

SU(4) model of high-temperature superconductivity and antiferromagnetismMike Guidry,¹ Lian-Ao Wu,^{2,*} Yang Sun,¹ and Cheng-Li Wu,^{3,4}¹*Department of Physics and Astronomy, University of Tennessee, Knoxville, Tennessee 37996*²*Department of Physics, Jilin University, Changchun, Jilin 130023, People's Republic of China*³*Department of Physics, Chung-Yuan Christian University, Chungli, Taiwan 320, Republic of China*⁴*National Center for Theoretical Science, Hsinchu, Taiwan 300, Republic of China*

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We present an SU(4) model of high-temperature superconductivity having many similarities to dynamical symmetries known to play an important role in microscopic nuclear structure physics and in elementary particle physics. Analytical solutions in three dynamical symmetry limits of this model are found: an SO(4) limit associated with antiferromagnetic order; an SU(2)×SO(3) limit that may be interpreted as a *d*-wave pairing condensate; and an SO(5) limit that may be interpreted as a doorway state between the antiferromagnetic order and the superconducting order. The model suggests a phase diagram in qualitative agreement with that observed in the cuprate superconductors. The relationship between the present model and the SO(5) unification of superconductivity and antiferromagnetic order proposed by Zhang is discussed.

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I. INTRODUCTION

There are compelling arguments that the mechanism leading to high-temperature superconductivity does not correspond to ordinary BCS *s*-wave pairing. Although experimental evidence implicates singlet (hole) pairs as the carriers of the supercurrent, the interaction leading to the formation of the singlet pairs appears not to be the traditional lattice phonon mechanism underlying the BCS theory, but rather seems to be a collective electronic interaction. Furthermore, the pairing gap is anisotropic, with nodes in the k_x - k_y plane strongly suggestive of *d*-wave hybridization in the two-particle wave functions, and the mechanism responsible for superconductivity in the cuprates is thought to be closely related to the unusual antiferromagnetic (AF) insulator properties of their normal states.

Contrary to the case for BCS superconductors, the formation of Cooper pairs and the formation of a superconducting (SC) condensate of those pairs in high- T_c compounds may be distinct, with pair formation corresponding to a higher temperature scale than the condensation of the pairs into the SC state. That is, there appear to be at least two distinct energy scales associated with the formation of the high-temperature SC state. This is reminiscent of grand unified theories in elementary particle physics, where qualitatively different physical phases result from a hierarchy of symmetry breakings occurring on different energy (temperature) scales. This finds its most natural explanation in a Lie group structure that is broken spontaneously (and perhaps explicitly) down to subgroups at different characteristic energy scales.

II. DYNAMICS AND SYMMETRIES

Such observations argue strongly for a theory based on continuous symmetries of the dynamical system that is capable of describing more sophisticated pairing than found in

the simple BCS picture (which is described by a single complex order parameter), and capable of unifying different collective modes and phases on a equivalent footing. Then such fundamentally different physics as antiferromagnetic order and superconductivity can emerge from the same effective Hamiltonian as concentration variables (e.g., doping parameters) are varied.

A. Fermion dynamical symmetries

For approximately the same period of time that the high- T_c compounds have been known, techniques based on dynamical symmetries in fermion degrees of freedom that are capable of satisfying the preceding conditions have been in development in the field of nuclear structure physics. There it has proven fruitful to ask the following questions: what are the most important collective degrees of freedom in the low-lying spectrum of complex nuclei, what are the microscopic many-body quantum operators that create and annihilate these modes, and what is the commutator algebra obeyed by this set of operators.

Systematic investigation of these questions has led to strong confirmation of the following set of conjectures about the nuclear many-body system. (1) Low-lying collective modes are in approximate one-to-one correspondence with dynamical symmetries in the fermion degrees of freedom. (A system possesses a dynamical symmetry if it has a Hamiltonian that can be expressed as a polynomial in the Casimir invariants of a subgroup chain.) (2) A dynamical symmetry associated with low-lying collective modes is associated with a Lie algebra and its subalgebras that are formed from a set of fermion operators closed under commutation. (3) Different dynamical symmetry subgroup chains arising from the same highest symmetry group are associated with fundamentally different phases of the theory. These dynamical symmetries are characterized by different collective modes and the corresponding phases are unified in the highest group, just as

grand unified theories are unified in the higher groups of the symmetry breaking chain. (4) The unification implied by the preceding point suggests that the many low-lying collective states formed by systematic filling of valence shells in heavy nuclei are in reality different projections in an abstract multidimensional space of the same state. Equivalently, the different states are transformed into each other by the generators of the symmetry. Thus, the systematics of collective modes and phase transitions as a function of concentration variables are specified by the group structure.

It has been demonstrated that dynamical symmetries of the type described in Ref. 1 are realized to remarkably high accuracy in the spectrum and the wavefunctions of large-scale numerical calculations using the Projected Shell model.² Since this model is known to give extremely good agreement with a broad range of experimental data (see, e.g., Ref. 3), this provides rather conclusive proof that these dynamical symmetries are strongly realized in the low-lying states of complex nuclei. This raises the issue of whether similar symmetries might be found in other complex many-body fermion systems such as those important in condensed matter. We may expect that the nuclear and condensed-matter systems have many similarities that could make this consideration fruitful: both are composed of interacting fermions and both are ultimately many-body systems that are only approximately describable by mean-field ideas.

