Superconducting fluctuations at low temperature

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The effect of fluctuations on the transport and thermodynamic properties of two-dimensional superconductors in a perpendicular magnetic field is studied at low temperature $T \ll T_{c0}$. The fluctuation conductivity is calculated in the framework of perturbation theory with the help of the usual diagram technique. It is shown that in the dirty case the Aslamazov-Larkin, Maki-Thompson, and density of states contributions are of the same order. At extremely low temperature $T/T_{c0} \ll [H - H_{c2}(0)]/H_{c2}(0)$ the total fluctuation correction to the normal conductivity is negative in the dirty limit and depends on the external magnetic field logarithmically $\delta\sigma \propto \ln[H - H_{c2}(0)]$. In the nonlocal clean limit, the Aslamazov-Larkin contribution to conductivity is evaluated with the aid of Helfand-Werthamer theory. The longitudinal and Hall conductivities are found. The fluctuating magnetization is calculated in the one-loop and two-loop approximations.

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I. INTRODUCTION

Over the last decade there has been continuing interest in quantum phase transitions. Particular attention has been focused on two-dimensional systems which possess some unusual properties at low temperatures.[1] It is remarkable that a phase transition at zero temperature is possible in the framework of the usual BCS theory of superconductivity. The transition temperature can be suppressed either by magnetic impurities or by a magnetic field. It is interesting to find the fluctuation conductivity as a function of the closeness to the transition in these cases. The impurity-driven quantum phase transition has been considered by Ramazashvili and Coleman.[2] Their consideration was based on the renormalization group analysis of the Aslamazov-Larkin correction to the conductivity. Fluctuations in an external magnetic field have been considered in different systems and various limiting cases.[3–9] However, up until now, there has been no consistent microscopic theory of superconducting fluctuations near $H_{c2}(0)$. The purpose of the present paper is to develop such a theory for two-dimensional superconductors in the dirty and clean limits.

We begin with a brief review of the studies of fluctuations in superconductors. The subject was initiated in the work of Aslamazov and Larkin.[3] The conductivity of fluctuating Cooper pairs was calculated in zero magnetic field. Maki[4] and Thomson[5] have included the effects of electron scattering off the fluctuations. It was found that there is another badly divergent contribution known as the anomalous Maki-Thomson correction. Physically, this correction is connected with the coherent scattering of the electrons by the impurities and analogous to the weak localization correction. The divergence can be removed by introducing a pair-breaking rate. Note that experimental results at $T \sim T_{c0}$ can be described by the Aslamazov-Larkin term only. This suggests that the pair-breaking rate is relatively large in real superconductors. Later, Thomson and Maki returned to the issue and evaluated the fluctuation correction to the normal conductivity in finite fields. Thomson[6] evaluated paraconductivity for small perpendicular fields $T \sim T_{c0}$ and large fields parallel to a two-dimensional superconducting sample. Ami and Maki[7] considered a dirty three-dimensional superconductor put in an arbitrarily strong magnetic field having calculated the diagrams numerically. However, some technical simplifications that had been made in the paper (namely, the dynamic fluctuations had been neglected) make the results inapplicable at very low temperature. Moreover, the three-dimensional case is very different from the two-dimensional one, as shown in the present paper. Let us mention some relatively recent results in this field. A few years ago, Aronov *et al.*[8] developed a theory of transport phenomena in the fluctuation region in the dirty, clean, and superclean $(\omega_c \tau \sim 1)$ limits. Their consideration was based on the Ginzburg-Landau equations and, thus, is applicable for relatively small fields $H \ll H_{c2}(0)$ only. Beloborodov *et al.*[9] have calculated the fluctuating conductivity of a three-dimensional granular superconductor in the region close to $H_{c2}(0)$.

Our paper is structured as follows. In Sec. II A, we consider a two-dimensional dirty sample $T_{c0}\tau \ll 1$ (where τ is the scattering time). We calculate the total fluctuation correction to the conductivity which is described by the standard set of diagrams (see Fig. 1). We derive an analytical expression for the fluctuation conductivity in the region close to the transition line at low temperatures, i.e., at $t = T/T_{c0} \ll 1$ and $h = [H - H_{c2}(T)]/H_{c2}(0) \ll 1$. It is shown that in the case $t \gg h$ the total correction is positive and has the usual form $\delta\sigma \propto T_{c0}[T - T_c(H)]^{-1}$, while at extremely low temperature $t \ll h$ (at zero temperature, in particular) the total correction becomes negative and logarithmically divergent $\delta\sigma \propto \ln h$.

In Sec. II B, we address the issue of fluctuations in the clean superconductors. This problem is more complex, since the elements upon which the diagrams are built (current vertices, cooperons, etc.) are nonlocal in the clean limit. We argue that the corresponding operators can be found on the basis of Helfand-Werthamer theory.[10] We apply this theory to our problem and calculate all the necessary values in the following limiting cases: $\omega_c \ll T$ or $\omega_c \tau \ll 1$ [where $\omega_c = eH_{c2}(0)/m \sim T_{c0}(T_{c0}/\varepsilon_F)$ is the cyclotron frequency]. This allows us to treat the magnetic field effects semiclassically. The curving of the classical trajectories is taken into

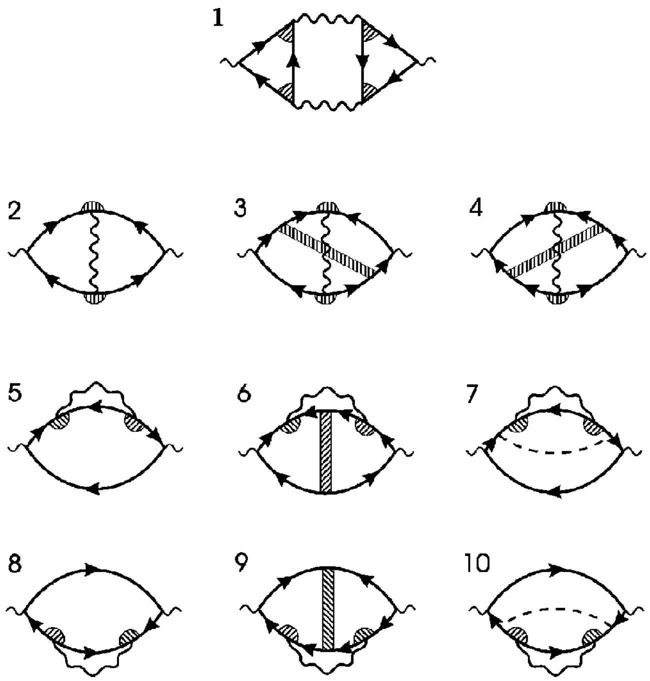

FIG. 1. Impurity averaging diagrams contributing to conductivity in the first (one-loop) approximation.

account by comparison with the Drude conductivity. The longitudinal and Hall conductivities are found. It is shown that the fluctuation correction to the conductivity in the clean limit is similar to the one in the dirty limit, except for an additional cyclotron-resonance-like pole of the second order which appears in the clean case. At the end of Sec. II B, we qualitatively discuss the effects of orbital quantization on the fluctuation conductivity, i.e., Shubnikov–de Haas oscillations which become essential at low temperatures $T \sim \omega_c$.

In Secs. III A and III B, we calculate the thermodynamic properties of a superconductor. We find that the magnetization is logarithmically divergent in the first approximation and exceeds Landau diamagnetism. It is found that in the clean case de Haas–van Alphen oscillations can become observable at high enough temperature. Under certain circumstances, the oscillating part of the fluctuating magnetization represents the dominant effect.

In Sec. IV, we calculate the free energy and magnetization in the two-loop approximation for a dirty superconductor. We find that the divergence becomes more severe in the higher orders in perturbation theory. We discuss the area of applicability of the results obtained. We find that the fluctuation region is determined by $h \lesssim N_{\rm Gi}$, where $N_{\rm Gi} \sim (\varepsilon_F \tau)^{-1}$, for low temperatures $t \ll h$ but it becomes wider $h \lesssim \sqrt{N_{\rm Gi}\, t}$ for relatively large temperatures $t \gg h$.

