Linear renormalization transformation for weakly interacting spin chains

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The linear renormalization-group transformation is proposed to study critical temperatures and temperature dependence of the thermodynamic values (free energy, specific heat) of the weakly interacting spin chains. The method is examined in two classical systems: the standard Ising model on a rectangular lattice and the Ising spin chains coupled by pair-pair interactions. In the latter case the transformation does not exhibit any nontrivial fixed point. This result agrees with the exactly determined free energy of the system under consideration. The method is also applied to the weakly interacting quantum *XY* spin chains. The critical temperature as a function of the interchain interactions and the temperature dependence of the specific heat are found.

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I. INTRODUCTION

Compounds made of spin chains with the interchain coupling weaker than intrachain coupling attract continuous attention both by theoreticians and experimentalists.[1–5] The reason is their model character as quasi-one-dimensional magnetic systems in which the interchain interactions can be neglected in a wide range of temperature. It allows to treat these compounds as really one-dimensional quantum [2,4] or classical[3,6] spin systems. However, at sufficiently low temperature, the interchain interaction that is responsible for an eventual magnetic ordering becomes crucial a one and, of course, it has to be taken into account.

In this paper we propose the linear real-space renormalization-group method that can be applied to study weakly interacting classical and quantum spin chains. The method is not based on the Migdal-Kadanoff[7] (MK) bond-moving approximation but uses the existence of the small parameter—the ratio of interchain to intrachain coupling.

The undisputable success of the Niemeijer and Van Leeuwen real-space renormalization-group (RSRG) method[8] as a tool for the investigation of two-dimensional Ising spin systems has shown that many attempts have been made to generalize this method for quantum spin systems. However, the first results were rather inconclusive not only regarding the character of the singularity near the transition temperature but also the existence of a phase transition at all.[9] The quantum extension of the MK approach was proposed by Suzuki and Takano (ST).[10] The ST method has been based on an approximate decimation for one-dimensional systems and generalized to quantum spin systems in higher-dimension by means of the MK transformations. The authors pointed out that the simple MK approach gives qualitatively reasonable results for two- and three-dimensional quantum spin models in the higher temperature region. The disadvantages of the latter method were discussed by Barma *et al.* [11] and Castellani *et al.*[12] Here, we wish only to remind that the MK approach gives rather poor quantitative results even for the two-dimensional Ising model both for the location of the critical point and critical indices and furthermore there is no possibility to construct any systematic approximation procedure within this method. For example, the MK procedure gives for the Ising model on the square lattice, the value of the inverse critical temperature $K_c \simeq 0.61$ whereas the exact value is $K_c \simeq 0.44$. Similarly, for the $s=1/2$ *XY* model ST approximation gives $K_c \simeq 1.2$ (Ref. 10) much larger than the value $K_c \simeq 0.64$ estimated from the high-temperature series expansion[13] or $K_c \simeq 0.71(0.67)$ found from Monte Carlo simulations by fitting to the exponential law[14] and power law,[15] respectively. Much better quantitative results concerning the location of the critical point of the two-dimensional $s=1/2$ *XY* model $K_c \simeq 0.62$ has been found by means of the rotationally invariant RSRG transformation.[16] However, in the case of weakly interacting spin chains the method based on the one-dimensional decimation seems to be a very natural approximation especially since it does not have to be connected with the MK bond-moving mechanism.

The outline of the remainder of this paper is as follows. In Sec. II, the linear-perturbation renormalization-group (LPRG) method for weakly interacting Ising spins chains is presented. In Sec. III, a generalization of the ST approximate decimation for one-dimensional quantum spin systems is discussed. In Sec. IV, the weakly interacting quantum spin chains are studied by means of the LPRG method. Our conclusion are given in Sec. V.

II. LINEAR-PERTURBATION RENORMALIZATION

In this section we present the LPRG method for weakly interacting Ising spins chains described by the Hamiltonian

$$H = K_1 \sum_{\langle ij \rangle} S_{i,j} S_{i,j+1} + K_2 \sum_{\langle ij \rangle} S_{i,j} S_{i+1,j}, \tag{1}$$

where the label i refers to rows and j to columns, the factor $-1/k_B T$ has already been absorbed in the Hamiltonian, and $K_2 < K_1$. We define the renormalization transformation by

$$\exp[H'(\sigma)] = \mathrm{Tr}_S P(\sigma, S) \exp[H(S)]. \tag{2}$$

The weight operator $P(\sigma,S)$ is chosen in the linear form

$$P(\sigma,S) = \frac{1}{2^{N/2}} \prod_{i,j=0} (1 + \sigma_{i+1,j+1} S_{2i+1,2j+1}), \tag{3}$$

so the transformation (2) is the decimation transformation and in the renormalization step, only every other spin from

every other row survives. For the Ising model in one dimension, i.e., for $K_2=0$ this decimation can be carried out exactly. So, it seems to be very natural to separate the Hamiltonian (1) in a manageable exactly unperturbed part H_0 containing all intrachain interaction (K_1) and a remainder V containing interchain interaction (K_2).