B. The Zhang SO(5) model

S. C. Zhang and collaborators have introduced ideas bearing many similarities to these into the high-temperature superconductor discussion.⁴ Motivated by a desire to unify AF and SC order parameters, Zhang *et al.* have assembled these into a five-dimensional vector order parameter, and have then constructed an SO(5) group that rotates the AF order parameters into the SC ones. This construction is based explicitly on the assumption of *d*-wave pairing in the SC state.

In this paper we proceed differently. We start, not from the desire to unify two particular phases, but from identification of a closed algebra associated with a general set of fermion pairing and particle-hole operators defined on a periodic lattice. Nevertheless, we shall find that we recover Zhang's SO(5) symmetry as a subgroup of a more general U(4) symmetry if certain commutators of the full SU(4) algebra are set to zero. Thus, much of the extensive recent discussion of the Zhang SO(5) symmetry applies directly to the results of this paper.

However, the present paper extends this discussion substantially: (1) The SO(5) subgroup is embedded in a larger algebra defined microscopically in the fermion degrees of freedom, which implies constraints on the SO(5) subgroup; (2) The SU(4) highest symmetry has subgroups in addition to SO(5) that may be relevant to the condensed-matter problem in general and the cuprate superconductors in particular; (3) we frame the discussion, not in terms of an approximate symmetry of a Hubbard or *t*-*J* Hamiltonian, but in terms of an exact *dynamical symmetry* constructed using the Casimir invariants of group chains. We shall present a more detailed discussion of the relationship between the present model and the Zhang SO(5) model.

III. DYNAMICAL SYMMETRY METHOD

The dynamical symmetry method applied here corresponds schematically to the following algorithm.

(0) Assume the following conjecture: *All strongly collective modes in fermion (or boson) many-body systems can be put into correspondence with a closed algebra defining a dynamical symmetry of the sort described below.* This is a conjecture, but there is so much evidence in support of it from various fields of physics that it is almost a theorem: Strongly correlated motion implies a symmetry of the dynamics described by a Lie algebra in the second-quantized operators implementing that motion.

(1) Identify, within a suitable "valence space," degrees of freedom that one believes are physically relevant for the problem at hand, guided by phenomenology, theory, and general principles. In the present case, that reduces to defining a minimal set of operators that might be important to describe superconductivity and antiferromagnetism on a spin lattice.

(2) Try to close a commutation algebra (of manageable dimension) with the second-quantized operators creating and annihilating the modes chosen in step (1). If necessary, approximate these operators, or add additional ones to the set if the algebra does not close naturally. In the present context, the simple $g(k)$ form factor introduced below is an example of simplifying things to close the algebra.

(3) Use standard Lie algebra theory to identify relevant subalgebra chains that end in algebras for conservation laws that one expects to be obeyed for the problem at hand. In the present example, we require all group chains to end in $U(1) \times SU(2)$, corresponding to an algebra implementing conservation of charge and spin.

(4) Construct dynamical symmetry Hamiltonians (Hamiltonian that are polynomials in the Casimir invariants of a group chain) for each chain. Each such group chain thus defines a wave-function basis labeled by the eigenvalues of chain invariants (the Casimirs and the elements of the Cartan subalgebras), and a Hamiltonian that is diagonal in that basis (since it is constructed explicitly from invariants). Thus, the Schoedinger equation is solved analytically for each chain, by construction.

(5) Calculate the physical implications of each of these dynamical symmetries by considering the wave functions, spectra, and transitional matrix elements of physical relevance. This is tractable, because the eigenvalues and eigenvectors were obtained in step (4). Consistency of the symmetry requires that transition operators be related to group generators; otherwise, transitions would mix irreducible multiplets and break the symmetry.

(6) If step (5) suggests that one is on the right track [meaning that a wise choice was made in step (1)], one can write the most general Hamiltonian for the system in the model space, which is just a linear combination of all the Hamiltonians for the symmetry group chains. Since the Casimir operators of different group chains do not generally commute with each other, a Casimir invariant for one group chain may be a symmetry-breaking term for another group chain. Thus the competition between different dynamical

symmetries and the corresponding phase transitions can be studied.

(7) The symmetry-limit solutions may be used as a starting point for more ambitious calculations that incorporate symmetry breaking. Although no longer generally analytical, such more realistic approximations may be solved by perturbation theory around the symmetry solutions (which are generally nonperturbative, so this is perturbation theory around a nonperturbative minimum), by numerical diagonalization of symmetry breaking terms, or by coherent state or other mean-field approximations.

Representative application of these ideas for both fermion and boson systems may be found in nuclear,^{1,5} particle,⁶ molecular,⁷ and polymer physics.⁸ We also note that the general idea of symmetry having dynamical implications lies at the heart of local gauge-field theories in particle physics, though the details and methodologies in that case differ from the ones used here.

The only approximation in the dynamical symmetry approach outlined above is the space truncation. If all degrees of freedom are incorporated, this defines an exact microscopic theory. Of course, in practical calculations only a few carefully selected degrees of freedom can be included and the effect of the excluded space must be incorporated by renormalized interactions in the truncated space. It follows that the validity of such an approach hinges on a wise choice of the collective degrees of freedom and sufficient phenomenological or theoretical information to specify the corresponding effective interactions of the truncated space.

