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Inhomogeneous superconducting state in quasi-one-dimensional systems

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We report on results of theoretical study of nonuniform superconducting states in quasi-one-dimensional systems, with attractive interactions and Zeeman splitting between electron spins. Using bosonization to treat intrachain electron-electron interactions, and a combination of renormalization group and mean-field approximation to tackle interchain couplings, we obtain the phase diagram of the system, and show that the transition between the uniform and nonuniform superconducting phases is a continuous transition of the commensurate-incommensurate type.

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The possibility of a superconducting state with inhomogeneous order parameter, stabilized by a sufficiently large Zeeman splitting between electrons with opposite spin orientations due to either an external magnetic or internal exchange field, was suggested more than thirty years ago by Fulde and Ferrell[1] and Larkin and Ovchinnikov.[2] Since then this Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state has been the subject of a number of theoretical studies, but no direct evidence of its existence has ever been found in conventional superconductors. More recently it has attracted renewed interest in the context of organic, heavy-fermion, and high-T_c cuprate superconductors,[3–22] as these new classes of superconductors are believed to provide conditions that are favorable to the formation of the FFLO state due to their quasi-one- or quasi-two-dimensionality as well as unconventional pairing symmetry. Indeed, some experimental evidence of its existence has been reported.[3,10,12,21]

The following picture emerged from the early theoretical studies (mostly of mean-field type) of conventional *s*-wave superconductors subject to a Zeeman field B. For a sufficiently high field, the system is in the normal state. As the field strength decreases, at low temperatures the system undergoes a second-order phase transition at $B=B_{c2}(T)$ into the FFLO superconducting state. As the field strength further decreases, another phase boundary is encountered at $B_{c1}(T)$, and the system goes through another phase transition into the usual BCS superconducting state with uniform superconducting order parameter. While it is much more difficult to locate the position of $B_{c1}(T)$ than $B_{c2}(T)$ (even in the mean-field theory), as well as to address the nature of the transition there, it has been widely assumed[23] that this is a first-order-phase boundary, across which the momentum of the order parameter and the magnetization change discontinuously. This viewpoint was disputed in Ref. 6, in which the authors argue that the transition at $B_{c1}(T)$ is of second order. Thus the nature of this transition is an unsettled issue.

In this paper we study quasi-one-dimensional (Q1D) superconductors subject to a Zeeman field, and the possibility of the formation of FFLO states in these systems. Our motivation comes from two considerations. First of all, some of the experimental candidates for FFLO state are made of weakly coupled chains and are therefore Q1D. Secondly, it is known that the fluctuations are much stronger in low-dimensional systems than in three-dimensional (3D) systems, and mean-field theories are much less reliable there.[24] On the other hand, the nonperturbative machinery developed for studying one-dimensional interacting electron systems (especially bosonization[25]) allows us to go beyond the mean-field theory and treat the intrachain electron-electron correlation exactly in Q1D systems. In this paper we will take an approach that is similar to the one used in Ref. 26, namely to treat the intrachain electron-electron interaction exactly using bosonization, and tackle the interchain couplings using a combination of renormalization group (RG) analysis and mean-field approximation. Using this approach we are able to make a number of quantitative and reliable predictions about the FFLO state in these systems. In particular, we will show that the phase transition at B_{c1} is *continuous* in these systems, and work out its critical properties. For the sake of simplicity and concreteness, we restrict our discussion to zero temperature throughout the paper.

We start by considering a one-dimensional electron gas with attractive interactions. In the bosonized form, the Hamiltonian reads

$$H=H_c+H_s+H_z, \tag{1}$$

where H_c and H_s are the Hamiltonian for the charge and spin sectors (which are decoupled, signaling the spin-charge separation):[26]

$$H_\alpha=\int dx\left\{\frac{v_\alpha}{2}\left[K_\alpha(\partial_x\theta_\alpha)^2+\frac{(\partial_x\phi_\alpha)^2}{K_\alpha}\right]+V_\alpha\cos(\sqrt{8\pi}\phi_\alpha)\right\}, \tag{2}$$

where $\alpha=c$ or s, and H_z is the Zeeman coupling:

$$H_z=g\mu_B B S_z^{tot}=\sqrt{\frac{1}{2\pi}}g\mu_B B\int dx\,\partial_x\phi_s(x). \tag{3}$$

