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Rotating superconductors and the London moment: Thermodynamics versus microscopics

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Comparing various microscopic theories of rotating superconductors to the conclusions of thermodynamic considerations, we traced their marked difference to the question of how some thermodynamic quantities (the electrostatic and chemical potentials) are related to more microscopic ones: The electron's work function, mean-field potential, and Fermi energy—certainly a question of general import. After the proper identification is established, the relativistic correction for the London moment is shown to vanish.

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I. INTRODUCTION

The defining property of superconductors, the well-known Meissner effect, is to expel an (undercritical) magnetic field from its bulk. This property reverses itself when the superconductor is rotated, and a spontaneous magnetic field appears—again in the bulk, one or two penetration lengths away from the surface. This is usually referred to as the London moment.[1] Its magnitude is[2]

$$\mathbf{B}=-[2m_e(1+\zeta)c/e]\mathbf{\Omega}, \tag{1}$$

where Ω is the rotational velocity, c the vacuum light velocity, m_e the bare mass, and $e<0$ the charge, of the electron; finally, ζ is a relativistically small correction. All theories and experiments agree that ζ is small, but strong disagreement exists with respect to its actual value and sign. The microscopic theories, by Anderson,[3] Brady,[4] Cabrera and co-workers,[5,6] and Baym[7] take the main correction to be positive and given by the Fermi velocity

$$\zeta\approx(v_F/c)^2\approx 2\times 10^{-4}. \tag{2}$$

The thermodynamic theory, on the other hand, finds ζ to be negative, and very much smaller:[8]

$$\zeta=\tilde{\mu}/c^2\approx -10^{-10}, \tag{3}$$

where $\tilde{\mu}$ is the chemical potential of the metal, the energy needed to add a unit mass to the superconducting solid. (Since the solid holds its atoms together, the chemical potential $\tilde{\mu}$ is necessarily a negative quantity.) The value of 10^{-10} makes the correction negligible for all conceivable purposes and renders Eq. (1), taking the values of e and c as given, a very precise expression for m_e—about three orders of magnitude more precise than any present experimental technique to determine the electron mass directly.[9]

Generally speaking, although the thermodynamic, macroscopic theory lacks details and is incapable of answering many quantitative questions, it is nevertheless a rigorous theory, and uniquely appropriate for understanding the London moment. This is because (i) the London moment is an equilibrium phenomenon, (ii) the measurement concerns a quotient between two macroscopic fields, B and Ω, and most importantly, (iii) the reason for the microscopic parameters of m_e and e to appear on the macroscopic level is due to simple symmetry principles, rather than some detailed properties of the interaction.

Comparing both types of theories in detail, we find their difference to be easy to bridge, if the relations between some thermodynamic and microscopic quantities had been clear beforehand. More specifically, it is the incorrect identification of electrostatic and chemical potentials on one hand, the work function, mean-field potential, and Fermi energy of the electron on the other that has led to the above discrepancy. The correct identification of these quantities is of rather general interest and transcends the understanding of the London moment alone. The proper identification is the most important result of the present paper.

Another problem clouding the understanding of the London moment is the disagreement between experiment and all theories. Although this remains a point we do not understand, we took a step towards its clarification by achieving understanding of the following point: The best present experiment measures the flux, $\oint\mathbf{A}\cdot d\mathbf{s}=\int\mathbf{B}\cdot d\mathbf{a}$, outside the rotating superconductor.[10] Two effects contribute to this flux, the London moment, and the so-called double layer. The latter is a result of the fact that a metal may be conceived of as an electrostatic potential of the square-well form, with the depth $\Delta\Phi$. As the metal rotates, this discontinuity in the rest-frame electrostatic potential produces a discontinuity in the laboratory-frame vector potential

$$\Delta\mathbf{A}=-(\mathbf{\Omega}\times\mathbf{r})\Delta\Phi/c. \tag{4}$$

And this also contributes to the flux $\oint\mathbf{A}\cdot d\mathbf{s}$, with a contribution much larger than the relativistic correction of the London moment. It has been generally suspected that both contributions are experimentally inseparable, that they are always measured in conjunction. (A double layer of opposite charge—or one layer of dipoles—produces a discontinuity in the electrostatic potential, hence the name.)

We disagree. In this paper, we shall discuss both effects separately, as only the first is universal—the second varies with materials, and is quite independent from superconduc-

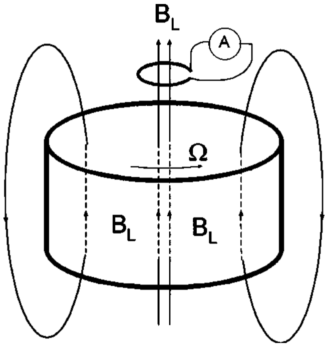

FIG. 1. Direct measurement of the London field B_L.

tivity. Besides, we believe that these effects may, in principle, be measured independently. Taking the superconductor to be a cylinder rotating around its axis, the London moment may be measured immediately outside its top or bottom surface, where any magnetic field **B** is continuous, see Fig. 1. (It is not continuous across the cylinder surface, where the field drops to zero within a few penetration lengths, due to the presence of persistent currents.) The flux from the double layer alone may be seen above T_λ, or in the experiment proposed in Sec. V.