IV. THE SU(4) MODEL

Let us now introduce a mathematical formalism that provides a systematic implementation of the dynamical symmetry procedure for a particular physically motivated choice of operators.

A. Choice of operators

The success of the dynamical symmetry method depends on a wise selection of the operators that describe the low-energy degrees of freedom for the system. In the case of cuprate superconductors, we know that (unlike for normal superconductors) antiferromagnetism and superconductivity lie very near each other in the phase diagram. Further, data suggest that the SC phases are associated with Cooper pairs of spin-singlet electron holes having d -wave geometry. Finally, we expect that the physical system must conserve both charge and spin. These observations suggest that at a minimum we need d -wave singlet pairs and operators associated with antiferromagnetism, spin operators, and charge operators entering the theory on an equivalent footing. Let us now construct a minimal closed algebra that incorporates these degrees of freedom.⁹

B. The algebra

We begin by defining the following lattice fermion operators:

$$p_{12}^\dagger = \sum_k g(k) c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger, \quad p_{12} = \sum_k g^*(k) c_{-k\downarrow} c_{k\uparrow},$$

$$q_{ij}^\dagger = \sum_k g(k) c_{k+Q,i}^\dagger c_{-k,j}^\dagger, \quad q_{ij} = (q_{ij}^\dagger)^\dagger, \quad (1)$$

$$Q_{ij} = \sum_k c_{k+Q,i}^\dagger c_{k,j}, \quad S_{ij} = \sum_k c_{k,i}^\dagger c_{k,j} - \frac{1}{2} \Omega \delta_{ij},$$

where $c_{k,i}^\dagger$ creates a fermion of momentum k and spin projection $i, j = 1$ or $2 = \uparrow$ or \downarrow , $Q = (\pi, \pi, \pi)$ is an AF ordering vector, Ω is the lattice degeneracy, and we approximate a d -wave form factor by

$$g(k) = \text{sign}(\cos k_x - \cos k_y) = \pm 1,$$

with $g(k+Q) = -g(k)$ and $|g(k)| = 1$ (see Refs. 10,11). Using the usual fermion anticommutators, we deduce the following commutation relations among the operators of Eq. (1):

$$[p_{12}, p_{12}^\dagger] = -S_{11} - S_{22},$$

$$[q_{ij}, q_{kl}^\dagger] = -\delta_{ik} S_{lj} - \delta_{il} S_{kj} - \delta_{kj} S_{li} - \delta_{jl} S_{ki},$$

$$[p_{ij}, q_{kl}^\dagger] = \delta_{ik} Q_{lj} + \delta_{il} Q_{kj} - \delta_{kj} Q_{li} - \delta_{jl} Q_{ki},$$

$$[S_{ij}, S_{kl}] = \delta_{jk} S_{il} - \delta_{il} S_{kj},$$

$$[Q_{ij}, Q_{kl}] = \delta_{jk} S_{il} - \delta_{il} S_{kj}, \quad (2)$$

$$[S_{ij}, p_{kl}^\dagger] = \delta_{ik} p_{kl}^\dagger - \delta_{jl} p_{ik}^\dagger,$$

$$[S_{ij}, q_{kl}^\dagger] = \delta_{ik} q_{il}^\dagger + \delta_{jl} q_{ik}^\dagger,$$

$$[Q_{ij}, p_{kl}^\dagger] = \delta_{jk} q_{il}^\dagger - \delta_{jl} q_{ik}^\dagger,$$

$$[Q_{ij}, q_{kl}^\dagger] = \delta_{jk} p_{il}^\dagger + \delta_{jl} p_{ik}^\dagger.$$

Thus, this set of 16 operators is closed under commutation and generates a Lie algebra. Detailed examination indicates that the algebra is associated with the group $U(4)$ and has the subgroup chains

$$\supset \text{SO}(4) \times \text{U}(1) \supset \text{SU}(2)_s \times \text{U}(1)$$

$$\text{U}(4) \supset \text{SU}(4) \supset \text{SO}(5) \supset \text{SU}(2)_s \times \text{U}(1) \quad (3)$$

$$\supset \text{SU}(2)_p \times \text{SU}(2)_s \supset \text{SU}(2)_s \times \text{U}(1)$$

that end in the subgroup $\text{SU}(2)_s \times \text{U}(1)$ representing spin and charge conservation. The physical interpretation is aided by rewriting the generators of the $U(4)$ algebra as

TABLE I. Properties of SU(4) and its subgroups (valid for representations with no broken pairs).

