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Quantum critical effects on transition temperature of magnetically mediated *p*-wave superconductivity

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We determine the behavior of the critical temperature of magnetically mediated *p*-wave superconductivity near a ferromagnetic quantum critical point in three dimensions, distinguishing universal and nonuniversal aspects of the result. We find that the transition temperature is nonzero at the critical point, raising the possibility of superconductivity in the ferromagnetic phase.

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Recent experiments have shown that superconductivity in strongly correlated electron systems is closely associated with proximity to magnetic quantum critical points,[1–3] suggesting it is mediated by critical spin fluctuations,[4] as indicated by theoretical calculations.[5,6] However, the interplay of superconductivity and criticality is not yet understood. In this paper we study the theoretically simplest case, namely *p*-wave superconductivity near a ferromagnetic quantum critical point in dimension $d=3$. Our work is complimentary to that of Ref. 7 which studied pairing near a two-dimensional antiferromagnetic quantum critical point.

We have two motivations. One is to know whether the superconducting T_c vanishes as the magnetic critical point is approached [as shown, for example, in Fig. 1(a) and as found in Refs. 7 and 8], or whether it does not [as shown in Fig. 1(b)]. This question has not been definitively theoretically settled, because numerical difficulties have prevented a straightforward attack.[6] The latter scenario raises the interesting possibility of the coexistence of superconductivity and magnetism noted by Fay *et al.*[8], who suggested a third scenario shown in Fig. 1(c).

Our second motivation is theoretical. Studies of magnetically mediated superconductivity have almost uniformly been based on the Eliashberg equations (defined below),[9,10] which are believed[11] to give the leading contributions to the low-energy behavior of systems near critical points. We wish to know which aspects of the observed T_c are controlled by low-energy physics.

We consider a three-dimensional metal with uniaxial anisotropy near a ferromagnetic quantum critical point. The magnetic susceptibility is[12,13]

$$\boldsymbol{\chi}^{-1}(q,\nu)=\begin{pmatrix}0&0&0\\0&\Delta&0\\0&0&\Delta\end{pmatrix}+\left[\frac{|\nu|p_F}{\Lambda q}\tan^{-1}\left(\frac{\Lambda q}{|\nu|p_F}\right)+\left(\frac{q}{2p_F}\right)^2+r\right]\boldsymbol{\delta}_{ab}+\dots, \tag{1}$$

where r is a parameter that measures the distance of the system from its quantum critical point (Fig. 1); a and b are Cartesian coordinates, Δ measures deviations from Heisenberg symmetry, and the ellipsis denotes less singular terms. Here p_F is a momentum scale of the order of the Fermi momentum and $\Lambda\sim v_F p_F$ is an energy scale of the order of the Fermi energy. We assume (following Refs. 5, 6, 9, and 11) that the coupling of spin fluctuations to the electron system is given by the Eliashberg equations for the electron self-energy $\Sigma(p,i\omega)=i\omega[1-Z_p(\omega)]$ and the anomalous self-energy $W(p,i\omega)$. We find T_c by solving the linearized Eliashberg equations, which are

$$i\omega[1-Z_p(\omega)]=\frac{\lambda}{16\pi^2}\int N(\Omega_{p'})d\Omega_{p'}\int_{-\infty}^{\infty}d\epsilon_{p'}\times\pi T\sum_{i\omega'}\mathrm{Tr}\,\boldsymbol{\chi}(p-p',i\omega-i\omega')\times\frac{1}{i\omega' Z_p(\omega')-\epsilon_p'}, \tag{2}$$

$$W(p,i\omega)=\frac{\lambda}{16\pi^2}\int N(\Omega_{p'})d\Omega_{p'}\int_{-\infty}^{\infty}d\epsilon_{p'}\times\pi T\sum_{i\omega'}\chi_{11}(p-p',i\omega-i\omega')\times\frac{-W(p',i\omega')}{[i\omega' Z_p(\omega')]^2-\epsilon_{p'}^2}. \tag{3}$$