In these equations ϕ_c and ϕ_s are bosonic charge and spin fields which are related to the (coarse-grained) charge and spin densities:

$$\rho(x)=\sqrt{\frac{2}{\pi}}\partial_x\phi_c(x),\quad S_z(x)=\sqrt{\frac{1}{2\pi}}\partial_x\phi_s(x); \tag{4}$$

while θ_α are their dual fields satisfying

$$[\phi_\alpha(x),\partial_{x'}\theta_\alpha(x')]=i\delta(x-x'). \tag{5}$$

For attractive interactions, we typically have the Luttinger liquid parameters $K_c > 1$ and $K_s < 1$ (for noninteracting electrons, we have $K_c = K_s = 1$). If the 1D electron gas is sufficiently far away from lattice commensuration, which we assume to be the case here, V_c (which measures the strength of $4k_f$ Umklapp scattering) may be set to zero. Thus H_c takes the form of free massless bosons. On the other hand, in the spin sector H_s has the form of 1+1D quantum sine-Gordon model, and for $K_s < 1$, V_s (which measures the strength of back scattering between electrons with opposite spins) is relevant in the RG sense; at low energies it opens up a gap $\Delta_s \sim v_s \Lambda [V_s / v_s \Lambda^2]^{1/(2-2K_s)}$ (Λ is the ultraviolet cutoff) for spin excitations.[26] The elementary spin excitations are massive solitons (kinks and antikinks) of the ϕ_s field, which carry spin $\pm 1/2$.[27] This spin gap Δ_s is the analog of quasiparticle gap in the BCS theory of higher dimensional superconductors. The fundamental difference here, however, is that in 1D there is *no* long-range superconducting order; instead the correlation function of the Cooper pair operator decays with a power law. The power-law exponent can be calculated using the explicit representation of electron operators in terms of boson fields:

$$\psi_{\lambda,\sigma} = N_\sigma \exp[i\lambda k_f x - i\Phi_{\lambda,\sigma}(x)], \tag{6}$$

where $\lambda = \pm 1$ represents left/right movers, $\sigma = \pm 1$ represents up/down spin particles, N_σ is the Klein factor that also includes a normalization constant, and

$$\Phi_{\lambda,\sigma} = \sqrt{\pi/2}[(\theta_c - \lambda\phi_c) + \sigma(\theta_s - \lambda\phi_s)]. \tag{7}$$

Thus the singlet pair correlation function (at $T=0$)

$$\begin{aligned}\langle \psi^\dagger_{+1+1}(x)\psi^\dagger_{-1-1}(x)\psi_{-1-1}(x')\psi_{+1+1}(x')\rangle \\ \propto \langle \exp[i\sqrt{2\pi}[\theta_c(x) - \theta_c(x')]]\rangle \\ \times \langle \exp[i\sqrt{2\pi}[\phi_s(x) - \phi_s(x')]]\rangle \\ \propto |x - x'|^{-2\xi_{sc}},\end{aligned} \tag{8}$$

where the scaling dimension

$$\xi_{sc} = \frac{1}{2K_c}. \tag{9}$$

Here we have used the fact that the spin field $\phi_s(x)$ is long-range ordered in the spin-gapped phase.

Let us now consider the effect of H_Z. In H_Z the Zeeman field couples to the soliton density and plays the role of the chemical potential of spin solitons. In fact, $H_s + H_Z$ takes exactly the form of the Pokrovsky-Talapov model[28,29] which was introduced to study the two-dimensional classical commensurate-incommensurate (CIC) transition. In our context, we thus expect a *continuous* CIC transition at

$$B = B_{c1} = 2\Delta_s / g\mu_B, \tag{10}$$

beyond which spin solitons start to proliferate in the ground state. Equation (10) is an *exact* result because the Zeeman field couples to S_z^{tot} which is a conserved quantity.[30] In the incommensurate phase, the spin solitons form a spinless Luttinger liquid with its own bosonized Hamiltonian, which describes the low-energy spin excitations of the system:

$$H_{sol} = \int dx \frac{v_{sol}}{2}[K_{sol}(\partial_x\theta_{sol})^2 + (\partial_x\phi_{sol})^2/K_{sol}]. \tag{11}$$

In the long-wave length limit, the soliton density field $\phi_{sol}(x)$ is related to the spin field ϕ_s through

$$\phi_s(x) = \phi_{sol}(x)/\sqrt{2} + \sqrt{\pi/2}n_{sol}(B)x + \text{const}, \tag{12}$$

where n_{sol} is the soliton density of the ground state. In the limit $B \to B_c + 0^+$, the solitons become extremely dilute and the repulsive interaction among them becomes irrelevant; they can be treated as spinless free fermions.[29] As a consequence of this we have (i) $K_{sol} = 1$ and (ii) $n_{sol}(B) \propto (B - B_c)^{1/2}$ in this limit. Using these results we find in the incommensurate phase the superconducting correlation function

$$\begin{aligned}\langle \psi^\dagger_{+1+1}(x)\psi^\dagger_{-1-1}(x)\psi_{-1-1}(x')\psi_{+1+1}(x')\rangle \\ \propto \exp[iQ(B)(x - x')]|x - x'|^{-2\xi'_{sc}},\end{aligned} \tag{13}$$