Group	Generators	Quantum numbers	Casimir operator	Casimir eigenvalue
SU(4)	$Q_+, \vec{S}, \vec{Q}, \vec{\pi}^\dagger$	$\sigma_1 = \frac{\Omega}{2}$ ($\sigma_2 = \sigma_3 = 0$)	$\vec{\pi}^\dagger \cdot \vec{\pi} + D^\dagger D + \vec{S} \cdot \vec{S}$	$\frac{\Omega}{2} \left(\frac{\Omega}{2} + 4 \right)$
	$\vec{\pi}, D^\dagger, D, M$		$+ \vec{Q} \cdot \vec{Q} + M(M-4)$	
SO(4)	\vec{Q}, \vec{S}	w ($F = G = w/2$)	$\vec{Q} \cdot \vec{Q} + \vec{S} \cdot \vec{S}$	$w(w+2)$
SO(5)	$\vec{S}, \vec{\pi}^\dagger, \vec{\pi}, M$	τ ($\omega = 0$)	$\vec{\pi}^\dagger \cdot \vec{\pi} + \vec{S} \cdot \vec{S} + M(M-3)$	$\tau(\tau+3)$
SU(2) _p	D^\dagger, D, M	v	$D^\dagger D + M(M-1)$	$\frac{1}{4}(\Omega-v)(\Omega-v+2)$
SU(2) _s	\vec{S}	S	$\vec{S} \cdot \vec{S}$	$S(S+1)$

$$Q_+ = Q_{11} + Q_{22} = \sum_k (c_{k+Q\uparrow}^\dagger c_{k\uparrow} + c_{k+Q\downarrow}^\dagger c_{k\downarrow}),$$

$$\vec{S} = \left(\frac{S_{12} + S_{21}}{2}, -i \frac{S_{12} - S_{21}}{2}, \frac{S_{11} - S_{22}}{2} \right),$$

$$\vec{Q} = \left(\frac{Q_{12} + Q_{21}}{2}, -i \frac{Q_{12} - Q_{21}}{2}, \frac{Q_{11} - Q_{22}}{2} \right), \quad (4)$$

$$\vec{\pi}^\dagger = \left(\frac{i}{2} (q_{11}^\dagger - q_{22}^\dagger), \frac{1}{2} (q_{11}^\dagger + q_{22}^\dagger), -\frac{i}{2} (q_{12}^\dagger + q_{21}^\dagger) \right),$$

$$\vec{\pi} = (\vec{\pi}^\dagger)^\dagger \quad D^\dagger = p_{12}^\dagger \quad D = p_{12} \quad M = \frac{1}{2}(n - \Omega),$$

where Q_+ generates charge density waves, \vec{S} is the spin operator, \vec{Q} is the staggered magnetization, and $\vec{\pi}^\dagger, \vec{\pi}$ create and annihilate triplet d -wave pairs (see Ref. 4), the operators D^\dagger and D are associated with singlet d -wave pairs, n is the electron number operator, and M is the charge operator. Properties of this group structure are summarized in Tables I and II, and Fig. 1.

Notice in this context that we require exact conservation of charge and spin for our dynamical symmetry solutions because we have not introduced approximations that violate these symmetries. Although it is common to refer to superconductivity as resulting from violation of particle number, this is a statement about an approximate solution. In the exact solution and in nature, particle number is conserved.¹² In

a later section we shall introduce useful *approximate solutions* through coherent state methods that lead to spontaneous symmetry breaking and to intrinsic states violating particle number, but our dynamical symmetry solutions conserve charge and spin exactly.

C. The collective subspace

The group SU(4) has a quadratic Casimir operator

$$C_{su(4)} = \vec{\pi}^\dagger \cdot \vec{\pi} + D^\dagger D + \vec{S} \cdot \vec{S} + \vec{Q} \cdot \vec{Q} + M(M-4). \quad (5)$$

The group is rank-3 and the irreducible representations (irreps) may be labeled by three weight-space quantum numbers, $(\sigma_1, \sigma_2, \sigma_3)$. We assume for the simplest implementation of the model a collective d -wave pair subspace spanned by the following vectors:

$$|S\rangle = |n_x n_y n_z n_s\rangle = (\pi_x^\dagger)^{n_x} (\pi_y^\dagger)^{n_y} (\pi_z^\dagger)^{n_z} (D^\dagger)^{n_s} |0\rangle. \quad (6)$$

This collective subspace is associated with irreps of the form

$$(\sigma_1, \sigma_2, \sigma_3) = \left(\frac{\Omega}{2}, 0, 0 \right), \quad (7)$$

where Ω is the number of lattice sites. The corresponding expectation value of the SU(4) Casimir evaluated in these irreps is a constant:

$$\langle C_{su(4)} \rangle = \frac{\Omega}{2} \left(\frac{\Omega}{2} + 4 \right). \quad (8)$$

TABLE II. The Hamiltonian, eigenstates, and spectra in three dynamical symmetry limits of the SU(4) model (assume N is even and $1/\Omega$ is negligible). $E_{g.s.}$ is the ground-state energy, ΔE the excitation energy, Dim the dimension of each representation for a given N , $N = n/2$ the pair number, $x = 1 - n/\Omega$, and $\kappa_{so4} = \kappa_{\text{eff}} + \chi_{\text{eff}}$.

