sec. III, then the p-wave channel becomes a 1D Fabry-Perot resonator with mirrors of matched reflectivity. Sweeping the round-trip phase then produces peaks of complete transmission (corresponding to complete p-wave absorption on the left side). The ratio of peak separation to peak width is the quality factor $Q \sim 1/|t_{11}|^2$. Such peaks, with heights much greater than the bare tunneling QPC conductance, were observed in the experiments of Katine *et al.*[8] However, Eq. (16) has not yet been tested quantitatively because of the difficulty of matching the Fabry-Perot reflectivities in a real 2DEG experiment. Note that the maximum conductance (16) also follows immediately from the Landauer formula when we realize that there can be complete transmission of the incoming $l = 1$ channel state (from Sec. III).

An interesting possibility arises when we realize[17] that higher l channels are still *slightly* transmitted by the bare QPC, when $ka \ll 1$, even though they are increasingly evanescent. If the resonator has a high enough reflectivity for these modes, then additional Fabry-Perot conductance peaks will be produced.[23,25] The peaks may be extremely narrow, but can carry a full quantum of conductance because they can transmit another incoming l channel. By careful arrangement of the cavity, one or more of these peaks could be brought into conjunction with an already-existing $l = 1$ peak at the Fermi energy. (For instance, the $l = 1$ and $l = 2$ resonances are in different symmetry classes in Fig. 2 so there can be an exact level crossing). Therefore, we have the surprising result that, in theory, a conductance of $(2e^2/h)n$ can pass through an arbitrarily small QPC hole if n resonances (from n different channels) coincide at the Fermi energy. However, due to their extremely small width, such large conductance peaks are unlikely to be observable in a real mesoscopic tunneling QPC due to finite dephasing length and finite-temperature smearing.[1]

Finally, we should not overlook the fact that our expressions for partial cross sections are a factor of 4 greater than those conventionally arising in 2D scattering theory from a radial potential,[14] because we are measuring cross section on the reflective boundary of a semi-infinite half plane. For instance, the maximum inelastic partial cross section for a single channel in free space[14] is $\sigma_r = 1/k$, compared to our maximum ''inelastic'' cross section per channel Eq. (15). Similarly, the maximum elastic result in free space is $\sigma_e = 4/k$, compared to our maximum (normal-incidence) ''elastic'' cross section per channel $\sigma_{\mathrm{R,max}} = 16/k$. This latter case occurs when $\delta = (\text{integer} + \frac{1}{2})\pi$.

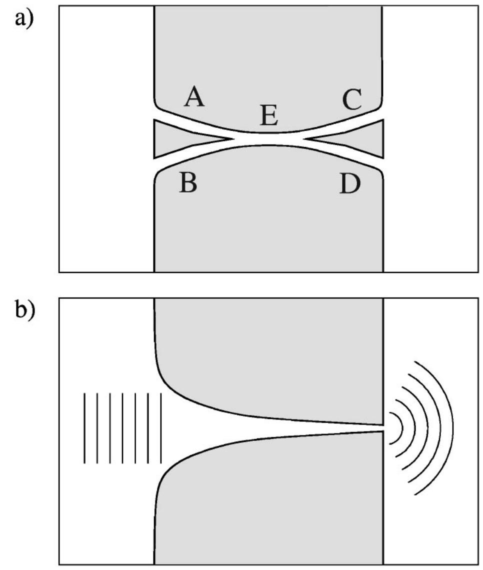

FIG. 3. (a) An attempt to increase conductance through a single channel by multiple connections feeding from the reservoirs. All channels are single mode and sufficiently long that the evanescent tunneling of higher modes is negligible. (b) An illustrative hard-walled exponential horn system that has differing acceptance angles on each side: very narrow on the left, and very wide on the right. Such a mesoscopic 2DEG system would exhibit symmetric conductance, however, in an atom beam context the conductance can become unsymmetric.

V. WHAT IS THE MAXIMUM CONDUCTANCE OF A SINGLE QUANTUM CHANNEL?

The surprising theoretical results of the previous section might lead one to question the conductance limit $2e^2/h$ for a single quantum channel (by which we mean a single transverse mode for which the longitudinal degree of freedom is a 1D Fermi gas; this includes both conventional and partial-wave basis sets). For this gedanken-experiment we will consider conventional electron waveguides that are single mode and long enough that evanescent waves are negligible, but which are also $\ll l_\phi$. We try to encourage more current to pass down a single-mode channel (E) by connecting it to a reservoir via multiple routes (A,B,C,D), as shown in Fig. 3(a), where two routes are used on each side. It is possible to match the junctions so that a wave entering down A,B,C, or D has no reflection back along the same lead. In this case we might guess that the hypothetical left-side observer (from the previous section) would see the single-mode entrances to guides A and B as two ''black dots,'' giving twice the effective absorption cross section, and therefore infer a conductance of twice $2e^2/h$. We might also justify this by saying that waves traveling down A and B will meet and continue down E, and since they have no particular phase relation, their currents will add to give a doubled current through E, as would be necessary.

