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Spin-orbit scattering as an experimental tool to measure spin currents

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Impurities with large spin-orbit scattering can be used to detect a finite spin polarization in a conductor. If one considers a Fermi sphere with only spin-up conduction electrons in the presence of an electric field in the x direction, then impurities with finite spin-orbit scattering scatter the conduction electrons asymmetrically. The current in the y direction is nonzero, even in the absence of a magnetic field. The magnitude of this ''anomalous'' Hall effect is calculated in terms of Friedel phase shifts.

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In a series of recent experiments[1,2] we measured the Hall resistance of quench-condensed potassium films. When such a film is covered with $\frac{1}{100}$ of a monolayer of Pb we observe (on top of the linear Hall resistance) a Hall curve that is very similar to the anomalous Hall resistance of a ferromagnet. The Pb impurities have a large spin-orbit scattering (SOS). We believe that the Pb impurities act as detectors of a finite spin polarization in the K films. This is due to the fact that impurities with large SOS scatter electrons with spin up or down asymmetrically.

The asymmetric scattering of polarized conduction electrons by spin-orbit scattering was already investigated by Ballentine and Huberman[3,4] when they tried to explain the Hall constant of heavy liquid metals such as Tl, Pb, and Bi. They performed the spin-orbit scattering in a perturbation calculation. This might not be applicable in the present cases. We investigated recently the SOS of Pb and Bi in a number of alkali metals, in particular in Cs.[5] The SOS cross section of Pb and Bi in Cs was remarkably large. In units of $4\pi/k_F^2$ we found $\sigma'_{SO}\approx 0.6$ for the dimensionless SOS cross section. The spin-orbit-scattering cross section is given by the formula

$$\sigma'_{SO}=\frac{\sigma_{SO}k_F^2}{4\pi}=\sum_{l=1}^{\infty}\frac{l(l+1)}{2l+1}\sin^2(\delta_{l,+}-\delta_{l,-}). \qquad (1)$$

Here we use the notation for the Friedel phase shifts: $\delta_{l+1/2,l}=\delta_{l,+}$ and $\delta_{l-1/2,l}=\delta_{l,-}$. ($\sigma_{SO}$ and σ'_{SO} include the non-spin-flip contribution of the SOS.)

If one assumes that the Pb is an (s,p) scatterer and neglects higher-angular-momentum l scattering then the value $\frac{2}{3}$ is the absolute maximum for σ'_{SO} of Pb impurities in Cs. The phase difference between $\delta_{1,+}$ and $\delta_{1,-}$ has to be of the order of $\pi/2$.

The purpose of this Brief Report is to calculate the strength of the asymmetric scattering and the resulting ''anomalous'' Hall resistance R_{yx} in terms of the Friedel phase shifts of the Pb impurity and the polarization of the conduction electrons and to establish Pb impurities as a detector for spin polarization. Such a calculation goes beyond perturbation theory and is exact.

I start by considering first a plane electron wave with momentum $\mathbf{k}$, spin up, and one SOS impurity at the origin, which scatters the plane wave. The asymptotic form of the total wave function, plane wave plus scattered wave (for large r), is given by

$$\Psi_{\mathbf{k},\mathrm{tot}}(\mathbf{r})\simeq\exp[i\mathbf{k}\cdot\mathbf{r}]\chi_+ + F_{\mathrm{nsf}}(\Omega_{\mathbf{r}})\chi_+\frac{e^{ikr}}{r} + F_{\mathrm{sf}}(\Omega_{\mathbf{r}})\chi_-\frac{e^{ikr}}{r}, \qquad (2)$$

where $F_{\mathrm{nsf}}(\Omega_{\mathbf{r}})$ give the non-spin-flip amplitude and $F_{\mathrm{sf}}(\Omega_{\mathbf{r}})$ the spin-flip amplitude of the scattered waves. $\Psi_{\mathbf{k},\mathrm{tot}}(\mathbf{r})$ is then an asymptotic (stationary) solution of the impurity potential.

