

Phase transitions in the one-dimensional spin- S J_1 - J_2 XY model

P. Lecheminant,¹ T. Jolicoeur,² and P. Azaria³

¹*Laboratoire de Physique Théorique et Modélisation, Université de Cergy Pontoise, 5 Mail Gay-Lussac, Neuville sur Oise, 95301 Cergy Pontoise Cedex, France*

²*Laboratoire de Physique de la Matière Condensée, Ecole Normale Supérieure, 24 rue Lhomond, 75005 Paris, France*

³*Laboratoire de Physique Théorique des Liquides, Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris, France*

(Received 25 July 2000; revised manuscript received 20 November 2000; published 10 April 2001)

The one-dimensional spin- S J_1 - J_2 XY model is studied within the bosonization approach. Around the two limits ($J_2/J_1 \ll 1$, $J_2/J_1 \gg 1$) where a field theoretical analysis can be derived, we discuss the phases as well as the different phase transitions that occur in the model. In particular, it is found that the chiral critical spin nematic phase, discovered by Nersesyan *et al.* [Phys. Rev. Lett. **81**, 910 (1998)] for $S=1/2$, exists in the general spin- S case. The nature of the effective-field theory that describes the transition between this chiral critical phase and a chiral gapped phase is also determined.

DOI: 10.1103/PhysRevB.63.174426

PACS number(s): 75.10.Jm

I. INTRODUCTION

The interplay between frustration and quantum fluctuations in low-dimensional spin systems has attracted much interest. One of the main reasons for this attention is that frustration is expected to lead to new exotic phases as well as unconventional spin excitations. In the one-dimensional case, powerful nonperturbative methods are available and the key features of frustration can then be analyzed in depth. In this respect, the phase diagram of the one-dimensional $S=1/2$ J_1 - J_2 XY model has been studied extensively over the years. The lattice Hamiltonian of this model is defined by

$$\mathcal{H} = J_1 \sum_n (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) + J_2 \sum_n (S_n^x S_{n+2}^x + S_n^y S_{n+2}^y), \quad (1)$$

where $S_n^\pm = S_n^x \pm iS_n^y$ is a spin-1/2 operator at site n and J_2 is a competitive antiferromagnetic interaction ($J_1, J_2 > 0$) which introduces frustration in the model. The phase diagram of the Hamiltonian (1) is expected to be rich. For a small value of J_2 , one has a spin fluid phase (XY phase) characterized by gapless excitations with central charge $c=1$ whereas for $J_2/J_1 \approx 0.32$ (Ref. 1) a phase transition of Kosterlitz-Thouless (KT) type occurs and the model enters a massive region with a twofold degenerate ground state (dimerized phase).² Interestingly enough, it was recently predicted by Nersesyan *et al.*,³ within the bosonization approach,⁴ that in the large J_2 limit, where the model can be viewed as a two-leg XY zigzag ladder, a critical spin nematic phase with chiral long-range order [$\langle (\vec{S}_n \wedge \vec{S}_{n+1})_z \rangle \neq 0$] should emerge. This unconventional phase with unbroken time-reversal symmetry is characterized by nonzero local spin currents polarized along the anisotropy z axis. The transverse spin-spin correlation functions are incommensurate and fall off with the distance as a power law with the exponent 1/4. However, this phase has not been reported in the numerical calculations of Refs. 5 and 6. In contrast, only the previous well-known phases (the spin fluid and dimerized phases) have been found. A numerical analysis of the model using an exact diagonalization method with twisted

boundary conditions has pointed out the presence of incommensuration in the large J_2 limit⁷ but within this approach it has not been possible to conclude on the criticality or not of this phase. Very recently, Nishiyama⁸ has investigated the existence of chiral order of the Josephson-junction ladder with half a flux quantum per plaquette by means of the exact diagonalization method. He was able to show that the critical phase predicted in Ref. 3 does exist in the range of the parameters of the model (1) in contrast to the numerical findings of Ref. 5.

The situation is less controversial in the $S=1$ case and the corresponding phase diagram has been determined by a density-matrix renormalization-group (DMRG) study.^{5,6} The model with $J_2=0$ is a critical spin fluid (the so-called $XY1$ phase⁹) and as soon as the next-nearest-neighbor interaction is switched on the Haldane phase is stabilized (KT transition). As noted in Ref. 5, this fact seems to be in contradiction with the bosonization result obtained in the small J_2 limit¹⁰ which suggests that the $XY1$ phase extends to finite J_2/J_1 . Increasing on the value of J_2/J_1 , the authors of Refs. 5 and 6 have reported the occurrence of two successive transitions: A first one at $(J_2/J_1)_{c1} \approx 0.473$ (Ising transition) from the Haldane phase to a gapped phase with chiral long-range order (chiral gapped phase) and a second transition at $(J_2/J_1)_{c2} \approx 0.49$ (presumably a KT transition) with a chiral critical phase which corresponds to the spin nematic phase discussed by Nersesyan *et al.*³ in the context of the two-leg $S=1/2$ XY zigzag ladder. In this latter phase, the spin-spin correlations are incommensurate and decay with a power law with an exponent approximately equal to 0.15.⁶ Recently, the phase diagram of the model (1) in the general spin- S case has been further discussed in Ref. 11 by means of a large- S approach. It has been found that the existence of the gapless and gapped chiral phases is not specific of $S=1$ but is rather a generic large- S feature. The predicted phase diagram has four different phases. First of all, one has the $XY1$ critical spin fluid phase and for a finite value of J_2 (KT transition) one enters the Haldane phase. As J_2/J_1 is further increased, there are the two successive phase transitions previously discussed: An Ising-type transition separating the Haldane

phase from the chiral gapped phase, followed by a KT transition corresponding to the transition between the chiral gapless phase and the chiral gapped phase. However, this large- S approach does not distinguish between integer and half-integer spins and also it does not take into account the possibility of a spontaneously dimerized phase. In addition, for the very special $S=1$ case, the numerical calculations of Refs. 5 and 6 predict rather that the critical phase $XY1$ does not extend for a finite value of J_2 .

In this paper, we shall investigate the phase diagram of the one-dimensional spin- S J_1 - J_2 XY model within the bosonization approach. Using the Abelian bosonization of a general spin- S operator introduced by Schulz,⁹ the low-energy physics of the Hamiltonian (1) can be studied in two different limits: in the weak-coupling limit when $J_2/J_1 \ll 1$ and in the ladder limit $J_2/J_1 \gg 1$. This enables us to determine the nature of the phase transitions that occur in the model from a field theoretical point of view together with a comparison with the numerical results^{5,6} as well as the large- S predictions.¹¹ The remainder of the paper is organized as follows: the weak-coupling analysis ($J_2/J_1 \ll 1$) is given in Sec. II whereas the zigzag limit of the model ($J_2/J_1 \gg 1$) is performed in Sec. III. Section IV presents our concluding remarks and finally the conventions and some technical details used in this work are described in the two appendices.

II. WEAK-COUPLING LIMIT

In this section, we shall investigate the low-energy physics of the one-dimensional spin- S J_1 - J_2 XY model in the limit $J_2/J_1 \ll 1$ within the bosonization approach. This enables us to study the stability of the critical $XY1$ phase (for $J_2=0$) upon switching on a small next-nearest-neighbor interaction. The Hamiltonian (1) in the $S=1$ case can be tackled within the bosonization framework by representing spin-1 operators as a sum of two spin-1/2 operators: $S_n^\pm = s_{1,n}^\pm + s_{2,n}^\pm$. It has been argued¹²⁻¹⁴ that the additional local singlets introduced will lead to extra levels with higher energy than the triplet states so that the ground state and the low-lying excitations are correctly captured. In particular, with this representation, Timonen and Luther¹⁵ and Schulz⁹ predicted the correct phase diagram of the one-dimensional anisotropic antiferromagnetic spin-1 Heisenberg model with single-ion anisotropy. Moreover, the effect of weak randomness on this latter model has also been analyzed within this bosonization approach.¹⁶ This procedure was further generalized by Schulz⁹ by representing a general spin- S operator S_n^\pm as the sum of $2S$ spin-1/2 operators $s_{a,n}^\pm$, $a=1, \dots, 2S$:

$$S_n^\pm = \sum_{a=1}^{2S} s_{a,n}^\pm, \quad (2)$$

from which the low-energy physics of the one-dimensional spin- S Heisenberg model can be captured and in particular the difference between half-integer and integer spins.¹⁷

Let us first review the continuum description of the spin- S XY chain obtained by Schulz.⁹ Using the decomposition (2), the Hamiltonian (1) for $J_2=0$ writes

$$\begin{aligned} \mathcal{H}_{XY1} = & \frac{J_1}{2} \sum_{a=1}^{2S} \sum_n (s_{a,n}^\dagger s_{a,n+1}^- + \text{H.c.}) \\ & + \frac{J_1}{2} \sum_{a \neq b} \sum_n (s_{a,n}^\dagger s_{b,n+1}^- + \text{H.c.}). \end{aligned} \quad (3)$$