In Sec. II, we shall present the simple yet stringent-macroscopic calculation leading to the London moment. This is followed in Sec. III by a comparison to the microscopic results, which are brought into agreement with the macroscopic ones by the appropriate identification of the above-mentioned thermodynamic and microscopic quantities. In Sec. IV, we consider the effect of the double layer, the contribution of which may be measured in the experiment described in Sec. V.

II. THE MACROSCOPIC APPROACH

A. Electrostatic and chemical potential

Given any neutral, macroscopic system we may either change its numbers of electrons and ions, N_- and N_+, or its mass and charge, M and Q. They are related as

$$M = m_i N_+ + m_e N_-, \tag{5}$$

$$Q = |e|(N_+ - N_-). \tag{6}$$

(m_i and m_e denote the bare mass of the ion and electron, respectively, and $e<0$ is the elementary charge.) Frequently, the number of atoms $N_a \equiv M/(m_i+m_e)$ is used instead of the mass M. The energy of these changes is given by the respective chemical potentials,

$$dE = \mu_+ \, dN_+ + \mu_- \, dN_- \tag{7}$$

$$= \mu \, dM + \Delta\Phi \, dQ \tag{8}$$

$$= \mu_a \, dN_a + \Delta\Phi \, dQ, \tag{9}$$

where employing Eqs. (5) and (6), we see the chemical potentials to be related as

$$\mu_+ = m_i \mu - e\Delta\Phi, \tag{10}$$

$$\mu_- = m_e \mu + e\Delta\Phi, \tag{11}$$

$$\mu = \mu_a/(m_i+m_e). \tag{12}$$

Because μ_-, μ_+, μ_a, μ, and $\Delta\Phi$, respectively, denote the energy needed to bring an electron, an ion, an atom, unit mass, and unit charge from infinity to inside the system, these are fixed quantities, and we are not at liberty to alter them by an additive constant. We especially have

$$\Delta\Phi = \Phi(\mathbf{r}) - \Phi(\infty), \tag{13}$$

where Φ is the electrostatic potential, which is fixed only up to an arbitrary gauge transformation, and which forms a four-vector with the vector potential **A**. Note that the energy and chemical potentials employed in this paper include the rest mass, because otherwise the relativistic transformation formulas of the following sections will not work. Hence we define the nonrelativistic chemical potentials with tilde,

$$\mu_+ = m_i c^2 + \tilde{\mu}_+, \tag{14}$$

$$\mu_- = m_e c^2 + \tilde{\mu}_-, \tag{15}$$

$$\mu = c^2 + \tilde{\mu}, \tag{16}$$

$$\mu_a = (m_i+m_e)c^2 + \tilde{\mu}_a. \tag{17}$$

Usually, both $\tilde{\mu}_a$ and $\tilde{\mu}_-$ are of the order of a few electron volts, this being the scale of atomic physics, compare an early model calculation.[11] Since $m_e/(m_i+m_e) \approx 10^{-5}$, Eqs. (11) and (12) imply, to great accuracy,

$$\tilde{\mu}_- = e\Delta\Phi, \tag{18}$$

with

$$\tilde{\mu}_-/m_e \approx 10^{-5}\, c^2, \quad \tilde{\mu} \approx 10^{-10}\, c^2. \tag{19}$$

B. Josephson equation and superfluid velocity

The Josephson equation is usually given as

$$(\hbar/2)\dot{\varphi} + \mu_- = 0, \tag{20}$$

though this equation is not gauge invariant. In view of Eqs. (11) and (13), the correct form for the Josephson equation is clearly

$$(\hbar/2)\dot{\varphi} + m_e \mu + e\Phi = 0, \tag{21}$$

implying that Eq. (20) is valid only if $\Phi(\infty)=0$, a special gauge choice. [Similarly, taking $\dot{\varphi}=0$ when considering Eq. (21) also constitutes a gauge choice.] Note that the form of Eq. (21) does presume rest frame of the crystal, and equilibrium. The superfluid velocity is also defined as a gauge-invariant quantity

$$\mathbf{v}^s \equiv \frac{1}{m_e}\left(\frac{\hbar}{2}\nabla\varphi - \frac{e}{c}\mathbf{A}\right). \tag{22}$$

Its equation of motion reads

$$\dot{\mathbf{v}}^s + \nabla\mu = e\mathbf{E}/m_e, \tag{23}$$

obtained by inserting Eq. (22) in (21) and employing $\mathbf{E} = -(\nabla\Phi + \dot{\mathbf{A}}/c)$. Although this pleasingly simple equation, independent from any coefficients, may be used to justify the definitions of Eqs. (21) and (22), we must be aware that $\mathbf{v}^s$ as defined need not transform as a velocity under a Galilean-Lorentz boost. The transformation behavior of $\mathbf{v}^s$ is given by the Josephson equation (21): Define a four-velocity

$$u_\alpha \equiv \frac{\hbar}{2m_e}\frac{\partial\varphi}{\partial x^\alpha} - \frac{e}{m_e c}A_\alpha \tag{24}$$

$$= \frac{\hbar}{2m_e}(\dot\varphi/c, \nabla\varphi) - \frac{e}{m_e c}(-\Phi, \mathbf{A}), \tag{25}$$

