Local field theory for disordered itinerant quantum ferromagnets

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An effective field theory is derived that describes the quantum critical behavior of itinerant ferromagnets in the presence of quenched disorder. In contrast to previous approaches, all soft modes are kept explicitly. The resulting effective theory is local and allows for an explicit perturbative treatment. It is shown that previous suggestions for the critical fixed point and the critical behavior are recovered under certain assumptions. The validity of these assumptions is discussed in the light of the existence of two different time scales. It is shown that, in contrast to previous suggestions, the correct fixed-point action is not Gaussian, and that the previously proposed critical behavior was correct only up to logarithmic corrections. The connection with other theories of disordered interacting electrons and, in particular, with the resolution of the runaway flow problem encountered in these theories, is also discussed.

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I. INTRODUCTION

The theory of quantum phase transitions, i.e., phase transitions at zero temperature ($T=0$), is an important problem that is relevant to many topics in condensed-matter physics.[1] Perhaps the most obvious example is the $T=0$ transition from a paramagnet to an itinerant ferromagnet as it occurs in, e.g., diluted Ni or in solid solutions like MnSi. Historically, this was the first quantum phase transition that was studied in detail. Hertz[2] showed how to treat this transition by means of renormalization group (RG) methods and he concluded that the transition in the physically interesting dimension $d=3$ was mean-field-like. This conclusion hinged on the observation that in quantum statistical mechanics the effective dimension of a system for scaling purposes is $d+z$, with d the spatial dimensionality and z the dynamical critical exponent. Since in a simple theory at tree level one has $z=3$ for clean itinerant quantum ferromagnets and $z=4$ for disordered ones,[2] this seemed to imply that the upper critical dimension d_c^+, above which one finds mean-field critical behavior, is $d_c^+=1$ and $d_c^+=0$ in the clean and disordered cases, respectively.

This conclusion was later challenged.[3–5] Reference 3 noted that Hertz's results for clean systems in $d=1-\epsilon$ dimensions were inconsistent with general scaling arguments. This left open the possibility of mean-field behavior in physical dimensions. However, in Refs. 4 and 5 it was shown that the critical behavior in $d>1$ (clean case) and $d>0$ (disordered case), respectively, is not mean-field-like after all. The salient point is that in itinerant electron systems at $T=0$, soft modes other than the order-parameter fluctuations exist. These modes are diffusive (in disordered systems) or ballistic (in clean ones) particle-hole excitations. Since they couple to the order-parameter fluctuations, they influence the critical behavior. Specifically, they lead to an effective long-range interaction between the order-parameter fluctuations. As a result, Refs. 4 and 5 found that the critical behavior is governed by a Gaussian fixed point that, however, does not yield mean-field exponents. The critical behavior determined in these references was claimed to be exact.

A separate, and seemingly unconnected, development in the many-electron problem has been the study of metal-insulator transitions of disordered interacting electrons.[6] For one of the universality classes that occur in this problem, a transition was found that is *not* a metal-insulator transition, but rather of magnetic nature.[7] While the order parameter, and the nature of the ordered phase, could not be identified with the methods employed, the critical behavior for all quantities other than the order parameter was determined.[8] Apart from differences in logarithmic corrections to power laws, this critical behavior turned out to be identical with the Gaussian critical behavior for the disordered ferromagnetic transition. This led, in Ref. 4, to the suggestion that the unidentified transition studied in Ref. 8 was the ferromagnetic transition. The discrepancy with respect to the logarithmic terms was explained as due to the fact that, of the two integral equations in Ref. 8 only one had been shown to be exact. The proposal thus was that the two approaches describe the same transition, and that the critical behavior found in Ref. 4 was exact while the one in Ref. 8 represented an approximation.

The theory developed in Refs. 4 and 5 suffers from one major drawback: Since the additional soft modes were integrated out in order to obtain a description entirely in terms of the order parameter, the effective field theory that was derived is nonlocal[9] and not very suitable for perturbatively calculating effects that depend on all of the soft modes in the system. The analysis in Refs. 4 and 5 therefore was restricted to power-counting arguments at tree level to show that all non-Gaussian terms are irrelevant in a RG sense. While this turned out to be true, relying entirely on tree-level power counting can be dangerous. Indeed, even a one-loop analysis of Hertz's action would have revealed that the mean-field fixed point is unstable, and the absence of such a calculation led to the instability that was not to be noticed for 20 years.[10] Furthermore, integrating out the fermionic degrees of freedom obscures the fact that the problem contains two time scales, a diffusive one and a critical one, which in itself makes power counting very subtle. This, combined with the suggestive relation between the ferromagnetic transition and the unidentified transition discussed above, and the puzzling logarithmic discrepancies between the critical behaviors found for the two transitions, makes it desirable to have a theoretical description of the quantum ferromagnetic transition that takes the form of a local field theory that facilitates a controlled loop expansion and keeps the two time scales explicitly. Another motivation for constructing a local field theory is that it will allow for a more explicit study of the effects of rare regions[11] than was possible within the framework of Refs. 4 and 5, although we will not pursue this issue in the present paper.

It is the purpose of the present paper, and a second one to be referred to as II,[12] to put these remaining questions to rest. We will focus on the disordered case, although we expect analogous conclusions to hold for clean systems. By using a local-field-theory description, we will show that Ref. 4 missed effects of the two time scales that lead to logarithmic corrections to the Gaussian critical behavior. Moreover, taking these effects into account leads to integral equations for the relevant vertex functions that are identical to the ones derived in Ref. 8. The current formulation makes it obvious that the transition described by these equations is the quantum ferromagnetic one, and it allows to determine the exponents in the ferromagnetic phase as well as those in the paramagnetic one. It, furthermore, shows that the integral equations that were first derived in Ref. 8 are exact and it elucidates many physical points that were rather obscure in Ref. 4, and to an even larger extent in Ref. 8.

This paper is organized as follows. In Sec. II we use methods developed in Ref. 13 to derive an effective theory for disordered itinerant quantum ferromagnets that systematically separates massive modes from soft ones and explicitly keeps all of the latter. In Sec. III we give a RG analysis of this model. We first show how Hertz's fixed point as well as the Gaussian fixed point of Ref. 4 emerge within this framework. We then show that Hertz's fixed point is unstable, and the Gaussian one is marginally unstable due to the existence of two separate time scales, viz., a critical time scale associated with the order-parameter fluctuations, and a diffusive one associated with the additional particle-hole excitations. We identify an effective action that contains a stable critical fixed point. This action is not Gaussian and its solution is therefore nontrivial and deferred to II. In Sec. IV we discuss our results, in particular, the relation of the present approach to previous ones and the complications that the presence of two time scales leads to in scaling considerations.

II. EFFECTIVE FIELD THEORY

In this section we start with a simple model for interacting electrons in a disordered environment. We then introduce the ferromagnetic order parameter and identify all other soft modes. Integrating out the massive modes leads to an effective field theory that describes all of the soft modes in the system. The general method employed here is the one that was developed in Ref. 13.

A. Model of itinerant electrons

Our starting point is a general field-theoretic representation of the partition function of a many-fermion system, which can be written in the form[14]

$$Z=\int D[\bar{\psi},\psi]\exp(S[\bar{\psi},\psi]). \tag{2.1a}$$

Here the functional integration measure is defined with respect to Grassmannian or anticommuting fields $\bar{\psi}$ and ψ and S is the action,

$$S=-\int_0^\beta d\tau\int d\mathbf{x}\bar{\psi}_a(\mathbf{x},\tau)\frac{\partial}{\partial\tau}\psi_a(\mathbf{x},\tau)-\int_0^\beta d\tau H(\tau). \tag{2.1b}$$

We denote the spatial position by $\mathbf{x}$ and the imaginary time by τ. $H(\tau)$ is the Hamiltonian in imaginary time representation, $\beta=1/T$ is the inverse temperature, $a=1,2$ denotes spin labels, and a summation over repeated spin indices is implied. We choose units such that $k_B=\hbar=e^2=1$. The Hamiltonian describes a fluid of interacting electrons moving in a static random potential $v(\mathbf{x})$,

$$\begin{aligned}H(\tau)=\int d\mathbf{x}\Big\{&\frac{1}{2m}\nabla\bar{\psi}_a(\mathbf{x},\tau)\cdot\nabla\psi_a(\mathbf{x},\tau)\\&+[v(\mathbf{x})-\mu]\bar{\psi}_a(\mathbf{x},\tau)\psi_a(\mathbf{x},\tau)\Big\}\\&+\frac{1}{2}\int d\mathbf{x}d\mathbf{y}u(\mathbf{x}-\mathbf{y})\bar{\psi}_a(\mathbf{x},\tau)\bar{\psi}_b(\mathbf{y},\tau)\\&\times\psi_b(\mathbf{y},\tau)\psi_a(\mathbf{x},\tau).\end{aligned} \tag{2.2a}$$

Here m is the electron mass, μ is the chemical potential, and $u(\mathbf{x}-\mathbf{y})$ is the electron-electron interaction potential. For simplicity, we assume that the random potential $v(\mathbf{x})$ is delta correlated and obeys a Gaussian distribution $P[v(\mathbf{x})]$ with second moment

$$\{v(\mathbf{x})v(\mathbf{y})\}_{\rm dis}=\frac{1}{2\pi N_F\tau_{\rm el}}\delta(\mathbf{x}-\mathbf{y}), \tag{2.2b}$$

where

$$\{\cdots\}_{\rm dis}=\int D[v]P[v](\cdots) \tag{2.2c}$$

denotes the disorder average, N_F is the bare density of states per spin at the Fermi level, and $\tau_{\rm el}$ is the bare electron elastic mean-free time. Our results will not be sensitive to the simplifications inherent in the assumptions that lead to Eq. (2.2b). We also mention that it would be possible to include more realistic features, e.g., band structure in the model. However, ultimately we will be interested in universal behavior at a phase transition that is independent of all microscopic details. For our purposes it therefore is sufficient to study the model defined in Eqs. (2.2).[15]

As in Ref. 4, and as is standard practice in the theory of magnetism, we break the interaction part of the action S, which we denote by $S_{\rm int}$, into spin-singlet and spin-triplet contributions $S^{(s,t)}$. For simplicity, we assume that the interactions are short ranged in both of these channels.[16] The spin-triplet interaction $S_{\rm int}^{(t)}$ describes interactions between spin-density fluctuations. This is the interaction that causes ferromagnetism and it therefore needs to be considered separately. We thus write

$$S=S_0+S_{\rm int}^{(t)}, \tag{2.3}$$

with

$$S_{\rm int}^{(t)}=\frac{\Gamma_t}{2}\int d\mathbf{x}d\tau\,\mathbf{n}_s(\mathbf{x},\tau)\cdot\mathbf{n}_s(\mathbf{x},\tau), \tag{2.4a}$$

where $\mathbf{n}_s$ is the electron spin-density vector with components

$$n_s^i(\mathbf{x},\tau)=\overline{\psi}_a(\mathbf{x},\tau)\sigma_i^{ab}\psi_b(\mathbf{x},\tau). \tag{2.4b}$$

Here σ_i $(i=1,2,3)$ are the Pauli matrices and Γ_t is the spin-triplet interaction amplitude that is related to the interaction potential u in Eq. (2.2a) via

$$\Gamma_t=\frac{1}{2}\int d\mathbf{x}u(\mathbf{x}). \tag{2.4c}$$

S_0 in Eq. (2.3) contains all other contributions to the action. It explicitly reads

$$\begin{aligned}S_0=&-\int_0^\beta d\tau\int d\mathbf{x}\Big[\overline{\psi}_a(\mathbf{x},\tau)\frac{\partial}{\partial\tau}\psi_a(\mathbf{x},\tau)-\overline{\psi}_a(\mathbf{x},\tau)\\&\times\left\{\frac{\nabla^2}{2m}+\mu\right\}\psi_a(\mathbf{x},\tau)\Big]-\frac{\Gamma_s}{2}\int_0^\beta d\tau\int d\mathbf{x}n_c(\mathbf{x},\tau)n_c(\mathbf{x},\tau)\\&-\int_0^\beta d\tau\int d\mathbf{x}v(\mathbf{x})\overline{\psi}_a(\mathbf{x},\tau)\psi_a(\mathbf{x},\tau),\end{aligned} \tag{2.5a}$$

where n_c is the electron charge or number density,

$$n_c(\mathbf{x},\tau)=\overline{\psi}_a(\mathbf{x},\tau)\psi_a(\mathbf{x},\tau), \tag{2.5b}$$

and Γ_s the spin-singlet interaction amplitude.