With the notation

$$z_0=\mathrm{Tr}_S P(\sigma,S)\exp[H_0(S)] \tag{4}$$

and

$$\langle A\rangle_0=\frac{1}{z_0}\mathrm{Tr}_S AP(\sigma,S)\exp[H_0(S)], \tag{5}$$

the transformation (2) can be rewritten as

$$H'(\sigma)=\ln[z_0]+\ln[\langle\exp(V)\rangle_0] \tag{6}$$

with the following cumulant expansion for $\langle\exp(V)\rangle_0$

$$\ln[\langle\exp(V)\rangle_0]=\langle V\rangle_0+\frac{1}{2!}(\langle V^2\rangle_0-\langle V\rangle_0^2)+\cdots. \tag{7}$$

To evaluate the cumulants (7) one has to know the averages $\langle S_i,\ldots,S_n\rangle$. In our case we have to consider the averages in, say, ''odd'' rows where in the renormalization step every other spin is decimated and in ''even'' rows where all spins are removed. Let $S_{2i+2,j}$ be a subset of decimated spins from ''even'' removed rows then

$$\langle S_{2i+2,j}\rangle=0,\quad \langle S_{2i+2,j}S_{2i+2,j+n}\rangle=t^n,$$

$$t=\tanh(K_1). \tag{8}$$

For the spins from ''odd'' rows the averages are

$$\langle S_{2i+1,2j+1}\rangle=\sigma_{i+1,j+1}, \tag{9}$$

$$\langle S_{2i+1,2j+2}\rangle=\frac{1}{2}r(\sigma_{i+1,j+1}+\sigma_{i+1,j+2}),$$

$$r=\tanh(2K_1), \tag{10}$$

$$\langle S_iS_j\rangle=\langle S_i\rangle\langle S_j\rangle. \tag{11}$$

It is seen from Eq. (8) that $\langle S_{2i+2,j}\rangle$ vanish so there is no contribution from the first-order cumulant expansion

$$\langle V\rangle_0=K_2\sum_{\langle ij\rangle}\langle S_{2i+1,j}\rangle\langle S_{2i+2,j}\rangle=0. \tag{12}$$

In the next order in V as usual, new interactions appear. Unfortunately, in our case the number of this new interactions already in the second order is infinite for an infinite system because all possible bilinear couplings between several spins from the adjoining renormalized rows come into play. Thus, in order to carry out the LPRG transformation we have to confine ourselves to a finite lattice. Notice that averages (9)–(11) are exact for any size of the chain provided one uses a closed ring. If we restrict our considerations to six spin clusters, then to the second order in the cumulant expansion, only one new interaction will be generated,

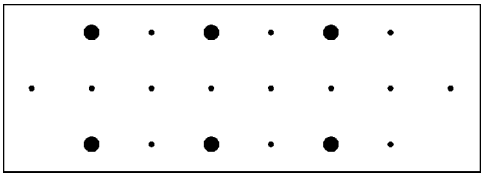

FIG. 1. The cluster used to get renormalized Hamiltonian (14). Small dots denote decimated spins.

$$K_3S_{i,j}S_{i+1,j+1}. \tag{13}$$

It should be noted that in our procedure we make two approximations. First, connected with the perturbation with regard to K_2, which is valid if $K_2<K_1$ and second, related to the truncation of the interactions generated in the LPRG procedure. The latter is better if $t=\tanh(K_1)\ll 1$, it means at higher temperatures.