and use Eq. (21) to yield

$$u_\alpha = (-\mu/c, \mathbf{v}^s). \tag{26}$$

Now, because u_α transforms as

$$u'_\alpha = (u_0 - \mathbf{u}\cdot\mathbf{v}/c, \mathbf{u} - u_0\mathbf{v}/c) \tag{27}$$

under a boost of v, to linear order in v/c, so does the four-vector $(-\mu/c, \mathbf{v}^s)$, leading to

$$\mu' = \mu + \mathbf{v}^s\cdot\mathbf{v}, \tag{28}$$

$$(\mathbf{v}^s)' = \mathbf{v}^s + (\mu/c^2)\mathbf{v}, \tag{29}$$

implying that it is the quantity

$$\mathbf{v}^{ts} \equiv (\mu/c^2)^{-1}\mathbf{v}^s = \mathbf{v}^s/(1+\zeta) \tag{30}$$

that transforms as a velocity, $(\mathbf{v}^{ts})' = \mathbf{v}^{ts} + \mathbf{v}$. We shall refer to $\mathbf{v}^{ts}$ as the true-superfluid velocity—although with

$$\zeta = \tilde\mu/c^2 = (\tilde\mu_- - e\Delta\Phi)/m_e c^2 \approx 10^{-10}, \tag{31}$$

cf. Eqs. (11), (16), and (19), the difference is very small indeed. Defining an effective mass m_e^* by

$$m_e^* \equiv m_e(1+\zeta), \tag{32}$$

$$\mathbf{v}^{ts} = \frac{1}{m_e^*}\left(\frac{\hbar}{2}\nabla\varphi - \frac{e}{c}\mathbf{A}\right), \tag{33}$$

we shall find in the next section that this is the mass appearing in the formula for the London moment, Eq. (1).

C. Thermodynamics and hydrodynamics

In a general inertial frame, the thermodynamics of a superconducting solid is given as

$$d\varepsilon = T ds + \mu d\varrho + \mathbf{v}^n\cdot d\mathbf{g} + \sigma_{ij} d\nabla_j u_i + \mathbf{E}\cdot d\mathbf{D} + \mathbf{H}\cdot d\mathbf{B} + \mathbf{j}^s\cdot d\mathbf{v}^s. \tag{34}$$

Notation and explanation: s is the entropy density, and T the temperature. $\mu\, d\varrho$ and $\mathbf{E}^0\cdot d\mathbf{D}$ are the respective local expressions for $\mu\, dM$ and $\Delta\Phi\, dQ$ of Eq. (8), where $\mathbf{E}\cdot d\mathbf{D}$ involves a partial integration, and is manifestly gauge invariant; $\mathbf{H}\cdot d\mathbf{B}$ is the magnetic counter term. $\mathbf{j}^s\cdot d\mathbf{v}^s$ characterizes the broken phase invariance of superconductivity, as $\sigma_{ij} d\nabla_j u_i$ the broken translational invariance of solids—with u_i the displacement vector. In a general inertial frame, the total, conserved-momentum density $\mathbf{g}^{\mathrm{tot}}$ is also a thermodynamic variable, though we employ $\mathbf{g} \equiv \mathbf{g}^{\mathrm{tot}} - \mathbf{D}\times\mathbf{B}$ instead[12], with $\mathbf{v}^n \equiv \partial\varepsilon/\partial\mathbf{g}$ being the equilibrium velocity of crystal points, atoms, and normal electrons. In the present case of interest,

$$\mathbf{v}^n = \mathbf{\Omega}\times\boldsymbol{r}. \tag{35}$$

Being a conjugate variable, $\mathbf{j}^s \equiv \partial\varepsilon/\partial\mathbf{v}^s$ is a function of these two thermodynamic variables $\mathbf{v}^s$ and $\mathbf{v}^n$, which are also odd under time reversal. In an expansion, to linear order of the variables, we have

$$j_i^s = (c^2/\mu)\rho_{ij}^s(v_j^s - \alpha_{jk}v_k^n), \tag{36}$$

where ρ_{ij}^s and α_{jk} are two expansion coefficients, while the prefactor (c^2/μ) simply renormalizes ρ_{ij}^s. A Maxwell relation then implies

$$\left(\frac{\partial g_i}{\partial j_j^s}\right)_{v_n} = \left(\frac{\partial v_j^s}{\partial v_i^n}\right)_{j_s} = \alpha_{ji}. \tag{37}$$

Confining ourselves to the local rest frame and disregarding dissipative terms (then $\dot s, \dot u_i = 0$), the hydrodynamic set of equations is given by the Josephson equation (23), the Maxwell equations, and the conservation laws for energy and mass

$$\dot\varepsilon = -\nabla\cdot\mathbf{Q}, \quad \dot\varrho + \nabla\cdot\mathbf{j}_\rho = 0, \tag{38}$$

$$\dot{\mathbf{B}} = -c\nabla\times\mathbf{E}, \quad \dot{\mathbf{D}} = c\nabla\times\mathbf{H} - \mathbf{j}_e, \tag{39}$$

where $\mathbf{Q}$, $\mathbf{j}_\rho$, and $\mathbf{j}_e$ are as yet unknown. Inserting the expressions into the temporal derivative of Eq. (34), $\dot\varepsilon = \mu\dot\varrho + \mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{j}^s\cdot\dot{\mathbf{v}}^s$, and insisting that all equations are satisfied simultaneously, we find