Before we proceed, we integrate out the quenched disorder by means of the replica trick.[17] Performing the disorder average as prescribed in Eq. (2.2c) replaces the last contribution to the action S_0, Eq. (2.5a), by

$$\begin{aligned}S_{\rm dis}=&\frac{1}{4\pi N_{\rm F}\tau_{\rm el}}\sum_{\alpha_1,\alpha_2=1}^{N}\int_0^\beta d\tau d\tau'\int d\mathbf{x}\overline{\psi}_a^{\alpha_1}(\mathbf{x},\tau)\\&\times\psi_a^{\alpha_1}(\mathbf{x},\tau)\overline{\psi}_b^{\alpha_2}(\mathbf{x},\tau')\psi_b^{\alpha_2}(\mathbf{x},\tau'),\end{aligned} \tag{2.6}$$

where α_1 and α_2 are replica indices, and $N\to 0$ is the number of replicas. Of course, all other terms in the action also are replicated N times.

B. Composite variables

We now proceed by rewriting our model in terms of variables that are more suitable for our purposes than the basic fermionic field. First of all, we decouple the spin-triplet interaction by means of a Hubbard-Stratonovich transformation. All other terms we rewrite in terms of bosonic matrix fields Q and $\tilde{\Lambda}$. The latter procedure exactly follows Ref. 13 and we refer the reader to this reference for details. Here we just mention that $\tilde{\Lambda}$ serves as a Lagrangian multiplier whose physical interpretation is self-energy, while Q is isomorphic to bilinear products of fermion fields. We perform a Fourier transform from imaginary time τ to Matsubara frequencies $\omega_n=2\pi T(n+1/2)$,

$$\begin{aligned}\psi_{n,a}(\mathbf{x})&=\sqrt{T}\int_0^\beta d\tau e^{i\omega_n\tau}\psi_a(\mathbf{x},\tau),\\ \overline{\psi}_{n,a}(\mathbf{x})&=\sqrt{T}\int_0^\beta d\tau e^{-i\omega_n\tau}\overline{\psi}_a(\mathbf{x},\tau).\end{aligned} \tag{2.7a}$$

For later reference, we also define a spatial Fourier transform

$$\begin{aligned}\psi_{n,a}(\mathbf{k})&=\frac{1}{\sqrt{V}}\int d\mathbf{x}e^{-i\mathbf{k}\cdot\mathbf{x}}\psi_{n,a}(\mathbf{x}),\\ \overline{\psi}_{n,a}(\mathbf{k})&=\frac{1}{\sqrt{V}}\int d\mathbf{x}e^{i\mathbf{k}\cdot\mathbf{x}}\overline{\psi}_{n,a}(\mathbf{x}),\end{aligned} \tag{2.7b}$$

and analogously for other position-dependent quantities. The isomorphism then takes the form[13]

$$Q_{12}\cong\frac{i}{2}\begin{pmatrix}-\psi_{1\uparrow}\overline{\psi}_{2\uparrow} & -\psi_{1\uparrow}\overline{\psi}_{2\downarrow} & -\psi_{1\uparrow}\psi_{2\downarrow} & \psi_{1\uparrow}\psi_{2\uparrow}\\ -\psi_{1\downarrow}\overline{\psi}_{2\uparrow} & -\psi_{1\downarrow}\overline{\psi}_{2\downarrow} & -\psi_{1\downarrow}\psi_{2\downarrow} & \psi_{1\downarrow}\psi_{2\uparrow}\\ \overline{\psi}_{1\downarrow}\overline{\psi}_{2\uparrow} & \overline{\psi}_{1\downarrow}\overline{\psi}_{2\downarrow} & \overline{\psi}_{1\downarrow}\psi_{2\downarrow} & -\overline{\psi}_{1\downarrow}\psi_{2\uparrow}\\ -\overline{\psi}_{1\uparrow}\overline{\psi}_{2\uparrow} & -\overline{\psi}_{1\uparrow}\overline{\psi}_{2\downarrow} & -\overline{\psi}_{1\uparrow}\psi_{2\downarrow} & \overline{\psi}_{1\uparrow}\psi_{2\uparrow}\end{pmatrix}. \tag{2.8}$$

Here all fields are understood to be taken at position $\mathbf{x}$, and $1\equiv(n_1,\alpha_1)$, etc., comprises both frequency and replica labels. It is convenient to expand the 4×4 matrix in Eq. (2.8) in a spin-quaternion basis,

$$Q_{12}(\mathbf{x}) = \sum_{r,i=0}^{3} (\tau_r \otimes s_i)_r^i Q_{12}(\mathbf{x}) \tag{2.9}$$

and analogously for $\tilde{\Lambda}$. Here $\tau_0 = s_0 = \mathbb{1}_2$ is the 2×2 unit matrix and $\tau_j = -s_j = -i\sigma_j$ $(j=1,2,3)$, with $\sigma_{1,2,3}$ the Pauli matrices. In this basis, $i=0$ and $i=1,2,3$ describe the spin-singlet and the spin-triplet, respectively. An explicit calculation reveals that $r=0,3$ corresponds to the particle-hole channel (i.e., products $\overline{\psi}\psi$), while $r=1,2$ describes the particle-particle channel (i.e., products $\overline{\psi\psi}$ or $\psi\psi$). For our purposes the latter will not be of importance and we therefore drop $r=1,2$ from the spin-quaternion basis.

Denoting the Hubbard-Stratonovich field by M, we now can exactly rewrite the partition function as an integral over the three fields Q, $\tilde{\Lambda}$, and M,

$$Z = \int D[Q,\tilde{\Lambda},M] e^{\mathcal{A}[Q,\tilde{\Lambda},M]}, \tag{2.10a}$$

with an action

$$\begin{aligned}\mathcal{A}[Q,\tilde{\Lambda},M] = {} & \mathcal{A}_{\rm dis}[Q] + \mathcal{A}_{\rm int}^{(s)}[Q] \\ & + \frac{1}{2}\mathrm{Tr}\ln(G_0^{-1} - i\tilde{\Lambda}) + \mathrm{Tr}(\tilde{\Lambda}Q) \\ & - \int d\mathbf{x} \sum_{\alpha} \sum_{n} \sum_{i=1}^{3} {}^{i}M_n^{\alpha}(\mathbf{x}) {}^{i}M_{-n}^{\alpha}(\mathbf{x}) \\ & + \sqrt{2T\Gamma_t} \int d\mathbf{x} \sum_{\alpha} \sum_{n} \sum_{i=1}^{3} {}^{i}M_n^{\alpha}(\mathbf{x}) \\ & \times \sum_{r=0,3} (\sqrt{-1})^r \sum_{m} \mathrm{tr}[(\tau_r \otimes s_i) Q_{m,m+n}^{\alpha\alpha}(\mathbf{x})].\end{aligned} \tag{2.10b}$$

Here Tr denotes a trace over all degrees of freedom, including the continuous real space position, while tr is a trace over all discrete degrees of freedom that are not summed over explicitly. The first two terms in Eq. (2.10b) explicitly read

$$\mathcal{A}_{\rm dis}[Q] = \frac{1}{2\pi N_{\rm F}\tau_{\rm el}} \int d\mathbf{x} \mathrm{tr}[Q(\mathbf{x})]^2, \tag{2.10c}$$

$$\begin{aligned}\mathcal{A}_{\rm int}^{(s)} = {} & \frac{T\Gamma^{(s)}}{2} \int d\mathbf{x} \sum_{r=0,3} (-1)^r \\ & \times \sum_{n_1,n_2,m} \sum_{\alpha} [\mathrm{tr}\{(\tau_r \otimes s_0) Q_{n_1,n_1+m}^{\alpha\alpha}(\mathbf{x})\}] \\ & \times [\mathrm{tr}\{(\tau_r \otimes s_0) Q_{n_2+m,n_2}^{\alpha\alpha}(\mathbf{x})\}].\end{aligned} \tag{2.10d}$$

They are simply the last two terms in Eq. (2.5a), rewritten in terms of Q. Finally,

$$G_0^{-1} = -\partial_\tau + \nabla^2/2m + \mu \tag{2.10e}$$

is the inverse Green operator. For later reference we also note that if $\tilde{\Lambda}$ in the Tr ln term in Eq. (2.10b) is replaced by its saddle-point value, one obtains the inverse saddle-point Green function $G_{\rm sp}$ as the argument of the logarithm.[13] For our purposes it suffices to treat the disorder contribution to the self-energy in the Born approximation, and to neglect the (Hartree-Fock) interaction contribution. $G_{\rm sp}$ then reads

$$G_{\rm sp}(\mathbf{k},\omega_n) \approx \left[i\omega_n - \frac{\mathbf{k}^2}{2m} + \mu + \frac{i}{2\tau_{\rm el}} \mathrm{sgn}\,\omega_n \right]^{-1}. \tag{2.10f}$$

Notice that Eqs. (2.10) can also be obtained from Eqs. (2.22)–(2.25) in Ref. 13 by subjecting the spin-triplet interaction term to a Gaussian transformation and by using the symmetry properties of the Q matrices. However, the above derivation makes it clear that the order-parameter field M is introduced in a standard way, and the only difference to a standard treatment is that the fermionic parts of the action have been rewritten in terms of Q and $\tilde{\Lambda}$.

C. Separation of soft and massive modes

The reason for our rewriting of the fermionic part of the action in terms of bosonic-matrix fields in the previous subsection was that this formulation is particularly well suited for a separation of soft and massive modes. For the purpose of analyzing a Fermi liquid, this separation was carried out in Ref. 13, and we briefly recall the most important points of that procedure.[18] One first uses group-theory arguments to show that the most general Q can be written in the form

$$Q = \mathcal{S} P \mathcal{S}^{-1}. \tag{2.11}$$

Here P is block diagonal in the Matsubara frequency space,

$$P = \begin{pmatrix} P^{>} & 0 \\ 0 & P^{<} \end{pmatrix}, \tag{2.12}$$

where $P^{>}$ and $P^{<}$ are matrices with elements P_{nm} where $n,m>0$ and $n,m<0$, respectively. For a system with N replicas and n Matsubara frequencies, the matrices $\mathcal{S}$ are elements of the homogeneous space $\mathrm{USp}(8Nn,\mathcal{C})/\mathrm{USp}(4Nn,\mathcal{C})\times\mathrm{USp}(4Nn,\mathcal{C})$.[19] As such they can be expressed in terms of matrices q whose elements q_{nm} are restricted to frequency labels $n\geq 0$, $m<0$,

$$\mathcal{S} = \begin{pmatrix} \sqrt{1-bb^{\dagger}} & b \\ -b^{\dagger} & \sqrt{1-b^{\dagger}b} \end{pmatrix}, \tag{2.13a}$$

where

$$b(q,q^{\dagger}) = \frac{-1}{2} q f(q^{\dagger}q), \tag{2.13b}$$

with

$$f(x) = \sqrt{\frac{2}{x}}(1-\sqrt{1-x})^{1/2}. \tag{2.13c}$$

Ward identities ensure that the P are massive while the q are massless. The latter are the diffusive modes or 'diffusons' that we mentioned in the Introduction. Similarly, if $\tilde{\Lambda}$ is transformed according to

$$\Lambda(\mathbf{x}) = \mathcal{S}^{-1}(\mathbf{x})\tilde{\Lambda}(\mathbf{x})\mathcal{S}(\mathbf{x}), \tag{2.14}$$

Λ can be shown to be massive. This representation thus achieves the desired separation of modes. In Ref. 13 the q were the only soft modes. In the present case, the order-parameter field M is massive in the paramagnetic phase but becomes soft at criticality, so it must be handled together with the diffusons.