Now it is relatively easy to find the renormalized Hamiltonian H' [Eq. (6)] for the cluster presented in Fig. 1 as

$$H'=\ln[z_0^{(1)}z_0^{(2)}z_0^{(3)}]+F_0+F_1\sum\ \sigma_{i,j}\sigma_{i,j+1}$$

$$+F_2\sum\ \sigma_{i,j}\sigma_{i+1,j}+F_3\sum\ \sigma_{i,j}\sigma_{i+1,j+1}, \tag{14}$$

where $z_0^{(n)}$ are exactly known renormalized interactions and partition functions of the odd and even chains, respectively,

$$z_0^{(odd)}=[\cosh(2K_1)+1]+[\cosh(2K_1)-1]\sigma_{i,j}\sigma_{i,j+1}, \tag{15}$$

$$z_0^{(even)}=2[\cosh(2K_1)+1], \tag{16}$$

and functions F can be written in the form

$$F_n=g_1^{(n)}K_2^2+g_2^{(n)}K_3^2+g_3^{(n)}K_2K_3. \tag{17}$$

The coefficients $g_m^{(n)}$ as functions of K_1 are presented in the Appendix.

Finally, the renormalized parameters are as follows:

$$K_1'=\frac{1}{2}\ln[\cosh(2K_1)+1]+F_1,\quad K_2'=F_2,\quad K_3'=F_3, \tag{18}$$

and the free energy per site

$$f=\sum_{n=1}^{\infty}\frac{G(K_i^{(n)})}{2^n}, \tag{19}$$

where

$$G(K_i)=\frac{1}{6}\{\ln[4\sqrt{\cosh(6K_1)+3\cosh(2K_1)}]+F_0\}. \tag{20}$$

We have evaluated numerically the renormalization transformation from the original set of coupling parameters K_α, ($\alpha=1,2,3$) to the set of the renormalized parameters K_α' [Eq. (18)] and have found two stable fixed points at $K_\alpha=0$ and $K_\alpha=\infty$, and the critical surface in the space of the param-

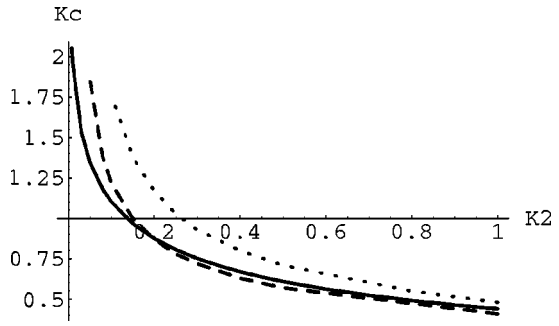

FIG. 2. Dashed curve, critical inverse temperature (K_c) found from the recursion relations (18); solid curve, exact result; dotted curve, the results found by using 3×3 spin cluster

eters K_1, K_2, and K_3. The critical inverse temperature $K_c=1/T_c$ as a function of $K2=K_2/K_1$ for $K_3=0$ is presented in Fig. 2 and compared with the exact results. In the same figure the critical inverse temperature found by using the LPRG method with a 3×3 spin cluster instead of the cluster presented in Fig. 1 is also shown.

The simple LPRG procedure in the lowest nontrivial approximation gives for $K_2=K_1$ (standard Ising model) the critical value $K_{1c}\simeq0.41$, which is in reasonable agreement with the exact value $K_{1c}\simeq0.44$. As mentioned above, the LPRG is rather a high-temperature approximation and can give much worse results for $K_1>1$. So, as one could expect, the deviation from the exact critical line is larger if a transition shifts to lower temperatures (large K_1) that occur for $K_2\ll K_1$. As can be seen from Fig. 2 for $0.15<K_2/K_1<1$ the deviations of the critical temperatures from the exact values are only a few percent. Of course, the thermodynamic values in the higher-temperature region ($K_1<1$) can be the better estimated the smaller the ratio K_2/K_1. Figures 3 and 4 show the numerical results for the specific heat and internal energy for several values of K_2/K_1 and $K_3=0$.

A. Four-spin interaction

As mentioned above, the LPRG transformation in the second-order calculation by using the cluster presented in Fig. 1 generates only one new bilinear interaction (13). However, using the same cluster one can include the four-spin interaction in the original Hamiltonian. Let us consider the

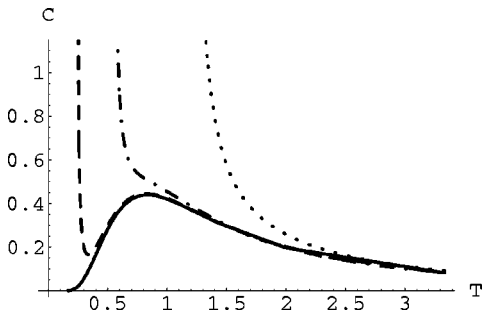

FIG. 3. Temperature dependence of the specific heat for several values of K_2/K_1. $K_2/K_1=0$ (solid line); 0.02 (dashed line); 0.05 (dashed-dotted line); 0.2 (dotted line).