$$\mathbf{j}_\rho = \mathbf{j}_s, \quad \mathbf{j}_e = e\mathbf{j}_s/m_e, \tag{40}$$

$$\mathbf{Q} = \mathbf{j}_s\mu + c\mathbf{E}\times\mathbf{H}. \tag{41}$$

Clearly, $\mathbf{j}_s \equiv \partial\varepsilon/\partial\mathbf{v}^s$ has the significance of being the persistent mass and electric current in the rest frame. The relation

$$\mathbf{j}_\rho = e\mathbf{j}_s/m_e, \tag{42}$$

is a necessary one, as the transfer of one electron is coupled to the transfer of the bare values of e and m_e. Starting from this relation and tracing it back, we would have found that the charge e in Eq. (22) must indeed be the bare charge, and that $\mathbf{v}^s$ must indeed be gauge invariant. Because of the symmetry of the energy stress four-tensor, $\boldsymbol{g}^{\mathrm{tot}} = \mathbf{Q}/c^2$, we have

$$\mathbf{g}^{\mathrm{tot}} = \mathbf{g} = \mathbf{j}_s\mu/c^2, \tag{43}$$

for $E, D = 0$ (and still in the local rest frame). This implies

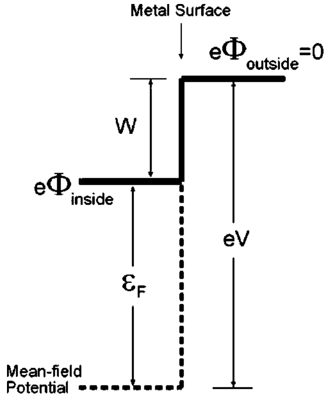

FIG. 2. The relations between the macroscopic electrostatic potential $e\Phi$, the work function W, the Fermi energy ε_F, and the mean-field potential eV, close to a metal surface.

$$\left(\frac{\partial g_i}{\partial j_j^s}\right)_{v_n}=\frac{\mu}{c^2}\delta_{ij}, \tag{44}$$

or in combination with Eq. (37),

$$j_i^s=\rho_{ij}^s[(c^2/\mu)v_j^s-v_j^n]=\rho_{ij}^s[v_j^{ts}-v_j^n]. \tag{45}$$

Inserting this expression into Eq. (38) with $\dot{\mathbf{D}}=0$, taking another curl on both sides, and denoting the matrix ρ_{ij}^s as $\hat{\boldsymbol{\rho}}^s$, we finally obtain

$$\begin{aligned}\nabla\times\left[\frac{m_e c}{e}(\hat{\boldsymbol{\rho}}^s)^{-1}\nabla\times\mathbf{H}\right]&=\nabla\times(\mathbf{v}^{ts}-\mathbf{v}^n)\\&=-\frac{e}{m_e(1+\zeta)c}\mathbf{B}-2\boldsymbol{\Omega}.\end{aligned} \tag{46}$$

This is the equation that accounts for the equilibrium behavior of the magnetic field in superconductors: $(m_e c/e)(\hat{\boldsymbol{\rho}}^s)^{-1}$ yields the square of the inverse-penetration depth, a quantity that depends on the crystal direction, while the vanishing of the right-hand expression gives the bulk value of the magnetic field, Eq. (1). Because of Eq. (31), we may generally neglect the factor $1+\zeta$.

III. THE SQUARE-WELL POTENTIAL

Consider a square-well potential (see Fig. 2), a popular model for metals. Taking the outside value of the energy as zero, the depth of the well is $eV(<0)$, with V the mean-field potential for electrons; filling the potential up to the Fermi energy $\varepsilon_F(>0)$, the gap still remaining may be identified as the work function of the electron $W(<0)$. Together, they satisfy

$$W-\varepsilon_F=eV. \tag{47}$$

When making contact with thermodynamics, there are no doubts that the identification

$$W=\tilde{\mu}_- \tag{48}$$

holds, because the physical significance of W and $\tilde{\mu}_-$, the energy needed to take out an electron, and the energy gained when putting one in, respectively, is the same. The other two scales, the Fermi energy ε_F and the mean-field potential V, are model-dependent quantities, with a large measure of arbitrariness attached to them. So we should not expect them to be directly measurable, or connected to thermodynamic quantities in a simple fashion. Nevertheless, all mentioned authors employing the microscopic approach to understand the London moment[3–7] adopt the identification

$$V=\Delta\Phi, \tag{49}$$

with some apparently plausible consequences. First, with Eqs. (31) and (32), we have $m_e^*=m_e(1+\zeta)$ with

$$\zeta(m_e c^2)=\tilde{\mu}_--e\Delta\Phi=W-eV=\varepsilon_F. \tag{50}$$

This makes the mass correction positive, and rather tangible: With the estimates $W\approx-4$ eV, $eV\approx-96$ eV, and $\varepsilon_F\approx 92$ eV (by averaging the electron wave functions over the Fermi surface), the value $\zeta\approx1.8\times10^{-4}$ was found[5] for the mass correction—as compared to $\zeta\approx-10^{-10}$ of the last section. Moreover, with $\varepsilon_F=\frac{1}{2}m_e v_F^2$, Eq. (50) delivers the simple, kinematic-relativistic interpretation for the mass correction,