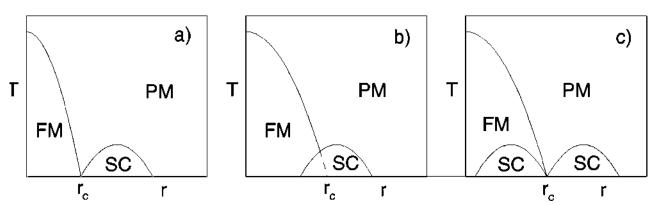

FIG. 1. Possible scenarios for the emergence of a superconducting state near a quantum critical point of a magnetic system: (a) the superconducting T_c is zero in the quantum critical point (QCP) and the superconducting phase does not extend into the magnetically ordered phase; (b) T_c is finite in QCP and there is a coexistence of the superconducting and ferromagnetic state; (c) as in (b) but the superconducting T_c vanishes in QCP. The parameter r measures the distance from the quantum critical point r_c (Refs. 12 and 13). In experimental realizations r corresponds to hydrostatic pressure (Ref. 1).

Here the momentum integration has been separated into integration in a direction perpendicular to the Fermi surface (ϵ_p integration) and integration over angles Ω_p of the Fermi surface; $N(\Omega_p)$ is the angle-dependent density of states of the quasiparticles on the Fermi surface divided by its average. All three spin components contribute to ωZ; but as stressed by Monthoux and Lonzarich[6] for spin triplet pairing only one combination can contribute to any given component of the gap function and T_c is maximized if we pick the one for which $\Delta=0$. λ may be experimentally defined from the singular (as $r\to 0$) behavior of the specific heat coefficient

$$\gamma=\lim_{T\to 0}\frac{C}{T}=\int d\Omega_p N(\Omega_p)Z_p(\pi T)$$
$$=N_0\left[1+\frac{1}{2}\lambda\left(\ln\frac{1}{r}+2\ln\frac{1}{r+\Delta}\right)\right]. \tag{4}$$

Equations (2) and (3) apply only for frequencies much less than the electron bandwidth and only if the momentum dependence of Z_p and W is negligible relative to the frequency dependence, conditions which are satisfied for the leading singular behavior as $r\to 0$. We therefore employ the Migdal approximation[14] $Z_p(\omega)\to Z(\omega)$, $W(p,\omega)\to W(\Omega_p,i\omega)$ and perform the integral over the magnitude of the momentum. To perform the remaining integration over angles we note that $i\omega Z(\omega)$ has the full symmetry of the lattice, while for *p*-wave superconductivity W corresponds to the $l=1$ irreducible representation of the lattice group $[W_l(\alpha\Omega_p)=\alpha^l W_l(\Omega_p)]$.

The momentum transfer q carried by the spin fluctuations in Eq. (1) is given by $q^2=(\mathbf{p}-\mathbf{p}')^2=2p_F^2(1-(\mathbf{p}\cdot\mathbf{p}')/|\mathbf{p}||\mathbf{p}'|)+\epsilon_{p'}^2/v_F^2$. The first term in q^2 is obtained by placing both momenta $\mathbf{p}$ and $\mathbf{p}'$ on the Fermi surface while the last term is a small correction δp^2 taking into account the fact that intermediate states can explore regions close to the Fermi surface and will be important as a cutoff. We perform the ϵ_p integral, use the angle dependences of Z and W and then follow Bergmann and Rainer,[15] defining an order parameter $\Phi_l(\omega)=W_l(i\omega)/|\omega Z(\omega)|$ and casting Eqs. (2) and (3) into an eigenvalue problem for an eigenvalue ρ

$$\sum_{\omega'}K_{\omega\omega'}\Phi_l(\omega')=\rho\Phi_l(\omega). \tag{5}$$

T_c corresponds to the solution $\rho(T_c)=0$. Here

$$K_{nm}=D_1(\omega_n-\omega_m)-\frac{|\omega_m Z(\omega_m)|}{\pi T}\delta_{nm}, \tag{6}$$

$$|\omega Z(\omega)|=|\omega|+\pi T\left(D_0(0)+2\sum_{\omega'=0}^{\omega}D_0(\omega-\omega')\right). \tag{7}$$