The next step is to expand the massive modes about their expectation values,

$$P = \langle P \rangle + \Delta P, \quad \Lambda = \langle \Lambda \rangle + \Delta\Lambda. \tag{2.15}$$

As explained in Ref. 13, the expectation values $\langle P \rangle$ and $\langle \Lambda \rangle$ can be replaced by simple saddle-point approximations. The saddle point used in this reference is also a saddle point of our current action, Eq. (2.10b), if it is supplemented by the saddle-point value of M, $\langle M \rangle = 0$. If P and Λ are integrated out in the saddle-point approximation, i.e., if one neglects all fluctuations of these fields, then one obtains for the fermionic degrees of freedom the nonlinear sigma model (NLσM) that was first proposed by Wegner[20] as an effective field theory for the disordered-electron problem, and later studied extensively by him and others.[6,21] Following the same procedure here we get the NLσM, but without the spin-triplet interaction term, from the first four terms on the right-hand side of Eq. (2.10b), and a coupling between the order-parameter field and both the soft and the massive fermionic modes from the last one. We will also need the corrections to the NLσM and thus rewrite the action in the form

$$\begin{aligned}
&\mathcal{A}[q, M, \Delta P, \Delta\Lambda] \\
&\quad = \mathcal{A}_{\mathrm{NL}\sigma\mathrm{M}}[q] + \delta\mathcal{A}[\Delta P, \Delta\Lambda, q] \\
&\qquad - \int d\mathbf{x} \sum_{\alpha} \sum_{n} \sum_{i=1}^{3} {}^{i}M_{n}^{\alpha}(\mathbf{x})\, {}^{i}M_{-n}^{\alpha}(\mathbf{x}) \\
&\qquad + \sqrt{\pi T K_t} \int d\mathbf{x} \sum_{\alpha} \sum_{n} \sum_{i=1}^{3} {}^{i}M_{n}^{\alpha}(\mathbf{x}) \\
&\qquad \times \sum_{r=0,3} (\sqrt{-1})^{r} \sum_{m} \mathrm{tr}(\tau_r \otimes s_i) \Big[\hat{Q}_{m,m+n}^{\alpha\alpha}(\mathbf{x}) \\
&\qquad + \frac{2}{\pi N_{\mathrm{F}}} (\mathcal{S}\Delta P \mathcal{S}^{-1})_{m,m+n}^{\alpha\alpha}(\mathbf{x}) \Big].
\end{aligned} \tag{2.16}$$

Here $K_t = \pi N_{\mathrm{F}}^2 \Gamma_t / 2$. $\mathcal{A}_{\mathrm{NL}\sigma\mathrm{M}}$ is the action of the nonlinear sigma model,

$$\begin{aligned}
\mathcal{A}_{\mathrm{NL}\sigma\mathrm{M}} = \frac{-1}{2G} \int d\mathbf{x}\, \mathrm{tr}[\nabla \hat{Q}(\mathbf{x})]^2 + 2H \int d\mathbf{x}\, \mathrm{tr}[\Omega \hat{Q}(\mathbf{x})] \\
+ \mathcal{A}_{\mathrm{int}}^{(s)}[\pi N_{\mathrm{F}} \hat{Q}/2],
\end{aligned} \tag{2.17a}$$

with $\mathcal{A}_{\mathrm{int}}^{(s)}$ from Eq. (2.10d), and

$$\hat{Q} = \begin{pmatrix} \sqrt{1 - qq^{\dagger}} & q \\ q^{\dagger} & -\sqrt{1 - q^{\dagger}q} \end{pmatrix}, \tag{2.17b}$$

in terms of q, and Ω is a frequency matrix with elements

$$\Omega_{12} = (\tau_0 \otimes s_0)\, \delta_{12} \omega_{n_1}. \tag{2.17c}$$

The coupling constants G and H are proportional to the inverse conductivity, $G \propto 1/\sigma$, and the specific heat coefficient, $H \propto \gamma \equiv \lim_{T \to 0} C_V / T$, respectively.[6,13,22]

$\delta\mathcal{A}$ contains the corrections to the nonlinear sigma model that were given in Ref. 13. We list explicitly the terms that are bilinear in the massive fluctuations ΔP and $\Delta\Lambda$, but do not contain couplings between the massive modes and q,

$$\begin{aligned}
\delta\mathcal{A}^{(2)} = \mathcal{A}_{\mathrm{dis}}[\Delta P] + \int d\mathbf{x}\, \mathrm{tr}[\Delta\Lambda(\mathbf{x}) \Delta P(\mathbf{x})] \\
+ \frac{1}{4} \int d\mathbf{x} d\mathbf{y}\, \mathrm{tr}[G(\mathbf{x}-\mathbf{y}) \Delta\Lambda(\mathbf{y}) G(\mathbf{y}-\mathbf{x}) \Delta\Lambda(\mathbf{x})].
\end{aligned} \tag{2.18}$$

D. Effective field theory for the soft modes

So far we have exactly rewritten the microscopic action in a form that separates the soft modes from the massive ones. In this subsection we approximately integrate out the massive modes to arrive at an effective action that is capable of describing the critical behavior at the ferromagnetic transition. We will also add some terms to the bare action that we will find later to be generated by the renormalization group. In Sec. III we will derive these terms and will also justify our approximations and show that they do not influence the asymptotic critical behavior.

1. Integrating out the massive modes

We now need to dispose of the massive modes. Clearly, we cannot just ignore the massive fluctuations. It is obvious from Eqs. (2.16) and (2.17) that they are needed to bring the purely M-dependent part of the action in a standard Landau-Ginzburg-Wilson (LGW) form. Let us first integrate out ΔP and $\Delta\Lambda$ in a Gaussian approximation, while neglecting the coupling between q and these massive fluctuations. We will consider the effects of this coupling later. From Eq. (2.18) and the last term in Eq. (2.16) with $\mathcal{S} = 1$ we obtain a contribution to the action that is quadratic in the order-parameter field M. Combining it with the M^2 term in Eq. (2.16) yields a term proportional to

$$\sum_{\mathbf{k}} \sum_{\alpha} \sum_{n} \sum_{i=1}^{3} {}^{i}M_{n}^{\alpha}(\mathbf{k}) [1 + 2\Gamma_t \tilde{\chi}(\mathbf{k}, \Omega_n)]\, {}^{i}M_{-n}^{\alpha}(-\mathbf{k}), \tag{2.19a}$$

where

$$\tilde{\chi}(\mathbf{k}, \Omega_n) = T \sum_{n_1, n_2} \Theta(n_1 n_2)\, \delta_{n_1 - n_2, n}\, \mathcal{D}_{n_1 n_2}(\mathbf{k}) \tag{2.19b}$$

is a restricted spin susceptibility that is given in terms of

$$\mathcal{D}_{nm}(\mathbf{k}) = \varphi_{nm}(\mathbf{k})\left[1 - \frac{1}{2\pi N_F \tau_{el}}\varphi_{nm}(\mathbf{k})\right]^{-1} \quad (2.19c)$$

with

$$\varphi_{nm}(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{p}} G_{sp}(\mathbf{p},\omega_n) G_{sp}(\mathbf{p}+\mathbf{k},\omega_m). \quad (2.19d)$$

Here G_{sp} is the saddle-point Green function from Eq. (2.10f). The Θ function in Eq. (2.19b), which restricts the frequency sum to frequencies that both have the same sign, results in $\tilde{\chi}$ being the nonhydrodynamic part of the spin susceptibility of noninteracting disordered electrons. For small frequencies and wavenumbers, it reads

$$\tilde{\chi}(\mathbf{k},\Omega_n) = -N_F + O(\mathbf{k}^2,\Omega_n). \quad (2.20a)$$

For later reference, we also note that for $nm<0$, $\mathcal{D}_{nm}$ is the basic diffusion propagator. In the limit of small frequencies and wavenumbers one finds

$$\mathcal{D}_{nm}(\mathbf{k}) = \frac{2\pi N_F}{D\mathbf{k}^2 + |\Omega_{n-m}|} \quad (nm<0),$$
$$\equiv 2\pi N_F GH \mathcal{D}_{n-m}(\mathbf{k}), \quad (2.20b)$$

with

$$\mathcal{D}_n(\mathbf{k}) = \frac{1}{\mathbf{k}^2 + GH|\Omega_n|}. \quad (2.20c)$$

Here $D = 1/GH$ is a bare diffusion coefficient, and $\Omega_n = 2\pi T n$ is a bosonic Matsubara frequency.

Within the approximation, Eq. (2.20a), the effective action has the form

$$\tilde{\mathcal{A}}_{eff} = -\sum_{\mathbf{k}}\sum_{\alpha}\sum_{n}\sum_{i=1}^{3} {}^iM_n(\mathbf{k})[t + a\mathbf{k}^2 + b|\Omega_n|]{}^iM_{-n}(-\mathbf{k})$$
$$+ \mathcal{A}_{NL\sigma M}[\hat{Q}] + \sqrt{\pi T K_t}\int d\mathbf{x}\sum_{\alpha}\sum_{n}\sum_{i=1}^{3} {}^iM_n^{\alpha}(\mathbf{x})$$
$$\times \sum_{r=0,3}(\sqrt{-1})^r \sum_{m} \mathrm{tr}[(\tau_r \otimes s_i)\hat{Q}^{\alpha\alpha}_{m,m+n}(\mathbf{x})]. \quad (2.21)$$

Here $t = 1 - 2N_F\Gamma_t$, a and b are constants, and in the first term we have truncated the gradient expansion after the leading wavenumber and frequency-dependent terms. Notice that the first term taken in isolation describes a ferromagnetic transition at $t=0$, which represents the Stoner criterion.

The following structure emerges. Our effective action takes the form of a LGW action for the order-parameter field M, a nonlinear sigma model for the soft fermionic modes, and a coupling between the order parameter and the composite fermion field $\hat{Q}$. In our current approximation, only the Gaussian part of the LGW action appears. However, keeping terms of higher order in ΔP would clearly produce terms of higher order in M, with coefficients that are numbers in the limit of zero wave numbers and frequencies. For our purposes it will suffice to keep the term of order M^4. We thus should add to the action a term

FIG. 1. M-ΔP and M-ΔP-q^2 vertices. Broken lines denote M, solid lines denote q, and wiggly lines denote ΔP.

$$\mathcal{A}^{(4,1)}_{LGW} = u_4 \int d\mathbf{x} T \sum_{n_1,n_2,n_3}\sum_{\alpha}[\mathbf{M}^{\alpha}(\mathbf{x},\omega_{n_1})\cdot\mathbf{M}^{\alpha}(\mathbf{x},\omega_{n_2})]$$
$$\times[\mathbf{M}^{\alpha}(\mathbf{x},\omega_{n_3})\cdot\mathbf{M}^{\alpha}(\mathbf{x},-\omega_{n_1}-\omega_{n_2}-\omega_{n_3})], \quad (2.22)$$

where u_4 is a number and $\mathbf{M}$ is the three-vector whose components are ${}^iM$. A detailed derivation shows that there is also a term of the third order in $\mathbf{M}$, with a coefficient that is proportional to either a gradient or a time derivative. This term is irrelevant for the critical behavior at the ferromagnetic transition[4] and we neglect it.