$$\zeta=\frac{1}{2}(v_F/c)^2\approx1.8\times10^{-4}. \tag{51}$$

So why do we claim that Eqs. (49)–(51) are in error? The identification of Eq. (49) is made by taking the macroscopic electrostatic potential $\Delta\Phi$ as the potential felt by superconducting electrons from *all charge distributions in the metal, such as surface dipoles…, the screening hole…, and charge inhomogeneities associated with atomic cores and valence electrons*.[4] (Sentences in italic are quotes, here and below.) Clearly, the one ostentatiously lacking in this list is the contribution from the other band electrons (including the superconducting ones), to the potential, which brings the potential energy, gained by the last electron to be added to the system, to

$$W=\tilde{\mu}_-=e\Delta\Phi, \tag{52}$$

cf. Eq. (18). This is the thermodynamic definition of the potential $\Delta\Phi$, see the discussion of the last section, and we are not at liberty to alter it. In fact, as already mentioned, it would have been highly surprising for the directly measurable macroscopic electrostatic potential Φ that forms a four-vector with the vector potential **A** to be simply related to the model-dependent mean-field potential V.

Clearly, inserting Eq. (52) into Eq. (50), we have $\zeta m_e c^2=\tilde{\mu}_--e\Delta\Phi=0$, reducing the mass correction to (essentially zero—as in the last section. Now, even with Eq. (49) false and the derivation of Eq. (51) incorrect, a relativistic-kinetic interpretation of the Fermi velocity may still appear to leave $\zeta=\frac{1}{2}(v_F/c)^2$ intact. As it is a rather popular preconception, and employed by two more recent papers without the detour via the Fermi energy,[13] we need to emphasize here that this simplistic interpretation is really only valid for the case of free particles, and does not at all work for strongly interacting systems.

To understand this, consider an atom consisting of an electron and an ion—the same way the metal consists of band electrons and the rest. We may write the energy of the atom as $E=(m_i^*+m_e^*)c^2$, or

$$(m_i^*+m_e^*-m_i-m_e)c^2=W, \tag{53}$$

where the superscript * denotes the effective mass including the mass defect, while W the binding energy between the electron and the ion, the analog of the work function above. Note that the binding energy W includes all contributions, especially the kinetic energy of the electron. Microscopically, we may divide W between m_i^* and m_e^* arbitrarily. But if the division is in proportion to the rest mass, a natural (if not obvious) procedure, the mass defect of the electron is

$$m_e^*=m_e\left(1+\frac{W}{m_ec^2}\,\frac{m_e}{m_i+m_e}\right), \tag{54}$$

in agreement with the result of the last section, see Eqs. (12), (17), and (31).

Baym considered the problem of mass correction within the Landau theory of Fermi liquid in his work.[7] The theory is beyond the usual mean-field approach and accounts for many-body effects. Because his arguments and calculations are complicated, we shall only quote his final expression

$$(m_e^*-m_e)c^2=\varepsilon_{kin}+\varepsilon_{int}-eV_c+\cdots, \tag{55}$$

where ε_{kin} is the kinetic energy of a Fermi electron, similar to ε_F above, while the last two terms are new: V_c is given by *the average of the electrostatic potential within a given unit cell* (e.g., the Wigner-Seitz cell), *arising from the charges within the cell.* And ε_{int} is *the interaction energy of an electron at the Fermi surface, containing electrostatic as well as exchange contributions* (but excluding the boundary dipole layer). The value of eV_c was given as -19.8 eV, while the value of ε_{int} remains unknown. Because ε_{int} not only includes many-body interactions, but also electrostatic contribution, it may well be an appreciable term, canceling other contributions to yield virtually vanishing total correction, in agreement with the thermodynamic result. Baym also points out that corrections from the presence of the lattice are to be expected, because the Landau Fermi liquid theory is confined to translationally invariant systems, of which metal electrons are not one. Only a generalized Landau theory that also considers the ions, would indeed be dealing with a system that is, in its totality, translationally invariant. This should result in more generally valid relations, replacing relations such as $\mathbf{g}=[\mu_-/(m_ec^2)]\mathbf{j}_s$ as given in Ref. 7, with $\mathbf{g}=[\mu_a/(m_e+m_i)c^2]\mathbf{j}_s$, or Eq. (43).

IV. THE BOUNDARY DOUBLE LAYER

Because of the presence of surface dipoles, the electrostatic potential is discontinuous at the surface of the superconductor, $\Delta\Phi\neq 0$. If the metal is in rotation, the vector potential will also be discontinuous, contributing to the total flux of the system in the laboratory frame. Since the magnitude of this is $\sim\Delta\Phi$, taking $e\Delta\Phi$ either as $\mu_-=W\approx -4$ eV, Eqs. (18) and (19), or as $eV\approx -96$ eV, Eq. (49), clearly makes a big difference. As argued at length in the last two sections, we believe the first is correct, as does the book[14] by Landau and Lifshitz, who unequivocally pronounced the equality between the work function and the potential's discontinuity, $W=e\Delta\Phi$, see Sec. 23.