The kernels D_0, D_1 are defined in terms of the fundamental integrals

$$d_l(r,\Delta)=\int_0^1 N_0(x)\frac{xP_l(1-2x^2)dx}{U_{r,\Delta}(U_{r,\Delta}+|\omega' Z(\omega')|/\Lambda)} \tag{8}$$

as $D_0=\lambda[d_0(r,0)+2d_0(r,\Delta)]$, $D_1=\lambda d_1(r,0)$ with $U_{r,\Delta}=[(|\nu|/\Lambda x)\tan^{-1}(\Lambda x/|\nu|)+x^2+r+\Delta]^{1/2}$, $\nu=\omega-\omega'$. $P_l(x)$ is a basis function of the lth irreducible representation of the lattice symmetry group and the cutoff term $|\omega Z|/\Lambda$ in Eq. (8) comes from the δp^2 correction in q^2. For definiteness we assume a spherical Fermi surface with a constant density of states $[N_0(x)\to 1]$ in which case $P_l(x)$ become Legendre polynomials but most of our results depend only on $\lim_{x\to 0}P_l(x)$.

In $\mathbf{K}$, the off-diagonal terms are from D_1 and give pairing; the diagonal terms come mainly from $|\omega Z|$ and represent depairing corresponding to scattering. At high temperatures the eigenvalues $\rho_n(T)$ are negative; at T_c the leading eigenvalue crosses 0. We solve the matrix system numerically; the size of the kernel is $\sim\Lambda/(2\pi T_c)$.

For *p*-wave pairing in systems with Heisenberg symmetry the critical temperatures are typically $\pi T_c\sim 10^{-5}\Lambda$ which translates into numerically unmanageable kernel sizes of $N\sim 50\,000$. We therefore use two alternative numerical approximations; a down-folding procedure and variants of the adaptive discretization proposed by Bickers.[17] In the down-folding procedure we separate $\Phi_l(\omega_n)$ in Eq. (5) into a low-frequency part $\Phi_l^{LOW}(\omega_n)$ with $0\leqslant|n|\leqslant N_{LOW}$ and high-frequency part $\Phi_l^{HIGH}(\omega_n)$ with $N_{LOW}<|n|\leqslant N$. Then Eq. (5) can be written as a block linear system and formally solved for Φ^{HIGH}, yielding $\mathbf{K}^{LOW}\cdot\Phi^{LOW}=\rho\Phi^{LOW}$ with

$$K_{nm}^{LOW}=K_{nm}+\sum_{|i|,|j|>N_{LOW}}K_{ni}(\rho-K_{ij})^{-1}K_{jm}.$$

This transformation is exact. The simplification is that for large N_{LOW} $\mathbf{K}$ is nearly diagonal so $\mathbf{K}^{-1}$ may easily be computed in the "high" subspace. In physical terms, e.g., in the Heisenberg case $\Delta=0$ and for $r=0$, this approximation retains only the diagonal scattering-dominated terms $K_{nn}\simeq-(2\lambda/3)\ln(\Lambda/\pi T)-(2n+1)[1+2\lambda+2\lambda\ln(\Lambda/2\pi Tn)]$ and drops all off-diagonal pairing terms $K_{nm}\simeq(\lambda/3)\ln(\Lambda/2\pi T|n-m|)$ for $n,m>N_{LOW}$ in the high-frequency kernel. In the adaptive discretization approximation we rearrange the high-frequency elements of the kernel K_{nm} $(n,m\geqslant N_{LOW})$ into rectangular blocks of quadratically or exponentially increasing size and represent each block by its average in the reduced kernel $\mathbf{K}^{LOW}$. We have verified that both approximations (where the matrix can be diagonalized exactly) reproduce faithfully the eigenvalues of Eq. (5) for large temperatures, and that our results are insensitive to the choice of N_{LOW}.

Results for $T_c(r,\Delta)$ are shown in Fig. 2. The inset of the figure demonstrates the convergence of the scaling procedure with reduced kernel size N_{LOW} for the numerically most difficult case, $\Delta=0$. Reduced kernel sizes $N_{LOW}\geqslant 500$ show satisfactory convergence, so we have used sizes $N_{LOW}=500$ in the cases when $N=\Lambda/(2\pi T_c)>1000$ and the full kernel $\mathbf{K}$ otherwise.