II. FLUCTUATING CONDUCTIVITY

A. Dirty superconductors

The fluctuation correction to the conductivity beyond the Ginzburg region can be found in perturbation theory. There are terms of three different types describing the fluctuation conductivity in the first (one-loop) approximation. The first one is the Aslamazov-Larkin (AL) term (see Fig. 1, diagram 1) which is connected with the direct conductivity of the fluctuating Cooper pairs. The AL contribution to conductivity is positive. Since some fluctuating pairs appear above the transition, the number of normal electrons decreases. According to the Drude formula this leads to some decrease in the conductivity of the normal electrons. This contribution is known as the density of states (DOS) term (see Fig. 1, diagram 5 and 8). It is clear that this correction must be negative. The third term is the Maki-Thomson (MT) contribution (see Fig. 1, diagram 2) which is connected with the coherent scattering of the normal electrons. The sign of the MT term is not prescribed.

In the presence of impurities, all these contributions must be averaged out over the impurities positions. This can be done in the framework of a diagram technique developed long ago.[11] There is a standard set of diagrams to be considered in our problem (see Fig. 1).

These diagrams are built of the following elements: A solid line represents the one-electron Green function which in zero field has the form (in the momentum representation)

$$\mathcal{G}_\varepsilon(\mathbf{p}) = \frac{1}{i\varepsilon - \xi_{\mathbf{p}} + \dfrac{i\,\mathrm{sgn}\;\varepsilon}{2\tau}}, \tag{1}$$

where $\varepsilon = (2n+1)\pi T$ is the fermion Matsubara frequency and $\xi_{\mathbf{p}} = \varepsilon(\mathbf{p}) - \varepsilon_F$ is the one-particle excitation spectrum. Here, we consider the quadratic spectrum.

In the presence of magnetic field $\mathbf{A}(\mathbf{r})$, the Green functions change and contain the effects of orbital quantization. However, in the presence of strong disorder $\omega_c \tau \ll 1$ or at relatively high temperatures $T \gg \omega_c$, the discrete Landau levels are smeared out and the effects of the magnetic field can be treated semiclassically. This means that the Green function in the coordinate representation can be written as

$$\mathcal{G}_\varepsilon(\mathbf{r_1}, \mathbf{r_2}) = \mathcal{G}_\varepsilon^{(0)}(\mathbf{r_1} - \mathbf{r_2}) \exp\left(-ie \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{A}(\mathbf{s})\, d\mathbf{s} \right), \tag{2}$$

where $\mathcal{G}_\varepsilon^{(0)}(\mathbf{r})$ is the Green function in zero field and the path of integration in Eq. (2) is a straight line. Let us note here that the system of units $\hbar = c = k_B = 1$ is used throughout the paper. The magnetic field $\mathbf{H}$ is considered in the Landau gauge $\mathbf{A} = (0, -Hx)$.

Another element is the fluctuation propagator or interaction in the Cooper channel (wavy line). It is a diagonal operator in the Landau representation. The corresponding matrix element has the form[12]

$$\mathcal{L}_n(\Omega) = \frac{1}{N(0)} \left[\ln \frac{T}{T_{c0}} + \psi\left(\frac{1}{2} + \frac{|\Omega| + \Omega_H \left(n + \frac{1}{2} \right)}{4\pi T} \right) - \psi\left(\frac{1}{2} \right) \right]^{-1}, \tag{3}$$

where n corresponds to the nth Landau level, $N(0)$ is the density of states per spin at the Fermi surface, $\Omega=2\pi mT$ is the bosonic Matsubara frequency, corresponding to the total energy in the Cooper channel, $\Omega_H=4e\mathcal{D}H$, and $\mathcal{D}=\frac{1}{2}v_F^2\tau$ is the diffusion coefficient. Note that Eq. (3) is obtained from the expression for the fluctuation propagator in zero field by the interchange of $\mathcal{D}q^2$ by $\Omega_H(n+1/2)$, with $\mathbf{q}$ being the total momentum in the Cooper channel.

The shaded vertices in the diagrams are Cooperons, which describe the coherent scattering of two particles off the impurities. The expression for this quantity has the following form:[12]

$$C_n(\varepsilon_1,\varepsilon_2)=\frac{1}{\tau}\frac{\theta(-\varepsilon_1\varepsilon_2)}{|\varepsilon_1-\varepsilon_2|+\Omega_H\left(n+\frac{1}{2}\right)},\tag{4}$$

where ε_1 and ε_2 are fermion Matsubara frequencies, corresponding to the electron energies.

To calculate the total fluctuation correction to the dc conductivity we have to evaluate all the diagrams 1–10 as functions of the external Matsubara frequency $\omega=2\pi\nu T$, perform an analytical continuation to the real frequency axis, take the limit $\omega\to 0$, and sum up all the contributions. The singular term, corresponding to $\omega=0$, is canceled out in the final result for the electromagnetic response tensor.

In the vicinity of T_{c0} (transition temperature in zero field) only the AL and anomalous MT terms are important. The typical arguments are as follows. The point of the superconducting transition is determined by the pole of the fluctuation propagator (wavy line). The AL diagram contains two such lines. Thus, close to the transition the corresponding contribution is the most singular one. Another singularity is due to the diffusionlike pole $(-i\omega+\mathcal{D}q^2)^{-1}$ which appears in the MT term[5] (recall that the MT process is connected with the coherent scattering of electrons). At small $\mathbf{q}$ and $\omega\to 0$ this yields a singular contribution.

Another simplification which can be made at $T\sim T_{c0}$ is the possibility to neglect the dynamic fluctuations in the MT and DOS terms. This means that instead of evaluating sum over the internal boson frequency Ω we can just take the first term $\Omega=0$, which gives the most singular contribution. In the AL term the Ω dependence is considered in the fluctuation propagators only and neglected in the current vertices.

The situation changes if a magnetic field is applied.[7] In this case, instead of integrating over $\mathbf{q}$, we have to trace the corresponding operators over the Landau levels. The AL diagram contains only one singular fluctuation propagator $\mathcal{L}_0$ corresponding to the lowest Landau level, since the current vertex is not a diagonal operator in the Landau representation. Moreover, the small terms $\mathcal{D}q^2$, which exist in zero field, have to be replaced by $\Omega_H(n+1/2)\sim T_{c0}$. Obviously, the anomalous MT term does not possess any additional singularity in this case. Thus, we conclude that different diagrams should give contributions of the same order if a large magnetic field is applied.

Let us now perform a representative calculation on the example of the AL term (see Fig 1, diagram 1). The corresponding expression for the longitudinal component of the electromagnetic response tensor has the following form:

$$Q_1(\omega)=-4e^2c^2\nu\sum_{n=0}^{\infty}\pi_{n\,n+1}^2T\sum_{\Omega}[\mathcal{L}_n(\Omega)\mathcal{L}_{n+1}(\Omega-\omega)+\mathcal{L}_n(\Omega-\omega)\mathcal{L}_{n+1}(\Omega)]\times\left[T\sum_{\varepsilon}C_n(\varepsilon,\Omega-\varepsilon)C_{n+1}(\varepsilon-\omega,\Omega-\varepsilon)\right]^2,\tag{5}$$

where the factor of 4 is due to the spin, the constant $c=4\pi N(0)\mathcal{D}\tau^2$ appears as a result of integration over ξ in the local current vertex [see Eq. (A9)], $\nu=eH/\pi$ is the number of states per unit area of a full Landau level, and $\pi_{n\,n+1}=\langle n|[-i\nabla+2e\mathbf{A}(\mathbf{r})]_x|n+1\rangle=\sqrt{(n+1)eH}$ are matrix elements of the kinetic momentum. ω is the Matsubara frequency corresponding to the frequency of the external electric field, and Ω and ε are the internal bosonic and fermionic Matsubara frequencies, respectively.

As we have already mentioned, the main singularity comes from the fluctuation propagator corresponding to the lowest Landau level. Close to the transition it can be written as

$$\mathcal{L}_0(\Omega)=\frac{1}{N(0)}\frac{1}{h+2|\Omega|/\Omega_H},\tag{6}$$

where $h=[H-H_{c2}(T)]/H_{c2}(0)$. Let us note that $\Omega_H=4e\mathcal{D}H_{c2}(0)=(2\pi/\gamma)T_{c0}$ and the bosonic frequency Ω is of the order of temperature. Thus, we conclude that at very low temperatures $t\ll h$ we can replace the sum over Ω in Eq. (5) by an integral. At relatively high temperatures $t\gg h$ we can keep the first term in the sum only. If $t\sim h$, we have to evaluate the sum. This also means that we have to consider the effects of quantum fluctuations as well.