The plane wave with spin up can be expressed asymptotically as

$$\exp[i\mathbf{k}\cdot\mathbf{r}]\chi_+\simeq\frac{4\pi}{2ikr}\sum_{l,m}Y_l^{m*}(\hat{\mathbf{k}})\times[e^{ikr}-(-1)^l e^{-ikr}]Y_l^m(\hat{\mathbf{r}})\chi_+ \qquad (3)$$

so that

$$\Psi_{\mathbf{k},\mathrm{tot}}(\mathbf{r})\simeq-(-1)^l\frac{4\pi}{2ik}\sum_{l,m}Y_l^{m*}(\hat{\mathbf{k}})Y_l^m(\hat{\mathbf{r}})\chi_+\frac{e^{-ikr}}{r} + \left(\frac{4\pi}{2ik}\sum_{l,m}Y_l^{m*}(\hat{\mathbf{k}})Y_l^m(\hat{\mathbf{r}})+F_{\mathrm{nsf}}(\Omega_{\mathbf{r}})\right)\chi_+\frac{e^{ikr}}{r} + F_{\mathrm{sf}}(\Omega_{\mathbf{r}})\chi_-\frac{e^{ikr}}{r}. \qquad (4)$$

For a sphere with a single SOS impurity in the center, the electronic states are eigenfunctions with total angular momentum $j=l\pm\frac{1}{2}$. Their wave function is $\Psi_{k,j,m_j}(\mathbf{r})=R_{k,l\pm1/2}(r)|l\pm\frac{1}{2},m+\frac{1}{2}\rangle$ where $|l\pm\frac{1}{2},m+\frac{1}{2}\rangle$ is the angular eigenstate with the quantum numbers $j=l\pm\frac{1}{2}$ and $j_z=m+\frac{1}{2}$. The radial part of the wave function $R_{k,l\pm1/2}(r)$ has the asymptotic form

$$R_{k,l\pm 1/2}(r)\simeq\frac{1}{kr}\sin\left(kr-\frac{l}{2}\pi+\delta_{l,\pm}\right)$$
$$=\frac{1}{kr}\frac{1}{2i}\left\{\exp\left[i\left(kr-\frac{l}{2}\pi+\delta_{l,\pm}\right)\right]-\exp\left[-i\left(kr-\frac{l}{2}\pi+\delta_{l,\pm}\right)\right]\right\}.$$

The wave function $\Psi_{\mathbf{k},\mathrm{tot}}(\mathbf{r})$ can be expressed in terms of the eigenfunctions,

$$\Psi_{\mathbf{k},\mathrm{tot}}(\mathbf{r})=4\pi\sum_{l,m}Y_l^{m*}(\hat{\mathbf{k}})\sum_{\pm}\alpha_{l\pm 1/2,m+1/2}\Psi_{k,l\pm 1/2,m+1/2}(\mathbf{r})$$
$$=4\pi\sum_{l,m}Y_l^{m*}(\hat{\mathbf{k}})\sum_{\pm}\alpha_{l\pm 1/2,m+1/2}\frac{1}{kr}\frac{1}{2i}\times(\exp\{i[kr-(l/2)\pi+\delta_{l,\pm}]\}-\exp\{-i[kr-(l/2)\pi+\delta_{l,\pm}]\})|l\pm\tfrac{1}{2},m+\tfrac{1}{2}\rangle. \tag{5}$$

Now one can express the states $|l\pm\frac{1}{2},m+\frac{1}{2}\rangle$ again in terms of $Y_l^m(\hat{\mathbf{r}})\chi_+$ and $Y_l^{m+1}(\hat{\mathbf{r}})\chi_-$ with

$$|l+\tfrac{1}{2},m+\tfrac{1}{2}\rangle=\left(\frac{\sqrt{l+m+1}}{\sqrt{2l+1}}Y_l^m\chi_++\frac{\sqrt{l-m}}{\sqrt{2l+1}}Y_l^{m+1}\chi_-\right),$$

$$|l-\tfrac{1}{2},m+\tfrac{1}{2}\rangle=\left(\frac{\sqrt{l-m}}{\sqrt{2l+1}}Y_l^m\chi_+-\frac{\sqrt{l+m+1}}{\sqrt{2l+1}}Y_l^{m+1}\chi_-\right).$$

By comparing the asymptotic forms of the incoming waves in Eqs. (4) and (5), one finds

$$\alpha_{l+1/2,m+1/2}=i^l e^{i\delta_+}\frac{\sqrt{l+m+1}}{\sqrt{(2l+1)}},$$

$$\alpha_{l-1/2,m+1/2}=i^l e^{i\delta_-}\frac{\sqrt{l-m}}{\sqrt{(2l+1)}}. \tag{6}$$