As previously noted[6] T_c is very low in the Heisenberg case, however, $T_c(r\to 0)>0$, raising the interesting possibility (already noted by Fay *et al.*[8]) of superconductivity extending into the magnetic phase, however, we do not agree with Ref. 8's claim that $T_c(r=0)=0$. The key point is the

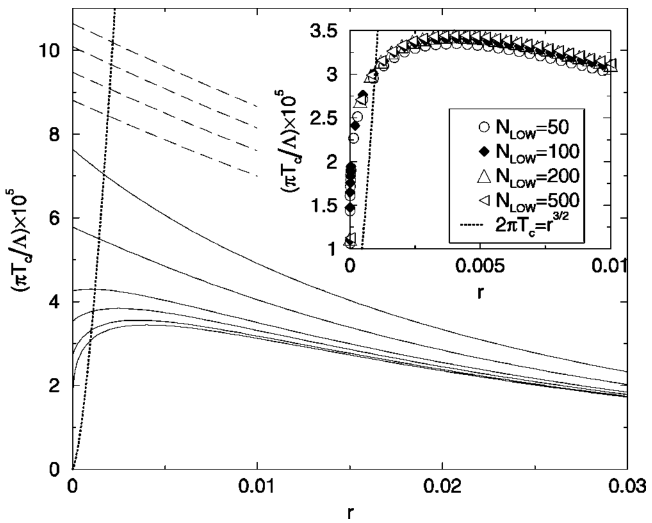

FIG. 2. Main figure: Superconducting transition temperature as a function of quantum critical control parameter r for Ising anisotropies $\Delta=0;0.0003;0.001;0.002;0.005;0.01;0.7;0.8;0.9;1.0$ going from the bottom curve up and coupling constant $\lambda=1.5$. For the $\Delta=0.7;0.8;0.9;1.0$ curves (dashed lines) we have plotted $T_c/10$ for better visual comparison with the Heisenberg ($\Delta=0$) case. $\Lambda \sim 2p_F v_F$ is the characteristic spin-fluctuation frequency; the dotted line is $2\pi T_c = r^{3/2}$. Inset: $T_c(r,\Delta=0)$ for different values of downfolding scale N_{LOW}.

cutoff $|\omega Z|/\Lambda$ in Eq. (8) which leads to the leading r dependence $D_0(0)\sim 3D_1(0)\sim 3\lambda \ln 1/(\sqrt{r}+|\omega Z|/\Lambda)$. If the cutoff is neglected (as in Ref. 8) the divergence of K_{nn} drives $T_c \to 0$ and produces the phase diagram shown in Fig. 1(c). Reference 7, which studied a two-dimensional antiferromagnetic problem, argued that the divergent mass enhancement associated with the critical fluctuation would drive the superfluid stiffness and thus T_c to zero. In their case the divergent mass occurs only at one point on the Fermi surface, so it seems to us the considerations of Hlubina and Rice[16] should imply a nonzero superfluid stiffness in the problem they studied. In any event, in the ferromagnetic problem of interest here the critical fluctuations have long wavelengths, and thus do not lead to divergences in the ''transport mass'' controlling the superfluid stiffness.

In the Ising case, as for s-wave pairing, $D_0(0)\sim D_1(0)$, the quasistatic depairing and pairing contributions to $\mathbf{K}$ cancel and T_c is substantially higher than in the Heisenberg case. The crossover to full Ising behavior has not been previously studied; we find T_c increases rapidly as the Ising anisotropy is increased.