The first term in this equation corresponds to $2S$ decoupled spin-1/2 XY chains. As recalled in the Appendix A, the low-energy physics of the spin-1/2 XY chain can be extracted from the introduction of a single $U(1)$ bosonic field φ with chiral components $\varphi_{R,L}$. As a consequence, the next step of the approach is to introduce $2S$ chiral decoupled bosonic fields $\varphi_{aR,L}$, $a=1, \dots, 2S$ and the Hamiltonian density of the first term in Eq. (3) in the continuum limit is given by

$$\mathcal{H}_{XY} \simeq \frac{v_0}{2} \sum_{a=1}^{2S} [(\partial_x \varphi_a)^2 + (\partial_x \vartheta_a)^2], \quad (4)$$

where $\varphi_a = \varphi_{aL} + \varphi_{aR}$, $\vartheta_a = \varphi_{aL} - \varphi_{aR}$ being the dual field, and $v_0 = J_1 a_0$ (a_0 being the lattice spacing) is the spin velocity. One can then derive the continuum limit of the second term in Eq. (3) using the bosonic description (A7) of the spin-1/2 operators s_a^\pm described in Appendix A:

$$\begin{aligned} s_a^+ = & \frac{(-1)^{x/a_0}}{\sqrt{2\pi a_0}} \exp(i\sqrt{\pi}\vartheta_a) + \frac{1}{\sqrt{8\pi a_0}} [\exp(i3\sqrt{\pi}\varphi_{aL} \\ & + i\sqrt{\pi}\varphi_{aR}) + \exp(-i3\sqrt{\pi}\varphi_{aR} - i\sqrt{\pi}\varphi_{aL})]. \end{aligned} \quad (5)$$

Notice that *a priori* this procedure has only a sense provided that the coupling constant associated to the second piece of Eq. (3) is much smaller than J_1 and this is clearly not the case here. However, one expects in this problem, on general grounds, a continuity between weak- and strong-coupling limits so that it is natural to bosonize the second term of Eq. (3) using the correspondence (5). The leading part of the density Hamiltonian associated to Eq. (3) in the continuum limit reads thus as follows:

$$\begin{aligned} \mathcal{H}_{XY1} \simeq & \frac{v_0}{2} \sum_{a=1}^{2S} [(\partial_x \varphi_a)^2 + (\partial_x \vartheta_a)^2] \\ & - \frac{J_1}{2\pi a_0} \sum_{a \neq b} \cos[\sqrt{\pi}(\vartheta_a - \vartheta_b)]. \end{aligned} \quad (6)$$

It is then suitable to switch to a basis to single out the degrees of freedom that will remain critical in the infrared limit. To this end, let us introduce a diagonal bosonic field $\Phi_{+R(L)}$ and $2S-1$ relative bosonic fields $\Phi_{mR(L)}$, $m=(1, \dots, 2S-1)$ as follows:

$$\begin{aligned} \Phi_{+R(L)} = & \frac{1}{\sqrt{2S}} (\varphi_1 + \dots + \varphi_{2S})_{R(L)} \\ \Phi_{mR(L)} = & \frac{1}{\sqrt{m(m+1)}} (\varphi_1 + \dots + \varphi_m - m\varphi_{m+1})_{R(L)}. \end{aligned} \quad (7)$$

The transformation (7) is canonical and preserves the bosonic commutation relations. This basis has been intro-

duced in Ref. 18 in the Abelian bosonization study of the one-dimensional Hubbard model with a SU(N) symmetry. The inverse transformation of Eq. (7) is easily found to be

$$\begin{aligned}\varphi_{1R(L)} &= \frac{1}{\sqrt{2S}}\Phi_{+R(L)} + \sum_{l=1}^{2S-1} \frac{\Phi_{lR(L)}}{\sqrt{l(l+1)}}, \\ \varphi_{aR(L)} &= \frac{1}{\sqrt{2S}}\Phi_{+R(L)} - \sqrt{\frac{a-1}{a}}\Phi_{(a-1)R(L)} \\ &\quad + \sum_{l=a}^{2S-1} \frac{\Phi_{lR(L)}}{\sqrt{l(l+1)}}, \quad a=2, \dots, 2S-1, \\ \varphi_{2SR(L)} &= \frac{1}{\sqrt{2S}}\Phi_{+R(L)} - \sqrt{\frac{2S-1}{2S}}\Phi_{(2S-1)R(L)}.\end{aligned}\quad (8)$$

Using Eq. (6), one observes that all relative dual fields Θ_m , $m=1, \dots, 2S-1$ are pinned whereas the diagonal bosonic field Φ_+ ($\Phi_+ = \Phi_{+L} + \Phi_{+R}$) is a strongly fluctuating field so that

$$\mathcal{H}_{XY1} \approx \frac{v_0}{2} [(\partial_x \Phi_+)^2 + (\partial_x \Theta_+)^2], \quad (9)$$

where $\Theta_+ = \Phi_{+L} - \Phi_{+R}$. From Eq. (5) and by integrating out the massive degrees of freedom, the expression of the effective spin- S density S^\pm in terms of the massless bosonic field in the $+$ sector can be deduced:

$$S^\pm \sim (-1)^{x/a_0} \exp(\pm i\sqrt{\pi/2S}\Theta_+). \quad (10)$$

The dual field Θ_+ is thus a compactified bosonic field with radius $\tilde{R}_S = \sqrt{2S/\pi}$. Using the general relation between the radius ($2\pi\tilde{R}_S R_S = 1$), we deduce that the compactified radius of the bosonic field Φ_+ is: $R_S = 1/\sqrt{8\pi S}$. Furthermore, one deduces from Eq. (10) that the transverse spin-spin correlation function has a power-law behavior with an exponent $\eta_\perp = 1/(4S)$. The value of this exponent coincides with the prediction of Alcaraz and Moreo¹⁹ who have analyzed the critical properties of the XXZ spin- S Heisenberg model by means of a combination of conformal invariance and exact diagonalizations techniques. It is worth noting that the value of the exponent in the $S=1$ case ($\eta_\perp = 1/4$) has been predicted by Kitazawa *et al.*²⁰ within a level spectroscopy analysis of the $S=1$ bond-alternating XXZ spin chain. Finally, one should observe that the uniform part of the spin density is a short-ranged piece since the massive modes that enter in this expression have a zero vacuum expectation value and thus give rise to an exponential decay in the uniform part of the spin-spin correlation function. However, as shown by Schulz,⁹ in the special half-integer case, higher orders of perturbation theory produce a strongly fluctuating piece in the uniform part of the spin density (10) with scaling dimension $2S+1/(8S)$ which is less relevant than the alternating contribution in Eq. (10) which has scaling dimension $1/(8S)$.

With all these results at hand, the stability of this critical XY1 phase with respect to a next-nearest-neighbor exchange

interaction $J_2/J_1 \ll 1$ can be analyzed. To this end, let us first rewrite the second term (\mathcal{H}_{XY2}) of the Hamiltonian (1) in terms of the $2S$ spin-1/2 operators:

$$\begin{aligned}\mathcal{H}_{XY2} &= \frac{J_2}{2} \sum_{a=1}^{2S} \sum_n (s_{a,n}^\dagger s_{a,n+2}^- + \text{H.c.}) \\ &\quad + \frac{J_2}{2} \sum_{a \neq b} \sum_n (s_{a,n}^\dagger s_{b,n+2}^- + \text{H.c.}).\end{aligned}\quad (11)$$

The first part of this equation corresponds to the sum of N decoupled next-nearest-neighbor $S=1/2$ XY chains. The resulting continuum limit has been obtained by Haldane in the erratum of Ref. 2 and is reviewed for completeness in Appendix B [see in particular Eq. (B4)]. The continuum limit of the second term in Eq. (11) can be obtained using the bosonized description (5) of the spin density s_a^\pm . The resulting continuum limit of the Hamiltonian (11) reads thus as follows:

$$\begin{aligned}\mathcal{H}_{XY2} &\approx -\frac{4J_2 a_0}{\pi} \sum_{a=1}^{2S} (\partial_x \vartheta_a)^2 + \frac{J_2}{\pi a_0} \\ &\quad \times \sum_{a < b} \cos[\sqrt{\pi}(\vartheta_a - \vartheta_b)] \\ &\quad - \frac{J_2}{\pi^2 a_0} \sum_{a=1}^{2S} \cos(\sqrt{16\pi}\varphi_a) \\ &\quad + \frac{J_2}{2\pi a_0} \sum_{a < b} \cos[\sqrt{4\pi}(\varphi_a + \varphi_b)] \\ &\quad \times \cos[\sqrt{\pi}(\vartheta_a - \vartheta_b)].\end{aligned}\quad (12)$$

Using the canonical transformation (8), one finally obtains the following effective Hamiltonian:

$$\mathcal{H} \approx \frac{v}{2} \left(K(\partial_x \Theta_+)^2 + \frac{1}{K}(\partial_x \Phi_+)^2 \right), \quad (13)$$

with

$$\begin{aligned}v &= v_0 \sqrt{1 - \frac{8J_2}{\pi J_1}}, \\ K &= \sqrt{1 - \frac{8J_2}{\pi J_1}}.\end{aligned}\quad (14)$$