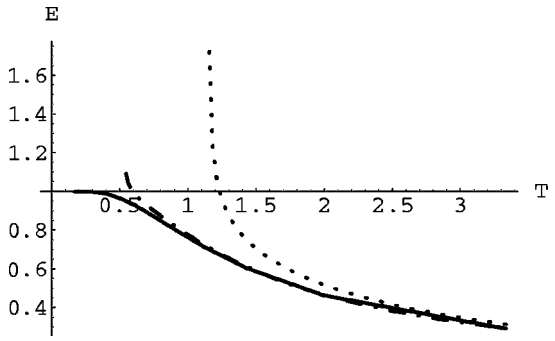

FIG. 4. Temperature dependence of the internal energy for several values of K_2/K_1. $K_2/K_1=0$ (solid line); 0.05 (dashed-dotted line); 0.2 (dotted line).

Ising spin chains interacting only via four-spin interactions with interacting spins located on a 2×2 square plaquette. This model corresponds to a Hamiltonian

$$H=K_1\sum_{\langle ij\rangle} S_{i,j}S_{i,j+1}+K_4\sum_{\langle ij\rangle} S_{i,j}S_{i,j+1}S_{i+1,j}S_{i+1,j+1}, \tag{21}$$

with $K_4<K_1$.

Following the same procedure as in the previous case one finds the renormalized Hamiltonian (6) in the form

$$H'=\ln[z_0^{(1)}z_0^{(2)}z_0^{(3)}]+F_0^{(4)}+F_1^{(4)}\sum\ \sigma_{i,j}\sigma_{i,j+1}$$

$$+F_4\sum\ \sigma_{i,j}\sigma_{i,j+1}\sigma_{i+1,j}\sigma_{i+1,j+1}, \tag{22}$$

where

$$F_1^{(4)}=rtK_4+\left[t^2(1-r^2)+\frac{1}{2}r^2(1-t^2)\right]K_4^2,$$

$$F_4=\frac{1}{2}r^2(1-t^2)K_4^2, \tag{23}$$

and

$$F_0^{(4)}=2rtK_4+\left[2(1-r^2t^2)+\frac{1}{2}r^2(1-t^2)\right]K_4^2. \tag{24}$$

Except for the contribution to the renormalized bilinear intrachain coupling, four-spin interaction K_4 generates only four-spin interaction. Finally, the second-order transformation equations read

$$K_1'=\frac{1}{2}\ln[\cosh(2K_1)+1]+F_1^{(4)},\quad K_4'=F_4. \tag{25}$$

It is easy to see that the transformation (25) has only trivial fixed points, and the two-dimensional Ising model made with spin chains coupled only by four-spin interaction (21) does not exhibit any finite-temperature phase transition. Figure 5 shows the numerical results obtained for the specific heat for several original values of K_4. These results can be compared

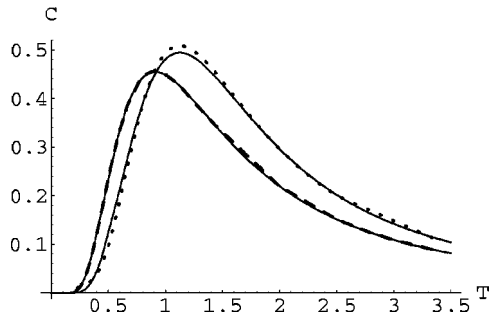

FIG. 5. Temperature dependence of specific heat for $K_4/K_1=0.05$ (dashed line); 0.2 (dotted line), solid lines denote the adequate exact results.

with the exact specific heat that can be easily found for the model described by the Hamiltonian (21).

Namely, after introducing a new variable

$$\eta_i^j = S_{ij}S_{ij+1}, \tag{26}$$

the Hamiltonian (21) can be rewritten in the form

$$H=\sum_j \left[K_1 \sum_i \eta_i^j + K_4 \sum_i \eta_i^j \eta_{i+1}^j \right], \tag{27}$$

giving in the thermodynamic limit *the exact free energy* per spin

$$f/T=\ln[\exp(\beta K_4)\cosh(\beta K_1) + \sqrt{\exp(2\beta K_4)\sinh^2(\beta K_1)+\exp(-2\beta K_4)}]. \tag{28}$$

From Eq. (28) we may, of course, calculate the exact specific heat that is presented in Fig. 5 for two values of $K_4/K_1=0.05$ and 0.2. It is seen that for sufficiently small K_4/K_1 the LPRG results are in very good agreement with the exact results.