We see from the inset of Fig. 2 that the Heisenberg case $T_c(r)$ displays a maximum at a small nonzero $r=r_{\max}$, but for Ising anisotropies Δ greater than $\sim r_{\max}^{3/2}$, T_c decreases monotonically as r increases. To understand this behavior we compute dT_c/dr using the Feynman-Hellman theorem:[15]

$$\frac{dT_c}{dr}=\left(\frac{d\rho}{dT_c}\right)^{-1}\frac{d\rho}{dr}=\left(\frac{d\rho}{dT_c}\right)^{-1}\frac{\langle\Phi|d\mathbf{K}/dr|\Phi\rangle}{\langle\Phi|\Phi\rangle}, \tag{9}$$

where Φ is the eigenvector defined in Eq. (5) and up to logarithms $\Phi(\omega)\sim 1/|\omega|$. The terms occurring in $d\mathbf{K}/dr$ may be seen from Eq. (8) to be infrared dominated, so the leading singular behavior may be expressed in terms of a scaling function:

$$F(r,\Delta,\omega_{nm})=\frac{1}{r+\Delta}f\left(\frac{\omega_{nm}}{(r+\Delta)^{3/2}}\right), \quad \omega_{nm}\neq 0$$

$$=\frac{1}{\sqrt{r+\Delta}}\cdot\frac{1}{\sqrt{r+\Delta}+|\omega Z|/\Lambda}, \quad \omega_{nm}=0,$$

where $f(x)=\int_0^\infty y\,dy/(x/y+y^2+1)^2$. [Note $f(0)=1/2$ and as $x\to\infty$ $f(x)\to(2\pi/9\sqrt{3})x^{-2/3}$.] Substituting, we find

$$\frac{dT_c}{dr}=\sum_{n=0}^{\infty}\frac{2F(r,\Delta,0)}{(2n+1)^2}+2\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\left[\frac{2F(r,\Delta,\omega_n-\omega_m)}{(2n+1)^2}+\frac{F(r,0,\omega_n-\omega_m)}{(2n+1)^2}-\frac{F(r,0,\omega_n-\omega_m)}{(2m+1)(2n+1)}\right], \tag{10}$$

where the $F(r,\Delta\neq 0,\omega_{nm})$ terms come from spin fluctuations in the two ''hard'' spin directions and are pairbreaking and the $\Delta=0$ terms come from the ''soft'' spin direction and are both depairing [the third term in Eq. (10)] and pairing [the last term in Eq. (10)]. For large Δ (strong Ising anisotropy) we may set $F(\Delta\neq 0)=0$; there are no $\omega_n=\omega_m$ terms; the off-diagonal terms are negative ($dT_c/dr<0$). The physical interpretation is that the pairing and depairing effects of quasistatic ($\omega<T$) spin fluctuations approximately cancel (as in the s-wave case[15]) while at $\omega>T$ the pairing effect wins. Thus T_c monotonically increases as $r\to 0$ because the spin fluctuations become stronger. At $r=0$, $dT_c/dr\sim -T^{-2/3}$, i.e., $T_c(r)$ is linear; for $r>T_c^{3/2}$ the derivative $\to(\ln 1/r)/r$, so we expect $T_c\sim \ln^2 1/r$.

As Δ decreases, the $\omega_n=\omega_m$ terms increase and as $\Delta\to 0$ dominate at small r. In this limit quasistatic spin fluctuations are strongly pairbreaking, and at $r=0$ T_c is set by the temperature at which the effect of these fluctuations becomes small enough to allow pairing. For $r<\lambda^2(\pi T_c)^2\ln^2\Lambda/T_c$, the ωZ term in F is important and $dT_c/dr\sim 1/(\sqrt{r}\lambda\pi T_c \ln\Lambda/T_c)$. For $\lambda^2(\pi T_c)^2\ln^2\Lambda/T_c<r<(\pi T_c)^{3/2}$; $dT_c/dr\sim 1/r$.

We now consider the variation of $T_c(r=0)$ with Δ (this could be varied experimentally, e.g., with uniaxial pressure). The arguments leading to dT_c/dr yield

$$\frac{dT_c}{d\Delta}\sim 2\sum_{n=0}^{\infty}\frac{F(0,\Delta,0)}{(2n+1)^2}+4\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{F(0,\Delta,\omega_n-\omega_m)}{(2n+1)^2}. \tag{11}$$

Both terms are positive; the first term is dominant leading to an approximately $\Delta\ln(1/\Delta)$ behavior at small Δ.