Of course, as usual, these latter identifications hold only in the vicinity of the Gaussian fixed point at $J_2/J_1 = 0$. However, one should carefully look at the higher-order corrections of perturbation theory that might generate an additional operator in the effective theory (13) and potentially destabilizes the XY1 phase. In this respect, integer and half-integer spins should be treated separately. For integer spins S , the last term in Eq. (12) leads to the following contribution at the S th order of perturbation theory:

$$\int \prod_{i=1}^S d^2x_i \prod_{i=1}^S \cos[\sqrt{4\pi}(\varphi_{2i-1} + \varphi_{2i})](x_i) \times \cos[\sqrt{\pi}(\vartheta_{2i-1} - \vartheta_{2i})](x_i), \quad (15)$$

which, after integrating out the short-ranged degrees of freedom and using the canonical transformation (8), gives rise to a fluctuating field in the + channel: $\cos(\sqrt{8\pi S}\Phi_+)$. In the same way, for half-integer spins S , one has the following contribution at the $2S$ th order of perturbation theory:

$$\int \prod_{i=1}^{2S} d^2x_i \prod_{i=1}^{2S} \cos[\sqrt{4\pi}(\varphi_i + \varphi_{i+1})](x_i) \times \cos[\sqrt{\pi}(\vartheta_i - \vartheta_{i+1})](x_i), \quad (16)$$

with the identification: $\varphi_{2S+1} = \varphi_1$ and $\vartheta_{2S+1} = \vartheta_1$. One then obtains the following operator: $\cos(\sqrt{32\pi S}\Phi_+)$ after averaging on the short-ranged degrees of freedom. The effective field theory associated to the spin- S J_1 - J_2 XY chain in the weak-coupling limit is thus

$$\mathcal{H} \simeq \frac{v}{2} \left(K(\partial_x \Theta_+)^2 + \frac{1}{K}(\partial_x \Phi_+)^2 \right) - \frac{g_{eff}}{a_0} \cos(\mu\sqrt{8\pi S}\Phi_+), \quad (17)$$

with $\mu=1$ (respectively 2) if S is integer (respectively half-integer). One should note that the Hamiltonian (17) corresponds to the effective-field theory of the spin- S XXZ Heisenberg chain derived by Schulz.⁹ In fact, the last operator in Eq. (17) can also be justified from a symmetry analysis. Indeed, under the one-step translation symmetry, the bosonic field φ_a transforms according to [see Eq. (A8) of Appendix A]: $\varphi_a \rightarrow \varphi_a + \sqrt{\pi}/2 + p_a\sqrt{\pi}$, p_a being an integer. From the definition (7) of the diagonal bosonic field Φ_+ , one thus has

$$\Phi_+ \rightarrow \Phi_+ + \sqrt{\frac{\pi S}{2}} + p \sqrt{\frac{\pi}{2S}}, \quad (18)$$

from which we conclude that the $\cos(\mu\sqrt{8\pi S}\Phi_+)$ term is the operator invariant under the translation symmetry with the smallest scaling dimension.

The phase diagram of the spin- S J_1 - J_2 XY chain in the small J_2/J_1 limit can then be deduced from the structure of the effective-field theory (17). For a small value of J_2/J_1 , the cosine operator in Eq. (17) is a strongly irrelevant contribution and the system is critical with central charge $c=1$ (Luttinger liquid): it is the spin fluid $XY1$ phase that extends to a finite value of J_2 . As J_2/J_1 increases, one expects from Eqs. (14) and (17) a KT phase transition from this spin fluid phase to a fully massive region (dimerized or Haldane phases depending on the nature of the spin S).²¹ At the transition, the Luttinger parameter K_c is equal to: $K_c = 1/(S\mu^2)$ and a very naive estimate of the critical value of $(J_2/J_1)_c$ can then be deduced from Eq. (14) within the bosonization approach:

$$\left(\frac{J_2}{J_1} \right)_c \simeq \frac{\pi(S^2\mu^4 - 1)}{8S^2\mu^4}. \quad (19)$$

In particular, for $S=1/2$, one finds $(J_2/J_1)_c = 3\pi/32 \simeq 0.2945$ which is not too bad in comparison to the value obtained in the numerical simulations of Ref. 1: $(J_2/J_1)_c \simeq 0.3238$. Moreover, in the $S=1$ case, the $XY1$ phase is destabilized upon switching on a nonzero value of J_2 in full agreement with the numerical findings of Refs. 5 and 6. The origin of the discrepancy noted in Ref. 5 between the DMRG study^{5,6} and the bosonization results obtained in Ref. 10 stems from the fact that the latter authors do not look at higher orders in perturbation theory as in this work. The $S=1$ case does not correspond to the generic situation since we observe from the estimate (19) that the size of the $XY1$ phase increases as S increases in the half-integer and integer cases. In this respect, one should note that the situation is in close parallel to the phase transition between the $XY1$ and the Haldane phases in the integer spin- S XXZ Heisenberg chain. In the $S=1$ case, the resulting phase transition occurs precisely at the $XY1$ point^{9,22,20} whereas the $XY1$ phase extends considerably as S increases.^{9,19,23,24}

III. ZIGZAG LADDER LIMIT

We shall now study the model (1) in the ladder limit $J_1 \ll J_2$ where it can be viewed as a two-leg spin- S XY ladder coupled in a zigzag way. For $S=1/2$ Heisenberg spins, the effect of a transverse zigzag interchain interaction has been extensively studied in Refs. 25,26,3,27,28 and also in Ref. 29 in the $S=1$ case. In the special case of $S=1/2$ XY spins, it has been found by Nersesyan *et al.*³ that the model is a critical spin nematic. In this section, we shall investigate the existence of such a phase in the general spin- S case and study its stability as the interchain interaction is further varied.

A. Critical spin nematic phase

The lattice Hamiltonian of the model (1), considered as a two-leg spin ladder, is defined now as follows:

$$\mathcal{H} = \frac{J_2}{2} \sum_n (S_{1,n}^\dagger S_{1,n+1}^- + S_{2,n-1/2}^\dagger S_{2,n+1/2}^- + \text{H.c.}) + \frac{J_1}{2} \sum_n [S_{1,n}^\dagger (S_{2,n-1/2}^- + S_{2,n+1/2}^-) + \text{H.c.}], \quad (20)$$

where $S_{1,n}^\pm$ (respectively $S_{2,n+1/2}^\pm$) is the spin- S operator of chain of index 1 (respectively 2) at site n (respectively $n+1/2$). It is more suitable to change the labeling of the second chain in the following way to perform the continuum limit of the model:

$$\mathcal{H} = \frac{J_2}{2} \sum_{a=1}^2 \sum_n (S_{a,n}^\dagger S_{a,n+1}^- + \text{H.c.}) + \frac{J_1}{2} \sum_n [S_{1,n}^\dagger (S_{2,n}^- + S_{2,n-1}^-) + \text{H.c.}]. \quad (21)$$

At this point, one should note that the interchain zigzag coupling can also be written as (using intrachain periodic boundary conditions):

$$\mathcal{H}'_{int} = \frac{J_1}{2} \sum_n [(S_{1,n}^\dagger + S_{1,n+1}^\dagger) S_{2,n}^- + \text{H.c.}] \quad (22)$$

Consequently, we shall thus write the interacting part of the Hamiltonian (21) in a symmetrized way for taking the continuum limit of the model:

$$\begin{aligned} \mathcal{H} = & \frac{J_2}{2} \sum_{a=1}^2 \sum_n (S_{a,n}^\dagger S_{a,n+1}^- + \text{H.c.}) \\ & + \frac{J_1}{4} \sum_n ((S_{1,n}^\dagger + S_{1,n+1}^\dagger) S_{2,n}^- \\ & + S_{1,n}^\dagger (S_{2,n}^- + S_{2,n-1}^-) + \text{H.c.}) \end{aligned} \quad (23)$$

In the absence of the interchain coupling ($J_1=0$), the model corresponds to two decoupled spin- S XY chains. As seen in Sec. II, it is critical with central charge $c=2$ and its low-energy physics can be obtained with the introduction of two decoupled chiral gapless bosonic fields $\Phi_{a+R,L}$ ($a=1,2$). The leading contribution of the spin density S_a^\pm comes from the alternating part [see Eq. (10)]:

$$S_a^\pm \approx \frac{\lambda}{\sqrt{a_0}} (-1)^{x/a_0} \exp(\pm i \sqrt{\pi/2S} \Theta_{a+}), \quad (24)$$