Let us discuss some simplifications that can be made in our case ($t\ll 1$). First of all, we can consider only the first term $n=0$ in the sum over Landau levels in Eq. (5). Only this term gives a singular contribution coming from $\mathcal{L}_0$. Next, we see that the sum over the internal frequency in Eq. (5) is determined by $\Omega\sim T\ll\Omega_H$. This allows us to make expansions with respect to $\Omega/\Omega_H\sim t$ everywhere except $\mathcal{L}_0$. With the same accuracy, we can replace the sum over the fermion energy ε in Eq. (5) by an integral.

Evaluating the integral over ε, we obtain from Eqs. (4) and (5)

$$T\sum_{\varepsilon}C_0(\varepsilon,\Omega-\varepsilon)C_1(\varepsilon-\omega,\Omega-\varepsilon)=\frac{1}{4\pi\tau^2}\left[\frac{1}{\Omega_H-\omega}\ln\left(\frac{3\Omega_H/2+|\Omega-\omega|}{\Omega_H/2+|\Omega-\omega|+|\omega|}\right)+\frac{1}{\Omega_H+\omega}\ln\left(\frac{3\Omega_H/2+|\Omega|+|\omega|}{\Omega_H/2+|\Omega|}\right)\right].\tag{7}$$

Now, we have to perform analytical continuation in the expression for the current response operator (5). In doing this, we can present the sum over the Matsubara frequency as an integral over the real frequency with the function coth(Ω/2T) which is chosen to generate poles at the points $2\pi imT$.[13] Making use of Eqs. (4)–(7) we obtain the following expression for the conductivity (within the logarithmic accuracy):

$$\delta\sigma_1 = \lim_{i\omega\to 0} \frac{Q_1(\omega\to i\omega)}{-i\omega} = \frac{e^2}{\pi^2}\left[\alpha_1 \int_0^{\Omega_{\max}} d\Omega \coth\frac{\Omega}{2T} \frac{\Omega}{\Omega^2 + \left(\frac{\Omega_H h}{2}\right)^2} + \beta_1 \int_0^{\infty} \frac{d\Omega}{2T} \frac{1}{\sinh^2 \frac{\Omega}{2T}} \frac{\Omega^2}{\Omega^2 + \left(\frac{\Omega_H h}{2}\right)^2}\right], \quad (8)$$

where $\alpha_1 = 4/3$ and $\beta_1 = 2$ are just numbers. One can see that the first integral in Eq. (8) is logarithmically divergent. This divergence appears as a result of our expansions in t. Thus, it has to be cut off at $\Omega_{\max} \sim T_{c0}$. The integrals in Eq. (8) can be easily calculated. The result is

$$\delta\sigma = \frac{e^2}{\pi^2}[\alpha I_\alpha(h,t) + \beta I_\beta(h,t)], \quad (9)$$

with

$$I_\alpha(h,t) = \ln\frac{r}{h} - \frac{1}{2r} - \psi(r) \quad (10)$$

and

$$I_\beta(h,t) = r\psi'(r) - \frac{1}{2r} - 1, \quad (11)$$

where $r = (1/2\gamma)\, h/t$ and $\gamma = 1.781$ is Euler's constant.

The other diagrams can be calculated analogously. The corresponding contributions to the conductivity can be written in the same form as Eqs. (8)–(11). Below we give the results in terms of the constants α and β:

$$\alpha_1 = \frac{4}{3}, \quad \beta_1 = 2, \quad (12)$$

$$\alpha_2 = -2, \quad \beta_2 = 2,$$

$$\alpha_3 = \alpha_4 = -\frac{2}{3}, \quad \beta_3 = \beta_4 = 0,$$

$$\alpha_5 = \alpha_8 = -\frac{3}{2}, \quad \beta_5 = \beta_8 = -\frac{3}{2},$$

$$\alpha_6 = \alpha_9 = \frac{5}{3}, \quad \beta_6 = \beta_9 = \frac{1}{3},$$

$$\alpha_7 = \alpha_{10} = \frac{1}{2}, \quad \beta_7 = \beta_{10} = \frac{1}{2},$$

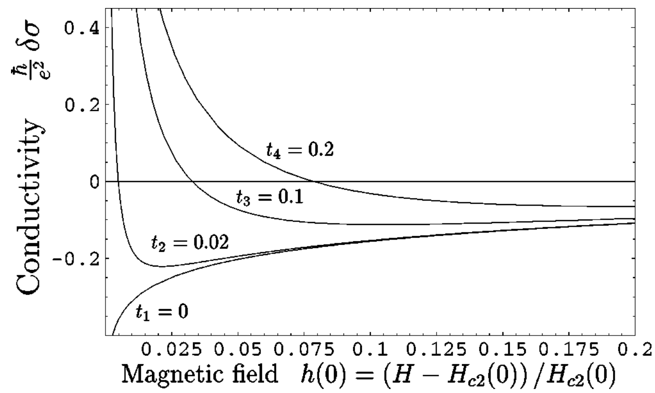

FIG. 2. Fluctuating conductivity (13) as a function of magnetic field is plotted for four different temperatures.

$$\alpha_{\rm tot} = -\frac{2}{3}, \quad \beta_{\rm tot} = \frac{8}{3},$$

where indexes correspond to a diagram number in Fig. 1 and $\alpha_{\rm tot}$ and $\beta_{\rm tot}$ describe the total correction to the conductivity, which can be written as

$$\delta\sigma = \frac{2e^2}{3\pi^2\hbar}\left[-\ln\frac{r}{h} - \frac{3}{2r} + \psi(r) + 4[r\,\psi'(r) - 1]\right]. \quad (13)$$

Let us consider some limiting cases. If the temperature is relatively large $t \gg h$, we obtain the following formula for the fluctuation conductivity:

$$\delta\sigma = \frac{2\gamma e^2}{\pi^2\hbar}\frac{t}{h}. \quad (14)$$

If $H < H_{c2}(0)$, we can introduce $T_c(H)$ and rewrite Eq. (14) in the usual way:

$$\delta\sigma = \frac{3e^2}{2\gamma\pi^2\hbar}\frac{T_{c0}}{T - T_c(H)}. \quad (15)$$

If $H > H_{c2}(0)$, in the low-temperature limit $t \ll h$ we have

$$\delta\sigma = -\frac{2e^2}{3\pi^2\hbar}\ln\frac{1}{h}. \quad (16)$$

One can see that even at zero temperature a logarithmic singularity remains and the corresponding correction is negative.

Let us note that the fluctuating conductivity depends on the magnetic field and temperature via their ratio h/t. The behavior of the conductivity in the vicinity of the critical point $H = H_{c2}(0)$, $T = 0$, depends on the way one approaches this point. If the transition is driven by the magnetic field and the temperature is zero, then the fluctuating correction is negative and logarithmically divergent. If the magnetic field is fixed and $H \leqslant H_{c2}(0)$, then the correction is positive and diverges as $[T - T_c(H)]^{-1}$. In the other cases, there is a crossover between these two regimes.

The magnetic field dependence of the fluctuating conductivity is presented on Fig. 2. One can see that if the magnetic

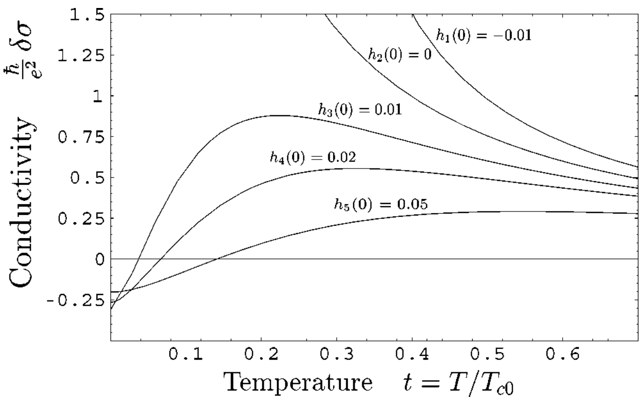

FIG. 3. Fluctuating conductivity (13) as a function of temperature is plotted for five different magnetic fields.

field is relatively large, then the total correction is negative. For any finite temperature, there is a region close to $H_{c2}(0)$ where the correction is positive.

The temperature dependence of the fluctuating conductivity is shown in Fig. 3. It is interesting that the conductivity is a nonmonotonous function of temperature if the magnetic field exceeds $H_{c2}(0)$.