Both from simple physical considerations and from Ref. 4 it is obvious that another term of $O(M^4)$ must exist. This is the ''random mass'' term that reflects spatial fluctuations in the location of the critical point, and it has the structure

$$\mathcal{A}^{(4,2)}_{LGW} = v_4 \int d\mathbf{x} \sum_{n_1,n_2}\sum_{\alpha,\beta}|\mathbf{M}^{\alpha}(\mathbf{x},\omega_{n_1})|^2|\mathbf{M}^{\beta}(\mathbf{x},\omega_{n_2})|^2, \quad (2.23)$$

where v_4 is a number. To see how this term arises in our present formulation, we go back to Eq. (2.16). If we expand the $\mathcal{S}$ in the last term in powers of q, the two lowest-order contributions[23,24] have the structures

$$\sqrt{T}\int d\mathbf{x} M(\mathbf{x})\Delta P(\mathbf{x}), \quad (2.24a)$$

$$\sqrt{T}\int d\mathbf{x} M(\mathbf{x})\Delta P(\mathbf{x}) q(\mathbf{x}) q(\mathbf{x}). \quad (2.24b)$$

Here we have dropped frequency labels and all other degrees of freedom that are irrelevant for power-counting purposes. Let us depict the fields M, ΔP, and q by dashed, wavy, and solid lines, respectively. Then the two vertices in Eqs. (2.24a) and (2.24b) have the structures shown in Fig. 1.

Contracting the wavy lines yields an effective vertex of the form

$$T\int d\mathbf{x} M^2(\mathbf{x}) q^2(\mathbf{x}), \quad (2.24c)$$

which is shown in Fig. 2.

This vertex can be used to construct the one-loop contribution to the term of $O(M^4)$ that is shown in Fig. 3.

Calculating the diagram yields Eq. (2.23). We will further discuss this term from a power-counting point of view in Sec. III C 2 below.

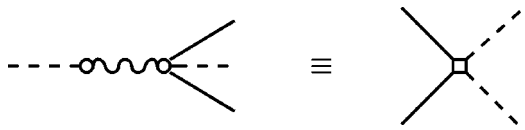

FIG. 2. Effective M^2-q^2 vertex obtained by contracting the ΔP fluctuations of the two vertices shown in Fig. 1.

2. Effective action and leading corrections

We now can assemble our effective action by collecting the various terms derived in the previous subsection. Before doing so, however, it is illustrative and convenient to add a term to the LGW part of the action that will be generated by the RG in Sec. III. As we will see, for dimensions $2<d<4$ the gradient-squared term in the first contribution to $\tilde{\mathcal{A}}$ is *not* the leading wavenumber dependence. Rather, as one would expect from Ref. 4, there is a term proportional to $|\mathbf{k}|^{d-2}$, which first appears at the one-loop order. We therefore add this term right away. Notice that the resulting action is still local in the sense of Ref. 9. It will also turn out that the term in the LGW action that is linear in frequency is not the leading frequency dependence. Rather, the coupling between M and $\hat{Q}$ effectively produces a term proportional to $|\Omega_n|/\mathbf{k}^2$.[2] Physically, this term reflects the fact that spin is a conserved variable, i.e., the characteristic frequency scale vanishes in the long-wavelength limit even away from the critical point. We can therefore drop the frequency dependence from the first term in Eq. (2.21).

Taking all of these points into account, we obtain the following effective action:

$$\mathcal{A}_{\rm eff}=\mathcal{A}_{\rm LGW}[M]+\mathcal{A}_{\rm NL\sigma M}[q]+\mathcal{A}_{\rm c}[M,q]. \quad (2.25a)$$

Here $\mathcal{A}_{\rm LGW}$ is the modified LGW part of the action,

$$\mathcal{A}_{\rm LGW}[M]=-\sum_{\mathbf{k}}\sum_{\alpha}\sum_{n}\sum_{i=1}^{3} {}^{i}M_n^{\alpha}(\mathbf{k})[t+a_{d-2}|\mathbf{k}|^{d-2} + a_2\mathbf{k}^2]{}^{i}M_{-n}^{\alpha}(-\mathbf{k})+\mathcal{A}_{\rm LGW}^{(4,1)}[M]+\mathcal{A}_{\rm LGW}^{(4,2)}[M], \quad (2.25b)$$

where a_{d-2} and a_2 are constants, and $\mathcal{A}_{\rm LGW}^{(4,1)}$ and $\mathcal{A}_{\rm LGW}^{(4,2)}$ are from Eqs. (2.22) and (2.23), respectively. The nonlinear sigma model part of the action, $\mathcal{A}_{\rm NL\sigma M}$, has been given in Eq. (2.17a), and $\mathcal{A}_{\rm c}$ represents the coupling between M and q,

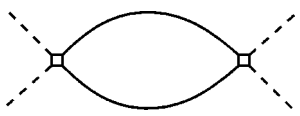

FIG. 3. One-loop renormalization of the M^4 vertex by means of the vertex in Fig. 2.

$$\mathcal{A}_{\rm c}[M,q]=\sqrt{\pi TK_t}\int d\mathbf{x}\sum_{\alpha}\sum_{n}\sum_{i=1}^{3} {}^{i}M_n^{\alpha}(\mathbf{x}) \times\sum_{r=0,3}(\sqrt{-1})^r\sum_{m}\mathrm{tr}[(\tau_r\otimes s_i)\hat{Q}_{m,m+n}^{\alpha\alpha}(\mathbf{x})]. \quad (2.25c)$$

It is useful to define a field

$$b_{12}(\mathbf{x})=\sum_{i,r}(\tau_r\otimes s_i)_r^i b_{12}(\mathbf{x}), \quad (2.26a)$$

with components

$${}_r^i b_{12}(\mathbf{k})=\delta_{\alpha_1\alpha_2}(-)^{r/2}\sum_{n}\delta_{n,n_1-n_2} \times[{}^{i}M_n^{\alpha_1}(\mathbf{k})+(-)^{r+1}{}^{i}M_{-n}^{\alpha_1}(\mathbf{k})]. \quad (2.26b)$$

In terms of b, the coupling part of the action can be written as

$$\mathcal{A}_{\rm c}[b,q]=-\frac{1}{2}\sqrt{\pi TK_t}\int d\mathbf{x}\,\mathrm{tr}[b(\mathbf{x})\hat{Q}(\mathbf{x})]. \quad (2.27)$$

We stress that we have added the $|\mathbf{k}|^{d-2}$ term in Eq. (2.25b) for convenience only. One could equally well work with the theory given by our action with $a_{d-2}=0$, but some effects that are included in the bare $\mathcal{A}_{\rm eff}$ would then appear only at the loop level. We also stress that the coefficients of all non-Gaussian terms in $\mathcal{A}_{\rm eff}$ are finite in the limit of zero wavenumbers and frequencies, so the theory is local in the sense of Ref. 9 in contrast to the situation in Ref. 4. More importantly, the physically relevant point is that the current formulation, as opposed to the one in Ref. 4, makes obvious the existence of two time scales as we will see in Sec. III below.

In order to justify the omission of the terms left out of our effective action, we also need to know the structure of the omitted terms. We therefore list these here using notation that leaves out anything that is not needed for power-counting purposes.

First of all we have the corrections to the nonlinear sigma model. In addition to Eq. (2.18), they consist of those terms in the Tr ln term in Eq. (2.10b) that contain cubic or higher orders of $\Delta\Lambda$, the part of $\mathcal{A}_{\rm int}^{(s)}$ that contains the massive fluctuation ΔP, and corrections to the saddle-point approximation for $\langle P\rangle$ in Eq. (2.15). These have all been discussed in Ref. 13 and the same discussion applies here.

Second, there are the terms that couple M, ΔP, and q, see the second term in the bracket in the last line of Eq. (2.16). In general, they have the structure

$$d_n\sqrt{T}\int d\mathbf{x}M(\mathbf{x})\Delta P(\mathbf{x})q^n(\mathbf{x}), \quad (2.28)$$

with coupling constants d_n and $n=0,2,3,\ldots$.[24] The first two terms in this expansion have already been given in Eqs. (2.24).

3. Observables

For a physical interpretation of any results obtained from our effective action we need to identify the appropriate observables in terms of the coupling constants of the theory. It is obvious and easily confirmed by keeping a source term for the electron spin-density that the expectation value of the order-parameter field M determines the magnetization m. Specifically,

$$m = \mu_B \sqrt{2T/\Gamma_t} \langle {}^i M^\alpha_{n=0}(\mathbf{x}) \rangle. \tag{2.29a}$$

The two-point M vertex is therefore proportional to the magnetic susceptibility.

Other relevant observables are the specific-heat coefficient

$$\gamma = \frac{4\pi}{3} H \tag{2.29b}$$

and the electrical conductivity

$$\sigma = \frac{8}{\pi G} \tag{2.29c}$$

(see Sec. II C). Also of interest is the electronic density of states per spin[13]

$$N(\epsilon_F + \omega) = N_F \langle {}^0_0 \hat{Q}^{\alpha\alpha}_{nn}(\mathbf{x}) \rangle_{i\omega_n \to \omega + i0}, \tag{2.29d}$$

where the energy or frequency ω is measured from the Fermi energy ϵ_F.

III. RENORMALIZATION GROUP ANALYSIS

In this section, we first consider low-order perturbation theory to see how those terms in Eqs. (2.25) that were not in the bare action are generated. We then do a power-counting analysis to determine the minimal effective action that needs to be analyzed in order to find the critical behavior at the ferromagnetic transition. Finally, we show that the terms that were omitted from the effective action are irrelevant, in the RG sense, for the critical behavior.

A. Perturbation theory

We first set up a standard perturbative expansion for our effective action, starting with the Gaussian theory.

1. Gaussian propagators

We start by expanding the effective action, Eqs. (2.25) to the bilinear order in M (or b) and q. We obtain for the Gaussian action

$$\begin{aligned} \mathcal{A}_G[M,q] = &- \sum_{\mathbf{k}} \sum_n \sum_\alpha \sum_{i=1}^{3} {}^i M^\alpha_n(\mathbf{k}) u_2(\mathbf{k}) {}^i M^\alpha_{-n}(-\mathbf{k}) \\ &- \frac{4}{G} \sum_{\mathbf{k}} \sum_{1,2,3,4} \sum_{i,r} {}^i_r q_{12}(\mathbf{k}) {}^i \Gamma^{(2)}_{12,34}(\mathbf{k}) {}^i_r q_{34}(-\mathbf{k}) \\ &+ 4\sqrt{\pi T K_t} \sum_{\mathbf{k}} \sum_{12} \sum_{i,r} {}^i_r q_{12}(\mathbf{k}) {}^i_r b_{12}(-\mathbf{k}), \end{aligned} \tag{3.1a}$$

where

$$u_2(\mathbf{k}) = t + a_{d-2} |\mathbf{k}|^{d-2} + a_2 \mathbf{k}^2. \tag{3.1b}$$

The bare two-point q vertex reads

$$\begin{aligned} {}^0\Gamma^{(2)}_{12,34}(\mathbf{k}) = &\delta_{13}\delta_{24}(\mathbf{k}^2 + GH\Omega_{n_1 - n_2}) \\ &+ \delta_{1-2,3-4} \delta_{\alpha_1\alpha_2} \delta_{\alpha_1\alpha_3} 2\pi T G K_s, \end{aligned} \tag{3.1c}$$

$${}^{1,2,3}\Gamma^{(2)}_{12,34}(\mathbf{k}) = \delta_{13}\delta_{24}(\mathbf{k}^2 + GH\Omega_{n_1 - n_2}), \tag{3.1d}$$

with $K_s = -\pi N_F^2 \Gamma_s / 2$.