λ being a nonuniversal constant. From Eq. (24), we deduce the continuum limit of the model (23) in the small $J_1 \ll J_2$ limit:

$$\mathcal{H} \approx \frac{v}{2} \sum_{a=\pm} [(\partial_x \bar{\Phi}_a)^2 + (\partial_x \bar{\Theta}_a)^2] + g \partial_x \bar{\Theta}_+ \sin\left(\sqrt{\frac{\pi}{S}} \bar{\Theta}_-\right), \quad (25)$$

where $g = J_1 \lambda^2 \sqrt{\pi/(4S)}$ and we have introduced the symmetric and antisymmetric combinations of the two bosonic fields:

$$\begin{aligned} \bar{\Phi}_\pm &= \frac{1}{\sqrt{2}} (\Phi_{1+} \pm \Phi_{2+}), \\ \bar{\Theta}_\pm &= \frac{1}{\sqrt{2}} (\Theta_{1+} \pm \Theta_{2+}). \end{aligned} \quad (26)$$

The Hamiltonian (25) describes a nontrivial field theory since the field with coupling constant g , called twist term in Ref. 3, is a parity symmetry-breaking perturbation with a nonzero conformal spin (equal to 1). The effect of such term is rather unclear since the usual irrelevant versus relevant criterion does not hold for such a nonscalar perturbation (see, for instance, Ref. 4). The simplest spin-1 conformal perturbation is the uniform part of the spin density ($\partial_x \Phi$) that couples to a uniform magnetic field along the z axis. In this case, this term leads to incommensuration as is well known. It is thus natural to expect some incommensurability effect in the model (25) due to the twist term as emphasized by Nersisyan *et al.*³ In particular, the presence of incommensuration in the system can be found by a direct mean-field analysis of

the model (25). Indeed, it is easy to see that the mean-field Hamiltonian separates into two commuting parts: $\mathcal{H}_{MF} = \mathcal{H}_+ + \mathcal{H}_-$ with

$$\begin{aligned} \mathcal{H}_+ &= \frac{v}{2} [(\partial_x \bar{\Phi}_+)^2 + (\partial_x \bar{\Theta}_+)^2] + \kappa \partial_x \bar{\Theta}_+, \\ \mathcal{H}_- &= \frac{v}{2} [(\partial_x \bar{\Phi}_-)^2 + (\partial_x \bar{\Theta}_-)^2] - \frac{\mu}{a_0} \sin\left(\sqrt{\frac{\pi}{S}} \bar{\Theta}_-\right), \end{aligned} \quad (27)$$

the mean-field parameters being

$$\begin{aligned} \kappa &= g \left\langle \sin\left(\sqrt{\frac{\pi}{S}} \bar{\Theta}_-\right) \right\rangle, \\ \frac{\mu}{a_0} &= -g \langle \partial_x \bar{\Theta}_+ \rangle. \end{aligned} \quad (28)$$

The Hamiltonian (\mathcal{H}_+) is easily solved by the redefinition $\bar{\Theta}_+ \rightarrow \bar{\Theta}_+ - \kappa x/v$. The $+$ sector displays thus criticality with a nonzero topological spin current in the ground state: $\langle \partial_x \bar{\Theta}_+ \rangle = -\kappa/v \neq 0$. In contrast, the Hamiltonian (\mathcal{H}_-) in the other sector is a standard sine-Gordon model at $\beta^2 = \pi/S$ which describes a massive theory with massive quantum solitons and their bound states (breathers) together with massive kinks. The dual field $\bar{\Theta}_-$ is locked at: $\langle \bar{\Theta}_- \rangle = \sqrt{\pi S/4} \text{sgn } \mu \pmod{\sqrt{4S\pi}}$. The mean-field analysis can then be closed using the fact that: $\langle \sin(\sqrt{\pi/S} \bar{\Theta}_-) \rangle = c(a_0 |\mu|/v)^{1/(8S-1)}$ (c being a constant that can be determined³⁰) and one easily finds

$$\begin{aligned} \mu &= \pm \frac{v}{a_0} \left(\frac{a_0 g}{v} \right)^{(8S-1)/(4S-1)} c^{(8S-1)/(8S-2)}, \\ \kappa &= \pm \frac{v}{a_0} \left(\frac{a_0 g}{v} \right)^{4S/(4S-1)} c^{(8S-1)/(8S-2)}. \end{aligned} \quad (29)$$

From the correspondence (24), one can then estimate the asymptotic behavior of the transverse spin-spin correlation functions of the model which display an incommensurate critical behavior:

$$\langle S_1^\dagger(x) S_a^-(0) \rangle \sim \frac{e^{iq_S x}}{|x|^{1/(8S)}}, \quad a=1,2, \quad (30)$$

with $q_S - \pi/a_0 \sim (J_1/J_2)^{4S/(4S-1)}$. The transverse spin-spin correlation functions fall off thus with the distance as a power law with the exponent $1/(8S)$. In the $S=1$ case, one should note that this exponent ($1/8=0.125$) found in this bosonization study is in good agreement with the numerical findings 0.15 of the DMRG analysis of Ref. 6.

Besides this incommensurate critical behavior observed in the spin-spin correlation functions (30), the physical picture of this phase obtained at the mean-field level corresponds to a spin nematic.³¹ Indeed, let us first introduce the z component of the spin current J_{as}^z associated to the a th spin- S XY chain ($a=1,2$):

$$J_{as}^z = -v \sqrt{\frac{2S}{\pi}} \partial_x \bar{\Theta}_{a+}. \quad (31)$$

The vacuum expectation value of this operator can then be computed since one has in the ground state of the mean-field Hamiltonian (27): $\langle \partial_x \bar{\Theta}_+ \rangle = -\kappa/v \neq 0$ and $\langle \partial_x \bar{\Theta}_- \rangle = 0$. This latter result stems from the fact that the Hamiltonian (\mathcal{H}_-) in Eq. (27) is a standard sine-Gordon model characterized by a ground state with zero topological charge. Using the redefinition (26), one finally obtains the following estimate:

$$\langle J_{1s}^z \rangle = \langle J_{2s}^z \rangle = -v \sqrt{\frac{S}{\pi}} \langle \partial_x \bar{\Theta}_+ \rangle = \sqrt{\frac{S}{\pi}} \kappa \neq 0. \quad (32)$$

These spin currents can also be expressed in terms of the original spin degrees of freedom of the lattice Hamiltonian (23) using the identification (24):

$$\langle (\vec{S}_{a,n} \wedge \vec{S}_{a,n+1})_z \rangle \approx -\lambda^2 \sqrt{\frac{\pi}{4S}} \langle \partial_x \bar{\Theta}_+ \rangle \neq 0, \quad a=1,2, \quad (33)$$

whereas similarly the (interchain) zigzag spin current along the z axis reads as follows:

$$J_1 \langle (\vec{S}_{1,n} \wedge \vec{S}_{2,n})_z \rangle \approx -2 \sqrt{\frac{S}{\pi}} g \left\langle \sin \left(\sqrt{\frac{\pi}{S}} \bar{\Theta}_- \right) \right\rangle = -2 \sqrt{\frac{S}{\pi}} \kappa \neq 0, \quad (34)$$

where Eq. (28) has been used.

The physical picture that emerges from this mean-field analysis is therefore a spin nematic phase that preserves the U(1) and time-reversal symmetries and displays long-range chiral ordering in its ground state (33) and (34). In the classification of Ref. 31, this phase corresponds to a p -type spin nematic. At this point, it is important to stress that this chiral ordering is different from the scalar chirality order operator:³² $\langle \vec{S}_{1,n} \cdot (\vec{S}_{2,n} \wedge \vec{S}_{2,n-1}) \rangle$ which breaks parity and time reversal symmetries. In our case, the spin nematic phase does not break the time-reversal symmetry but spontaneously breaks a Z_2 symmetry of the model which, as it will be shown in the next section, is a tensor product of a site-parity and link-parity symmetries on the two chains. As a result, as first discovered in the $S=1/2$ case in Ref. 3, this produces a picture of local nonzero spin currents (32) and (34) polarized along the z -anisotropy axis circulating around the triangular plaquettes of the two-leg zigzag spin ladder.

B. Stability of the chiral critical phase

It is important to study further the stability of this critical spin nematic phase (chiral critical phase) in the $+$ channel with respect to various operators that will be generated in higher orders of perturbation theory or equivalently terms consistent with the symmetries of the original lattice model. Indeed, on general grounds, one expects that some operators in the $+$ sector should destroy the criticality of the phase at least for some finite value of J_1/J_2 . First of all, in the mean-field approach, the twist term acts like a sort of magnetic

field. As is well known, a magnetic field along the anisotropy axis is a source of incommensuration but also leads to a renormalization of the compactification radius of the bosonic field. This last effect was not found in the previous approach as seen in the *universal* behavior of the spin-spin correlations (30). From a symmetry point of view [continuous U(1) symmetry], there are no reasons to expect such universal behavior. It is a first sign that higher-order terms in perturbation theory might be important here. On the other hand, as seen in Sec. II, there is at least a massive region (the dimerized or Haldane phases) in the phase diagram when increasing the value of J_1 at fixed J_2 . It is therefore likely that a vertex operator, generated in the renormalization-group flow, in the $+$ channel will kill the critical phase at least for a critical value $(J_1/J_2)_c$.