The jump of the vector potential follows from the Lorentz-transformation property of the electromagnetic four-potential A_α. Consider a frame comoving with the metal. If A_0' denotes the electric potential outside the metal, the potential inside the metal will be $A_0'-\Delta\Phi$. So the four-potential A_α' in the local rest frame is $(A_0',\mathbf{A}')$ outside the metal, and $(A_0'-\Delta\Phi,\mathbf{A}')$ inside it. In the laboratory frame, the four-potential A_α becomes, in linear order of v/c,

$$(A_0'-\mathbf{v}\cdot\mathbf{A}'/c,\mathbf{A}'-A_0'\mathbf{v}/c)$$

outside the metal, and

$$(A_0'-\Delta\Phi-\mathbf{v}\cdot\mathbf{A}'/c,\mathbf{A}'-A_0'\mathbf{v}/c+\Delta\Phi\mathbf{v}/c)$$

inside the metal, where $\mathbf{v}$ is the velocity of the boundary. So the discontinuity in the vector potential is

$$\Delta\mathbf{A}\equiv\mathbf{A}^{out}-\mathbf{A}^{int}=-(\Delta\Phi/c)\mathbf{v}. \tag{56}$$

For a metal cylinder uniformly rotating about the z axis $\hat{\boldsymbol{e}}_z$, we have $\mathbf{v}=\Omega\hat{\mathbf{e}}_z\times\mathbf{R}$, with $\mathbf{R}$ the position vector of the boundary. So the magnetic flux of a nonsuperconducting, rotating metal is

$$\oint\Delta\mathbf{A}\cdot d\mathbf{s}=-2\pi R^2\Delta\Phi\Omega/c. \tag{57}$$

Since the experiment[10] measures both this boundary flux and the London field simultaneously, it is convenient to introduce an observed mass $m_{\rm obs}$ (if the effect due to the penetration depth is negligible, as is the case in[10]). We write it as

$$m_{\rm obs}=(1+\zeta)(1-\alpha)m_e=(1-\alpha)m_e^*, \tag{58}$$

where α accounts for the flux from the boundary double layer. In the cited microscopic considerations,[3,7,10] it is the Fermi energy $\varepsilon_F\approx 92$ eV that enters ζ, and the mean-field potential $eV\approx -96$ eV that enters α, though in opposite directions. Combined, the observed mass is corrected only by the work function $W=eV+\varepsilon_F\approx -4$ eV, cf. Eq. (47), leading to a total correction of about 8×10^{-6}. Thermodynamically, although $\zeta\approx -10^{-10}$ is negligible, yet since the work function W does enter α, the measured effect is again given by a value around -4 eV.

Since the same value for $m_{\rm obs}$ is predicted by all theories, one may conjecture that the values of ζ or α individually are unimportant, because any experiment can only observe $(m_{\rm obs}/m_e)-1=\zeta-\alpha$. This does not seem right to us. First, both the strength of surface double layer of a metal and the London field are well-defined physical quantities. And the parameters ζ and α are unambiguously related to the chemi-

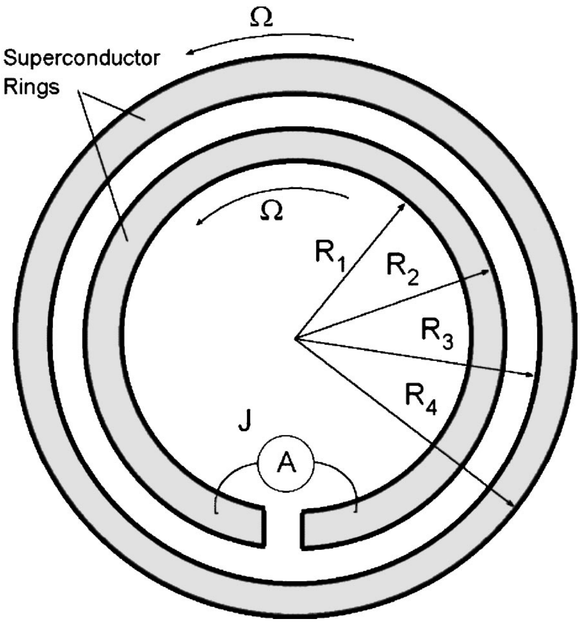

FIG. 3. Apparatus for measuring the boundary flux in the superconducting state. Superconducting metal is depicted by shaded area.

cal potential and the work function, respectively. Second, both effects are (at least in principle) independently measurable. The London field could be observed directly by measuring the flux near the top center of a rotating superconductor (see Fig. 1), and the flux from the double layer only may be observed in the normal state, or even in the superconducting state, by the method given in the next chapter.

V. A PROPOSED EXPERIMENT

Consider two concentric, corotating, hollow cylinders made of the same superconducting metal. The electric current J flowing through the inner cylinder is measured by a superconducting quantum interference device (SQUID), see Fig. 3. The cylinders rotate with the angular velocity Ω. We calculate the magnetic field and the current in the apparatus. For simplicity we assume that the cylinders are of infinite height, and no surface charges (or electric field) are present.