The quadratic form defined by this Gaussian action is easily inverted. For the order parameter correlations we find

$$\langle {}^i M^\alpha_n(\mathbf{k}) {}^j M^\beta_m(\mathbf{p}) \rangle = \delta_{\mathbf{k},-\mathbf{p}} \delta_{n,-m} \delta_{ij} \delta_{\alpha\beta} \frac{1}{2} \mathcal{M}_n(\mathbf{k}), \tag{3.2a}$$

$$\begin{aligned} \langle {}^i_r b_{12}(\mathbf{k}) {}^j_s b_{34}(\mathbf{p}) \rangle = &- \delta_{\mathbf{k},-\mathbf{p}} \delta_{rs} \delta_{ij} \delta_{\alpha_1\alpha_2} \delta_{\alpha_1\alpha_3} \mathcal{M}_{n_1 - n_2}(\mathbf{k}) \\ &\times [\delta_{1-2,3-4} - (-)^r \delta_{1-2,4-3}] \end{aligned} \tag{3.2b}$$

in terms of the paramagnon propagator

$$\mathcal{M}_n(\mathbf{k}) = \frac{1}{t + a_{d-2}|\mathbf{k}|^{d-2} + a_2 \mathbf{k}^2 + \dfrac{GK_t |\Omega_n|}{\mathbf{k}^2 + GH|\Omega_n|}}. \tag{3.2c}$$

Notice that the coupling between the order-parameter field and the fermionic degrees of freedom has produced the dynamical piece of $\mathcal{M}$ that is characteristic of disordered itinerant ferromagnets.[2]

For the fermionic propagators we find

$$\langle {}^i_r q_{12}(\mathbf{k}) {}^j_s q_{34}(\mathbf{p}) \rangle = \delta_{\mathbf{k},-\mathbf{p}} \delta_{rs} \delta_{ij} \frac{G}{8} {}^i \Gamma^{(2)-1}_{12,34}(\mathbf{k}), \tag{3.3a}$$

in terms of the inverse of Eq. (3.1c),

$$\begin{aligned} {}^0\Gamma^{(2)-1}_{12,34}(\mathbf{k}) = &\delta_{13}\delta_{24} \mathcal{D}_{n_1 - n_2}(\mathbf{k}) \\ &- \delta_{1-2,3-4} \delta_{\alpha_1\alpha_2} \delta_{\alpha_1\alpha_3} 2\pi T G K_s \\ &\times \mathcal{D}_{n_1 - n_2}(\mathbf{k}) \mathcal{D}^{(s)}_{n_1 - n_2}(\mathbf{k}), \end{aligned} \tag{3.3b}$$

and

FIG. 4. One-loop renormalization of the vertex u_2.

$$
{}^{1,2,3}\Gamma^{(2)-1}_{12,34}(\mathbf{k}) = \delta_{13}\delta_{24}\mathcal{D}_{n_1-n_2}(\mathbf{k}) - \delta_{1-2,3-4}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3} 2\pi T G K_t \times [\mathcal{D}_{n_1-n_2}(\mathbf{k})]^2 \mathcal{M}_{n_1-n_2}(\mathbf{k}). \quad (3.3c)
$$

Here $\mathcal{D}^{(s)}$ is the spin-singlet propagator, which in the limit of long wavelengths and small frequencies reads[6]

$$
\mathcal{D}^{(s)}_n(\mathbf{k}) = \frac{1}{\mathbf{k}^2 + G(H+K_s)\Omega_n}. \quad (3.3d)
$$

Finally, due to the coupling between M and q we have a mixed propagator

$$
\langle {}^i_r q_{12}(\mathbf{k}) {}^j_s b_{34}(\mathbf{p})\rangle = -\delta_{\mathbf{k},-\mathbf{p}}\delta_{rs}\delta_{ij}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\frac{G}{2}\sqrt{\pi T K_t} \times \mathcal{D}_{n_1-n_2}(\mathbf{k})\mathcal{M}_{n_1-n_2}(\mathbf{k})[\delta_{1-2,3-4} + (-)^{r+1}\delta_{1-2,4-3}]. \quad (3.4)
$$

2. One-loop order

Let us now consider the renormalization of the M^2 vertex u_2, Eq. (3.1b). At one-loop order, the relevant diagram is the one shown in Fig. 4. While the complete result is rather involved, it can be simplified by means of the following observation. The structure of the diagram leads to a frequency-momentum integral over diffusion poles multiplied by one or more paramagnon propagators. Inspection shows that the frequency in this integral scales like a wavenumber squared. To leading order in the distance from the critical point and for the purpose of obtaining leading infrared singularities, we therefore can use the following approximation in the integrand:

$$
G K_t \Omega_l \mathcal{D}_l(\mathbf{p})\mathcal{M}_l(\mathbf{p}) = 1 + O(t, |\mathbf{p}|^d). \quad (3.5a)
$$

At zero external frequency the diagram then yields

$$
\delta u_2(\mathbf{k}) = -\frac{G^2}{2}\frac{K_t}{V}\sum_{\mathbf{p}} 2\pi T \sum_{l=0}^{\infty} \mathcal{D}_l(\mathbf{p})\mathcal{D}_l(\mathbf{p}+\mathbf{k}). \quad (3.5b)
$$

Performing the integrals shows that this diagram provides a finite renormalization of the coefficient a_{d-2} of the $|\mathbf{k}|^{d-2}$ term. In particular, it would generate this term if it were not present in the bare action. This is the reason why we added this term by hand in Sec. II D 2.

We now consider the one-loop renormalizations of the other vertices in the effective action. Of particular interest are the coupling constants H and G in the two-point q vertex, which determine the specific-heat coefficient and the conductivity, respectively, see Sec. II D 3 above. The topological structure of the relevant diagrams is shown in Fig. 5.

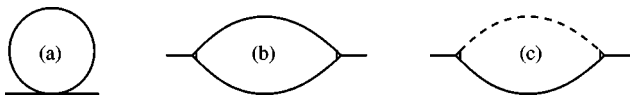

FIG. 5. One-loop diagrams that renormalize H and G.

From extensive work on the nonlinear sigma model it is known that Fig. 5(b), combined with suitable contributions from Fig. 5(a), is finite for $d>2$.[6] A calculation of Fig. 5(c) and the remaining parts of Fig. 5(a) reveals that their contribution to the one-loop renormalization of G consists of two pieces, which at zero external frequency are given by

$$
(\delta G)_1 = \frac{3}{8}G^3\frac{K_t}{V}\sum_{\mathbf{p}} 2\pi T\sum_{l=0}^{\infty}\mathcal{D}_l^2(\mathbf{p})\mathcal{M}_l(\mathbf{p}) \quad (3.6a)
$$

and

$$
(\delta G)_2 = \frac{3}{8}G^3 K_t \frac{\partial}{\partial \mathbf{k}^2}\bigg|_{\mathbf{k}=0} \frac{1}{V}\sum_{\mathbf{p}} 2\pi T\sum_{l=0}^{\infty}\mathcal{M}_l(\mathbf{p}) \times [\mathcal{D}_l(\mathbf{p}) - \mathcal{D}_l(\mathbf{p}+\mathbf{k})], \quad (3.6b)
$$

respectively. Both of these integrals are finite in $d>2$. A simple calculation shows that the one-loop correction to the density of states is given by the same integral as $(\delta G)_1$. For the H renormalization, Fig. 5(b) is again finite while the other two contributions yield

$$
\delta H = \frac{3}{8}G K_t \frac{1}{\Omega_n} 2\pi T\sum_{l=0}^{n}\frac{1}{V}\sum_{\mathbf{p}}\mathcal{D}_l(\mathbf{p})\mathcal{M}_l(\mathbf{p}), \quad (3.7a)
$$

where n is the external frequency label. For later reference we note that Figs. 5(a) and 5(c) each contribute one half of this result. Individually, each of these diagrams also contributes pieces that diverge like $1/\Omega_n$; these contributions cancel between the two diagrams. [The same is true for Fig. 5(b) and the corresponding contributions from Fig. 5(a).] Notice that the frequency structure is slightly different than in the case of the G-renormalization. This leads to a finite frequency sum in Eq. (3.7a) and as a result the integral is logarithmically infrared divergent for all dimensions $2\leqslant d<4$,

$$
\delta H = \frac{3}{4}\bar{G}K_t \ln(1/\Omega_n). \quad (3.7b)
$$

Here $\bar{G} = G S_d/(2\pi)^d$, with S_d the surface area of the unit $(d-1)$ sphere. While this is consistent with the result of Ref. 4 that the specific-heat coefficient at the quantum ferromagnetic transition is logarithmically divergent for all $2<d<4$, it is obvious from the present formulation of the theory that this is very unlikely to be the exact critical behavior as claimed in that reference. The reason is that an insertion of the one-loop H into the one-loop diagrams yields a $\ln^2\Omega$ singularity and so on. Similarly, inserting the one-loop result for the q propagator in the diagram shown in Fig. 3.5 generates a logarithmic correction to the vertex $u_2(\mathbf{k})$. Unless these insertions are exactly canceled by skeleton-diagram

contributions, the exact critical behavior must therefore involve a more complicated function of $\ln\Omega$. We will come back to this point in the next subsection and in II.

Finally, the diagrams shown in Fig. 5 also determine the renormalization of K_s. Again, Fig. 5(b) yields a finite result, while from the other two diagrams one expects

$$\delta K_s = -\delta H, \tag{3.7c}$$

where δH is the divergent part of the H renormalization, Eq. (3.7b). This result follows from what is known about the nonlinear sigma model as we will discuss in Sec. IV A.

Notice that the effects we have discussed above at the one-loop level would of course also occur if we had worked with the theory that one obtains if one puts the bare a_{d-2} equal to zero. However, in that theory their derivation would have required a renormalization of the paramagnon propagator. In a diagrammatic language, by adding the $|\mathbf{k}|^{d-2}$ term to our bare action, we have made use of skeleton diagrams that contain certain infinite resummations. We note in passing that one might worry about higher orders in the loop expansion producing even stronger nonanalyticities than $|\mathbf{k}|^{d-2}$. We will show in II that this is not the case.

B. Naive fixed points and their instability

We now proceed to perform a power-counting analysis of our effective action, Eqs. (2.25). Our goal is to understand the perturbative results of the preceding subsection from a more general point of view, and to determine the minimal effective action which, when solved, will yield the *exact* critical behavior. For this purpose it is convenient to rewrite the action in a schematic form that suppresses everything that is not necessary for power counting,

$$\begin{aligned}\mathcal{A}_{\rm eff}[M,q] = &-\int d\mathbf{x} M[t + a_{d-2}\partial_{\mathbf{x}}^{d-2} + a_2\partial_{\mathbf{x}}^2]M \\ &+ O(\partial_{\mathbf{x}}^4 M^2, M^4) - \frac{1}{G}\int d\mathbf{x}(\partial_{\mathbf{x}} q)^2 + H\int d\mathbf{x}\Omega q^2 \\ &+ K_s T\int d\mathbf{x} q^2 - \frac{1}{G_4}\int d\mathbf{x}\partial_{\mathbf{x}}^2 q^4 + H_4\int d\mathbf{x}\Omega q^4 \\ &+ O(Tq^3, \partial_{\mathbf{x}}^2 q^6, \Omega q^6) + \sqrt{T}c_1\int d\mathbf{x} M q \\ &+ \sqrt{T}c_2\int d\mathbf{x} M q^2 + O(\sqrt{T} M q^4).\end{aligned} \tag{3.8}$$

Here the fields are understood to be functions of position and frequency and only quantities that carry a scale dimension are shown. The bare values of G_4 and H_4 are proportional to those of G and H.[25] The term of order Tq^3, which arises from the interacting part of $\mathcal{A}_{\rm NL\sigma M}$, will not be of importance for our purposes although its coupling constant squared has the same scale dimension as $1/G_4$ and H_4. It is therefore not shown explicitly. The c_1, c_2, etc. are the coupling constants of the terms contained in $\mathcal{A}_c$, Eq. (2.25c).