III. LINEAR RSRG FOR ONE-DIMENSIONAL QUANTUM SYSTEM

Below in this section, we consider a one-dimensional quantum spin system defined by the Hamiltonian

$$H=K_x\sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + K_z \sum_i S_i^z S_{i+1}^z. \tag{29}$$

For a quantum system, because of the noncommutativity of several terms of the Hamiltonian the decimation transformation cannot be carried out exactly even for a one-dimensional lattice. However, Suzuki and Takano[10] proposed some approximate decimation that leads to reasonable values of the free energy for the one-dimensional quantum anisotropic Heisenberg model, especially in the high-temperature region. In fact, the authors applied the standard decimation procedure.[7] Dividing the chain into l-spin clusters and considering only one cluster, they found the renormalized nearest-neighbor interaction between the remaining spins $\vec{S}_0$ and $\vec{S}_n$ by using the transformation

$$\exp[H'(\vec{S}_0,\vec{S}_l)]=\left(\prod_{i=1}^{l-1} \mathrm{Tr}_{\vec{S}}\right)\exp\left[\sum_{i=0}^{l-1} H(\vec{S}_i,\vec{S}_{i+1})\right]. \tag{30}$$

This procedure takes quantum effect into account within a single cluster and neglects the effects of noncommutativity of several clusters. In the previous paper[17] we have proposed a simple extension of the Suzuki and Takano idea generalizing the linear RSRG [Eq. (3)]. The linear RSRG transformation can be used to study more general models, for example, models with many-spin interactions, and allows a systematic improvement of the ST approximation. We divide the Hamiltonian (29) into six spin clusters, and consider only one cluster—six spins on a ring. The renormalized Hamiltonian is defined as

$$\exp[H'(\vec{\sigma}_1,\vec{\sigma}_2,\vec{\sigma}_3)]=\frac{1}{8}\mathrm{Tr}_{\vec{S}}\prod_{i=0}^{2}(1+\vec{\sigma}_{i+1}\vec{S}_{2i+1}) \times\exp\left[\sum_{k=1}^{5} H(\vec{S}_k,\vec{S}_{k+1})\right]. \tag{31}$$

Similarly as in the ST procedure the quantum effects are taking into account only within a single cluster. In the ST decimation the renormalization transformation with scale factor l (l-spin clusters) is used to find the effective interaction between the first and the last spin of the cluster [Eq. (30)]. In the transformation (31) every other spin is removed and the renormalized interaction depends on three spins (generally on $n/2$, where n denotes number of spins in a single cluster).

Applying the transformation (31) to the Hamiltonian (29), one obtains the transformed Hamiltonian H' in the same form as the original Hamiltonian for new spin operators $(\vec{\sigma})$ with new parameters K_x' and K_z',[17]

$$K_x'=\frac{1}{6}(\lambda_1-\lambda_2), \quad K_z'=\frac{1}{12}(3\lambda_3-2\lambda_2-\lambda_1), \tag{32}$$

where

$$\lambda_1=\ln(F_0+4F_x-F_z),$$
$$\lambda_2=\ln(F_0-2F_x-F_z),$$
$$\lambda_3=\ln(F_0+3F_z), \tag{33}$$

with

$$F_0=\mathrm{Tr}_{\vec{S}}(\mathbf{P}\exp[H]),$$
$$F_\alpha=\mathrm{Tr}_{\vec{S}}(\mathbf{P}S_1^\alpha S_3^\alpha \exp[H]), \quad (\alpha=x,z), \tag{34}$$

and

$$\mathbf{P}=\frac{1}{8}\prod_{i=0}^{2}(1+\vec{\sigma}_{i+1}\vec{S}_{2i+1}). \tag{35}$$

As usual in each step of the transformation a constant term independent of σ^α appears,

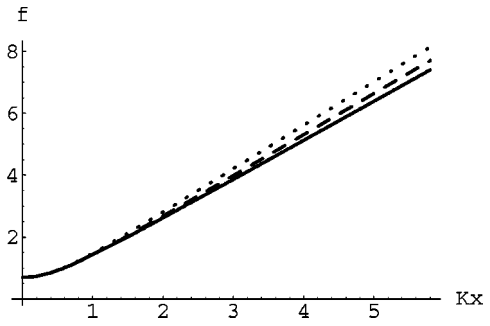

FIG. 6. The free energy of the quantum XY model obtained by decimation (dotted line); linear RSRG (dashed line); and the exact free energy obtained by Katsura (solid line).