where $Q(B) = \pi n_{sol}(B)$; approaching the phase boundary: $B \to B_c + 0^+$, we have $Q(B) \propto (B - B_c)^{1/2}$ and

$$\xi'_{sc} = \frac{1}{2K_c} + \frac{1}{4} = \xi_{sc} + \frac{1}{4}. \tag{14}$$

This incommensurate phase (in the spin sector) is the 1D analog of the FFLO phase in higher dimensional systems as the appearance of the spin solitons in the ground state induces an oscillatory phase in the superconducting correlation function, Eq. (13). Also the ground state now has a finite magnetization as in the FFLO phase, and the additional fluctuation due to the soliton liquid makes the superconducting correlation function decay faster in the incommensurate phase in an (loose) analogy to the fact that the appearance of unpaired quasiparticles reduces the size of the superconducting order parameter in the FFLO phase. We emphasize again here that there is *no* long-range superconducting order in either the commensurate or incommensurate phases; also the CIC transition is continuous as n_{sol} increases continuously from zero as B crosses B_{c1}; both the magnetization and wave vector of oscillation Q are proportional to n_{sol}.

We now turn to the discussion of interchain couplings. The three leading potentially relevant perturbations to the decoupled Luttinger liquid (dLL) fixed point are single electron hopping H_e, Cooper pair hopping (or Josephson tunneling) H_J, and interchain $2k_f$ back scatterings $H_{C/SDW}$.[31] For attractive interactions ($K_c > 1$), $H_{C/SDW}$ is less relevant than H_J in both the commensurate and incommensurate phases. H_e is irrelevant in the commensurate phase due to the presence of a spin gap. Since in this case the scaling dimension for H_J is

$$\xi_J = 2\xi_{sc} = 1/K_c < 2, \tag{15}$$

we conclude that Cooper pair hopping is the leading relevant perturbation at the dLL fixed point, and the system flows

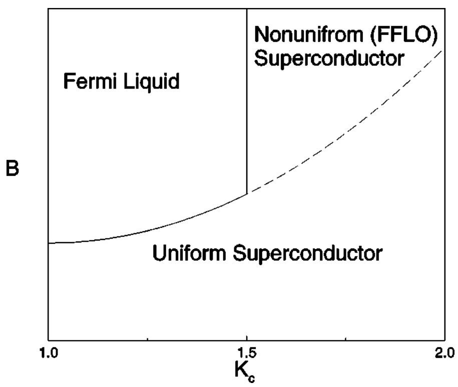

FIG. 1. Schematic phase diagram of coupled Luttinger liquids subject to a Zeeman field. The solid lines are first-order-phase boundaries while the dashed line is a second-order-phase boundary.

toward a superconducting phase with long-range superconducting order once interchain coupling is turned on in the commensurate phase.

Now let us consider the incommensurate phase. Right after the system enters the incommensurate phase ($B \to B_{c1} + 0^+$), we have

$$\xi_J' = 2\xi_{sc}' = 1/K_c + 1/2 < 2, \tag{16}$$

thus H_J is still relevant, albeit having a higher scaling dimension than that in the commensurate phase. However, in this case H_e may also be relevant, as there is no longer a spin gap in this case. We find in this case the scaling dimension of H_e to be

$$\xi_e' = \frac{1}{4}(K_c + 1/K_c) + \frac{5}{8}. \tag{17}$$

We thus find that for $K_c > 3/2$, we have $\xi_J' < \xi_e'$, and H_J is the leading relevant perturbation at the dLL fixed point which drives the system to the Q1D superconducting FFLO phase once interchain coupling is turned on. On the other hand, for $1 < K_c < 3/2$, we have $\xi_e' < \xi_J' < 2$, and H_e is the leading relevant perturbation at the dLL fixed point; in this case the system flows toward the high-dimensional Fermi-liquid fixed point.[32] These results are summarized in a schematic phase diagram (Fig. 1). The phase boundary separating the Fermi liquid and the two superconducting phases are likely to be first order since they are determined by the crossing of the scaling dimensions of two different relevant operators at the dLL fixed point; on the other hand, as we will argue below, the transition from uniform to FFLO superconducting phases is continuous. We emphasize in this phase diagram we assume the Zeeman field B is not too strong; if the Zeeman splitting is so strong as to be comparable to, say the Fermi energy, the continuum Luttinger liquid description of Q1D systems breaks down.