Finally, we consider which frequencies are important for pairing.[18] Figure 3 shows results obtained by simply truncating the pairing kernel (setting $K_{nm}=0$ for $|n|,|m|>N_{\text{trunc}}$) for values of N_{trunc} between 50 and 500. We have chosen to plot the data in scaled form, as relative change in T_c vs energy of upper cutoff. The reasonable data collapse (despite two order of magnitude variations in T_c) indicates that for the parameters considered, pairing comes basically from a

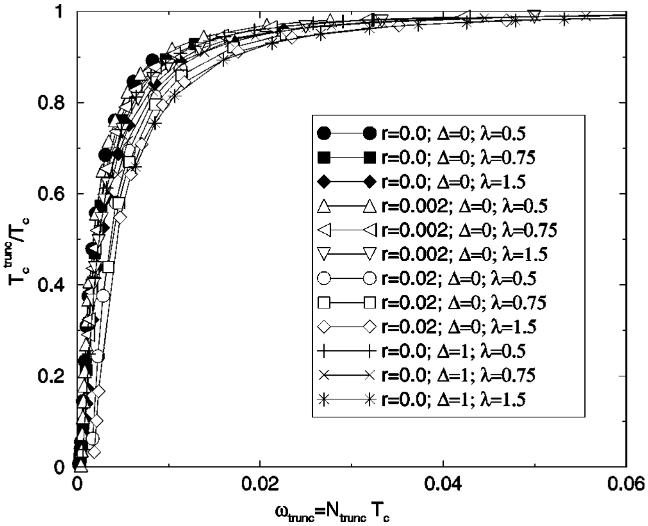

FIG. 3. $T_c^{\rm trunc}$, scaled by the true T_c, vs the truncation energy $N_{\rm trunc}T_c$.

fixed, small energy range ($\omega \lesssim 0.02\Lambda$) which does not change much with T_c, anisotropy or coupling constant. (Note that for the r values considered the energy scale set by r, $\omega_r = r^{3/2}$, is much less than the critical energy scale.)

Belitz, Kirkpatrick, and Vojta[19] have argued that the generally accepted quantum critical form $\chi(q,0)^{-1} = q^2 + r$ is not correct, and in $d=3$ should be $\chi^{-1}(q,0) = Aq^2 \ln(q_F/q) + r$ with A an unknown coefficient. We find that this form (with $A=1$) leads to substantially lower T_c's (factor of 20) and shrinks the r dependence towards $r=0$. The same effect can be achieved by absorbing the singular q dependence into the coefficient A: $A \ln(q_F/q) \to A = \ln(1/T^{1/3})$, and thus rescaling the parameters of the model: $r \to r/A$, $\lambda \to \lambda/A$, $\omega \to \omega/A$.

Fay and Appel,[8] used a BCS approximation with a pairing interaction determined by the static susceptibility and found $T_c(r=0)=0$. Our more complete Eliashberg treatment shows that this is an artifact of their approximation. However, their important predictions of a $T_c > 0$ in the ferromagnetic state and of a minimum (in the Heisenberg case) of T_c in the vicinity of $r=0$ seem consistent with our results.

To summarize, we have presented a theory of the variation of a p-wave superconducting T_c near a ferromagnetic quantum critical point. We have shown that within the model the value of T_c is determined by low but fixed energy spin fluctuations and the variation of T_c with distance from criticality is controlled by spin fluctuations on the scale of T_c. We have also demonstrated the crucial role played by the symmetry of the magnetic fluctuations. We have found generically that $T_c > 0$ at the magnetic critical point, raising the interesting possibility of superconductivity within the ordered phase. From this analysis we see that (up to an overall amplitude) the small r behavior of T_c is universal, in the sense that it is determined only by the long-wavelength susceptibility and coupling constant.

Note added: As the revised version of this manuscript was in preparation we became aware of a publication reporting superconductivity in the ferromagnetic phase of a heavy fermion material,[20] however, a direct application of our theory is not possible because the magnetic transition in this material is first order.

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