We now assign to a length L a scale dimension $[L] = -1$. Under a renormalization-group transformation that involves a length rescaling by a factor b, all quantities will then change according to $A \to b^{[A]}A$, with $[A]$ the scale dimension of A. In particular, imaginary time τ and temperature T or frequency Ω have scale dimensions $[\tau] \equiv -z$, and $[T] = [\Omega] \equiv z$, respectively.

1. Hertz's fixed point

To illustrate an important point, let us first show how one recovers Hertz's mean-field fixed point[2] within the present formalism. Let us look for a fixed point where the coefficients a_2 and c_1 are marginal, $[a_2] = [c_1] = 0$. This choice[26] is motivated by the desire to find a fixed point with mean field-like static critical behavior and with dynamics given by the frequency dependence of the standard paramagnon propagator $\mathcal{M}$, Eq. (3.2c). (Recall that the frequency dependence of $\mathcal{M}$ was produced by the vertex with coupling constant c_1.) From the condition that the action be dimensionless we then obtain the scale dimensions of the order-parameter field,

$$[M_n(\mathbf{x})] = (d-2)/2, \tag{3.9a}$$

and we find t to be relevant with scale dimension

$$[t] = 2. \tag{3.9b}$$

We expect the correlations of the q field to describe the diffusive dynamics of the fermions, so we choose[13]

$$[q_{nm}(\mathbf{x})] = (d-2)/2, \tag{3.9c}$$

and G, H, and K_s are all dimensionless. The marginality of the coupling constant c_1 then implies

$$z = [T] = 4. \tag{3.9d}$$

This is the fixed point proposed by Hertz,[2] which leads to mean-field critical behavior. It is unstable because the coupling a_{d-2} is relevant with respect to this fixed point as has been pointed out in Ref. 4. While this is obvious from the action as formulated here, the following interesting question arises. Suppose we had not added the term with coupling constant a_{d-2} to our bare action. Since this term was generated by means of the Mq^2 vertex with coupling constant c_2, see Fig. 4, c_2 should be relevant with respect to Hertz's fixed point. However, power counting with the above scale dimensions yields

$$[c_2] = \frac{-1}{2}(d + z - 6), \tag{3.9e}$$

so with the above value $z=4$, c_2 seems to be irrelevant. The resolution of this paradox lies in the fact that there is more than one time scale in the problem, and hence all factors of T do not carry the same scale dimension z. This is obvious if we consider the fermionic sector of our action; the factors of T in the q^2 vertex carry a scale dimension $[T] = z_{\rm diff} = 2$, which corresponds to the diffusive time scale that describes the dynamics of the electronic soft modes and which is distinct from the critical time scale that corresponds to $[T] = z_c = 4$. The scale dimensions of the factors of $\sqrt{T}$ in the coupling part of the action are therefore not *a priori* clear,

and they may depend on the diagrammatic context a vertex is used in. Consider the diagram in Fig. 4 again. In this context, the two factors of $\sqrt{T}$ contribute to the frequency measure of a fermionic loop and hence they carry the diffusive time scale. Indeed, with $z=2$ Eq. (3.9e) shows that c_2 is relevant for $2<d<4$ and its scale dimension is consistent with that of a_{d-2}.

It is worthwhile to mention that one can tell *a priori* that the scale dimension for t, Eq. (3.9b), cannot correspond to a stable fixed point since it corresponds to a correlation-length exponent $\nu=1/[t]=1/2$, which violates the Harris criterion inequality $\nu \geqslant 2/d$.[27] The relevance of c_2 provides an explicit mechanism for the instability.

We also note that the above discussion is oversimplified in that it pretends that M has always the same scale dimension, independent of the context the order-parameter fluctuations appear in. As we will see in the next subsection, this is not quite true. However, since this point is not crucial for the instability of Hertz's fixed point we have suppressed it.

2. A marginally unstable fixed point

Given the presence of the term with coupling constant a_{d-2}, an obvious attempt to find a stable fixed point is to choose a_{d-2} and c_1 to be marginal instead of a_2 and c_1.[28] A slight complication, however, lies in the fact that due to the existence of two time scales, a_{d-2} will not necessarily be marginal under all circumstances. Namely, if the frequency in the paramagnon propagator, Eq. (3.2c), is diffusive, i.e., if it scales like $\mathbf{k}^2$ then a_{d-2} will be irrelevant. As we will see below, this can happen if the paramagnon propagator appears as an internal propagator in perturbation theory although in the critical paramagnon a_{d-2} is marginal. In general, we therefore demand only that c_1 be marginal, that the scale dimension of the q field be consistent with a diffusive $\langle qq \rangle$ propagator,

$$[q_{nm}(\mathbf{x})]=\frac{1}{2}(d-2), \tag{3.10a}$$

and that the diffusive time scale be represented by a dynamical critical exponent

$$z_{\mathrm{diff}}=2. \tag{3.10b}$$

Equation (3.8) then implies

$$[G]=[H]=[K_s]=0. \tag{3.10c}$$

The marginality of c_1 implies for the scale dimension of the order-parameter field

$$[M_n(\mathbf{x})]=1+(d-z)/2, \tag{3.11a}$$

where z is the dynamical exponent associated with the $\sqrt{T}$ prefactor in the c_1 vertex. In the critical paramagnon propagator we expect a_{d-2} to be marginal, which implies $[M_n(\mathbf{x})]=1$, and hence a critical time scale characterized by

$$z_c=d \tag{3.11b}$$

and a critical exponent η, defined by $[M_n(\mathbf{x})]=(d-2+\eta)/2$,

$$\eta=4-d. \tag{3.11c}$$

This makes a_2 irrelevant, while t is relevant with scale dimension

$$[t]=d-2, \tag{3.12a}$$

which leads to a correlation-length exponent

$$\nu=1/[t]=1/(d-2). \tag{3.12b}$$

Notice that in contrast to the situation at Hertz's fixed point, this result respects the Harris criterion.[27] Here and in the remainder of this paper we restrict ourselves to the range of dimensions $2<d<4$, which includes the physically interesting case $d=3$. For the behavior in higher dimensions, see Ref. 4.

The preceding results characterize the Gaussian fixed point that was discussed in Ref. 4. If all other terms in the action were irrelevant or marginal leading to finite renormalizations only, then this fixed point would be stable. To check this, we need to consider the corrections to the Gaussian action. We start with c_2, whose scale dimension is

$$[c_2]=1-z/2, \tag{3.13}$$

where z is the scale dimension of the factor of T in that vertex. If this temperature represents the critical time scale, then c_2 is irrelevant. However if it represents the diffusive time scale, then it is marginal. This can indeed happen as we have discussed in Sec. III B 1 above. The example we used, viz., the diagram in Fig. 4 just leads to a finite renormalization of the coefficient a_{d-2}, which is part of our effective action anyway. If this were the only effect of c_2, then we could neglect it. However, this is not the case. The one-loop renormalization of H that was discussed in Sec. III A 2 provides an example of how operators that appear irrelevant by naive power counting can be effectively marginal due to the existence of two time scales, lead to logarithms, and therefore need to be kept. Consequently, c_2 is not necessarily harmless even if $z=d$ in Eq. (3.13). This is an important point that we now discuss in detail.

Consider Figs. 5(a) and 5(c). They both lead to a correction to the two-point q vertex that is of the form, at zero external wavenumber and frequency,

$$\delta\Gamma^{(2)} \propto \frac{1}{V}\sum_{\mathbf{p}} T\sum_{l=1}^{\infty} \mathcal{D}_l(\mathbf{p})\mathcal{M}_l(\mathbf{p}). \tag{3.14a}$$

For scaling purposes, let us cut off the momentum integral in the infrared by $1/b$, where b is a RG-length scale factor. Doing the integral then shows that it is given by a constant plus a term proportional to $b^{-d}\ln b$. The constants cancel between the two diagrams, and we have

$$\delta\Gamma^{(2)} \sim b^{-d}\ln b. \tag{3.14b}$$

Notice that the frequency in the above integral scales like a wavenumber to the power d, so $(c_2)^2$ in Fig. 5(c) has a negative scale dimension $-(d-2)$, and so does the quartic vertex in Fig. 5(a). The salient point is now as follows. For the purpose of the renormalization of G, i.e., the wave

number-dependent part of $\Gamma^{(2)}$, we need to replace $1/b$ by $\mathbf{k}$. We then obtain the gradient squared of the bare vertex times a factor $|\mathbf{k}|^{d-2}$. The contribution is therefore irrelevant, in agreement with the negative scale dimensions of the vertices and the result of the explicit perturbative calculation. However, for the purpose of the renormalization of H we need to replace $1/b$ by an appropriate power of the frequency. This can be $\Omega^{1/2}$, if Ω represents the diffusive frequency scale or $\Omega^{1/d}$, if it represents the critical one. Since the frequency in the integral scales like $|\mathbf{p}|^d$, the latter applies and we have

$$\delta\Gamma^{(2)} \propto \Omega \ln \Omega, \tag{3.14c}$$

in agreement with Eq. (3.7b).

The point illustrated above is as follows. Due to the existence of two different time scales, the fact that an operator has a negative scale dimension by naive power counting, which is based on the consideration of length scales, does not necessarily imply that it will be irrelevant. Rather, operators with scale dimensions between zero and $-(d-2)$ may act as marginal operators with respect to frequency scaling. Notice, however, that for this mechanism to be operative it is crucial that the vertex being renormalized is proportional to frequency. Therefore, the seemingly irrelevant operators become effectively marginal with respect to H but not with respect to G or any other coupling constant. In the Appendix we discuss another aspect of this phenomenon.

We conclude that the Gaussian fixed point of Ref. 4 is not stable since there are operators that are effectively marginal with respect to it. If these operators just led to finite renormalizations, this would still not change the conclusions of the earlier paper. However, as we have seen above, they lead to logarithmic corrections to power-law scaling and hence need to be kept. The problem is less severe than in the case of Hertz's fixed point, however, since now there are no relevant operators. If one can show that all other terms are truly irrelevant, then the conclusion would be that to determine the exact critical behavior it suffices to keep the Gaussian action plus the Mq^2 coupling and all terms up to $O(q^4)$. We investigate this hypothesis next.