However there is a fundamental flaw in the above reasoning. The *ABE* junction can be designed so that if waves come down *A* and *B* in phase, they will be adiabatically transformed into the lowest transverse mode of *E*, so will propagate through to the right side without reflection, carrying a current of twice that of a usual single-mode guide. However, if *A* and *B* are π out of phase, the same adiabatic transformation must result in the second transverse mode, which is evanescent. So this latter wave will reflect perfectly back out of the left side, and carry no current. Plane waves are impinging from the left reservoir uniformly over all angles, and because of the $>\lambda$ separation of the entrances, an average over angles gives an average over relative phase in *A* and *B*. Thus we are left with no increase above the single-channel conductance. This property of the *ABE* junction is not merely practical; rather, it is easy to show that its 3×3 *S* matrix cannot be unitary if a junction is to couple both $A\rightarrow E$ and $B\rightarrow E$ with unity transmissions. Such an appealing junction is therefore ruled out on the grounds of flux conservation. A consequence is that the entrances to *A* and *B* can at most appear ''half black'' to the observer, due to waves that enter *A* then exit *B* and vice versa.

This suggests another way to try and defeat the conductance limit: direct the incoming plane waves in a narrow enough angular distribution so that waves *always* come down *A* and *B* in phase, and this will double the conductance. (This is similar to experiments[26] where the series resistance of two QPC's was found to be less than the sum of the individual QPC resistances, because collimation at the exit of the first QPC illuminated the second with a narrow beam, increasing its conductance). However, this beam is no longer a *thermal* occupation of incoming states. This illustrates the inextricable link between thermal Fermi occupation of reservoir states and the universal quantum of conductance. At $T=0$, thermal occupation at a given chemical potential difference implies that *all* quantum states lying in the appropriate energy range are filled in the left reservoir and empty in the right. Semiclassically, this corresponds to a uniform distribution in phase space, or when projected into momentum states, uniform in angle, as exemplified by Eq. (11). The semiclassical viewpoint allows one to see that since transformations in phase space cannot change the phase-space density (Liouville's theorem), the universal conductance per quantum channel cannot be changed. This reminds us that unitarity in quantum mechanics is analogous to Liouville's theorem in classical mechanics.

VI. RECIPROCITY AND ''CONDUCTANCE'' OF ATOM WAVES

We can ask if the conductance (11) computed using transmission of left-side reservoir plane-wave states through the QPC is equal to that using right-side reservoir states. Since the two directions correspond to opposite signs of δV, then in order to have linear response (well-defined constant *G* around $\delta V=0$) we would hope that they are equal. That the angular average of transmission cross section is equal from the left and right sides is not immediately apparent in a general asymmetric system. For instance, consider Fig. 3(b) that has a small acceptance angle from the left but a large from the right, therefore very different forms of the transmission cross sections $\sigma_{\mathrm{T}}^{\mathrm{L}\rightarrow\mathrm{R}}(k,\phi)$ and $\sigma_{\mathrm{T}}^{\mathrm{R}\rightarrow\mathrm{L}}(k,\phi)$.

If we assume *classical* motion then we can imagine a map from a Poincaré section (PS) (y,p_y) at a vertical slice at $x=-x_0$ to another PS (y',p_y') at $x=+x_0$. At each PS we consider only rightwards-moving $(p_x>0)$ particles, and take $x_0>L$. A certain area of phase space (y,p_y) is transmitted and is mapped to an *equal area*[27] in phase space (y',p_y'). Time-reversal invariance holds since we consider magnetic field $B=0$, so we can negate the momenta (now considering $p_x<0$) and find that the *same* phase-space area is transmitted right to left. When it is realized that the angle-averaged cross section is proportional to the transmitted phase-space area on a PS, then the symmetry of the angle-averaged *classical* cross sections follows.

The same symmetry is not obvious for quantum cross sections, but it also holds true. Comparing Eq. (11a) with Eq. (13) gives

$$\int_{-\pi/2}^{\pi/2} d\phi\ \sigma_{\mathrm{T}}^{\mathrm{L}\rightarrow\mathrm{R}}(k_{\mathrm{F}},\phi)=\lambda_{\mathrm{F}}\ \mathrm{Tr}(t^{\dagger}t), \tag{17}$$

where *t* is measured from left-to-right states. It is instructive to derive this directly.[17] This relation ties together the cross section and Landauer views of conductance. Time-reversal invariance and flux conservation together imply[28] that $\mathrm{Tr}(t^{\dagger}t)$ is unchanged by swapping the labeling of the leads,[1,20] thus we immediately have from Eq. (17) the reciprocity of angle-integrated quantum cross section

$$\int_{-\pi/2}^{\pi/2} d\phi\ \sigma_{\mathrm{T}}^{\mathrm{L}\rightarrow\mathrm{R}}(k,\phi)=\int_{-\pi/2}^{\pi/2} d\phi\ \sigma_{\mathrm{T}}^{\mathrm{R}\rightarrow\mathrm{L}}(k,\phi). \tag{18}$$

So in Fig. 3(b) it is now clear that the ratio of acceptance angles must be balanced by the ratio of effective areas.