We shall now discuss the bosonic representation of the different discrete lattice symmetries of the model (23) to find the nature of the operator that will be generated in the $+$ sector by the renormalization-group flow. Let us first consider the one-step translation ($t_{a_0}^{(a)}$), site parity ($P_S^{(a)}$), and link parity ($P_L^{(a)}$) corresponding to the chain of index $a=1,2$. Using the definition (7) of the diagonal bosonic field that accounts for the criticality of the XY1 spin fluid phase in the decoupling limit ($J_1=0$) and the bosonic representations (A8)–(A10) in the $S=1/2$ case described in Appendix A, one obtains the following identifications respectively for the one-step translation, site parity, and link parity:

$$\begin{aligned} \Phi_{a+} &\rightarrow \Phi_{a+} + \sqrt{\frac{\pi S}{2}} + p_a \sqrt{\frac{\pi}{2S}}, \\ \Theta_{a+} &\rightarrow \Theta_{a+} + \sqrt{2S\pi} + p'_a \sqrt{8\pi S}, \end{aligned} \quad (35)$$

$$\begin{aligned} \Phi_{a+}(x) &\rightarrow -\Phi_{a+}(-x) + \sqrt{\frac{\pi S}{2}} + q_a \sqrt{\frac{\pi}{2S}}, \\ \Theta_{a+}(x) &\rightarrow \Theta_{a+}(-x) + q'_a \sqrt{8\pi S}, \end{aligned} \quad (36)$$

and

$$\begin{aligned} \Phi_{a+}(x) &\rightarrow -\Phi_{a+}(-x) + n_a \sqrt{\frac{\pi}{2S}}, \\ \Theta_{a+}(x) &\rightarrow \Theta_{a+}(-x) + \sqrt{2\pi S} + n'_a \sqrt{8\pi S}, \end{aligned} \quad (37)$$

where $p_a, p'_a, q_a, q'_a, n_a, n'_a$ are integers. From these correspondences, one can deduce the bosonic representations of the discrete symmetries of the Hamiltonian (23). The translation symmetry acts on the symmetric and antisymmetric combinations (26) of the bosonic fields Φ_{a+} as follows:

$$\begin{aligned}
\bar{\Phi}_+ &\rightarrow \bar{\Phi}_+ + \sqrt{\pi S} + \sqrt{\frac{\pi}{4S}}(p_1 + p_2), \\
\bar{\Theta}_+ &\rightarrow \bar{\Theta}_+ + \sqrt{4S\pi} + \sqrt{4\pi S}(p'_1 + p'_2), \\
\bar{\Phi}_- &\rightarrow \bar{\Phi}_- + \sqrt{\frac{\pi}{4S}}(p_1 - p_2), \\
\bar{\Theta}_- &\rightarrow \bar{\Theta}_- + \sqrt{4\pi S}(p'_1 - p'_2). \tag{38}
\end{aligned}$$

A second type of discrete symmetry of the Hamiltonian (23) s_1 consists of a vertical axial symmetry combined by an one-step translation symmetry $t_{a_0}^{(1)}$ along the lower chain (labeled 1 in the following):

$$\begin{aligned}
\vec{S}_{1,n} &\rightarrow \vec{S}_{1,-n+1}, \\
\vec{S}_{2,n} &\rightarrow \vec{S}_{2,-n}, \tag{39}
\end{aligned}$$

namely in the continuum limit:

$$\begin{aligned}
\vec{n}_1(x) &\rightarrow -\vec{n}_1(-x), \\
\vec{n}_2(x) &\rightarrow \vec{n}_2(-x), \tag{40}
\end{aligned}$$

which corresponds to a tensor product of a link-parity transformation on chain 1 and a site-parity transformation on chain 2 ($s_1 = P_L^{(1)} \otimes P_S^{(2)}$) when the model is viewed as a zigzag ladder [Eq. (20)]. The bosonic representation of this discrete symmetry is thus

$$\begin{aligned}
\bar{\Phi}_+(x) &\rightarrow -\bar{\Phi}_+(-x) + \frac{\sqrt{\pi S}}{2} + \sqrt{\frac{\pi}{4S}}(n_1 + q_2), \\
\bar{\Theta}_+(x) &\rightarrow \bar{\Theta}_+(-x) + \sqrt{S\pi} + \sqrt{4\pi S}(n'_1 + q'_2), \\
\bar{\Phi}_-(x) &\rightarrow -\bar{\Phi}_-(-x) - \frac{\sqrt{\pi S}}{2} + \sqrt{\frac{\pi}{4S}}(n_1 - q_2), \\
\bar{\Theta}_-(x) &\rightarrow \bar{\Theta}_-(-x) + \sqrt{S\pi} + \sqrt{4\pi S}(n'_1 - q'_2). \tag{41}
\end{aligned}$$

In the same way, the Hamiltonian (23) is also invariant under the transformation (s_2 symmetry):

$$\begin{aligned}
\vec{S}_{1,n} &\rightarrow \vec{S}_{1,-n}, \\
\vec{S}_{2,n} &\rightarrow \vec{S}_{2,-n-1}, \tag{42}
\end{aligned}$$

which can be viewed as a $P_L^{(2)} \otimes P_S^{(1)}$ transformation. In terms of the bosonic fields of the basis (26), this latter symmetry is realized through

$$\begin{aligned}
\bar{\Phi}_+(x) &\rightarrow -\bar{\Phi}_+(-x) + \frac{\sqrt{\pi S}}{2} + \sqrt{\frac{\pi}{4S}}(n_2 + q_1), \\
\bar{\Theta}_+(x) &\rightarrow \bar{\Theta}_+(-x) + \sqrt{S\pi} + \sqrt{4\pi S}(n'_2 + q'_1),
\end{aligned}$$

$$\begin{aligned}
\bar{\Phi}_-(x) &\rightarrow -\bar{\Phi}_-(-x) + \frac{\sqrt{\pi S}}{2} + \sqrt{\frac{\pi}{4S}}(-n_2 + q_1), \\
\bar{\Theta}_-(x) &\rightarrow \bar{\Theta}_-(-x) - \sqrt{S\pi} + \sqrt{4\pi S}(-n'_2 + q'_1). \tag{43}
\end{aligned}$$

There is a second family of discrete symmetries of the Hamiltonian (23): $s_3 = P_{12} \otimes t_{a_0}^{(1)}$ or $s_4 = P_{12} \otimes t_{-a_0}^{(2)}$ which corresponds to an interchange of the chains combined with a translation symmetry along the lower or upper chain. In terms of the original spin degrees of freedom, the s_3 and s_4 symmetries respectively write:

$$\begin{aligned}
\vec{S}_{1,n} &\rightarrow \vec{S}_{2,n}, \\
\vec{S}_{2,n} &\rightarrow \vec{S}_{1,n+1}, \tag{44}
\end{aligned}$$

$$\begin{aligned}
\vec{S}_{1,n} &\rightarrow \vec{S}_{2,n-1}, \\
\vec{S}_{2,n} &\rightarrow \vec{S}_{1,n}, \tag{45}
\end{aligned}$$

so that in the continuum limit, one has

$$\begin{aligned}
\vec{n}_1(x) &\rightarrow \vec{n}_2(x), \\
\vec{n}_2(x) &\rightarrow -\vec{n}_1(x), \tag{46}
\end{aligned}$$

and

$$\begin{aligned}
\vec{n}_1(x) &\rightarrow -\vec{n}_2(x), \\
\vec{n}_2(x) &\rightarrow \vec{n}_1(x). \tag{47}
\end{aligned}$$

The bosonic representation of these last discrete symmetries of Eq. (23) is then respectively given by

$$\begin{aligned}
\bar{\Phi}_+ &\rightarrow \bar{\Phi}_+ + \frac{\sqrt{\pi S}}{2} + \sqrt{\frac{\pi}{4S}}p_1, \\
\bar{\Theta}_+ &\rightarrow \bar{\Theta}_+ + \sqrt{S\pi} + \sqrt{4\pi S}p'_1, \\
\bar{\Phi}_- &\rightarrow -\bar{\Phi}_- - \frac{\sqrt{\pi S}}{2} - \sqrt{\frac{\pi}{4S}}p_1, \\
\bar{\Theta}_- &\rightarrow -\bar{\Theta}_- - \sqrt{S\pi} - \sqrt{4\pi S}p'_1, \tag{48}
\end{aligned}$$

$$\begin{aligned}
\bar{\Phi}_+ &\rightarrow \bar{\Phi}_+ + \frac{\sqrt{\pi S}}{2} + \sqrt{\frac{\pi}{4S}}p_2, \\
\bar{\Theta}_+ &\rightarrow \bar{\Theta}_+ + \sqrt{S\pi} + \sqrt{4\pi S}p'_2, \\
\bar{\Phi}_- &\rightarrow -\bar{\Phi}_- + \frac{\sqrt{\pi S}}{2} + \sqrt{\frac{\pi}{4S}}p_2, \\
\bar{\Theta}_- &\rightarrow -\bar{\Theta}_- + \sqrt{S\pi} + \sqrt{4\pi S}p'_2. \tag{49}
\end{aligned}$$