SU(2) limit: $ \psi(SU2)\rangle = N, v, S, m_s\rangle$	SO(4) limit: $ \psi(SO4)\rangle = N, w, S, m_S\rangle$	SO(5) limit: $ \psi(SO5)\rangle = \tau, N, S, m_S\rangle$
$\langle C_{SU(2)_p} \rangle = \frac{1}{4}(\Omega - v)(\Omega - v + 2)$	$\langle C_{SO(4)} \rangle = w(w + 2), \quad w = N - \mu$	$\langle C_{SO(5)} \rangle = \tau(\tau + 3), \quad \tau = \Omega/2 - N + \lambda$
$\text{Dim}(v, N) = (v + 1)(v + 2)/2$	$\text{Dim}(w, N) = (w + 1)^2$	$\text{Dim}(\tau, N) = (\lambda + 1)(\lambda + 2)/2$
$H = H'_0 - G_{\text{eff}}^{(0)} D^\dagger D + \kappa_{\text{eff}} \vec{S} \cdot \vec{S}$	$H = H'_0 - \chi_{\text{eff}} \vec{Q} \cdot \vec{Q} + \kappa_{\text{eff}} \vec{S} \cdot \vec{S}$	$H = H'_0 - G_{\text{eff}}^{(0)} \vec{Q} \cdot \vec{Q} + D^\dagger D + \kappa_{\text{eff}} \vec{S} \cdot \vec{S}$
$\langle D^\dagger D \rangle = \langle C_{SU(2)_p} - M(M - 1) \rangle$	$\langle \vec{Q} \cdot \vec{Q} \rangle = \langle C_{SO(4)} - \vec{S} \cdot \vec{S} \rangle$	$\langle \vec{Q} \cdot \vec{Q} + D^\dagger D \rangle = \langle C_{SU(4)} + M - C_{SO(5)} \rangle$
$E_{g.s.} = H'_0 - \frac{1}{4} G_{\text{eff}}^{(0)} \Omega^2 (1 - x^2)$	$E_{g.s.} = H'_0 - \frac{1}{4} \chi_{\text{eff}} \Omega^2 (1 - x^2)$	$E_{g.s.} = H'_0 - \frac{1}{4} \chi_{\text{eff}} \Omega^2 (1 - x)^2$
$\Delta E = \nu G_{\text{eff}}^{(0)} \Omega + \kappa_{\text{eff}} S(S + 1), \quad \nu = v/2$	$\Delta E = \mu \chi_{\text{eff}} (1 - x) \Omega + \kappa_{so4} S(S + 1)$	$\Delta E = \lambda G_{\text{eff}}^{(0)} x \Omega + \kappa_{\text{eff}} S(S + 1)$
$\nu = 0, 1, 2, \dots, N; \quad S = \nu, \nu - 2, \dots, 0$	$\mu = 0, 2, \dots, N; \quad S = w, w - 1, \dots, 0$	$\lambda = 0, 1, 2, \dots, N; \quad S = \lambda, \lambda - 2, \dots, 0$

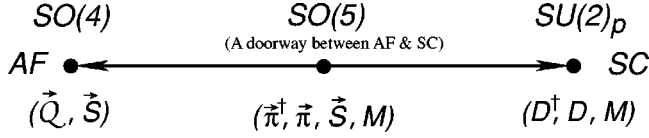


FIG. 1. Dynamical symmetries associated with the U(4) symmetry. The generators are listed for each subgroup.

The operator Q_+ defined in Eq. (4) is the generator of the U(1) factor in $U(4) \supset U(1) \times SU(4)$. Physically, it is associated with charge-density wave excitations in the system. We note that Q_+ commutes with all generators so it annihilates the state $|S\rangle$:

$$Q_+|S\rangle=0, \quad \langle S|Q_+|S\rangle=0. \quad (9)$$

Thus, the collective subspace considered in isolation is associated with an eigenvalue $Q_+=0$. Physically, this corresponds to exclusion of charge-density wave excitations in the low-lying collective subspace of the effective theory.¹³

The dimensionality of the full space is $2^{2\Omega}$. The dimensionality of the collective subspace is much smaller, scaling approximately as Ω^4 :

$$\text{Dim}\left(\frac{\Omega}{2}, 0, 0\right) = \frac{1}{12} \left(\frac{\Omega}{2} + 1\right) \left(\frac{\Omega}{2} + 2\right)^2 \left(\frac{\Omega}{2} + 3\right). \quad (10)$$

Thus for small lattices it is possible to enumerate all states of the collective subspace in a simple model where observables can be calculated analytically.

D. SU(4) model Hamiltonian

The most general two-body Hamiltonian within the d -wave pair space consists of a linear combination of (quadratic) Casimir operators C_g for all subgroups g :¹⁴

$$H = H_0 + \sum_g H_g C_g,$$

where H_0 and H_g are parameters and the Casimir operators C_g are (see Table I)

$$\begin{aligned} C_{SO(5)} &= \vec{\pi}^\dagger \cdot \vec{\pi} + \vec{S} \cdot \vec{S} + M(M-3), \\ C_{SO(4)} &= \vec{Q} \cdot \vec{Q} + \vec{S} \cdot \vec{S}, \\ C_{SU(2)_p} &= D^\dagger D + M(M-1), \\ C_{SU(2)_s} &= \vec{S} \cdot \vec{S}, \\ C_{U(1)} &= M \quad \text{and} \quad M^2. \end{aligned} \quad (11)$$