B. Clean limit

In this section, we investigate the fluctuation correction to the conductivity in the limit $T_c\tau\gg 1$. In this case, the usual expressions for the particle-particle bubble, fluctuation propagator, and current vertices are inapplicable. To calculate the diagrams we have to find these quantities in the presence of the magnetic field while taking into account their nonlocal structure. There are several effects associated with the magnetic field applied. First of all, the superconducting transition itself is governed by the magnetic field at low temperatures. Another effect is Shubnikov–de Haas oscillations in the conductivity due to the quantization of the energy levels. However, if $\omega_c\tau\ll 1$ or $T\gg\omega_c$, the oscillating terms are exponentially small and can be neglected. Note that

$$\omega_c=\frac{eH_{c2}(0)}{m}\sim T_{c0}\left(\frac{T_{c0}}{\varepsilon_F}\right)\ll T_{c0}. \tag{17}$$

In our formal derivation, we assume that either $\omega_c\tau\ll 1$ or $\omega_c\ll T\ll T_{c0}$. This allows us to consider low temperatures without dealing with de Haas oscillations in the Green functions. The effect of the orbital quantization on the fluctuation conductivity will be briefly discussed at the end of this section. Moreover, there is a purely classical effect due to the Lorentz force acting on the electrons forming fluctuating pairs. Namely, the magnetic field results in a curving of the classical trajectories. This curving leads to the cyclotron resonance and Hall effect in the fluctuation conductivity. First, we consider fluctuations neglecting the curving, which is eligible if $\omega_c\tau\ll 1$. Using the result obtained, we will be able to derive the formula valid in the superclean case $\omega_c\tau\sim 1$ as well.

We now proceed to calculate different blocks in the diagrams. Our calculation is based on the well-known Helfand-Werthamer theory developed long ago. In a seminal paper,[10] Helfand and Werthamer evaluated the matrix element C_0 for the Cooperon in a magnetic field, which determines the upper critical field $H_{c2}(T)$. They proved the following mathematical statement which we will refer to as the Helfand-Werthamer (HW) theorem throughout the paper.

Let us consider an operator $\hat{O}$. Suppose that its kernel in coordinate representation has the following form:

$$O(\mathbf{r},\mathbf{r}')=\tilde{O}(\mathbf{r}-\mathbf{r}')\exp\left(-2ie\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{A}(\mathbf{s})d\mathbf{s}\right). \tag{18}$$

Then, the operator can be written as

$$\hat{O}=\int \tilde{O}(\mathbf{r})\, e^{-i\mathbf{r}\cdot\hat{\boldsymbol{\pi}}}d^D r, \tag{19}$$

where $\hat{\boldsymbol{\pi}}=[\hat{\mathbf{p}}-2ie\mathbf{A}(\hat{\mathbf{r}})]$ is the kinetic momentum, which can be expressed in terms of the creation and annihilation operators in the Landau representation, and D is the dimensionality of the system ($D=2$ in our case).

One can see that all the operators involved in our calculations satisfy the HW theorem. Namely, the particle-particle bubble $\hat{\Pi}_\varepsilon(\Omega)$, current vertex $\hat{\gamma}_\alpha(\Omega,\omega)$, and the four Green function blocks $\hat{B}_{\alpha\beta}(\Omega,\omega)$ in coordinate representation can be written as a product of a function of the coordinate difference and the gauge factor. In the temperature range under consideration, we can treat the magnetic field effects semiclassically which means that the first factor $\tilde{O}$ in Eq. (18) can be considered in zero field.

To calculate the matrix elements of interest we will do the following. First, we calculate an operator in zero field in the momentum representation $\tilde{O}(\mathbf{q})$. We apply the Fourier transformation to this function and put the value obtained $\tilde{O}(\mathbf{r})$ in Eq. (19). Then, we evaluate the matrix elements for this operator and perform the integration over $\mathbf{r}$. Finally, we perform the frequency summation left (over the fermion energy ε).

Let us start with the calculation of the nonlocal fluctuation propagator which has the form

$$\hat{\mathcal{L}}(\Omega)=\frac{1}{g^{-1}-\hat{\Pi}(\Omega)}, \tag{20}$$

where g is the interaction constant and

$$\tilde{\Pi}(\mathbf{q},\Omega)=T\sum_{\varepsilon}\tilde{\Pi}_\varepsilon(\mathbf{q},\Omega), \tag{21}$$

with the particle-particle bubble $\tilde{\Pi}_\varepsilon(\mathbf{q},\Omega)$ defined by Eq. (A11). Note that in the clean limit we can neglect the impurity dependence in $\hat{\Pi}_\varepsilon(\Omega)$ and in the fluctuation propagator.

The matrix elements can be calculated by expressing $\hat{\boldsymbol{\pi}}$ in terms of the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ and expanding the exponentials.[14] One obtains

$$\exp(-i\mathbf{r}\hat{\boldsymbol{\pi}}) = e^{-\rho^2/2} \sum_{k,\, l=0}^{+\infty} \frac{(-i\rho)^{k+l}}{k!\, l!} (\hat{a}^\dagger)^k \hat{a}^l e^{-i\phi(l-k)}, \tag{22}$$

where $\rho = r/\sqrt{2}r_H$ and $r_H = \sqrt{2eH}$ is the magnetic length. Due to the integration over ϕ, only the diagonal matrix elements survive and we have the following expression:

$$\Pi_n(\Omega) = (-1)^n r_H^2 T \sum_\varepsilon \int_0^\infty dq^2 \tilde{\Pi}_\varepsilon(q,\Omega) e^{-q^2 r_H^2} L_n(2q^2 r_H^2), \tag{23}$$

where L_n is the Laguerre polynomial of the nth order.

At low temperature, we can replace the sum over ε by an integral and we have

$$\Pi_n(\Omega) = N(0)\Bigg[\ln(2\Lambda) - (-1)^n \times \int_0^\infty dx \ln(\lambda + \sqrt{\lambda^2 + x}) e^{-x} L_n(2x)\Bigg], \tag{24}$$

where we have introduced $\lambda = |\Omega| r_H / v_F$ being the lower limit of integration over ε and $\Lambda = 2 r_H \omega_D / v_F$, which is the BCS high-energy cutoff. Obviously, $\lambda \sim T/T_{c0} \ll 1$ and $\Lambda \gg 1$.

Let us realize that to find the most singular contribution to the conductivity we need to know $\Pi_0(\Omega)$ and $\Pi_1(\Omega)$ only. Making expansions with respect to λ in Eq. (24), one obtains

$$\Pi_0(\Omega) = N(0)[\ln(2\sqrt{\gamma}\Lambda) - \sqrt{\pi}\lambda] \tag{25}$$

and

$$\Pi_1(\Omega) = N(0)[1 + \ln(2\sqrt{\gamma}\Lambda) + \lambda^2]. \tag{26}$$

Thus, the fluctuation propagator corresponding to the lowest Landau level can be written in the vicinity of the transition as follows:

$$\mathcal{L}_0(\Omega) = \frac{2}{N(0)} \frac{1}{h + \sqrt{\dfrac{\gamma}{\pi}} |\Omega| / T_{c0}}, \tag{27}$$

where T_{c0} is the transition temperature in zero field, $h = [H - H_{c2}(T)]/H_{c2}(0)$, and

$$eH_{c2}(0) = \frac{2\pi^2}{\gamma} \left(\frac{T_{c0}}{v_F}\right)^2$$

is the upper critical field at zero temperature. Let us note that the corresponding relation in the three-dimensional case is

$$eH_{c2}^{3D}(0) = \frac{\pi^2 \mathbf{e}^2}{2\gamma} \left(\frac{T_{c0}}{v_F}\right)^2,$$

with $\mathbf{e} = 2.718$. As one can see, it differs by a constant from the two-dimensional one.

The current vertex can be evaluated in the same fashion as the fluctuation propagator. However, the corresponding calculation is more cumbersome (see the Appendix). In the momentum representation the vertex has the form

$$\boldsymbol{\gamma}(\mathbf{q};\Omega,\omega,\varepsilon) = \frac{2\pi N(0)}{\omega} \frac{\mathbf{q}}{q^2} \sum_{i=1}^4 \eta_i(\varepsilon,\Omega,\omega) f(\epsilon_i, q)\,. \tag{28}$$

The corresponding functions and constants are defined in the Appendix by formulas (A2)–(A8). Equation (28) is large mostly due to the θ functions. However, its q-dependent part has a simple form $(\mathbf{q}/q^2) f(q)$, where $f(0) = 0$ and $f(\infty)$ is finite.