$$G(K_i)=\frac{1}{12}(\lambda_1+2\lambda_2+\lambda_3), \tag{36}$$

and the free energy per site can be calculated by using formula (19). The found free energy f for the XY model is shown in Fig. 6 and compared with the exact result obtained by Katsura[18] and the results obtained by Suzuki and Takano.[10] For example, for inverse temperature $K_x=1$, our method leads to $f\simeq 1.4266$ that differs by less than 1% from the exact value $f\simeq 1.4152$. It is interesting to note that even the ground-state internal energy found by using transformation (28), $E=4/3$, is in a good agreement with the exact result $E=4/\pi\simeq 1.2732$ (ST approximation gives $E\simeq 1.4142$).

Figure 7 shows the results for the specific heat of the isotropic-Heisenberg, XY, and anisotropic-Heisenberg model obtained from Eq. (36).

IV. LPRG FOR COUPLED QUANTUM CHAINS

We consider the Heisenberg spins chains defined by the Hamiltonian (29) and coupled with adjacent chains by the interaction

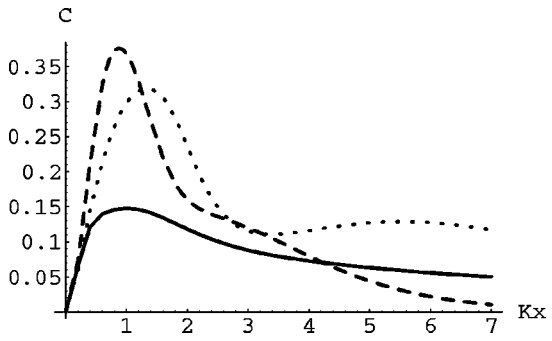

FIG. 7. The specific heat of the quantum spin models: the isotropic-Heisenberg model (solid line), the anisotropic-Heisenberg model $K_z/K_x=0.5$ (dotted line), and the XY model (dashed line).

$$V=K_{1x}\sum_{\langle ij\rangle}(S^x_{i,j}S^x_{i+1,j}+S^y_{i,j}S^y_{i+1,j})+K_{1z}\sum_{\langle ij\rangle}S^z_{i,j}S^z_{i+1,j}$$
$$+K_{2x}\sum_{\langle ij\rangle}(S^x_{i,j}S^x_{i+1,j+1}+S^y_{i,j}S^y_{i+1,j+1})$$
$$+K_{2z}\sum_{\langle ij\rangle}S^z_{i,j}S^z_{i+1,j+1}, \tag{37}$$

with $K_{n\alpha}<K_\alpha$ $(n=1,2\ \ \alpha=x,z)$. In comparison with the Ising model the application of the LPRG method to interacting quantum spins chains encounters some additional difficulties connected with lack of the exact solutions for a one chain, necessity of a decomposition of the exponential operators, and additional proliferation of the new interactions due to the vector character of the spin operator. Consequently, one has to use supplementary approximations.

We apply the cluster made with three six-spin rings. According to the procedure described in the Sec. II in the rings 1 and 3 (odd chains) every other spin survives whereas in the ring 2 all spins are removed. Because we do not know the exact form of the partition functions for several chains, we use the results of the previous section. Thus, the partition function is given by

$$z_0^{(odd)}=F_0+F_x\sum_{i=1}^{i=3}(\sigma^x_i\sigma^x_{i+1}+\sigma^y_i\sigma^y_{i+1})+F_z\sum_{i=1}^{i=3}\sigma^z_i\sigma^z_{i+1}, \tag{38}$$

where functions F are defined in Eq. (34) and

$$z_0^{(even)}=\mathrm{Tr}_{\tilde{S}}\exp\left[\sum_{k=1}^{5}H(\vec{S}_k,\vec{S}_{k+1})\right]. \tag{39}$$

If we confine ourselves to the second order in the cumulant expansion, we can use the following second-order decomposition:[19]

$$\exp[xA+yB]=\exp\left[\frac{x}{2}A\right]\exp[yB]\exp\left[\frac{x}{2}A\right]+O(x^2y,xy^2), \tag{40}$$

and then

$$\mathrm{Tr}[Pe^{(H+V)}]\approx z_0+\overline{V}+\frac{1}{2}\overline{V^2}, \tag{41}$$

with

$$z_0=\mathrm{Tr}[Pe^H],\quad \overline{V}=\frac{1}{2}\mathrm{Tr}[P(e^HV+Ve^H)],$$
$$\overline{V^2}=\frac{1}{4}\mathrm{Tr}[P(e^HV^2+2Ve^HV+V^2e^H], \tag{42}$$

and the average of V is defined as

$$\langle V\rangle=\frac{1}{2}(z_0^{-1}\overline{V}+\overline{V}z_0^{-1}). \tag{43}$$