C. Effective action for the critical behavior

1. The effective action

From the discussion in the preceding subsection we infer an educated guess for an effective action that contains only the terms needed for a description of the critical fixed point and the associated critical behavior. This action should contain all of the terms that are shown explicitly in Eq. (3.8), except that for $2<d<4$ one can drop the gradient-squared term in the LGW part of the action. Notice that we need to keep the terms of $O(\partial_{\mathbf{x}}^2 q^4, \Omega q^4)$ in the expansion of the nonlinear sigma model in powers of q, as they give rise to in Fig. 5(a). These terms appear irrelevant by naive power counting but contribute to the leading-frequency dependence by means of the mechanism discussed in Sec. III B 2 and in the Appendix. By the same argument one should keep the terms of order $O(q^3)$ and $O(q^4)$ that arise from the spin-singlet interaction. However, by themselves these vertices give only rise to diagrams that are finite in $d>2$,[6] and combined with c_2 or other vertices that contain M they lead to mixed $\langle bq \rangle$ propagators, Eq. (3.4), which are less infrared divergent then the second term on the right-hand side of Eq. (3.3c). These terms can therefore safely be neglected. This leaves the spin-singlet interaction constant K_s entering the effective theory via the vertex $\Gamma^{(2)}$ only. Since $K_s \neq 0$ does not change the diffusive structure of the noninteracting q propagator, it can be dropped there as well. Restoring all indices, the suggested effective action for describing the critical fixed point reads[29]

$$\begin{aligned}
\mathcal{A}_{\mathrm{FP}} = & -\sum_{\mathbf{k},n,\alpha} \sum_{i=1}^{3} {}^{i}M_{n}^{\alpha}(\mathbf{k})[t + a_{d-2}|\mathbf{k}|^{d-2}]{}^{i}M_{-n}^{\alpha}(-\mathbf{k}) - \frac{4}{G} \sum_{\mathbf{k}} \sum_{1,2,3,4} \sum_{r,i} {}_{r}^{i}q_{12}(\mathbf{k}) {}^{i}\Gamma_{12,34}^{(2)}(\mathbf{k}) {}_{r}^{i}q_{34}(-\mathbf{k}) \\
& - \frac{1}{4G} \sum_{1,2,3,4} \sum_{r,s,t,u} \sum_{i_1,i_2,i_3,i_4} \frac{1}{V} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} {}_{rstu}^{i_1 i_2 i_3 i_4}\Gamma_{1234}^{(4)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) {}_{r}^{i_1}q_{12}(\mathbf{k}_1) {}_{s}^{i_2}q_{32}(\mathbf{k}_2) {}_{t}^{i_3}q_{34}(\mathbf{k}_3) {}_{u}^{i_4}q_{14}(\mathbf{k}_4) \\
& + c_1\sqrt{T} \sum_{\mathbf{k}} \sum_{12} {}_{r}^{i}b_{12}(\mathbf{k}) {}_{r}^{i}q_{12}(-\mathbf{k}) + c_2\sqrt{T} \frac{1}{\sqrt{V}} \sum_{\mathbf{k},\mathbf{p}} \sum_{n_1,n_2,m} \sum_{r,s,t} \sum_{i=1}^{3} \sum_{j,k} \sum_{\alpha,\beta} {}_{r}^{i}b_{n_1 n_2}^{\alpha\alpha}(\mathbf{k}) \\
& \times [{}_{s}^{j}q_{n_2 m}^{\alpha\beta}(\mathbf{p}) {}_{t}^{k}q_{n_1 m}^{\alpha\beta}(-\mathbf{p}-\mathbf{k}) \mathrm{tr}(\tau_r \tau_s \tau_t^{\dagger}) \mathrm{tr}(s_i s_j s_k^{\dagger}) - {}_{s}^{j}q_{m n_2}^{\beta\alpha}(\mathbf{p}) {}_{t}^{k}q_{m n_1}^{\beta\alpha}(-\mathbf{p}-\mathbf{k}) \mathrm{tr}(\tau_r \tau_s^{\dagger} \tau_t) \mathrm{tr}(s_i s_j^{\dagger} s_k)],
\end{aligned} \tag{3.15a}$$

with $\Gamma^{(2)}$ from Eqs. (3.1c) and (3.1d) with $K_s = 0$, and

$$\begin{aligned}
{}_{rstu}^{i_1 i_2 i_3 i_4}\Gamma_{1234}^{(4)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = & -\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4,0} \mathrm{tr}(\tau_r \tau_s^{\dagger} \tau_t \tau_u^{\dagger}) \\
& \times \mathrm{tr}(s_{i_1} s_{i_2}^{\dagger} s_{i_3} s_{i_4}^{\dagger})(\mathbf{k}_1 \cdot \mathbf{k}_3 + \mathbf{k}_1 \cdot \mathbf{k}_4 \\
& + \mathbf{k}_1 \cdot \mathbf{k}_2 + \mathbf{k}_2 \cdot \mathbf{k}_4 - GH\Omega_{n_1 - n_2}.
\end{aligned} \tag{3.15b}$$

The bare values of the coupling constants c_1 and c_2 are related, and given by

$$c_1 = 16 c_2 = 4\sqrt{\pi K_t}. \tag{3.15c}$$

Notice that this action is *not* Gaussian, and therefore the critical behavior is not easy to determine.

We will solve the effective model given by Eqs. (3.15) in II,[12] where we will show that the exact critical behavior differs from the Gaussian one by logarithmic corrections only. In the remainder of this paper we show that the action given by Eqs. (3.15) really is sufficient for describing the critical behavior in $2<d<4$.

2. Corrections to the effective action

We now show that all terms that were neglected in writing Eqs. (3.15) are irrelevant by power counting, keeping in mind the complications due to the two time scales that were discussed in Sec. III B above. In addition to the scale dimensions of M and q given in Eqs. (3.10a) and (3.11a), we need for this purpose the scale dimension of the massive fields ΔP and $\Delta\Lambda$. The correlations of ΔP are short ranged, and of the same nature as at the Fermi-liquid fixed point that was discussed in Ref. 13. We thus choose

$$[\Delta P(\mathbf{x})]=[\Delta\Lambda(\mathbf{x})]=d/2. \tag{3.16}$$

Power counting now proceeds as usual. The nonlinear sigma model action we have kept up to $O(q^4)$. Higher-order corrections have the same scale dimensions as at the Fermi-liquid fixed point in Ref. 13. They thus are all irrelevant with scale dimensions that are smaller than $-(d-2)$ and are therefore harmless. The couplings between M, q, and the massive modes given in Eq. (2.28) for even powers of q have scale dimensions

$$[d_n]=-\frac{n-1}{2}(d-2)-\frac{z}{2}\leqslant\frac{-1}{2}(d+z-2) \tag{3.17}$$

for $n\geqslant 2$, and thus can safely be neglected. For odd powers of q, the couplings contain an effective external frequency, and therefore are even less irrelevant than Eq. (3.17) suggests. In particular we confirm that d_1, which we have dropped,[24] has a scale dimension $[d_1]=-3z/2$ and is thus more irrelevant then d_2. $[d_0]=(d-2-z)/2$, which becomes marginal in $d=4$ if $z=z_{\rm diff}=2$. However, $d=4$ is a special dimension anyway, and for $d>4$ one obtains a different fixed point since the $|\mathbf{k}|^{d-2}$ term in the LGW action is no longer leading. A remaining question is whether the formally irrelevant d_0 can be promoted to marginal or relevant status by the same mechanism that is operative for, e.g., c_2. The answer is negative since the mechanism works only for the renormalization of H, and in order to renormalize H, d_0 needs to be combined with some d_n with $n\geqslant 2$. However, $[d_0 d_2]=-z<-(d-2)$ for $d<4$. Therefore, all of the d_n can be safely neglected. Similarly, all terms of order higher than quadratic order in M are irrelevant. We mention, however, that in the ordered phase the term of $O(M^4)$ becomes dangerously irrelevant and needs to be kept. This will be important in II.

Finally the random mass term $\mathcal{A}^{(4,2)}_{\rm LGW}$, Eq. (2.23), deserves an extra discussion. The scale dimension of the coupling constant v_4 in Eq. (2.23) is $[v_4]=d-4$, while the scale dimension of the combination of d_0 and d_2 in Eq. (2.28) that produce v_4 in perturbation theory (see Fig. 2) is $[(d_0 d_2)^2]=-2z=-4$. $[v_4]$ is thus much less irrelevant than one might expect from naive power counting. The resolution of this discrepancy is as follows. The v_4 vertex shown in Fig. 2 at zero wavenumber has the schematic structure

$$\int d\mathbf{y}\int d\omega\,\omega\langle q^2(\mathbf{x})q^2(\mathbf{y})\rangle, \tag{3.18}$$

which has a naive scale dimension of d (with $z=2$). However, the integral is a finite number and so its actual scale dimension is zero. If we consider the vertex function at a finite wavenumber $|\mathbf{k}|$ and perform a gradient expansion, then we obtain an expansion of the form

$$\text{const}+\mathbf{k}^2+|\mathbf{k}|^d. \tag{3.19}$$

What happens here is that power counting yields the scale dimension of the first *nonanalytic* term in the gradient expansion, but misses more dominant analytic contributions. This is of no consequence as long as the latter just renormalize existing terms in the action. Here, however, they produce a *new* term in the action, viz., the random mass term and therefore need to be taken into account. The difference between the naive scale dimension of the integral, Eq. (3.18), viz., d and its actual scale dimension, viz., zero is precisely the difference between $[v_4]=d-4$ and $[(d_0 d_2)^2]=-4$.

We finally come back to the simplifications inherent in our starting point, Eqs. (2.2), which describe the paramagnetic phase as a disordered electron fluid while neglecting band structure and other features of solids. The justification for these simplifications is as follows. The disordered Fermi-liquid fixed point is characterized by relatively few parameters.[13] This is in contrast to a clean Fermi liquid, which requires an infinite number of Fermi-liquid parameters or a whole function to completely characterize the fixed point.[30] The crucial physical distinction is that for the disordered case the slowest, and therefore dominant, modes are diffusive and arise only from electron number density, spin density, and particle-particle density variables. In contrast, in the clean case there are an infinite number of soft single-particle and two-particle modes. This simplification for the disordered case carries over to the description of the ferromagnetic quantum phase transition.

IV. DISCUSSION

As we have seen in Sec. III C, the effective action for the critical behavior is not Gaussian, and therefore a determination of the critical behavior is nontrivial. It turns out that the critical behavior in all dimensions $d>2$ can nevertheless be determined exactly and is given by the power laws found in Ref. 4 with additional logarithmic corrections to scaling. This solution of the effective action will be deferred to II.[12] Here we restrict ourselves to a discussion of some general features of our effective theory and of its relation to previous approaches to the problem.

A. Relation to other approaches

Let us briefly discuss the relation between our current approach and previous theories. This is most easily done by starting from Eq. (2.10b). By formally integrating out the

fermions, i.e., the fields Q and $\tilde{\Lambda}$ from this formulation of the action one obtains an LGW theory or action entirely in terms of the order-parameter field. If the fermions are integrated out in the tree approximation, one recovers Hertz's theory.[2] If they are integrated out formally exactly, the vertices of the LGW functional are given in terms of spin-density correlation functions for a ''reference ensemble'' or fictitious electron system that has no bare spin-triplet interaction. This is the theory that was analyzed in Ref. 4. The disadvantage of that approach is that the reference ensemble contains soft modes, viz., the q and integrating them out produces effective vertices in the LGW theory that diverge in the limit of small wave numbers and frequencies. That is, one obtains a nonlocal field theory. Furthermore, Ref. 4 performed a power-counting analysis only, and integrating out the fermionic degrees of freedom obscured the subtleties that arise in this context due to the existence of the diffusive time scale in addition to the critical one. As a result, the power-counting analysis of Ref. 4 was insensitive to the logarithmic corrections that we found by means of explicit perturbative calculations in Sec. III A 2 and explained in Sec. III B 2 in terms of a more sophisticated scaling analysis than the pure LGW theory allowed for. Notice that in some other respects Ref. 4 was actually more sophisticated than the present theory. For instance, it included in the bare action, effects that require a one-loop analysis in the present approach, e.g. the $|\mathbf{k}|^{d-2}$ term in the vertex u_2. However, the insensitivity to logarithmic corrections is hard to overcome within the framework of the nonlocal theory.

The relation between the present theory and Ref. 8 is less obvious. To see it, consider Eqs. (2.25) and integrate out M. This yields a nonlinear sigma model with a triplet interaction amplitude that is given by the static paramagnon propagator. We have performed explicit calculations within this theory, and ascertained that it yields the same results as the coupled M-q theory discussed above, as it should. This equivalence between the M-q theory and the sigma model is the basis for Eq. (3.7c), since within the sigma model $H+K_s$ is not singularly renormalized.[6] The bare nonlinear sigma model in Ref. 8 had a pointlike spin-triplet interaction amplitude, but under renormalization the $|\mathbf{k}|^{d-2}$ that is characteristic of the static paramagnon is generated. It is thus plausible that the pure nonlinear sigma model should contain the critical fixed point for the ferromagnetic transition. However, since the order parameter has been integrated out, the nature of the transition is completely obscured within this approach, and a description of the ordered phase is not possible. This is the reason why Ref. 8 could only conclude that the transition is of magnetic nature.[31] We will come back to the detailed connection between the two approaches in II. Here we just mention that the present analysis positively identifies the runaway flow that is encountered in the nonlinear sigma model in the absence of any spin-flip mechanisms[6] as signaling the ferromagnetic transition.[7]

We also mention that the fixed point identified in Sec. III C above violates some of the general scaling laws obtained by Sachdev.[3] As has been discussed in Ref. 4 in some detail, this can be traced to the presence of dangerous irrelevant variables, which can always invalidate general scaling arguments.[32]

We conclude that all of the previous approaches to the problem break down at some level, and that the basic problem is always the same, namely a lack of explicitness. Only a local field theory that correctly identifies and keeps all of the soft modes allows for the explicit calculations necessary to check more general arguments that may break down because of the failure of hidden assumptions. Interestingly, as we will show in II, the problem was solved technically correctly in Ref. 8, but the missing physical interpretation rendered this result of limited value at the time.