Making use of the Helfand-Werthamer theorem, we obtain the following expression for the operator $\boldsymbol{\gamma}$:

$$\hat{\boldsymbol{\gamma}}(\Omega,\omega,\varepsilon) = 2\pi i N(0) \sum_{i=1}^4 \frac{\eta_i(\varepsilon,\Omega,\omega)}{\omega} \int_0^\infty dq\, f(\epsilon_i, q) \times \int_0^\infty dr\, r J_1(qr) \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{\mathbf{r}}{r} \exp(-i\mathbf{r}\hat{\boldsymbol{\pi}}). \tag{29}$$

Evaluating matrix elements and performing integration over ϕ and r, one can see that the current vertex possesses only near-diagonal nonzero matrix elements and they have the form

$$\langle n | \hat{\gamma}^x(\Omega,\omega,\varepsilon) | n+1 \rangle = \sqrt{\frac{2}{n+1}} \pi N(0) r_H^3 (-1)^n \sum_{i=1}^4 \frac{\eta_i(\varepsilon,\Omega,\omega)}{\omega} \times \int_0^\infty dq^2 f(\epsilon_i, q) e^{-r_H^2 q^2} L_n^{(1)}(2 r_H^2 q^2). \tag{30}$$

In order to calculate the most singular contribution to the conductivity we have to know γ_{01} only. Taking the corresponding integral by parts one obtains

$$\gamma_{01}(\Omega,\omega,\varepsilon) = \sqrt{2}\pi N(0) r_H \sum_{i=1}^4 \frac{\eta_i(\epsilon,\Omega,\omega)}{\omega} \times \int_0^\infty dq \frac{\partial f(\epsilon_i, q)}{\partial q} e^{-r_H^2 q^2}. \tag{31}$$

To evaluate the remaining integrals we have to use explicit expressions for the functions $f(\epsilon,q)$ and $\eta_i(\varepsilon,\Omega,\omega)$ (see the Appendix). Using formula (A2), one obtains after elementary integration over q

$$\gamma_{01}(\Omega,\omega,\varepsilon) = \sqrt{2}\pi N(0) r_H \sum_{i=1}^4 \frac{\eta_i(\epsilon,\Omega,\omega)}{\omega} \times [1 - \sqrt{\pi} |\delta_i| e^{\delta_i^2} \operatorname{erfc} |\delta_i|], \tag{32}$$

where we have introduced $\delta_i = \epsilon_i r_H / v_F$. Now, we have to perform the summation over the fermion frequency ε. For $T \ll T_{c0}$ we can replace this sum by an integral over δ taken in the appropriate limits well defined by the θ functions in

Eqs. (A3)–(A8). The corresponding indefinite integral can be easily evaluated and we finally obtain

$$\gamma_{01}(\Omega,\omega)=\frac{1}{2\pi}\int d\varepsilon\ \gamma_{01}(\Omega,\omega,\varepsilon)$$
$$=\frac{\sqrt{2\pi}N(0)v_F}{8}\sum_{i=0}^{4}\frac{\eta_i}{\omega}\operatorname{sgn}\delta_i\, e^{\delta_i^2}\operatorname{erfc}|\delta_i|\Big|_{\text{limits}}, \tag{33}$$

where the limits of integration are determined by the θ functions in η_i. Let us note that at $\delta=\infty$ the expression written above vanishes, while the other limits of integration are such that $\delta\sim T/T_{c0}\ll 1$.

There are four terms in Eq. (A3). The last two contain the factor $\omega^{-1}\operatorname{sgn}\varepsilon$, while the first two are proportional to the following factor:

$$\frac{\tau}{1+\omega\tau\operatorname{sgn}\varepsilon}=\frac{\operatorname{sgn}\varepsilon}{\omega}\left[1-\frac{1}{1+\omega\tau\operatorname{sgn}\varepsilon}\right].$$

Using Eq. (33), one can see that the singular ω^{-1} terms are canceled out exactly. Thus, the current vertex can be written (we keep only the linear terms with respect to the frequencies)

$$\gamma_{01}(\Omega,\omega)=-\frac{N(0)r_H}{\sqrt{2}}\frac{1}{1+|\omega|\tau}\times\left[1-\frac{\sqrt{\pi}}{2}\frac{r_H}{v_F}(|\Omega|+|\Omega-\omega|+|\omega|)+o(t)\right]. \tag{34}$$

We see that the current vertex is proportional to $(1+|\omega|\tau)^{-1}$ and its frequency dependence is determined by the two pairs of θ functions in Eq. (A1). These terms exist when the poles ε and $(\varepsilon-\omega)$ are located in the opposite half-planes of the complex plane ξ. Let us note that a similar situation takes place when calculating the Drude conductivity of the normal metal. Getting Eqs. (25), (26), and (34) together and using the following formula for the current response operator,

$$Q(\omega)=8\nu e^2T\sum_{\Omega}\ \gamma_{01}^2(\Omega,\omega)\mathcal{L}_0(\Omega)\mathcal{L}_1, \tag{35}$$

we can calculate the AL contribution to the conductivity. Note that in the framework of our approximation, we can treat $\mathcal{L}_1$ as a constant, since it does not have any linear Ω dependence [see Eq. (26)]. Analytical continuation yields the following expression for the conductivity (valid within the logarithmic accuracy):

$$\delta\sigma=\frac{\overline{\delta\sigma}}{(1-i\omega\tau)^2}=\frac{e^2}{\pi^2}\frac{1}{(1-i\omega\tau)^2}[I_\alpha(h,t)+I_\beta(h,t)], \tag{36}$$

where the functions $I_\alpha(h,t)$ and $I_\beta(h,t)$ are defined by Eqs. (10) and (11) with parameter $r=(\sqrt{\pi/4\gamma})\,h/t$, which differs by a constant from the one in the dirty case.

Equation (36) is valid for $\omega\ll T_{c0}$. Recall that the Drude conductivity has the following form:

$$\sigma_0=\frac{\overline{\sigma_0}}{1-i\omega\tau}=\frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}. \tag{37}$$

Thus, the total longitudinal resistivity reads

$$\rho_{xx}=\frac{m}{ne^2}\left(\frac{1}{\tau}-i\omega\right)-\frac{\overline{\delta\sigma}}{\overline{\sigma_0}^2}. \tag{38}$$

We see that the fluctuation correction to the resistivity does not depend on the external frequency (unless $\omega\sim T_{c0}$) and can be considered as a correction to the collision integral $1/\tau$. Physically, this means that the ac electric field acts on the normal electrons, rather than superconducting fluctuations.

In the superclean case $\omega_c\tau\lesssim 1$, we have to take into account the curving of the classical trajectories. This curving results in the Hall term in conductivity and cyclotron-resonance-like effects. The Hall term can be written as

$$\rho_{xy}=\frac{m}{ne^2}\omega_c+\delta\rho_{xy},$$

where the second term is due to superconducting fluctuations. The reasonings described above suggest that this term, which describes the curving of fluctuating pairs, is of the order of ω_c/T_{c0} and can be neglected. Hence, calculating the inverse matrix $\hat{\rho}^{-1}$ we find the following formula for the fluctuation conductivity:

$$\delta\sigma_{\pm}=\delta\sigma_{xx}\pm i\,\delta\sigma_{xy}=\frac{1}{[1-i(\omega\pm\omega_c)\tau]^2}\overline{\delta\sigma}, \tag{39}$$

where $\overline{\delta\sigma}$ is defined by Eq. (36). Let us note that $\overline{\delta\sigma}$ represents the longitudinal conductivity with no respect to the curving. The corresponding Hall term $\overline{\delta\sigma_{xy}}$ can only appear in the presence of a particle-hole asymmetry. It does not exist in the framework of our approximation. In the paper of Aronov *et al.*[8] this additional Hall term was controlled by the phenomenological parameter $T_c\partial\ln T_c/\partial\varepsilon_F$.