For fixed electron number the terms in M and M^2 in Eq. (11) are constant. The term H_0 is a quadratic function of particle number and may be parametrized as

$$H_0 = \varepsilon n + \frac{1}{2} \nu n(n-1),$$

where ε and ν are the effective single-electron energy and the average two-body interaction in zero-order approximation, respectively. Thus the Hamiltonian can be written as

$$H = H_0 + V = \varepsilon n - \frac{1}{2} \nu n(n-1) + V, \quad (12)$$

$$V = -G_0 D^\dagger D - G_1 \vec{\pi}^\dagger \cdot \vec{\pi} - \chi \vec{Q} \cdot \vec{Q} + \kappa \vec{S} \cdot \vec{S}, \quad (13)$$

where G_0 , G_1 , χ , and κ are the interaction strengths of d -wave singlet pairing, d -wave triplet pairing, staggered magnetization, and spin-spin interactions, respectively. Since $\langle C_{su(4)} \rangle$ is a constant, by using Eq. (5) we can eliminate the $\vec{\pi}^\dagger \cdot \vec{\pi}$ term and after renormalizing the interaction strengths the SU(4) Hamiltonian can be expressed as

$$H = H'_0 - G[(1-p)D^\dagger D + p\vec{Q} \cdot \vec{Q}] + \kappa_{\text{eff}} \vec{S} \cdot \vec{S}, \quad (14)$$

$$H'_0 = \varepsilon_{\text{eff}} n + \frac{1}{2} \nu_{\text{eff}} n(n-1), \quad (15)$$

with $(1-p)G = G_{\text{eff}}^0$, $pG = \chi_{\text{eff}}$, and κ_{eff} as the effective interaction strengths, and where $0 \leq p \leq 1$ for the parameter p . Since in this paper we primarily address the ground-state properties where $S=0$, the last term in Eq. (14) will generally not enter into the later discussion.

V. THE DYNAMICAL SYMMETRY LIMITS

As we have already noted, there are three subgroup chains of the SU(4) symmetry that conserve spin and charge. These define three dynamical symmetry limits with clear physical meanings. The three dynamical symmetry limits SU(2), SO(4), and SO(5), correspond to the choices $p=0$, 1, and 1/2, respectively, in Eq. (14). The Hamiltonian, eigenfunctions, energy spectrum, and the corresponding quantum numbers of these symmetry limits are listed in Tables I and II, where we introduce a doping parameter x that is related to particle number and lattice degeneracy through

$$x = 1 - \frac{n}{\Omega}. \quad (16)$$

The pairing gap Δ (measure of pairing order) and the staggered magnetization (measure of AF order) Q ,

$$\Delta = G_{\text{eff}}^0 \langle D^\dagger D \rangle^{1/2}, \quad Q = \langle \vec{Q} \cdot \vec{Q} \rangle^{1/2}, \quad (17)$$

may be used to characterize the states in these symmetry limits. As we shall now see, each limit represents a different possible low-energy phase of the SU(4) system.

A. The SO(4) limit

The dynamical symmetry chain

$$SU(4) \supset SO(4) \times U(1) \supset SU(2)_s \times U(1),$$

which we shall term the SO(4) limit, corresponds to long-range AF order. This is the symmetry limit of Eq. (14) when $p=1$. The SO(4) subgroup is locally isomorphic to $SU(2)_F \times SU(2)_G$, where the product group is generated by the linear combinations

$$\vec{F} = \frac{1}{2}(\vec{Q} + \vec{S}), \quad \vec{G} = \frac{1}{2}(\vec{Q} - \vec{S}), \quad (18)$$

of the original SO(4) generators \vec{Q} and \vec{S} . We may interpret the new generators \vec{F} and \vec{G} physically by noting that if we transform Q_{ij} and S_{ij} defined in Eq. (1) to the physical coordinate lattice,

$$Q_{ij} = \sum_r (-)^r c_{ri}^\dagger c_{rj} = \sum_{r=\text{even}} c_{ri}^\dagger c_{rj} - \sum_{r=\text{odd}} c_{ri}^\dagger c_{rj}, \quad (19)$$

$$S_{ij} = \sum_r c_{ri}^\dagger c_{rj} = \sum_{r=\text{even}} c_{ri}^\dagger c_{rj} + \sum_{r=\text{odd}} c_{ri}^\dagger c_{rj},$$

implying that \vec{F} is the generator of total spin on even sites and \vec{G} is the generator of total spin on odd sites. Thus, we may interpret the SO(4) group as being generated by two independent spin operators: one that is the total spin on all sites and one that is the difference in spins on even and odd sites of the spatial lattice. This clearly is an algebraic version of the physical picture associated with AF long-range order.

The SO(4) Casimir operator may be expressed as

$$C_{so(4)} = 2(\vec{F}^2 + \vec{G}^2). \quad (20)$$

The SO(4) representations can be labeled by the spinlike quantum numbers ($F = w/2, G = w/2$), where $w = N - \mu$ with $\mu = 0, 2, \dots, N$. The eigenstates are labeled by w and the spin S , $\psi(\text{SO4}) = |N, w, S, m_s\rangle$, and are of dimension $(w+1)^2$.