With these coefficients one obtains the outgoing wave. This yields for the scattered wave

$$\frac{4\pi}{2ik}\sum_{l,m}Y_l^{m*}(\hat{\mathbf{k}})\frac{e^{ikr}}{r}\times\left[\left(\frac{l+1}{2l+1}e^{2i\delta_{l,+}}+\frac{l}{2l+1}e^{2i\delta_{l,-}}-1\right)Y_l^m(\hat{\mathbf{r}})\chi_+ +\frac{m}{2l+1}(e^{2i\delta_{l,+}}-e^{2i\delta_{l,-}})Y_l^m(\hat{\mathbf{r}})\chi_+ +\frac{\sqrt{(l+m+1)(l-m)}}{2l+1}(e^{2i\delta_{l,+}}-e^{2i\delta_{l,-}})Y_l^{m+1}(\hat{\mathbf{r}})\chi_-\right]. \tag{7}$$

The first term in the square brackets of Eq. (7) is the potential scattering. It is independent of m and depends only on the angle between $\mathbf{k}$ and $\mathbf{r}$. The second term is the non-spin-flip part of the SOS and the third term is the spin-flip part of the SOS. Both SOS terms vanish if $(\delta_{l,+}-\delta_{l,-})$ is zero.

Next it has to be shown that this scattering is asymmetric. Such an asymmetry results from the interference of the non-spin-flip parts. The spin-flip scattering does not contribute to the anomalous Hall resistivity. Therefore we calculate the scattering intensity of the non-spin-flip parts. The non-spin-flip amplitude consists of potential scattering

$$F_0(\hat{\mathbf{r}})=\sum_{l,m}\Lambda_l Y_l^{m*}(\hat{\mathbf{k}})Y_l^m(\hat{\mathbf{r}})\frac{e^{ikr}}{r},$$

$$\Lambda_l=\frac{4\pi}{2ik}\left(\frac{(l+1)}{2l+1}e^{2i\delta_{l,+}}+\frac{l}{2l+1}e^{2i\delta_{l,-}}-1\right)$$

and spin-orbit scattering

$$F_{\mathrm{SO}}(\hat{\mathbf{r}})=\sum_{l,m}\lambda_l m Y_l^{m*}(\hat{\mathbf{k}})Y_l^m(\hat{\mathbf{r}})\frac{e^{ikr}}{r},$$

$$\lambda_l=\frac{4\pi}{2ik}\left(\frac{1}{2l+1}(e^{2i\delta_{l,+}}-e^{2i\delta_{l,-}})\right).$$

The SOS depends on the direction of the incident wave vector $\mathbf{k}$. Therefore it has to be averaged over all directions of $\mathbf{k}$ (with $|\mathbf{k}|=\mathrm{const}$). To simulate a shifted Fermi sphere in the x direction the weight of the plane wave $\mathbf{k}=(k,\theta_{\mathbf{k}},\phi_{\mathbf{k}})$ is chosen as

$$w(\theta_{\mathbf{k}},\phi_{\mathbf{k}})d\Omega_{\mathbf{k}}=\frac{3}{4\pi}\sin\theta_{\mathbf{k}}\cos\phi_{\mathbf{k}}d\Omega_{\mathbf{k}}$$

so that the averaged flow density is $(k,0,0)$, the same as for the plane wave $\exp[ikx]$ in the x direction.

We are interested here in the flow (due to scattering) perpendicular to the direction of $(k,0,0)$, i.e., in the y direction. Therefore we have to multiply the outgoing intensity of the scattered wave by $k\sin\theta_{\mathbf{r}}\sin\phi_{\mathbf{r}}d\Omega_{\mathbf{r}}$ (and integrate over $d\Omega_{\mathbf{r}}$).