Let us now discuss the contributions coming from the MT and DOS diagrams. The electromagnetic response tensor can be written in the following form:

$$Q_{\alpha\beta}(\omega)=2e^2T^2\sum_{\Omega,\,\varepsilon}\operatorname{Tr}[\hat{B}_{\alpha\beta}(\varepsilon,\omega,\Omega)\hat{\mathcal{L}}(\Omega)], \tag{40}$$

where $\hat{B}_{\alpha\beta}$ represents a four-Green-function block. Let us consider this quantity on the example of the MT term. In coordinate representation within the semiclassical approximation it has the form

$$B_{\alpha\beta}(\varepsilon,\omega,\Omega;\mathbf{r},\mathbf{r}')=\tilde{B}_{\alpha\beta}(\varepsilon,\omega,\Omega;\mathbf{r}-\mathbf{r}')\times\exp\left(-2ie\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{A}(\mathbf{s})d\mathbf{s}\right), \tag{41}$$

where

$$\tilde{B}_{\alpha\beta}(\varepsilon,\omega,\Omega;\mathbf{r}) = -\frac{r_\alpha r_\beta}{r^2}\int_0^\infty \frac{dp\,dk}{(2\pi)^2} J_1(pr)J_1(kr) \times \mathcal{G}_\varepsilon(p)\mathcal{G}_{\varepsilon-\omega}(p)\mathcal{G}_{\Omega-\varepsilon}(k)\mathcal{G}_{\Omega+\omega-\varepsilon}(k). \tag{42}$$

Putting this expression into the HW theorem (19), evaluating the diagonal matrix element for $n=0$, and performing integration over $\mathbf{r}$ we obtain the following expression:

$$\langle 0|\hat{B}_{xx}(\varepsilon,\omega,\Omega)|0\rangle = -\int_0^\infty \frac{dp\,dk}{2\pi} p^2 k^2 \times \mathcal{G}_\varepsilon(p)\mathcal{G}_{\varepsilon-\omega}(p)\mathcal{G}_{\Omega-\varepsilon}(k)\mathcal{G}_{\Omega+\omega-\varepsilon} \times (k) I_{00}(p,k), \tag{43}$$

where

$$I_{00}(p,k) = r_H^2 \exp[-r_H^2(p^2+k^2)] I_1\left[\frac{1}{2} r_H^2 pk\right] \tag{44}$$

and I_1 is a modified Bessel function of the first order. Since $p\sim k\sim p_F$, we see that $r_H^2 pk \sim \varepsilon_F/\omega_c \gg 1$; we can take the asymtpotical form of the Bessel function and then approximate the resulting exponent $\exp[-r_H^2(p-k)^2]$ by the δ function. Thus, we have

$$I_{00}(p,k) \approx \frac{1}{p}\delta(p-k). \tag{45}$$

Performing integration with respect to $\mathbf{k}$ and introducing the density of states at the Fermi surface $N(0)$ we obtain the usual expression for the four-Green-function block:

$$\langle 0|B_{xx}(\varepsilon,\omega,\Omega)|0\rangle = -\frac{1}{2} v_F^2 N(0) \times \int_{-\infty}^{+\infty} d\xi_{\mathbf{p}} \mathcal{G}_\varepsilon(p)\mathcal{G}_{\varepsilon-\omega}(p)\mathcal{G}_{\Omega-\varepsilon} \times (p)\mathcal{G}_{\Omega+\omega-\varepsilon}(p). \tag{46}$$

One can see that the derivation of this expression does not depend on the purity of a superconductor. It is valid for the dirty and clean limits and any magnetic fields applied, unless $\omega_c \sim \varepsilon_F$.

Let us note that Eq. (46) for the MT diagram and similar expressions for the DOS diagrams are identical to the ones in the vicinity of T_{c0} and do not involve a magnetic field at all. It is known that DOS and MT terms are strongly compensated in the clean limit[15] and this compensation takes place at the level of the Green functions (i.e., before integrals over $\mathbf{q}$). This suggests that in the clean limit the only remaining diagram is the AL term even in the case of a strong magnetic field.

Let us now discuss quantum oscillations in the fluctuation conductivity. At very low temperature these oscillations become important.[16,17] In this case, each current vertex contains an oscillating part. This oscillating part can be found by comparison with the Drude conductivity and can be written as $\gamma=\gamma_0+\gamma_{\rm osc}$, where $\gamma_{\rm osc}/\gamma_0 \sim \sigma_{\rm osc}/\sigma_0$ with $\sigma_{\rm osc}$ being the oscillating part of the normal conductivity (see, e.g., Ando *et al.*[18]). However, there are other ''sources'' of quantum oscillations. The transition temperature $T_c(B)$ oscillates as well[19,20] and affects the fluctuation conductivity. Let us also realize that the oscillations of magnetization (de Haas–van Alphen oscillations) can influence Shubnikov–de Haas oscillations. Under certain conditions, this effect may be dominant. Moreover, magnetization fluctuates as well and in the vicinity of the transition the fluctuations can exceed Landau diamagnetism (see Sec. 3 and Ref. 21). We see that the oscillating part of the fluctuating conductivity has a complicated structure and can differ significantly from the usual Shubnikov–de Haas oscillations.

III. THERMODYNAMICS: FLUCTUATING MAGNETIZATION

A. Dirty case

Considering the thermodynamic properties of a film, we can calculate the free energy directly. In the one-loop approximation, the free energy can be written as[14]

$$F_1 = -T\sum_\Omega \operatorname{Tr}\ln[1-g\,\hat{C}(\Omega)], \tag{47}$$

where $\hat{C}(\Omega)$ is the Cooperon.

Using Eqs. (3), (4), and (47), one can easily obtain the magnetization

$$M_1 = -\frac{1}{V}\frac{\partial F_1}{\partial H} = \frac{\nu}{2\pi d}\frac{\Omega_H}{H_{c2}(0)} I_\alpha(h,t), \tag{48}$$

where d is the thickness of the film or the interlayer distance, $\nu = eH/\pi$ is the number of states of a Landau level, and the function $I_\alpha(h,t)$ is defined in Eq. (10). Thus, at low temperature $t \ll h$ the susceptibility takes the form

$$\chi_1 = -\frac{\partial M_1}{\partial H} = \frac{e^2}{\pi^2\hbar c^2}\frac{v_F^2\tau}{d} h^{-1}. \tag{49}$$

One can see that the fluctuation susceptibility (49) is large compared to the magnetic susceptibility of the normal metal $\chi_{\rm L}$ even far from the transition:

$$\chi_1 \sim \frac{1}{N_{\rm Gi}\, h}\chi_{\rm L}, \tag{50}$$

where $N_{\rm Gi} = (\varepsilon_F\tau)^{-1}$ is the Ginzburg parameter.

B. Clean case

The calculation of magnetization in the clean limit can be done in the same fashion as in the dirty limit. However, there are some features specific for the clean case. As we have already mentioned, de Haas oscillations become essential at low temperature in pure samples. These quantum oscillations appear in all quantities including Green functions, transition temperature $T_c(H)$, fluctuating conductivity etc. The oscillating terms are proportional to the factors $\exp(-\pi/\omega_c\tau)$ and $\exp(-2\pi^2 T/\omega_c)$. Hence, the oscillations are strongly suppressed, unless $\omega_c\tau\sim 1$ and $T/\omega_c\sim 1$. Let us note that de Haas–van Alphen oscillations in the magnetization can reveal themselves much earlier than the quantum oscillations in the other quantities. This is because magnetization is a derivative of the free energy with respect to the magnetic field. Even though the oscillating terms in the Green functions are small, they contain fast-oscillating functions $\cos(2\pi\varepsilon_F/\omega_c)$ which may lead to observable effects in the oscillating magnetization. It is easy to calculate the fluctuating magnetization with respect to these effects.

We can use Eq. (47),

$$F_1=-T\sum_{\Omega}\ \mathrm{Tr}\ln[1-g\,\hat{\Pi}(\Omega)], \tag{51}$$

where $\hat{\Pi}=\hat{\Pi}_0+\hat{\Pi}_{\mathrm{osc}}$ is the particle-particle bubble which contains an oscillating part. The matrix element for the monotonous part of the particle-particle bubble corresponding to the lowest Landau level was found in Sec. II B [see Eq. (25)]. The oscillating part has been considered in a number of papers and has the form[19,22]

$$\Pi_{\mathrm{osc}}=-8\pi^{3/2}N(0)\frac{T}{\sqrt{\varepsilon_F\omega_c}}\cos\left(2\pi\frac{\varepsilon_F}{\omega_c}\right)\times\cos\left(6\pi\frac{\mu_e H}{\omega_c}\right)\exp\left(-\frac{\Delta}{\omega_c}\right), \tag{52}$$

where $\Delta=6\pi(\pi T+1/2\tau)$ and μ_e is the magnetic moment of an electron. For the sake of simplicity, we keep the first oscillating term only.