The ground state corresponds to $\omega = N$ and $S = 0$, and has $n/2$ spinup electrons on the even sites ($F = N/2$) and $n/2$ spin-down electrons on odd sites ($G = N/2$), or vice versa. Thus it has maximal staggered magnetization

$$Q = \frac{1}{2}\Omega(1-x) = \frac{1}{2}n \quad (21)$$

and a large energy gap (associated with the correlation $\vec{Q} \cdot \vec{Q}$),

$$\Delta E = 2\chi_{\text{eff}}(1-x)\Omega, \quad (22)$$

that inhibits electronic excitation and favors magnetic insulator properties at half filling. In addition, the pairing gap

$$\Delta = \frac{1}{2}G_{\text{eff}}^0\Omega\sqrt{x(1-x)} \quad (23)$$

is small near half filling ($x=0$). We conclude that these SO(4) states are identified naturally with an AF insulating phase of the system.

B. The SU(2) limit

The dynamical symmetry chain

$$\text{SU}(4) \supset \text{SU}(2)_p \times \text{SU}(2)_s \supset \text{SU}(2)_s \times \text{U}(1),$$

which we shall term the SU(2) limit, corresponds to SC order and is the $p=0$ symmetry limit of Eq. (14). The eigenstates are labeled by v and spin S , $\psi(\text{SU2}) = |N, v, S, m_s\rangle$, and are of dimension $(v+1)(v+2)/2$. The senioritylike quantum number v is the number of particles that do not form singlet

d pairs (see Table II). The ground state has $v=0$, implying that all electrons are singlet paired. In addition, there exists a large pairing gap

$$\Delta E = G_{\text{eff}}^{(0)}\Omega \quad (24)$$

(see Table II), the pairing correlation is the largest among the three symmetry limits, and the staggered magnetization vanishes in the ground state:

$$\Delta = \frac{1}{2}G_{\text{eff}}^0\Omega\sqrt{1-x^2}, \quad Q = 0. \quad (25)$$

Thus we propose that this state is a pair condensate associated with a d -wave SC phase of the cuprates.

C. The SO(5) Limit

The dynamical symmetry chain

$$\text{SU}(4) \supset \text{SO}(5) \supset \text{SU}(2)_s \times \text{U}(1),$$

which we shall term the SO(5) limit, corresponds to a phase with the nature of a transitional or *critical dynamical symmetry*. This symmetry limit appears when $p=1/2$ in Eq. (14). The SO(5) irreps are labeled by a quantum number τ and the eigenstates may be labeled by τ and the spin S , $\psi(\text{SO5}) = |N, \tau, S, m_s\rangle$ with $N = \Omega/2 - \tau + \lambda$, where λ is the number of π pairs. The irreducible representation dimensionality for given N is $(\lambda+1)(\lambda+2)/2$ and the ground state has $\lambda=0$ and $S=0$.

The SO(5) dynamical symmetry has very unusual behavior. Although the expectation values of Δ and Q for ground state in this symmetry limit are the same as that of Eq. (25) for the SU(2) case, there exist a huge number of states with different values of λ (the number of π pairs) that can mix easily with the ground state when x is small because the excitation energy in this symmetry limit is

$$\Delta E = \lambda G_{\text{eff}}^{(0)}\Omega x \quad (26)$$

(see Table II). In particular, at half filling ($x=0$) the ground state is highly degenerate with respect to λ and mixing different numbers of π pairs in the ground state costs no energy. The π pairs must be responsible for the antiferromagnetism in this phase, since within the model space only π pairs carry spin. Thus the ground state in this symmetry limit has large-amplitude fluctuation in the AF order (and SC order). This indicates that the SO(5) symmetry limit is associated with phases in which the system is extremely susceptible to fluctuations between AF and SC order.

D. Energy Surfaces

The soft nature of the SO(5) phase is seen most clearly if we introduce approximate solutions in terms of SU(4) coherent states.¹⁵ We shall discuss the interpretation of SO(5) using coherent states in a separate paper,¹⁶ but we quote one result of that study here. In Fig. 2, ground-state energy surfaces for various particle number n (or doping x) are plotted as a function of a quantity β , which is related directly to the AF order parameter (see figure caption). Three plots are associated with the symmetry limits discussed above (p

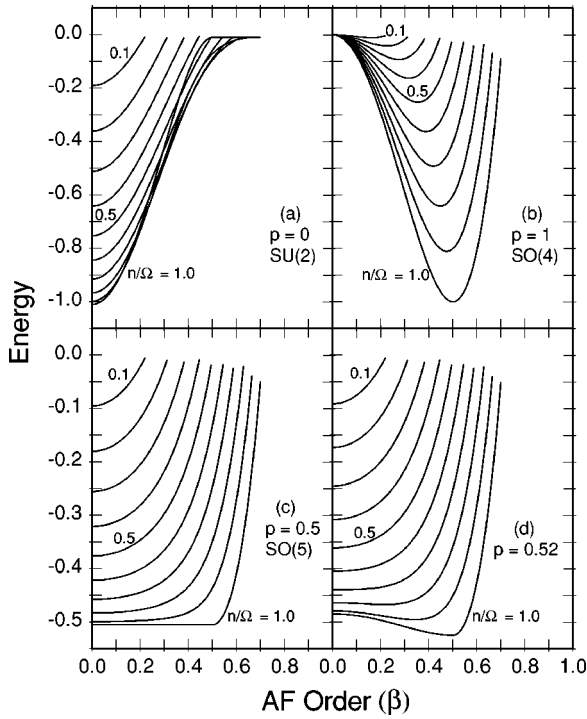


FIG. 2. Coherent-state energy surfaces for Eq. (14). The energy unit is $G\Omega^2/4$ [see Eq. (14)]. The staggered magnetization Q is related to β by $\langle Q \rangle = 2\Omega\beta_0(n/2\Omega - \beta_0^2)^{1/2}$, where β_0 is the value of β at the stable point, so β measures the AF order. Numbers on curves are the lattice occupation fractions, with $n/\Omega=1$ corresponding to half filling and $0 < n/\Omega < 1$ to finite hole doping.