With all these identifications, one observes that the continuum limit (25) of the lattice Hamiltonian (23) is invariant under all discrete symmetries (38), (41), (43), (48), and (49) as it should be. However, the mean-field Hamiltonian (27) is invariant under Eqs. (38), (48), and (49) but breaks the $s_{1,2}$ symmetries (41) and (43). These latter Z_2 discrete symmetries are spontaneously broken in the ground state of the critical spin nematic phase and account for the formation of nonzero local spin currents polarized along the z axis circulating around the triangular plaquette of the two-leg zigzag spin ladder. The operator, that occurs in the $+$ sector of the mean-field Hamiltonian (27), with the smallest scaling dimension and consistent with the symmetries (38), (48), and (49) without breaking the continuous $U(1)$ diagonal symmetry of the model turns out to be: $\cos(\mu\sqrt{16\pi S}\bar{\Phi}_+)$, with $\mu = 1$ (respectively $\mu = 2$) if S is integer (respectively half-integer). The stable effective field theory in the $+$ channel is thus

$$\mathcal{H}_+ \simeq \frac{v}{2} \left[K(\partial_x \bar{\Theta}_+)^2 + \frac{1}{K}(\partial_x \bar{\Phi}_+)^2 \right] + \kappa \partial_x \bar{\Theta}_+ - \frac{g_{eff}}{a_0} \cos(\mu\sqrt{16\pi S}\bar{\Phi}_+), \quad (50)$$

where the value of the Luttinger parameters v, K cannot be determined within this bosonization approach. For a small value of J_1/J_2 (i.e., $K \simeq 1$), the cosine operator in Eq. (50) is a strongly irrelevant contribution and the system displays a critical phase with incommensuration generated by the $\partial_x \bar{\Theta}_+$ field. This chiral critical phase, predicted in the $S=1/2$ case in Ref. 3, is thus a generic phase in the large J_1/J_2 limit of the model (1) in the general spin- S case. In particular, it is worth stressing that, in the $S=1/2$ case, the operator $\cos(\sqrt{8\pi}\bar{\Phi}_+)$, which opens a mass gap in the $+$ channel and thus destroys the chiral critical phase found in Ref. 3, is not generated by the renormalization-group flow. Indeed, while this latter operator is permitted by the translation symmetry (38), it is odd under the s_3 and s_4 discrete symmetries (48) and (49) which forbid its presence in the low-energy effective-field theory. This result leads us to expect that the chiral critical phase does exist in the certain range of the parameter of the lattice model for $S=1/2$ in full agreement with the very recent numerical study.⁸ As J_1/J_2 is further increased, it is natural to expect that the effective theory (50) describes a phase transition of KT type from the chiral gapless phase at $g_{eff}=0$ to a chiral gapped phase. Indeed, there will be a critical value $(J_1/J_2)_c$ [the Luttinger parameter at the transition being equal to $K_c = 1/(2S\mu^2)$], which cannot be obtained within this bosonization approach, above which the cosine operator $\cos(\beta\bar{\Phi}_+)$ becomes relevant and a mass gap opens in the $+$ sector (KT transition) *without* killing the incommensuration stemming from the $\partial_x \bar{\Theta}_+$ operator. In this respect, this mechanism of generation of incommensuration is different from the usual commensurate-incommensurate scenario³³ since, in this latter case, there is a *competition* between the uniform spin density $\partial_x \bar{\Phi}_+$ field and the cosine

operator $\cos(\beta\bar{\Phi}_+)$ leading to a threshold above which the incommensuration settles in the system. One should note that the existence of this incommensurate gapful phase when the cosine operator in Eq. (50) becomes relevant can also be seen using a Luther-Emery or Toulouse limit of the Hamiltonian (50) as it has been used to explain the origin of the incommensuration found in the phase diagram of the quantum axial next-nearest-neighbor Ising chain.³⁴

The full characterization of the intermediate phase (chiral gapped phase) depends on whether S is integer or half-integer. Indeed, for $J_1/J_2 > (J_1/J_2)_c$, the bosonic field $\bar{\Phi}_+$ of Eq. (50) is locked in one of the minima of the potential $-g_{eff} \cos(\mu\sqrt{16\pi S}\bar{\Phi}_+)$ which for $g_{eff} > 0$ Ref. 35) are located at: $\langle \bar{\Phi}_+ \rangle = p \sqrt{\pi/4S}/\mu$, p being an integer. Moreover, the value of the compactification radius of the bosonic field $\bar{\Phi}_+$ is equal to: $\bar{R}_S = 1/\sqrt{16\pi S}$. This follows from the redefinition (26) and the fact that the compactification radius of the bosonic field that accounts for the critical properties of the spin- S XY chain is $R_S = 1/\sqrt{8\pi S}$ as it has been found in Sec. II. From the precise knowledge of \bar{R}_S , one deduces the following identification:

$$\bar{\Phi}_+ \sim \bar{\Phi}_+ + 2\pi\bar{R}_S = \bar{\Phi}_+ + \sqrt{\frac{\pi}{4S}}. \quad (51)$$

From this equivalence and the position of the minima corresponding to the pinning of the bosonic field $\bar{\Phi}_+$, we thus conclude that in the integer spin case ($\mu=1$) the ground state of the massive phase is nondegenerate whereas for half-integer spins ($\mu=2$) there is a twofold degenerate ground state.³⁶ Therefore the chiral gapful phase corresponds to a massive phase with a coexistence of incommensuration and a Haldane phase (respectively dimerized phase) in the integer (respectively half-integer) spin case. From the identification of the massive phase found at large J_1/J_2 in the weak-coupling analysis (see Sec. II), we then expect an Ising (Z_2) transition between the chiral gapped phase and the Haldane or dimerized phases as J_1/J_2 is further increased. At this Ising critical point, the total spin current $\langle \partial_x \bar{\Theta}_+ \rangle$ vanishes, i.e., the disappearance of the incommensurate behavior and the system enters a commensurate massive phase: Haldane or dimerized phases depending on the spin. At this point, one has to mention that the existence of this intermediate incommensurate massive phase, within our mean-field approach, relies on the decoupling of the degrees of freedom in the two channels $+$ and $-$ as in Eq. (27). We cannot rule out a different scenario that might occur in the system nonperturbatively due to the effect of the interactions in the two sectors: a single phase transition between the chiral critical phase and the Haldane or dimerized phases. At this critical point, one has *simultaneously* the appearance of a mass gap in the spectrum as well as the cancelation of the spin current so that the chiral gapful phase shrinks to zero in this case.

IV. CONCLUDING REMARKS

In the present work, we have investigated the low-energy physics of the one-dimensional spin- S J_1 - J_2 XY model within the bosonization approach. Around the two limits ($J_2/J_1 \ll 1$, $J_1/J_2 \ll 1$) where a field theoretical analysis can be performed, we have described the nature of the different phases that occurs as well as the determination of the effective-field theories of the resulting phase transitions. The critical $XY1$ spin fluid phase at $J_2=0$ is generically stable upon switching on a nonzero value of the next-nearest-neighbor interaction except for the very special $S=1$ case where the Haldane phase is immediately stabilized in full agreement with the DMRG study of Refs. 5 and 6. As the exchange interaction J_2 is further varied, the model exhibits a KT phase transition described by a standard sine-Gordon model between the $XY1$ spin fluid phase and a fully massive dimerized or Haldane phases depending on the value of the spin. In the zigzag ladder limit ($J_1/J_2 \ll 1$), we have shown that, whatever the value of the spin, the chiral critical phase, first predicted in the $S=1/2$ case by Nersesyan *et al.*,³ should exist in a certain range of the parameters of the model. This interesting spin nematic phase preserves the $U(1)$ and time-reversal symmetries but spontaneously breaks a Z_2 symmetry ($P_L^{(1)} \otimes P_S^{(2)}$) resulting on the formation of nonzero local spin currents in the ground state polarized along the anisotropy z axis. Furthermore, the transverse spin-spin correlation functions are incommensurate with a wave vector $q_S - \pi/a_0 \sim (J_1/J_2)^{4S/(4S-1)}$ and decay algebraically with the distance with an exponent $1/(8S)$ obtained within the mean-field approach used here. As the interchain J_1/J_2 is further increased, one expects the existence of a KT phase transition between the chiral critical spin nematic phase and an incommensurate gapful phase (chiral gapped phase). In particular, the effective-field theory corresponding to this transition has been determined in this work. The nature of this chiral gapped phase corresponds to a coexistence of incommensuration stemming from the presence of nonzero spin currents in the ground state and a Haldane or dimerized phases depending on whether the spin S is an integer or half-integer. We then expect an Ising phase transition associated to the disappearance of the spin current between the chiral gapped phase and the standard Haldane or dimerized phases. The phase diagram found in this work is consistent with the predictions of the large- S study of Kolezhuk¹¹ except for the special $S=1$ case where the $XY1$ spin fluid phase shrinks to zero. It will be very interesting if some extended DMRG studies can be performed in the $S>1$ case to further shed light on the physical properties of the model as well as the possibility to extract the Luttinger parameters of the effective-field theory (50). The different phase transitions reported in this work could also be investigated by means of a level spectroscopy analysis as in the one-dimensional spin- S XXZ Heisenberg model.²⁴

Note added. When this work was completed, we became aware of a very recent work by Hikihara *et al.*³⁹ who have investigated the $S=1/2, 3/2, 2$ J_1 - J_2 XY chain using a DMRG analysis. They have found that the chiral critical phase appears in a broad region of the phase diagram in the

general spin- S case in agreement with our work. Furthermore, the prediction on the decay of the spin-spin correlation [$1/(8S)$] in the chiral critical phase found within the bosonization approach has been numerically verified. Finally, for integer spins ($S=1, 2$), the authors of Ref. 39 have reported the existence of the chiral gapped phase in a very narrow region of the phase diagram whereas in the half-integer case ($S=1/2, 3/2$) it has not been identified within the numerical precision of the work.³⁹

ACKNOWLEDGMENTS

The authors would like to thank D. Allen, E. Boulat, A. K. Kolezhuk, and A. A. Nersesyan for valuable discussions related to this work.