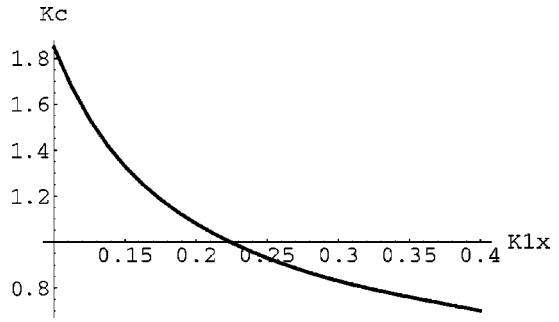

FIG. 8. Critical inverse temperature K_c of the XY model.

The expressions for the average of the spins are now more complicated than in the Ising case (9) and, for example, for z component of the spin from odd rows one obtains

$$\langle s_{1z}\rangle = a_1\sigma_{1z} + a_2(\sigma_{2z}+\sigma_{3z}) + a_z\sigma_{1z}\sigma_{2z}\sigma_{3z} + a_{1x}\sigma_{1z}(\sigma_{2x}\sigma_{3x}+\sigma_{2y}\sigma_{3y}) + a_{2x}[\sigma_{2z}(\sigma_{1x}\sigma_{3x} + \sigma_{1y}\sigma_{3y}) + \sigma_{3z}(\sigma_{1x}\sigma_{2x}+\sigma_{1y}\sigma_{2y})], \quad (44)$$

where coefficients a_α are numerically determined functions of the interaction parameters K_i. For the spins from even rows the averages of single operators vanish $\langle S_{i\alpha}\rangle = 0$, and consequently there is no contribution from the first-order cumulant expansion.

It is easy to see from Eq. (44) that in the second-order calculation in V the transformation (31), contrary to the Ising case, generates four- and six-spin couplings. The evaluation of the contributions to the four- and six-spin interactions is rather straightforward but labor consuming and these interactions should not considerably affect the critical temperature of the XY model. Thus, we will neglect them in the following. The critical inverse temperature and the temperature dependence of the free energy and specific heat of the XY weakly coupled chains are shown in Figs. 8 and 9, respectively.

The value of the critical temperature found for the standard XY model, e.g., for $K_{1x}=K_x$, $K_c \approx 0.5$ is about 20% smaller than expected, but of course the LPRG approximation should be better for $K_{1x}<K_x$.

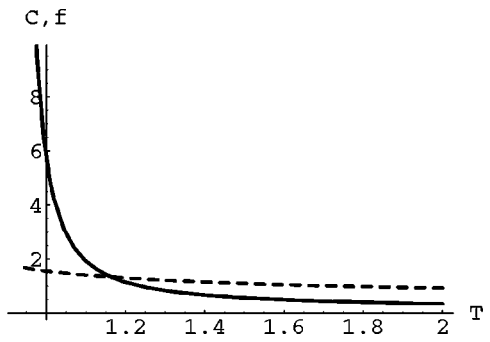

FIG. 9. Free energy (dashed line) and specific heat (solid line) of the XY model with $K_{1x}=0.2K_x$.

V. CONCLUSION

We have proposed a simple perturbation method, LPRG, for a broad class of classical and quantum lattice systems made with weakly interacting chains. The method is based on the linear renormalization-group transformation, which for the Ising spins is the standard decimation procedure and for the quantum spins the generalization of the Suzuki-Takano approximate decimation. This generalization resolves itself into construction of the effective interactions between n effective spins in a cluster by performing a partial trace over every other spin. In the original ST approximation the effective interaction between the first and the last spin in the cluster is constructed by taking the trace over all other spins of the cluster. In the first case the scale factor l is always 2 whereas in the second one it can be, in principle, arbitrary. It has been shown that for the XY quantum spin chain our method gives the free energy in better agreement with the exact results than the ST one. Moreover, the results of the ST method cannot be improved by enlarging the scale factor l.[10]

To find the renormalized Hamiltonian we have used the perturbation theory with the interchain interactions as the perturbation parameters, which seems to be very natural choice for weakly interacting chains. The calculations have been performed in the second-order cumulant expansion for a finite cluster. It should be emphasized that such an approximation is only reliable for small values of the ratio interchain and intrachain interactions and in higher temperatures. So, the method can be used to evaluate the temperature dependence of the thermodynamic values in the high temperatures and to find the critical line if a transition does not take place in a very low temperature. In fact, the method based on the linear transformation is not appropriate to define the character of the critical singularities, i.e., to calculate the values of the critical indices.