In the vicinity of the transition we can present the magnetization in the following way:

$$M_1=\frac{T\nu}{H_{c2}(0)}\sum_{\Omega}\ \frac{1}{\mathcal{L}_0(\Omega)^{-1}-\Pi_{\mathrm{osc}}}\frac{\partial}{\partial h}[\mathcal{L}_0(\Omega)^{-1}-\Pi_{\mathrm{osc}}], \tag{53}$$

where for $\mathcal{L}_0(\Omega)$ see Eq. (27).

From Eqs. (27), (47), and (52), we obtain the following expression for the fluctuating magnetization:

$$M_1=\frac{1}{\sqrt{\pi\gamma}}\frac{T_{c0}\nu}{H_{c2}(0)}\left[\ln\frac{1}{t}-\psi\left(\frac{1}{\sqrt{4\pi\gamma}}\frac{h}{t}\right)-\sqrt{\pi\gamma}\frac{t}{h}\right]\times\left[1+32\pi^{5/2}\frac{T\sqrt{\varepsilon_F}}{\omega_c^{3/2}}\sin\left(2\pi\frac{\varepsilon_F}{\omega_c}\right)\cos\left(6\pi\frac{\mu_e H}{\omega_c}\right)\times\exp\left(-\frac{\Delta}{\omega_c}\right)\right]. \tag{54}$$

Let us note that if $T\sim\omega_c$, then $T\sqrt{\varepsilon_F}/\omega_c^{3/2}\sim\varepsilon_F/T_{c0}\gg 1$ and the numerical factor in the oscillating term in Eq. (54) is very large. Thus, we conclude that de Haas–van Alphen oscillations in magnetization may exist even in the absence of the Shubnikov–de Haas oscillations and oscillations of the transition temperature. It is worth mentioning that the fluctuation effects exceed Landau diamagnetism in the clean limit as well [formula (50) is valid with $\mathrm{Gi}=T_{c0}/\varepsilon_F$]. Thus, under certain circumstances ($\Delta\sim 1$) the oscillating part of the fluctuating magnetization may be more important than the monotonous part of magnetization and the oscillating part in the Landau term.

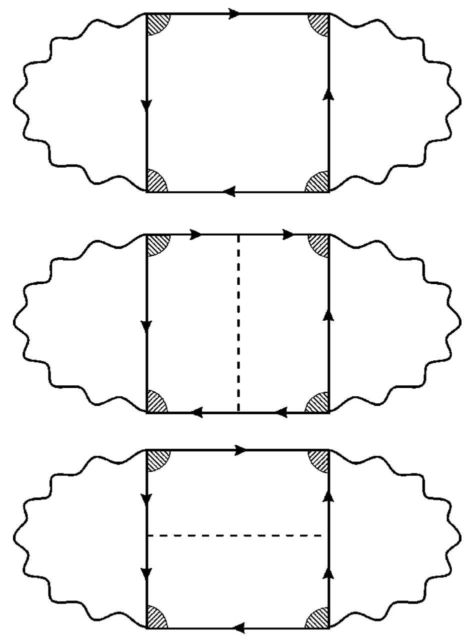

FIG. 4. Diagrams contributing to the free energy in the two-loop approximation. Similar diagrams appear in the derivation of the Ginzburg-Landau equations from the microscopic theory.

IV. TWO-LOOP APPROXIMATION: APPLICABILITY OF THE RESULTS

In the previous sections we found the fluctuation correction to the transport and thermodynamic properties of a superconductor in a magnetic field in the first (one-loop) approximation. The purpose of the given section is to find the order of the subleading corrections. This will determine the area of applicability of the results obtained. We shall calculate the magnetization in the two-loop approximation for a dirty superconductor. This correction can be easily found in view of the simplifications described above.

In the two-loop approximation, we have to deal with diagrams presented in Fig. 4. The corresponding contribution can be written in the coordinate representation in the following way:

$$F_2=T^3\sum_{\varepsilon,\Omega,\Omega'}\int d^2r_1\,d^2r_2\,d^2r_3\,d^2r_4\times K_\varepsilon(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3,\mathbf{r}_4)\mathcal{L}_\Omega(\mathbf{r}_1,\mathbf{r}_2)\mathcal{L}_{\Omega'}(\mathbf{r}_3,\mathbf{r}_4), \tag{55}$$

where K_ε is the operator corresponding to the square blocks in the diagrams presented in Fig. 4. This operator is familiar from the usual BCS theory. It has been calculated by Maki[23] and Caroli *et al.*[24] and has the form

$$K_\varepsilon(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3,\mathbf{r}_4)=\frac{\pi N(0)}{2}\,\delta(\mathbf{r}_1-\mathbf{r}_2)\,\delta(\mathbf{r}_1-\mathbf{r}_3)\,\delta(\mathbf{r}_1-\mathbf{r}_4)\times\left\{\prod_{k=1}^{4}\frac{1}{|\varepsilon|+\frac{1}{2}\mathcal{D}\partial_{(k)}^2}\right\}\times\left[|\varepsilon|+\frac{1}{8}\mathcal{D}([\partial_{(1)}-\partial_{(3)}]^2+[\partial_{(2)}-\partial_{(4)}]^2)\right], \quad (56)$$

where we make use of the Maki's notation:

$$\partial_{(k)}=-i\nabla-2e(-1)^k\mathbf{A}(\mathbf{r}).$$

In the coordinate representation, the fluctuation propagator can be expanded on the basis of the eigenfunctions in the magnetic field and has the form

$$\mathcal{L}_\Omega(\mathbf{r},\mathbf{r}')=\int_{-\infty}^{+\infty}\frac{dp_y}{2\pi}\sum_{n=0}^{\infty}\mathcal{L}_n(\Omega)\,\psi^*_{np_y}(\mathbf{r})\,\psi_{np_y}(\mathbf{r}'), \quad (57)$$

where $\mathcal{L}_n(\Omega)$ are matrix elements of the fluctuation propagator in the magnetic field [see Eq. (3)], $\psi_{np_y}(\mathbf{r})$ is the eigenfunction for an electron in a magnetic field in the Landau gauge, and p_y is the y component of the momentum, which determines the orbit's center. Again, in the vicinity of the transition line we keep the $n=0$ term only in Eq. (57). From Eqs. (55)–(57), we obtain the free energy per unit volume:

$$\frac{F_2}{V}=\frac{\pi N(0)}{2d}\,\nu^2T^3\left(\sum_\Omega L_0(\Omega)\right)^2\sum_\varepsilon\frac{1}{(|\varepsilon|+\Omega_H/4)^3}. \quad (58)$$

Thus, the magnetization takes the form

$$M_2=\frac{\nu^2}{\pi^2 dN(0)}\,\frac{1}{H_{c2}(0)}\,\frac{\partial I_\alpha^2(h,t)}{\partial h}. \quad (59)$$

At low temperatures $t\ll h$ we have

$$M_2=-\frac{2\nu^2}{\pi^2 dN(0)}\,\frac{1}{H_{c2}(0)}\,\frac{1}{h}\ln\frac{1}{h}. \quad (60)$$

We see that the second order correction is negative.

From Eqs. (48) and (59) we obtain the ratio

$$\frac{M_2}{M_1}=\frac{N_{\rm Gi}}{\pi}\left[2\gamma\frac{t}{h^2}-\frac{1}{\gamma t}\psi'\left(\frac{1}{2\gamma}\frac{h}{t}\right)\right], \quad (61)$$

where $N_{\rm Gi}$ is the Ginzburg parameter. The one-loop approximation is valid unless this ratio becomes of the order of unity. At low temperatures $t\ll h$, Eq. (61) yields the following condition:

$$h\gg N_{\rm Gi}. \quad (62)$$

If $t\gg h$, we have

$$h\gg\sqrt{N_{\rm Gi}\,t}. \quad (63)$$

This indicates that at large enough temperatures the fluctuation region becomes wider.

These results stand for the kinetic coefficients as well. In the clean case the formulas (62) and (63) are valid with $N_{\rm Gi}\sim T_{c0}/\varepsilon_F$. However, the explicit calculations are more complicated due to the nonlocal structure of the K operator.