The sum of the potential and the non-spin-flip spin-orbit scattering is proportional to $(\Lambda_l+m\lambda_l)$. So the flow of the scattered wave(s) is given by the integral

$$J_y=\frac{3}{4\pi}k\int\int\left|\sum_{l,m}(\Lambda_l+m\lambda_l)Y_{l,m}^*(\hat{\mathbf{k}})Y_{l,m}(\hat{\mathbf{r}})\right|^2\times\sin\theta_{\mathbf{k}}\cos\phi_{\mathbf{k}}\sin\theta_{\mathbf{r}}\sin\phi_{\mathbf{r}}d\Omega_{\mathbf{k}}d\Omega_{\mathbf{r}}.$$

Only the interference between the two parts contributes to the anomalous resistivity. It is proportional to $(\Lambda_l^* m'\lambda_{l'}+m\lambda_l^*\Lambda_{l'})$.

After a lengthy calculation we obtain for the current in the y direction

$$J_y=-i\frac{1}{8\pi}k\sum_l \{\Lambda_l^*\lambda_{l+1}(l+2)(l+1)+\lambda_l^*\Lambda_{l+1}l(l+1)$$
$$-\Lambda_l^*\lambda_{l-1}l(l-1)-\lambda_l^*\Lambda_{l-1}l(l+1)\}.$$

The ratio J_y divided by the flow in the x direction $j_x=k$ yields the Hall cross section of the SOS impurity,

$$\sigma_{xy}=-i\frac{1}{8\pi}\sum_l [(l+2)(l+1)(\Lambda_l^*\lambda_{l+1}-\lambda_{l+1}^*\Lambda_l)$$
$$+l(l+1)(\lambda_l^*\lambda_{l+1}-\Lambda_{l+1}^*\lambda_l)].$$

In the next step we evaluate $(\Lambda_l^*\lambda_{l+1}-\lambda_{l+1}^*\Lambda_l)$ and $(\lambda_l^*\Lambda_{l+1}-\Lambda_{l+1}^*\lambda_l)$. This yields for the Hall cross section

$$\sigma_{xy}=\frac{2\pi}{k^2}\sum_l\left(\frac{\sin(\delta_{l+1,+},-\delta_{l+1,-})}{(2l+1)(2l+3)}(l+2)(l+1)[-(l+1)\sin(\delta_{l,+})\cos(\delta_{l+1,+}+\delta_{l+1,-}-\delta_{l,+})\right.$$
$$-l\cos\sin(\delta_{l,-})\cos(\delta_{l+1,+}+\delta_{l+1,-}-\delta_{l,-})]+\frac{\sin(\delta_{l,+}-\delta_{l,-})}{(2l+1)(2l+3)}l(l+1)[+(l+2)\sin(\delta_{l+1,+})\cos(\delta_{l,+}+\delta_{l,-}$$
$$\left.-\delta_{l+1,+})+(l+1)\sin(\delta_{l+1,-})\cos(\delta_{l,+}+\delta_{l,-}-\delta_{l+1,-})]\right). \tag{8}$$

If we restrict ourselves to (s,p) scattering then the dominant term is

$$\sigma_{xy}^{(1)}=-\frac{4\pi}{3k^2}\sin(\delta_{1,+}-\delta_{1,-})\sin\delta_0\cos(\delta_{1,+}+\delta_{1,-}-\delta_0). \tag{9}$$

(For $l=0$ we have $\delta_{0,+}=\delta_{0,-}=\delta_0$.)

In the next step I calculate the Hall resistance due to the SOS. For this purpose one has to treat the current self-consistently. I choose the direction of the current as the x direction. The displacement of the Fermi sphere is $\boldsymbol{\kappa}=(\kappa_x,0,0)$. The electric field forms the angle α with the current. The rate equation for the average momentum is

$$\hbar\frac{d\kappa_x}{dt}=(-e)E\cos\alpha-\hbar\frac{\kappa_x}{\tau_0},$$
$$\hbar\frac{d\kappa_y}{dt}=\hbar\frac{\kappa_x}{\tau_{xy}}+(-e)E\sin\alpha, \tag{10}$$

where

$$\frac{1}{\tau_{xy}}=n_i\sigma_{xy}v_F.$$

(The contributions of κ_y on the right side are dropped because κ_y is zero.) The elastic scattering time τ_0 generally has contributions from other scattering centers besides the SOS. (Note that the sign of τ_{xy} determines whether the scattering is in the positive or negative y direction.)