This geometry was first analyzed by Brady,[4] who estimated the effect of penetration depths λ by simply assuming constant magnetic fields in the regimes: $0 \leqslant r < R_1 + \lambda$ and $R_2 - \lambda < r < R_3 + \lambda$, in other words, by assuming that the actual internal spaces, $0 < R_1$ and $R_2 < r < R_3$, are increased by λ at the superconducting boundaries. Moreover, because the magnitudes of the field was estimated by dividing the magnetic flux by the respective area, the contributions from the boundary layer have also been neglected.

The current J in the inner cylinder as determined by Brady is proportional to the observed mass m_{obs} and the difference of the angular velocities of the two cylinders. This was given as a method to measure the London field. In our case, both angular velocities are the same, so J would vanish in his approximation. Our more detailed calculation, however, shows that J is finite and observable if the distance between the two cylinders $R_3 - R_2$ is small enough. In fact, it is proportional to $\alpha\Omega$, rendering α, or the flux from the boundary layer, measurable.

The magnetic fields and the current density inside the cylinders can be obtained by solving Eq. (46) in the cylindrical symmetry. Considering the isotropic case $\rho_{ij}^s = \varrho_s \delta_{ij}$ with ϱ_s the superfluid density, we write the vector fields as

$$\mathbf{B} = B\mathbf{e}_z, \quad \mathbf{A} = A(\mathbf{e}_z \times \mathbf{r}/r),$$

$$\mathbf{j}^{\mathrm{e}} = j^{\mathrm{e}}(\mathbf{e}_z \times \mathbf{r}/r). \tag{59}$$

The general solution for $\mathbf{H} = \mathbf{B}$ is

$$B = 2\gamma^* \Omega + C_1 I_0 + C_2 K_0. \tag{60}$$

Notations: $\gamma^* \equiv -m_e^* c/e$; C_{1-5} are integration constants; $\lambda = (m_e c/|e|)\sqrt{(1+\zeta)/\varrho_s}$ is the penetration depth; I_N, K_N ($N = 0,1$) are Bessel functions with the arguments r/λ; $\phi_0 = -\hbar c/2e$ is the quantum fluxoid.

Inserting Eq. (60) into Eqs. $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, $\mathbf{j}^e = c \boldsymbol{\nabla} \times \mathbf{B}$, and noting the quantization property $\oint \boldsymbol{\nabla}\varphi \cdot d\mathbf{s} = 2\pi n$ for the phase in Eq (33), we obtain the magnetic potential and current density

$$A = \gamma^* \Omega r - n\phi_0/r + \lambda(C_1 I_1 - C_2 K_1), \tag{61}$$

$$j^e = -(c/\lambda)(C_1 I_1 - C_2 K_1). \tag{62}$$

The integer n denotes quantum state of the cylinder. The induction B is constant outside the rings, while the potential is of the form

$$A = Br/2 + C/r \tag{63}$$

with C again a constant. Using the above solutions, it can be easily seen that the field of our apparatus is

$$B = \begin{cases} B_1, & r < R_1 \\ 2\gamma^*\Omega + C_1 I_0 + C_2 K_0, & R_1 < r < R_2 \\ B_2, & R_2 < r < R_3 \\ 2\gamma^*\Omega + C_3 I_0 + C_4 K_0, & R_3 < r < R_4, \end{cases} \tag{64}$$

and the potential is

$$A = \begin{cases} B_1 r/2, & r < R_1 \\ \gamma^*\Omega r - n_1\phi_0/r + \lambda(C_1 I_1 - C_2 K_1), & R_1 < r < R_2 \\ B_2 r/2 + C_5/r, & R_2 < r < R_3 \\ \gamma^*\Omega r - n_2\phi_0/r + \lambda(C_3 I_1 - C_4 K_1), & R_3 < r < R_4. \end{cases} \tag{65}$$

Note that n_1, n_2 may be different, because the two cylinders can be in different quantum states. The current per unit length in the inner cylinder is

$$\begin{aligned} J &= \int_{R_1}^{R_2} j^e \, dr \\ &= -c[C_1(I_{02} - I_{01}) + C_2(K_{02} - K_{01})]. \end{aligned} \tag{66}$$

Here and below we will use the notations

$$I_{NM} = I_N(R_M/\lambda), \quad K_{NM} = K_N(R_M/\lambda),$$

where the integers $N=0,1$, $M=1-4$. The unknown constants C_{1-5}, B_1, B_2 are determined from the boundary conditions. At R_1, according to Eqs. (64) and (65), we have

$$B_1 = 2\gamma^*\Omega + C_1 I_{01} + C_2 K_{01}, \tag{67}$$

$$B_1 R_1/2 = (\gamma^* - \Delta\Phi/c)\Omega R_1 - n_1\phi_0/R_1 + \lambda(C_1 I_{11} - C_2 K_{11}), \tag{68}$$

where Eq. (67) shows that the induction varies continuously at the surface. Equation (68) is obtained from the jump of the potential Eq. (56). Similarly, at R_2 we have

$$B_2 = 2\gamma^*\Omega + C_1 I_{02} + C_2 K_{02}, \tag{69}$$

$$B_2 R_2/2 + C_5/R_2 = (\gamma^* - \Delta\Phi/c)\Omega R_2 - n_1\phi_0/R_2 + \lambda(C_1 I_{12} - C_2 K_{12}), \tag{70}$$

and at R_3,

$$B_2 = 2\gamma^*\Omega + C_3 I_{03} + C_4 K_{03}, \tag{71}$$

$$B_2 R_3/2 + C_5/R_3 = (\gamma^* - \Delta\Phi/c)\Omega R_3 - n_2\phi_0/R_3 + \lambda(C_3 I_{13} - C_4 K_{13}). \tag{72}$$