Let us note that at an exponentially low temperature some other effects may reveal themselves. In the dirty case, the mesoscopic fluctuations may be important.[25,26] Really, the upper critical field depends on disorder. The distribution of impurities is random. There are some regions where the concentration of the impurities is such that the upper critical field is smaller than the bulk value. These regions may form superconducting islands weakly coupled one with another. At extremely low temperature the proximity effect and the Josephson coupling can make these mesoscopic fluctuations observable. The effects due to the mesoscopic fluctuations will be considered elsewhere.

V. CONCLUSION

The central result of the paper is the existence of the logarithmic correction to the conductivity which persists down to zero temperature. This correction is shown to be negative in the dirty case. The minus sign comes from the DOS diagrams as well as from the MT term. The AL contribution is positive but numerically smaller. Let us note that similar results (negative fluctuation correction to the conductivity) exist for the granular and layered superconductors.[9,27] In these cases, the AL and MT contributions are parametrically small compared to the DOS term.

The fluctuating magnetization exceeds conventional Landau diamagnetism for a very large range of fields. It is shown to be logarithmically divergent as well at $T\to 0$.

Let us note that the singular behavior of the transport and thermodynamic quantities at low temperature is due to the low dimensionality of the system. In the three-dimensional case the leading correction to the conductivity is not singular $\delta\sigma_{3D}\propto\sqrt{h}$.

The results obtained in the present paper can be checked experimentally by measuring the fluctuation conductivity in two-dimensional and quasi-two-dimensional systems. The results obtained in the dirty limit can be checked by measuring the magnetoresistance in the dirty superconducting films at low temperatures. In this case, there could be some experimental difficulties connected with the H_{c3} effects that can screen the bulk properties of a film. The edge effects can be excluded, for example, by putting a sufficient amount of magnetic impurities on the edge of the film.

Let us mention some recent experiments of Gantmakher *et al.*[28,29] In these experiments the magnetic-field-tuned quantum phase transition has been studied in dirty In-O films at low temperatures. It was found that in the vicinity of the transition, the magnetoresistance reaches a maximum. It is possible that the theory developed in the present paper can give an explanation for the observed effects.

The clean case may be relevant to high-T_c superconductors[30] and, probably, to the recently discovered two-dimensional organic superconductors.[31] Let us note that our results assume s pairing and an isotropic Fermi surface which is not true for high-T_c superconductors. However, it can be shown that the logarithmic singularity remains for any pairing type (with a coefficient different from our case). It is worth mentioning that in the overdoped high-T_c superconductors the Ginzburg parameter N_{Gi} is small and, thus, the fluctuations are negligible. In the underdoped superconductors the fluctuations are extremely large and they lead to a large pseudogap which makes the conventional Fermi-liquid theory inapplicable. Hence, optimally doped superconductors should be used to check the results obtained.

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APPENDIX: CURRENT VERTEX

In this appendix we derive the formula for the current vertex depending on two frequencies Ω and ω in the nonlocal clean limit. The corresponding result is used when calculating the Aslamazov-Larkin contribution to the conductivity (see Sec. II B).

The current vertex is the triangle block in the AL diagram (see Fig. 1 diagram 1). It consists of three Green functions. In the momentum representation, it can be written as

$$\begin{aligned}\boldsymbol{\gamma}_\varepsilon(\mathbf{q};\Omega,\omega) &= \int \frac{d^2p}{(2\pi)^2}\mathbf{v}\,\mathcal{G}_\varepsilon(\mathbf{p})\mathcal{G}_{\varepsilon-\omega}(\mathbf{p})\mathcal{G}_{\Omega-\varepsilon}(\mathbf{q}-\mathbf{p})\\ &= -N(0)\int d\xi \frac{1}{\xi - i\tilde{\varepsilon}}\frac{1}{\xi - i(\widetilde{\varepsilon-\omega})}\\ &\quad\times\left\langle \frac{\mathbf{v}}{\xi - \mathbf{vq} - i(\widetilde{\Omega-\varepsilon})}\right\rangle, \end{aligned}\tag{A1}$$

where $\tilde{\varepsilon} = \varepsilon + (i/2\tau)\,\mathrm{sgn}\,\varepsilon$ and the angular brackets imply averaging over the Fermi line. To perform this averaging one can use the following identity:

$$\left\langle \frac{\mathbf{v}}{\mathbf{vq} - i\epsilon}\right\rangle = \frac{\mathbf{q}}{q^2}\left(1 - \frac{|\epsilon|}{\sqrt{\epsilon^2 + v_F^2q^2}}\right) \equiv \frac{\mathbf{q}}{q^2}f(\epsilon,q). \tag{A2}$$

There are six possible configurations of the poles which give nonzero contributions to the integral over ξ in Eq. (A1). Straightforward calculation yields the following expression for the current vertex:

$$\boldsymbol{\gamma}_\varepsilon(\mathbf{q};\Omega,\omega) = \frac{2\pi N(0)}{\omega}\frac{\mathbf{q}}{q^2}\sum_{i=1}^{4}\eta_i(\varepsilon,\Omega,\omega)f(\epsilon_i,q), \tag{A3}$$

where

$$\begin{aligned}\eta_1(\varepsilon,\Omega,\omega) = \frac{\omega\tau}{1+\omega\tau\,\mathrm{sgn}\,\varepsilon}[&\theta(\varepsilon)\theta(\omega-\varepsilon)\theta(\varepsilon-\Omega)\\ &+\theta(-\varepsilon)\theta(\varepsilon-\omega)\theta(\Omega-\varepsilon)],\end{aligned}\tag{A4}$$

$$\begin{aligned}\eta_2(\varepsilon,\Omega,\omega) = \frac{\omega\tau}{1+\omega\tau\,\mathrm{sgn}\,\varepsilon}[&\theta(\varepsilon)\theta(\omega-\varepsilon)\theta(\Omega-\varepsilon)\\ &+\theta(-\varepsilon)\theta(\varepsilon-\omega)\theta(\varepsilon-\Omega)],\end{aligned}\tag{A5}$$

$$\begin{aligned}\eta_3(\varepsilon,\Omega,\omega) = -\eta_4(\varepsilon,\Omega,\omega) = [&\theta(\varepsilon)\theta(\varepsilon-\omega)\theta(\varepsilon-\Omega)\\ &-\theta(-\varepsilon)\theta(\omega-\varepsilon)\theta(\Omega-\varepsilon)],\end{aligned}\tag{A6}$$

and

$$\epsilon_1 = \epsilon_4 = 2\varepsilon - \Omega + \tau^{-1}\,\mathrm{sgn}\,\varepsilon, \tag{A7}$$

$$\epsilon_2 = \epsilon_3 = 2\varepsilon - \Omega - \omega + \tau^{-1}\,\mathrm{sgn}\,\varepsilon. \tag{A8}$$

This presentation is convenient when we calculate the current vertex in the magnetic field (Sec. II B).

In the dirty limit, Eq. (A3) reduces to the local equation, since in the limit $\tau\to 0$

$$f(\epsilon,q) \approx \mathcal{D}q^2\tau$$

and thus the current vertex reads

$$\boldsymbol{\gamma}(\mathbf{q}) = c\,\mathbf{q} \equiv -4\pi N(0)\mathcal{D}\tau^2\mathbf{q}. \tag{A9}$$

Here we keep only the terms which do not contradict the θ functions in the Cooperons, i.e., the third and fourth terms in Eq. (A3). We see that in the coordinate representation the vertex has the form

$$\boldsymbol{\gamma}(\mathbf{r}) = -c\nabla\delta(\mathbf{r}).$$

This δ-functional behavior implies the locality of the current vertex in the dirty case.

Let us also note that if the external frequency ω is zero, the current vertex is easily connected with the particle-particle bubble for any τ:

$$\lim_{\omega\to 0}\boldsymbol{\gamma}_\varepsilon(\mathbf{q};\Omega,\omega) = \frac{\partial\Pi_\varepsilon(\mathbf{q};\Omega)}{\partial\mathbf{q}}, \tag{A10}$$

where

$$\begin{aligned}\Pi_\varepsilon(\mathbf{q};\Omega) &= N(0)\left\langle\int d\xi\,\mathcal{G}_\varepsilon(\mathbf{p})\mathcal{G}_{\Omega-\varepsilon}(\mathbf{q}-\mathbf{p})\right\rangle\\ &= 2\pi N(0)\frac{\theta(\varepsilon(\varepsilon-\Omega))}{\sqrt{\left(2\varepsilon-\Omega+\frac{1}{\tau}\mathrm{sgn}\,\varepsilon\right)^2 + v_F^2q^2}}.\end{aligned}\tag{A11}$$

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