APPENDIX A: XY CHAIN IN THE CONTINUUM LIMIT

In this appendix, we shall recall some well-known facts on the continuum limit of the XY chain to fix the notations that will be used throughout this paper. The Hamiltonian of the antiferromagnetic spin- $1/2$ XY chain is ($J_1>0$)

$$\mathcal{H}_0 = J_1 \sum_n (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y), \quad (\text{A1})$$

where \vec{S}_n is a spin- $1/2$ operator at site n . As is well known, this model can be written in terms of lattice fermions c_n using the Jordan-Wigner transformation:

$$S_n^z = c_n^\dagger c_n - \frac{1}{2},$$

$$S_n^+ = (-1)^n c_n^\dagger \exp\left(i\pi \sum_{j=1}^{n-1} c_j^\dagger c_j\right). \quad (\text{A2})$$

The continuum limit of the model (A1) can then be performed with the introduction of right- and left-moving fermion fields R, L : $c_n/\sqrt{a_0} \rightarrow R(x)(i)^{x/a_0} + L(x)(-i)^{x/a_0}$, $x = na_0$, a_0 being the lattice spacing. Using the fermion-boson correspondence (see, for instance, Refs. 37 and 4):

$$R = \frac{1}{\sqrt{2\pi a_0}} \exp(i\sqrt{4\pi}\Phi_R),$$

$$L = \frac{1}{\sqrt{2\pi a_0}} \exp(-i\sqrt{4\pi}\Phi_L), \quad (\text{A3})$$

the Hamiltonian (A1) can be expressed in terms of a bosonic field Φ and its dual field Θ in the continuum limit:

$$\mathcal{H}_0 = \frac{v_0}{2} \int dx [(\partial_x \Theta)^2 + (\partial_x \Phi)^2], \quad (\text{A4})$$

where $v_0 = J_1 a_0$ is the spin velocity and we work with the following conventions:

$$\begin{aligned}\Phi &= \Phi_L + \Phi_R, \\ \Theta &= \Phi_L - \Phi_R, \\ [\Phi_R, \Phi_L] &= i/4.\end{aligned}\quad (\text{A5})$$

This latter commutation relation is necessary to insure the anticommutation between the right and left fermion operators [see Eq. (A3)]. The bosonic field is compactified with the radius $R = 1/\sqrt{4\pi}$: $\Phi \sim \Phi + \sqrt{\pi}$ whereas the dual field is compactified with the radius $\tilde{R} = 1/(2\pi R)$: $\Theta \sim \Theta + 2\sqrt{\pi}$. The spin density operator in the continuum limit decomposes into uniform and alternating parts:

$$\vec{S} \simeq \vec{J} + (-1)^{x/a_0} \vec{n}, \quad (\text{A6})$$

which can also be expressed in terms of the bosonic fields as follows:

$$n^z = \frac{-1}{\pi a_0} \sin(\sqrt{4\pi}\Phi), \quad (\text{A7})$$

$$n^\dagger = \frac{1}{\sqrt{2\pi a_0}} \exp(i\sqrt{\pi}\Theta),$$

$$J^z = \frac{1}{\sqrt{\pi}} \partial_x \Phi,$$

$$J^\dagger = \frac{1}{\sqrt{8\pi a_0}} (\exp[i3\sqrt{\pi}\Phi_L + i\sqrt{\pi}\Phi_R] + \exp[-i3\sqrt{\pi}\Phi_R - i\sqrt{\pi}\Phi_L]),$$

$$= \frac{1}{\sqrt{2\pi a_0}} \exp(i\sqrt{\pi}\Theta) \sin(\sqrt{4\pi}\Phi).$$

We end this appendix by giving the bosonic representation of the discrete symmetries of the XY Hamiltonian (A1) that will be very useful when investigating the stability of the chiral critical phase in Sec. III B. Under a one-step translation symmetry, the bosonic fields transform according to:

$$\begin{aligned}\Phi &\rightarrow \Phi + \frac{\sqrt{\pi}}{2} + p\sqrt{\pi}, \\ \Theta &\rightarrow \Theta + \sqrt{\pi} + p'\sqrt{4\pi},\end{aligned}\quad (\text{A8})$$

p, p' being integers since from Eq. (A6) the alternating part (\vec{n}) of the spin density should be odd under the one-step translation symmetry. Under the site parity $P_s(\vec{S}_n \rightarrow \vec{S}_{-n})$, the uniform and staggered parts of the spin density should be even so that

$$\begin{aligned}\Phi(x) &\rightarrow -\Phi(-x) + \frac{\sqrt{\pi}}{2} + q\sqrt{\pi}, \\ \Theta(x) &\rightarrow \Theta(-x) + q'\sqrt{4\pi},\end{aligned}\quad (\text{A9})$$

where q, q' are integers. The link parity $P_L(n \rightarrow 1-n)$ is a combination of a site parity and a translation symmetry so that under P_L the bosonic fields Φ and Θ transform as

$$\begin{aligned}\Phi(x) &\rightarrow -\Phi(-x) + n\sqrt{\pi}, \\ \Theta(x) &\rightarrow \Theta(-x) + \sqrt{\pi} + n'\sqrt{4\pi},\end{aligned}\quad (\text{A10})$$

n, n' being integers.

APPENDIX B: ONE-DIMENSIONAL $S=1/2$ J_1 - J_2 XY MODEL

In this appendix, we derive the continuum limit of the model (1) in the $S=1/2$ case in the weak-coupling limit $J_2 \ll J_1$. This calculation has been done several times^{2,1,38} with different bosonized expressions. This discrepancy stems from the fact that one has to be extremely careful when deriving the continuum limit and in particular for obtaining the correct velocity renormalization. We shall redo here this calculation for completeness and also since it will be needed in Sec. II when deriving the bosonization approach of the J_1 - J_2 spin- S XY chain in the $J_2 \ll J_1$ limit.

The first step of the computation is to express the interacting part of Hamiltonian (1) in terms of the lattice fermions using the Jordan-Wigner transformation (A2):

$$\mathcal{H}_{int} = -J_2 \sum_n [c_{n+2}^\dagger (c_{n+1}^\dagger c_{n+1} - 1/2) c_n + \text{H.c.}]. \quad (\text{B1})$$

Using the continuum limit of the fermions and the bosonization correspondence (A3) described in Appendix A, one has

$$\begin{aligned}\mathcal{H}_{int} &= \frac{J_2 a_0}{2\pi} \int dx [(-i)^{x/a_0} : e^{-i\sqrt{4\pi}\Phi_R} : (x+2a_0) \\ &\quad + (i)^{x/a_0} : e^{i\sqrt{4\pi}\Phi_L} : (x+2a_0)] \left(\frac{1}{\sqrt{\pi}} \partial_x \Phi(x+a_0) \right. \\ &\quad \left. + \frac{(-1)^{x/a_0}}{\pi} : \sin(\sqrt{4\pi}\Phi) : (x+a_0) \right) \\ &\quad \times [(i)^{x/a_0} : e^{i\sqrt{4\pi}\Phi_R} : (x) + (-i)^{x/a_0} : e^{-i\sqrt{4\pi}\Phi_L} : (x)] \\ &\quad + \text{H. c.}\end{aligned}\quad (\text{B2})$$