In the stationary state (with $\kappa_y=0$) we have

$$\hbar\frac{\kappa_x}{\tau_0}=(-e)E_x=(-e)E\cos\alpha,$$
$$\hbar\frac{\kappa_x}{\tau_{xy}}=-(-e)E_y=eE\sin\alpha,$$
$$\tan\alpha=-\frac{\tau_0}{\tau_{xy}}.$$

The current density is

$$\mathbf{j}=(j_x,0,0),$$
$$j_x=n(-e)\frac{\hbar\kappa_x}{m}.$$

The normal resistivity is

$$\rho_{xx}=\frac{E_x}{j}=\frac{E_x}{n(-e)\tau_0(-e)E_x/m}=\frac{m}{e^2n}\frac{1}{\tau_0}.$$

The Hall resistivity is

$$\rho_{xy}=\frac{E_y}{j}=\frac{-\hbar\kappa/(-e)\tau_{xy}}{n(-e)\hbar\kappa/m}$$
$$=-\frac{m}{e^2n}\frac{1}{\tau_{xy}}$$
$$=-\frac{m}{e^2n}n_i\sigma_{xy}v_F=-\frac{\hbar k_F}{e^2n}n_i\sigma_{xy}.$$

It is interesting to note that the Hall resistance does not depend on the relaxation time of the conduction electron. (Of course, the Hall angle depends on τ_0.)

For the conduction electrons with the opposite spin direction one obtains analogous expressions, containing the density n, the relaxation rate τ, and the Hall cross section σ_{xy} for the other spin direction. In the final step one has to add the conductance (conductivity) matrices of both spins, which are given by

$$\sigma_{\uparrow\downarrow}=\begin{pmatrix} \left(\dfrac{\rho_{x,x}}{\rho_{x,x}^2+\rho_{x,y}^2}\right)_{\uparrow\downarrow} & -\left(\dfrac{\rho_{x,y}}{\rho_{x,x}^2+\rho_{x,y}^2}\right)_{\uparrow\downarrow} \\ \left(\dfrac{\rho_{x,y}}{\rho_{x,x}^2+\rho_{x,y}^2}\right)_{\uparrow\downarrow} & \left(\dfrac{\rho_{x,x}}{\rho_{x,x}^2+\rho_{x,y}^2}\right)_{\uparrow\downarrow} \end{pmatrix}. \qquad (11)$$

If the relaxation rates do not depend on the spin and only the density of the spins is different then the result is rather simple:

$$\sigma_{\uparrow}=n_{\uparrow}\frac{e^2/m}{(1/\tau_0)^2+(1/\tau_{x,y})^2}\begin{pmatrix} \dfrac{1}{\tau_0} & \dfrac{1}{\tau_{x,y}} \\ -\dfrac{1}{\tau_{x,y}} & \dfrac{1}{\tau_0} \end{pmatrix},$$

$$\sigma_{\downarrow}=n_{\downarrow}\frac{e^2/m}{(1/\tau_0)^2+(1/\tau_{x,y})^2}\begin{pmatrix} \dfrac{1}{\tau_0} & -\dfrac{1}{\tau_{x,y}} \\ \dfrac{1}{\tau_{x,y}} & \dfrac{1}{\tau_0} \end{pmatrix},$$

which yields

$$\sigma=\frac{e^2/m}{(1/\tau_0)^2+(1/\tau_{x,y})^2}\times\begin{pmatrix} (n_{\uparrow}+n_{\downarrow})\dfrac{1}{\tau_0} & (n_{\uparrow}-n_{\downarrow})\dfrac{1}{\tau_{x,y}} \\ -(n_{\uparrow}-n_{\downarrow})\dfrac{1}{\tau_{x,y}} & (n_{\uparrow}+n_{\downarrow})\dfrac{1}{\tau_0} \end{pmatrix}.$$

In this case the Hall conductivity is proportional to the polarization of the conduction electrons. If only the (s,p) scattering has to be included then one finds for the Hall conductivity

$$\sigma_H=-n_i v_F\frac{e^2(n_{\uparrow}-n_{\downarrow})}{m}\tau_0^2 \times\frac{4\pi}{3k^2}\sin(\delta_{1,+}-\delta_{1,-})\sin\delta_0\cos(\delta_{1,+}+\delta_{1,-}-\delta_0).$$

The Hall resistivity is given by the inverse matrix of the conductivity (which can yield rather lengthy expressions).

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