The induction is zero beyond R_4. (We consider the case of no externally applied field, i.e., all the sources of the field are from supercurrents in the rings.) We have then

$$2\gamma^*\Omega + C_3 I_{04} + C_4 K_{04} = 0. \tag{73}$$

Solving Eqs. (67)–(73) for C_{1-5}, B_1, B_2 and inserting them into Eq. (66), we obtain the current. The result can be written as

$$J = [f_3 n_1 + f_4(n_2 - n_1)]J_0 + c\gamma^*(f_2 - f_1\alpha)\Omega, \tag{74}$$

with

$$\alpha = -\frac{e\Delta\Phi}{m_e^* c^2}, \quad J_0 = -\frac{\hbar c^2}{2eR_1^2}. \tag{75}$$

Here the factors f_{1-4} depend only on the penetration depth and geometry of the system. For observing the boundary effect, it is convenient to choose the geometry such that $|f_2| \ll |\alpha f_1|$. One example satisfying the requirement is: $R_1 = 2.5$ cm, $R_2 - R_1 = 1\,\mu$m, $R_3 - R_2 = 40$ nm, $R_4 - R_3 = 1\,\mu$m, $\lambda = 40$ nm. We have obtained, by the numerical computation, the values

$$f_1 = 1.333\,33, \quad f_2 = 1.852 \times 10^{-11},$$

$$f_3 = -1.999\,996\,8, \quad f_4 = 2.083 \times 10^7.$$

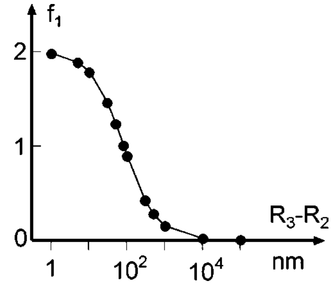

FIG. 4. Variation of the factor f_1 with the separation distance of the two rings.

Because f_4 is very large, it is not probable for the two rings to lay in different quantum states. So within a good approximation (and also neglecting $\zeta \ll 1$) we have

$$J = -2J_0 n_1 - f_1(\Delta\Phi)\Omega, \tag{76}$$

with $f_1 = 4/3$ for the geometry considered. This shows that the strength of the surface double layer $\Delta\Phi$ can be measured by measuring the change of the current with the rotation. Note that the factor f_1 decreases with increasing distance $R_3 - R_2$ between the two cylinders, see Fig. 4. When the distance is large, $f_1 \to 0$, the last term in Eq. (74) is negligible and we return to the result by Brady.[4]

VI. CONCLUSIONS

The conclusion of this paper is that the relativistic correction to the London field is not the result of the Fermi velocity. Instead, it is given by the chemical potential μ of the metal, which quantifies the complicated interaction among all the particles, including that between electrons and ions. Because the interaction energy is typically several electron volt per atom, while the mass of an atom is around 10^4 MeV, the relativistic correction is tiny, of order 10^{-10}, and beyond the scope of any present experimental techniques. When previous microscopic theories considered the London moment, they considered the kinetic contribution to the correction ζ of the electron's mass, but neglected other contributions, especially from the interaction between the electrons and the lattice. In addition, the discontinuity of the macroscopic electrostatic potential at the metal surface was incorrectly taken to be the mean-field potential, or the sum of the kinetic energy and the work function. As soon as these errors are revised, ζ is found to essentially vanish. Unfortunately, the reason for the discrepancy between the experiment[10] and all theories remains unclear.

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[1] F. London, *Superfluids* (Wiley, New York, 1950), Vol. I.

[2] The Heaviside-Lorentz units, or the so-called *rational* units, employed in this paper render the formulas and their manipulations simple: In this system, all four fields have the same dimension, square root of the energy density, i.e., $\sqrt{\mathrm{J/m^3}}$ in MKS and $\sqrt{\mathrm{erg/cm^3}}$ in cgs, sensibly with $H=B$ and $D=E$ in vacuum. As a result, the ubiquitous factors of 4π, ϵ_0 and μ_0—the actual reason why we have to continually look up the formulas—simply vanish. To revert to MKSA ($\hat{E}, \hat{H}, \ldots$), employ $\hat{H} = H/\sqrt{\mu_o}$, $\hat{B} = B\sqrt{\mu_o}$, $\hat{E} = E/\sqrt{\epsilon_o}$, $\hat{D} = D\sqrt{\epsilon_o}$, and $\hat{\varrho}_e = \varrho_e\sqrt{\epsilon_o}$, $\hat{j}_e = j_e\sqrt{\epsilon_o}$. To revert to the Gaussian system ($\tilde{E}, \tilde{H}, \ldots$), increase all four fields, and decrease all four sources, by the factor of

$\sqrt{4\pi}$: $B=\tilde{B}/\sqrt{4\pi}$ (similarly for D,H,E); $\varrho_e=\sqrt{4\pi}\tilde{\varrho}_e$ (similarly for j,P,M).

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