To derive the continuum expression of this Hamiltonian, we need the following operator product expansions in a standard Gaussian $c=1$ theory:

$$\begin{aligned}: e^{-i\sqrt{4\pi}\Phi_R} : (\bar{z}) \partial_x \Phi(w, \bar{w}) &\sim : \partial_x \Phi_L e^{-i\sqrt{4\pi}\Phi_R} : (w, \bar{w}) \\ &\quad - \frac{1}{\sqrt{4\pi}(\bar{z}-\bar{w})} : e^{-i\sqrt{4\pi}\Phi_R} : (\bar{w}) \\ &\quad - \frac{1}{\sqrt{4\pi}} : \bar{\partial} e^{-i\sqrt{4\pi}\Phi_R} : (\bar{w}),\end{aligned}$$

$$\begin{aligned}
& :e^{i\sqrt{4\pi}\Phi_L}(z)\partial_x\Phi(w,\bar{w})\sim:\partial_x\Phi_R e^{i\sqrt{4\pi}\Phi_L}(w,\bar{w}) \\
& -\frac{1}{\sqrt{4\pi}(z-w)}:e^{i\sqrt{4\pi}\Phi_L}(w) \\
& -\frac{1}{\sqrt{4\pi}}:\partial e^{i\sqrt{4\pi}\Phi_L}(w),
\end{aligned}$$

$$\begin{aligned}
& :e^{-i\sqrt{4\pi}\Phi_R}(\bar{z})\sin(\sqrt{4\pi}\Phi)(w,\bar{w}) \\
& \sim-\frac{\bar{z}-\bar{w}}{2}:e^{-i\sqrt{16\pi}\Phi_R}(\bar{w})e^{-i\sqrt{4\pi}\Phi_L}(w): \\
& +\frac{1}{2(\bar{z}-\bar{w})}:[1-i\sqrt{4\pi}(\bar{z}-\bar{w})\bar{\partial}\Phi_R \\
& -2\pi(\bar{z}-\bar{w})^2(\bar{\partial}\Phi_R)^2]e^{i\sqrt{4\pi}\Phi_L}(w,\bar{w}),
\end{aligned}$$

$$\begin{aligned}
& :e^{i\sqrt{4\pi}\Phi_L}(z):\sin(\sqrt{4\pi}\Phi)(w,\bar{w}) \\
& \sim-\frac{z-w}{2}:e^{i\sqrt{16\pi}\Phi_L}(w)e^{i\sqrt{4\pi}\Phi_R}(\bar{w}): \\
& +\frac{1}{2(z-w)}:[1+i\sqrt{4\pi}(z-w)\partial\Phi_L \\
& -2\pi(z-w)^2(\partial\Phi_L)^2]e^{-i\sqrt{4\pi}\Phi_R}(w,\bar{w}) \quad (\text{B3})
\end{aligned}$$

with the convention $w=v_0\tau+ix$ and $\partial_x=i(\partial-\bar{\partial})$. Using these results and keeping only nonoscillatory contributions in Eq. (B2), we finally obtain

$$\mathcal{H}_{int}\simeq-\frac{J_2}{\pi^2 a_0}\int dx\cos(\sqrt{16\pi}\Phi)-\frac{4J_2 a_0}{\pi}\int dx(\partial_x\Theta)^2 \quad (\text{B4})$$

which is in perfect agreement with the earlier derivation made by Haldane (see the erratum)² and is in contradiction with some recent ones in the literature.^{1,38}

- ¹K. Nomura and K. Okamoto, J. Phys. Soc. Jpn. **62**, 1123 (1993); S. Hirata and K. Nomura, Phys. Rev. B **61**, 9453 (2000).
²F. D. M. Haldane, Phys. Rev. B **25**, 4925 (1982); **26**, 5257(E) (1982).
³A. A. Nersesyan, A. O. Gogolin, and F. H. L. Essler, Phys. Rev. Lett. **81**, 910 (1998).
⁴A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, England, 1998).
⁵M. Kaburagi, H. Kawamura, and T. Hikihara, J. Phys. Soc. Jpn. **68**, 3185 (1999).
⁶T. Hikihara, M. Kaburagi, H. Kawamura, and T. Tonegawa, J. Phys. Soc. Jpn. **69**, 259 (2000).
⁷A. A. Aligia, C. D. Batista, and F. H. L. Essler, Phys. Rev. B **62**, 3259 (2000).
⁸Y. Nishiyama, Eur. Phys. J. B **17**, 295 (2000).
⁹H. Schulz, Phys. Rev. B **34**, 6372 (1986); H. J. Schulz, G. Cuniberti, and P. Pieri, cond-mat/9807366 (unpublished).
¹⁰H. Shimaoka and K. Kuboki, J. Phys. Soc. Jpn. **62**, 3257 (1993).
¹¹A. K. Kolezhuk, Phys. Rev. B **62**, R6057 (2000).
¹²A. Luther and D. J. Scalapino, Phys. Rev. B **16**, 1153 (1977).
¹³M. P. M. den Nijs, Physica A **111**, 273 (1982).
¹⁴J. Sólyom and J. Timonen, Phys. Rev. B **34**, 487 (1986); **38**, 6832 (1988); **40**, 7150 (1989).
¹⁵J. Timonen and A. Luther, J. Phys. C **18**, 1439 (1985).
¹⁶V. Brunel and T. Jolicoeur, Phys. Rev. B **58**, 8481 (1998).
¹⁷F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983); Phys. Lett. **93A**, 464 (1983).
¹⁸R. Assaraf, P. Azaria, M. Caffarel, and P. Lecheminant, Phys. Rev. B **60**, 2299 (1999).
¹⁹F. C. Alcaraz and A. Moreo, Phys. Rev. B **46**, 2896 (1992).
²⁰A. Kitazawa, K. Nomura, and K. Okamoto, Phys. Rev. Lett. **76**, 4038 (1996).
²¹When the cosine term $\cos(\mu\sqrt{8\pi S}\Phi_+)$ in Eq. (17) becomes relevant, the bosonic field Φ_+ becomes pinned in one of the

minima of the sine-Gordon model at $\beta^2=\mu\sqrt{8\pi S}$. The effective coupling constant g_{eff} that enters in Eq. (17) cannot be fixed within our approach. In some situations, the sign of this coupling constant is crucial to fully determine the nature of the discrete symmetry breaking that may occur in a massive phase but here it will not be important. The position of the minima of the interacting part of Eq. (17) for $g_{eff}>0$ reads as follows: $\langle\Phi_+\rangle=p\sqrt{\pi/(2S)}/\mu$, p being an integer. However, the bosonic field Φ_+ has a compactification radius equal to $R_S=1/\sqrt{8\pi S}$ so that one has the following equivalence: $\Phi_+\sim\Phi_++\sqrt{\pi/(2S)}$. From the position of the minima where the Φ_+ field is locked, we thus deduce that in the integer spin case ($\mu=1$) the ground state in the massive phase is nondegenerate whereas in the half-integer case ($\mu=2$) there is a twofold degenerate ground state as it should be respectively for the Haldane and dimerized phases. In the latter case, there is a Z_2 symmetry breaking corresponding to the symmetry breaking of one-site lattice translation. Finally, if one has $g_{eff}<0$, it is easy to see that the same physical characterization of the massive phase can be obtained.

- ²²M. den Nijs and K. Rommelse, Phys. Rev. B **40**, 4709 (1989).
²³U. Schollwöck and Th. Jolicoeur, Europhys. Lett. **30**, 493 (1995).
²⁴K. Nomura and A. Kitazawa, J. Phys. A **31**, 7341 (1998).
²⁵S. R. White and I. Affleck, Phys. Rev. B **54**, 9862 (1996).
²⁶D. Allen and D. Sénéchal, Phys. Rev. B **55**, 299 (1997).
²⁷D. C. Cabra, A. Honecker, and P. Pujol, Eur. Phys. J. B **13**, 55 (2000).
²⁸C. Itoi and S. Qin, cond-mat/0006155, Phys. Rev. B (to be published).
²⁹D. Allen and D. Sénéchal, Phys. Rev. B **61**, 12 134 (2000).
³⁰S. Lukyanov and A. Zamolodchikov, Nucl. Phys. B **493**, 571 (1997).
³¹A. F. Andreev and I. A. Grishchuk, Zh. Éksp. Teor. Fiz. **87**, 467 (1984) [Sov. Phys. JETP **60**, 267 (1984)].
³²X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11 413 (1989).
³³G. I. Japaridze and A. A. Nersesyan, Pis'ma Zh. Éksp. Teor. Fiz.

- 27**, 356 (1978) [JETP Lett. **27**, 334 (1978)]; V. L. Pokrovsky and A. L. Talapov, Phys. Rev. Lett. **42**, 65 (1979).
- ³⁴D. Allen, P. Azaria, and P. Lecheminant, cond-mat/0101404 (unpublished).
- ³⁵The sign of this coupling constant cannot be fixed within our approach since the cosine term that occurs in Eq. (50) was obtained through a symmetry analysis. However, the precise sign of g_{eff} is irrelevant for the determination of the nature of the incommensurate gapful phase.
- ³⁶Here we do not take into account the additional twofold degeneracy of the ground state that stems from the spontaneous breaking of a Z_2 discrete symmetry (39) of the lattice model associated to the formation of nonzero spin currents in the system.
- ³⁷I. Affleck, in *Fields, Strings and Critical Phenomena*, edited by E. Brézin and J. Zinn-Justin (Elsevier, Amsterdam, 1989), p. 564.
- ³⁸J. Zang, A. R. Bishop, and D. Schmeltzer, Phys. Rev. B **52**, 6723 (1995).
- ³⁹T. Hikihara, M. Kaburagi, and H. Kawamura, cond-mat/0007095 (unpublished); cond-mat/0010283, Phys. Rev. B (to be published).