The LPRG method has been examined in the two-dimensional (2D) Ising-type spin models. For the standard Ising model on the rectangular lattice with the interaction parameter K_1 in the vertical direction larger than the parameter K_2 in the horizontal direction we have found the critical line as a function of the ratio K_2/K_1. For $0.15<K_2/K_1<1$ this line is in very good agreement with the exact result (Fig. 2) The results are worse for $K_2/K_1<0.15$ but then the phase transition takes place at low temperatures where the present approach is invalid. We have also considered a two-dimensional Ising model made with the chains coupled only via pair-pair (four-spin) interactions of the adjacent chains [Eq. (21)]. In this case the LPRG transformation does not exhibit any nontrivial fixed point. In other words, the 2D Ising model described by Hamiltonian (21) undergoes no phase transition at any nonzero temperature. This result agrees with the exactly determined free energy of the model under consideration. The LPRG approximation gives also proper temperature dependence of the specific heat for the sufficiently small four-spin interaction (Fig. 5).

The application of the LPRG method to quantum systems needs some additional approximations and consequently our approach deteriorates especially at low temperatures. How-

ever, for sufficiently high temperature and for small interchain interactions we have found reasonable results for the critical line and the temperature dependence of the free energy of the two-dimensional quantum XY model. Because, contrary to the nonlinear renormalization-group transformation, the proposed method does not require the choice of the weight operator, it can be used to study more complicated classical and quantum systems. The calculations can be made, for example, for the systems with higher values of spins or for systems of interacting fermions. Of course the approximation can be improved by taking into account the higher orders in the cumulant expansion and by increasing the used cluster.

APPENDIX

In this appendix we present the coefficients $g_m^{(n)}$ [Eq. (17)] as functions of K_1 using the notations introduced in Sec. II, $t=\tanh(K_1)$ and $r=\tanh(2K_1)$:

$$g_1^{(0)}=3+\frac{5}{2}rt+\frac{1}{2}r^2t^2+\frac{1}{4}r^2t^4\frac{1}{2}rt^5,$$

$$g_2^{(0)}=6(1+t^2)+\frac{5}{2}t(3+t^2)r+\frac{1}{2}(1+t^2)^2r^2+t^2(1+t^2)^2r^2/4+\frac{1}{2}t^3(1+t^2)^2r,$$

$$g_3^{(0)}=12t+5(1+t^2)r+t(1+t^2)r^2+\frac{1}{2}t^3(1+t^2)r^2+t^4(1+t^2)r; \tag{A1}$$

$$g_1^{(1)}=rt+t^2+rt^3+\frac{1}{2}r^2t^2+\frac{1}{4}r^2t^4,$$

$$g_2^{(1)}=(1+t^2)^2\left(1+rt+\frac{1}{2}r^2+\frac{1}{4}r^2t^2\right)+rt(3+t^2),$$

$$g_3^{(1)}=(1+t^2)\left(2t+2r+r^2t+2rt^2+\frac{1}{2}r^2t^3\right); \tag{A2}$$

$$g_1^{(2)}=\frac{1}{2}(r^2+4rt+r^2t^2+2),$$

$$g_2^{(2)}=\frac{1}{2}[2r^2(1+t^2)+4rt(3+t^2)+r^2(1+t^2)^2+4(1+t^2)],$$

$$g_3^{(2)}=4t+2r^2t+4r(1+t^2)+r^2t(1+t^2); \tag{A3}$$

$$g_1^{(3)}=\frac{1}{2}\left(\frac{1}{2}r^2+2rt+r^2t^2+2t^2+2rt^3+\frac{1/2^2}{r}t^4\right),$$

$$g_2^{(3)}=\frac{1}{2}\left[r^2(1+t^2)^2+2rt(1+t^2)^2+2rt(3+t^2)+2(1+t^2)^2+r^2(1+t^2)+\frac{1}{2}r^2t^2(1+t^2)^2\right],$$

$$g_3^{(3)}=(1+t^2)\left(r^2t+2rt^2+2r+2t+\frac{1}{2}r^2t^3\right)+r^2t. \tag{A4}$$

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