We now discuss a case in which nonthermal occupation of incoming states is possible: the rapidly developing field of coherent matter-wave optics, in which potentials are defined by microfabricated structures.[29–31] There is a recent proposal[30] for observation of quantization of atomic flux through a micron-sized 3D QPC defined by the Zeeman effect potential of a magnetic field. The device is illuminated by a beam of atoms passing through a vacuum, whose angular distribution is an experimental parameter (for instance, a collimated oven source or a dropped cloud of cold atoms[32]). The atomic flux transmitted (per unit *k*, at wave vector *k*) will be $F(k)=G_{\mathrm{atom}}(k)J_0(k)$ where $J_0(k)$ is the flux incident per unit wall area, and we define the atomic ''conductance'' by

$$G_{\mathrm{atom}}(k)\equiv\int d\Omega\ w(k,\Omega)\sigma_{\mathrm{T}}(k,\Omega). \tag{19}$$

As before, the quantum-transmission cross section is $\sigma_{\mathrm{T}}(k,\Omega)$, but now there is a *weighting function* $w(k,\Omega)$ which defines the angular distribution of the incident beam.[33] The weight has the normalization $\int d\Omega\ w(k,\Omega)\cos(\theta)=1$. [All integrals over solid angle $\Omega\equiv(\theta,\phi)$ are over a range of 2π appropriate for the half-sphere.] Following the analogy of Thywissen,[30] $F(k)$ plays the role of current, $J_0(k)$ that of bias voltage. However, the name ''conductance'' does not imply any definite chemical potential difference as in the 2DEG case. For classical particles, the ''conductance'' of an aperture of area A_{eff} in a thin wall is simply $G_{\mathrm{atom}}(k) = A_{\mathrm{eff}}$, regardless of the incident angular distribution. Thus $G_{\mathrm{atom}}(k)$ gives the effective area A_{eff} of a QPC, in an analogous fashion to a_{eff} in 2D.

For an integer number of quantum channels, the 2D quantization of a_{eff} in units of $\lambda/2$ becomes in 3D the quantization[30,1,34] of A_{eff} in units of λ^2/π, a result well known from work on 3D metallic point contacts.[35] As stated by Thywissen,[30] this accurate flux quantization requires the incident beam width to be much larger than the QPC acceptance angle.

Equation (19) is the matter-wave equivalent of Eq. (11a), with the important difference that it has a general weight function. Possible nonuniformity of this weight function leads to a key result: that *asymmetry* of the conductance is possible given identical illumination on either side, even though the (center of mass) motion is time-reversal invariant. For example, if the incident flux used to illuminate the horn QPC of Fig. 3(b) is narrow in angular spread, then the left-to-right conductance will be much larger than the right-to-left conductance. This contrasts with the 2DEG case where the conductance is always symmetric.

Finally, it is interesting to note that for the nonthermal incident (reservoir) distributions discussed above, the Landauer formula takes the modified form

$$G \propto \mathrm{Tr}(t^{\dagger} t \rho), \tag{20}$$

where ρ is the density matrix of the incident beam.

VII. CONCLUSIONS

Quantum-scattering theory in the 2D half plane can provide an alternative description of the mesoscopic conductance of noninteracting particles. It is especially useful in ''open'' systems (e.g., those with nearby scatterers in the reservoir regions) where the usual transverse-channel approach is inappropriate. We have considered elastic potentials in zero magnetic field, in linear response in the low-temperature limit. Conductance is proportional to the transmission cross section integrated over all incident angles, Eq. (11). We also define a half-plane partial-wave basis applicable with the usual Landauer formula, and relate this to our transmission cross section result. A difference between this and previous work is the ability to treat a direct ''lead-less'' connection to the reservoir.

Using the example of a slit QPC combined with an open-cavity structure, we show that an arbitrarily small QPC can carry up to a single quantum of conductance via resonant tunneling (equal to the limit in the closed-dot resonant tunneling case). This requires a resonance at the Fermi energy. If n coincident resonances occur for different incoming channels, then n conductance quanta can in theory be achieved through this same tunneling QPC, a result that we believe has not been noted until now.

We emphasize that conductance is proportional to phase-space density of the reservoir states. Therefore the universal quantum of conductance e^2/h per spin in Fermi gas systems is a direct result of the uniform phase-space density (angular distribution) in a thermal occupation of the Fermi sea. This insight is supported by discussion of attempts to exceed this universal value. When the reservoir occupation differs from thermal, the conductance formula requires generalization: an angle-dependent weight is included in the cross section integral (19); equivalently for 2DEG systems the Landauer formula requires inclusion of the incoming ensemble (20). This result, and our approach in general, is relevant to the emerging field of matter-wave conductance by microfabricated structures (for instance, a quantum-point contact in 3D), under general illumination by atom waves. We hope this work provides new tools for the study of coherent electron and matter-wave systems.

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