To address the nature of the transition between the uniform and FFLO superconducting phases, we focus on the pair-hopping process and neglect other perturbations that are less relevant:

$$\begin{aligned} H_J &= -\tilde{t}_J \sum_{\langle ij\rangle} \int dx [\bar{\psi}^i_{+1+1}\bar{\psi}^i_{-1-1}\psi^j_{-1-1}\psi^j_{+1+1} \\ &\quad + \bar{\psi}^i_{+1-1}\bar{\psi}^i_{-1+1}\psi^j_{-1+1}\psi^j_{+1-1} + \text{H.c.}] \\ &= -t_J \sum_{\langle ij\rangle} \int dx \cos[\sqrt{2\pi}(\theta_c^i - \theta_c^j)]\cos[\sqrt{2\pi}(\phi_s^i - \phi_s^j)], \end{aligned} \tag{18}$$

where i and j are chain indices, $\tilde{t}_J$ is the pair hopping matrix element (or Josephson coupling strength), $\langle ij\rangle$ stands for neighboring chains, $t_J \propto \tilde{t}_J$, and H.c. stands for Hermitian conjugate. In the case of decoupled Luttinger liquids, there is spin-charge separation and the CIC transition occurs in the spin sector. As we see in Eq. (18), interchain pair hopping couples the spin and charge fields. On the other hand, since the system is in the superconducting phase (uniform or nonuniform) in which the charge field θ_c is long-range ordered, in studying the transition driven by B we may use a mean-field approximation and replace $\cos[\sqrt{2\pi}(\theta_c^i - \theta_c^j)]$ in Eq. (18) by its expectation value: $\langle \cos[\sqrt{2\pi}(\theta_c^i - \theta_c^j)]\rangle = C$. Clearly this expectation value depends on B and it will also develop a dependence on x in the incommensurate phase; however, as long as the dependence is smooth across the transition (which would be the case if the transition is continuous as we will show to be the case), we can treat it as a constant. Thus in the mean-field approximation H_J becomes

$$H_J^{MF} = -Ct_J \sum_{\langle ij\rangle} \int dx \cos[\sqrt{2\pi}(\phi_s^i - \phi_s^j)]. \tag{19}$$

Equation (19) can also be obtained more formally by integrating out the fluctuations of the θ_c field on top of its expectation value in the Lagrangian formalism, which will yield a slightly renormalized coupling C. The quantum Hamiltonian of $H_s + H_Z + H_J^{MF}$ can be mapped onto the problem of classical CIC transition driven by B at finite temperatures, in $d+1$ dimensions (d is the physical dimension of the quantum problem we study here). It is known that the CIC transition in higher dimensions is still continuous, but the critical behavior is very different from the $d=1$ case considered earlier; in this case the density of domain walls (that consist of solitons of individual chains aligned with true long-range order) depends logarithmically on the distance from criticality:[33]

$$n_{wall} \propto 1/\log(|B - B_c|^{-1}), \tag{20}$$

as $B \to B_c + 0^+$. The wave vector of the inhomogeneous superconducting order parameter Q and the magnetization are both proportional to n_{wall} and thus have the same dependence on B near criticality. We note that while we obtained these results by making a mean-field approximation to the (long-range ordered) charge fields, the main conclusion that the transition is continuous should be robust; this follows simply from the fact that the domain walls (whose appearance drives the transition) repel each other, which is clearly the case here. The logarithmic dependence of n_{wall} on B

$-B_c$ then follows from the exponentially weak repulsion between the domain walls. These in turn justify the validity of the mean-field approximation employed.

To summarize, we studied the formation of the nonuniform superconducting state in quasi-one-dimensional systems. Among our results include a phase diagram in terms of the Zeeman field and Luttinger liquid parameter. We also showed that the transition between the uniform and nonuniform superconducting states is continuous.

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[24] As a matter of fact in the past a number of authors have studied the FFLO state in superconductors in the 1D limit using the mean-field theory and a strictly 1D electron dispersion; see, A. I. Buzdin and V. V. Tugushev, Zh. Éksp. Teor. Fiz. **85**, 735 (1983) [Sov. Phys. JETP **58**, 428 (1983)]; K. Machida and H. Nakanishi, Phys. Rev. B **30**, 122 (1984); A. I. Buzdin and S. V. Polonskii, Zh. Éksp. Teor. Fiz. **93**, 747 (1987) [Sov. Phys. JETP **66**, 422 (1987)]. In these mean-field theories one assumes a long-range ordered superconducting order parameter. However, it is known (also see later) that quantum as well as thermal fluctuations do not allow long-range superconducting order in 1D, at zero or finite temperature, and thus put the validity of mean-field theory in question in 1D.

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[32] In principle, there may be residual attractive interaction among quasiparticles that can destabilize the Fermi-liquid fixed point that the system flows toward, and trigger a superconducting instability. An analysis of such possibilities is beyond the scope of the approach used here since we can only analyze the possible instabilities of the dLL fixed point in this approach.

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