B. Scaling issues

Let us finally come back to the issue of the two different time scales, which has been crucial for a correct application of scaling ideas to the problem. As we have seen in Sec. III B 1, the implicit assumption of the existence of only one time scale, namely the critical one can lead to wrong conclusions if one relies strictly on power-counting arguments. Explicit loop calculations, on the other hand, reveal the fallaciousness of the assumption by producing terms in the action that are inconsistent with the power counting. The point is that the diffusive modes, whose time scale is different from the critical one, produce long-range correlations *everywhere*, not just at the critical point as has been discussed in detail elsewhere.[33] These long-range correlations are reflected, for instance, in the $|\mathbf{k}|^{d-2}$ term in the LGW part of the action, Eq. (2.25b), which is responsible for the instability of Hertz's fixed point.

For the instability of the Gaussian fixed point of Ref. 4 a similar mechanism applies, although it is weaker and less obvious. As we have seen in the context of Eqs. (3.14), non-Gaussian terms that formally have a negative scale dimension can effectively become marginal with respect to frequency-dependent coupling constants. This ''counting accident'' can only happen for vertices that vanish at zero frequency, and it has been analyzed from a RG point of view in Sec. III B 2 and in the Appendix.

We will come back to these arguments in II, where we will provide both a resummation of perturbation theory to all orders and a complete scaling description of the exact critical behavior, including all logarithmic corrections.

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APPENDIX: CONSEQUENCES OF TWO DIFFERENT TIME SCALES

In this appendix we discuss two additional aspects of the crucial point made in Sec. III B 2.

Let us first take a phenomenological scaling point of view. The scaling equation for the two-point q-vertex function $\Gamma^{(2)}$ reads

$$\Gamma^{(2)}(\mathbf{k}=0,\Omega)=b^{-2}\gamma^{(2)}(\Omega b^2,\Omega b^d,c_2 b^{-(d-2)/2},\ldots). \tag{A1}$$

Here the ellipses denote the dependence of the scaling function $\gamma^{(2)}$ on all operators that are not shown explicitly. Among these are $1/G_4$ and H_4, which play the same role for scaling as c_2^2 does. For simplicity we restrict the discussion to the effect of the latter. In writing Eq. (A1), we have allowed for a dependence on both the diffusive and the critical frequency scale. At the zero-loop order, $\gamma^{(2)}$ depends only on the former, and by putting $b=1/\sqrt{\Omega}$ we have $\Gamma^{(2)}(\mathbf{k}=0,\Omega) \propto \Omega$. At one-loop order, it depends on the critical frequency as well, which opens the possibility of a stronger frequency dependence proportional to $\Omega^{2/d}$. However, the one-loop contribution has the property $\gamma^{(2)}(x,y,z)=f(yz^2)$, which restores the linear-frequency behavior of $\Gamma^{(2)}$. This is the same phenomenon that we have discussed within the context of explicit perturbation theory in connection with Eqs. (3.14). Logarithmic corrections to scaling are neglected in this simple argument.

To illustrate the same point from a RG flow-equation point of view, and at the same time see the origin of the logarithms, we absorb the frequency or temperature factors multiplying H and c_1 in Eq. (3.8) into these coupling constants by defining $\tilde{H}=H\Omega$ and $\tilde{c}_1=c_1\sqrt{T}$. For $\tilde{H}$, $\tilde{c}_1$, and c_2 we then have the flow equations

$$\frac{d\tilde{H}}{d\ln b}=2\tilde{H}+\text{const}\times\tilde{c}_1^2 c_2^2, \tag{A2a}$$

$$\frac{d\tilde{c}_1}{d\ln b}=d\tilde{c}_1/2, \tag{A2b}$$

$$\frac{dc_2}{d\ln b}=-(d-2)c_2/2 \tag{A2c}$$

plus higher-loop orders. Again, $1/G_4$ and H_4 play a role analogous to c_2^2 and we have suppressed them for simplicity. The solution of this system of flow equations is

$$\tilde{H}(b)=\tilde{H}(b=1)b^2+\text{const}\times b^2\ln b. \tag{A3}$$

In this picture, the positive scale dimensions of $\tilde{H}$ and $\tilde{c}_1$ reflect the fact that frequency or temperature is a relevant variable. The critical frequency is more relevant than the diffusive one but this difference is made up for by the fact that the critical frequency is always multiplied by c_2^2. In this way the formally irrelevant c_2 effectively acquires a marginal status. The logarithm, at one-loop order, reflects a resonance between the scale dimensions of $\tilde{c}_1$ and c_2, and represents one of the possibilities in Wegner's classification of logarithmic corrections to scaling[34] as was already pointed out in Ref. 4. At higher-loop order, however, additional logarithmic terms appear as we will show in II.

[1] See, e.g., S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).

[2] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976), and references therein. The finite temperature properties of this model were discussed by A. J. Millis, *ibid.* **48**, 7183 (1993).

[3] S. Sachdev, Z. Phys. B: Condens. Matter **94**, 469 (1994).

[4] T. R. Kirkpatrick and D. Belitz, Phys. Rev. B **53**, 14 364 (1996).

[5] T. Vojta, D. Belitz, R. Narayanan, and T. R. Kirkpatrick, Z. Phys. B: Condens. Matter **103**, 451 (1997).

[6] For a review, see D. Belitz and T. R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).

[7] In low-order perturbative RG treatments, the existence of this transition is signaled by runaway flow. This has led to widespread confusion, and even to a popular belief that the entire metal-insulator transition problem in the presence of interactions is pathological. To avoid exacerbating this situation, we emphasize that (1) the runaway flow occurs in only one universality class, while all others show a perturbative fixed point that describes a metal-insulator transition and (2) the technical problem disappears and a critical fixed point is found if a resummation of the perturbation theory to all orders is performed (Refs. 6 and 8). However, the nature of the transition could not be identified with the methods employed in these papers.

[8] T. R. Kirkpatrick and D. Belitz, Phys. Rev. B **45**, 3187 (1992).

[9] We call a theory ''local'' if the vertices in a LGW functional exist in the limit of zero frequencies and wave numbers. We stress that this definition in general makes sense only at the level of a bare theory as renormalization usually produces singular wave number dependences of sufficiently high-order vertices.

[10] To avoid misunderstandings, we point out that the Gaussian theory of Refs. 4 and 5 contains effects whose incorporation within Hertz's approach requires loops.

[11] R. Narayanan, T. Vojta, D. Belitz, and T. R. Kirkpatrick, Phys. Rev. B **60**, 10 150 (1999).

[12] D. Belitz, T. R. Kirkpatrick, Maria Teresa Mercaldo, and Sharon L. Sessions, following paper, Phys. Rev. B **63**, 174428 (2001).

[13] D. Belitz and T. R. Kirkpatrick, Phys. Rev. B **56**, 6513 (1997); D. Belitz, F. Evers, and T. R. Kirkpatrick, *ibid.* **58**, 9710 (1998).

[14] See, e.g., J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York, 1988).

[15] It is important to keep in mind that microscopic details neglected in our model can prevent the ferromagnetic transition from occurring altogether. For instance, nesting of the Fermi surface can lead to antiferromagnetic behavior instead. However, we expect that if a ferromagnetic transition occurs in a realistic system, then it will be in the same universality class as our simple model.

[16] For the spin-triplet interaction, this approximation is standard and easy to justify. In the spin-singlet channel, screening in a metallic system also ensures a short-ranged effective interaction. As in Ref. 4, we assume that the disorder is not so strong as to put the system in the vicinity of a metal-insulator transition, which

would cause screening to break down.

[17] For a pedagogical explanation of the replica trick, see, e.g., G. Grinstein, in *Fundamental Problems in Statistical Mechanics VI*, edited by E. G. D. Cohen (North-Holland, Amsterdam, 1985).

[18] The method employed in Ref. 13 was in turn adapted from L. Schäfer and F. Wegner, Z. Phys. B: Condens. Matter **38**, 113 (1980).

[19] The physically interesting limit is of course $N \to 0$, $n \to \infty$. The properties of the homogeneous space in this limit have not been investigated.

[20] F. Wegner, Z. Phys. B **35**, 207 (1979).

[21] More precisely, this is the fermionic version of the matrix NLσM, K. B. Efetov, A. I. Larkin, and D. E. Khmelnitskii, Zh. Éksp. Teor. Fiz. **79**, 1120 (1980) [Sov. Phys. JETP **52**, 568 (1980)]; A. M. Finkelstein, *ibid.* **84**, 168 (1983) [**57**, 97 (1983)], while Wegner's original model was a bosonic formulation that is adequate for noninteracting electrons.

[22] C. Castellani and C. Di Castro, Phys. Rev. B **34**, 5935 (1986).

[23] It is possible to integrate out ΔP in Gaussian approximation in closed form while keeping q to all orders. The result, however, is not very elucidating and we therefore expand in powers of q before integrating out the massive fluctuations.

[24] There is also a term proportional to $M \Delta P q$. However, its frequency structure is such that it is effectively proportional to the frequency carried by the M field. As a result, this term is less important for our purposes than the $M \Delta P q^2$ term and we neglect it.

[25] It is furthermore expected that G and G_4, and H and H_4, as well as all higher coupling constants in the expansion of the nonlinear sigma model in powers of q renormalize in the same way. For the sigma model in the absence of interactions this is known to be the case, see, e.g., J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989), and for the sigma model with pointlike interactions it is very likely to be true (see, e.g., Ref. 6). Since the present model can be reformulated as an interacting sigma model, albeit with a nonpointlike triplet interaction amplitude by integrating out the field M, the same property is expected to hold.

[26] Here we use Ma's method for identifying simple fixed points. Accordingly, we use physical arguments to determine which coupling constants should be marginal and then check whether this choice leads self-consistently to a stable fixed point. See S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, MA, 1976).

[27] A. B. Harris, J. Phys. C **7**, 1671 (1974); J. L. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).

[28] We note that one also obtains this fixed point within a theory where the bare value of a_{d-2} is zero if one chooses c_1 and c_2 to be marginal. This is an interesting alternative since it *forces* one to introduce two different time scales otherwise c_1 and c_2 cannot both be marginal. The problem with the formulation of the theory in Ref. 4 was not so much the nonlocality of the theory *per se* as the fact that this nonlocality obscured the existence of two time scales.

[29] Strictly speaking, $\delta \mathcal{A}^{(2)}$, Eq. (2.18), is part of the effective action. However, none of the quantities we are interested in are given in terms of correlations of ΔP or $\Delta \Lambda$ and the couplings between these fluctuations and M or q are all irrelevant. We therefore neglect this part of the effective action.

[30] R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).

[31] The arguments given in Ref. 8 for an absence of long-range magnetic order were incorrect.

[32] See, e.g., M. E. Fisher, in *Advanced Course on Critical Phenomena*, edited by F. W. Hahne (Springer, New York, 1983).

[33] T. R. Kirkpatrick and D. Belitz, J. Stat. Phys. **87**, 1307 (1997).

[34] F. Wegner, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6.