$=0, 1/2$), and one corresponds to a situation with a slight SO(5) symmetry breaking ($p=0.52$). For all doping values x the energy minimum lies at $\beta=0$ (implying that $Q=0$) in the SU(2) limit [Fig. 2(a)], while it lies at

$$\beta = \sqrt{\frac{1}{4}(1-x)}$$

(implying that $Q=n/2$) for the SO(4) limit [Fig. 2(b)]. In Fig. 2(c), there is a broad range of doping in which the SO(5) energy surface is almost flat in the parameter β , implying large excursions in the AF (and SC) order: one can fluctuate into the other at negligible energy cost. Notice in Fig. 2(d) that as doping varies the SO(5) Hamiltonian with slight symmetry breaking interpolates smoothly between AF order at half filling and SC order at smaller filling. Thus SO(5) acts as a kind of doorway between SU(2)_p and SO(4) symmetries and this gives a precise meaning to the assertion⁴ that the SO(5) symmetry rotates between AF and SC order.

However, the present discussion shows that the relation between the AF and SC phases is more complex than a simple SO(5) rotation because of the non-Abelian nature of the SU(4) parent group of SO(5). Such dynamical symmetries that define a phase of the theory but that also interpolate between two other dynamical symmetries are known in nuclear structure physics where they have been termed *critical dynamical symmetries*.¹⁷

The soft SO(5) energy surface could lead to “spin-glass-like” phases (by which we mean phases with local AF or SC

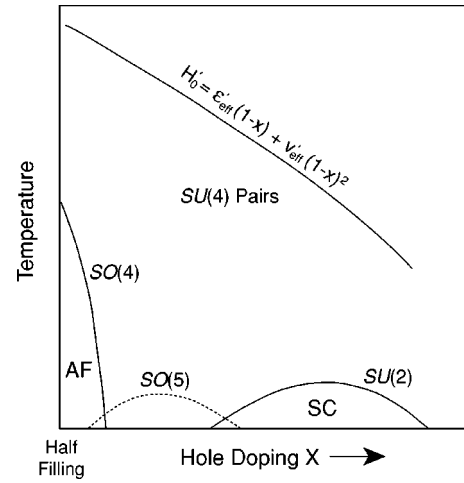


FIG. 3. Schematic phase diagram for SU(4) symmetry based on Fig. 1(d). The H'_0 in Eq. (14) is expressed in terms of hole doping x with $x = 1 - n/\Omega$; $\epsilon'_{\text{eff}} = \epsilon_{\text{eff}}\Omega/2$, and $v'_{\text{eff}} = v_{\text{eff}}\Omega^2/2$.

order but with large fluctuations in both). It could also lead to inhomogeneous structurelike stripes if there is a periodic spatial modulation of the system, since the soft nature of the energy surface implies that relatively small perturbations can shift an SO(5) system between AF and SC behavior. As noted in an earlier footnote, perturbations of the symmetry by a charge-density wave could be one source of such a spatial modulation.

VI. PHASES AND PHASE TRANSITIONS

In quantitative tests of the SU(4) model with parameters determined by fitting to pairing gap and pseudogap data, a cuprate phase diagram has been predicted. It is found that for the symmetry-mixing parameter p close to but larger than 0.5 [antiferromagnetically perturbed SO(5) symmetry] the experimental data are described quite well. The results of this study will be published separately.^{18,19} In this paper, we concentrate on general features and use the preceding discussion to construct the qualitative phase diagram illustrated in Fig. 3. This diagram, which contains essential features of the realistic phase diagram, is constructed based on the following arguments.

A. Phase diagram

Consider the Hamiltonian (14) and, to simplify this discussion, let us assume the spin term $k_{\text{eff}}\vec{S} \cdot \vec{S}$ to be neglected. In that case, there are two fundamental energy scales in the Hamiltonian (14): H'_0 and $H - H'_0 \sim G\Omega^2$. The term H'_0 originates in the single-particle energies $\epsilon_{\text{eff}}n$ and the SU(4) invariant $C_{SU(4)}$. This term depends only on particle number and is isotropic: it has the same expectation value for any state in the SU(4) representation space. In contrast, the other terms $H - H'_0$ (with characteristic energy scale $G\Omega^2$) are anisotropic in the SU(4) space: states associated with different dynamical symmetries may have quite different expectation values. (Thus, if we make a mean-field approximation to the

present many-body theory—like the method of coherent states—these anisotropic terms will lead to spontaneously broken symmetries.)

The term H'_0 in the Hamiltonian (14) may be regarded as the energy scale for the $U(4) \supset U(1) \times SU(4)$ symmetry, which describes a fermion system in which electrons are all paired but with no distinction among the d pairs and π pairs. We may expect this symmetry to hold while the thermal energy is less than H'_0 . When the system is cooled, the thermal energy eventually drops below the anisotropic scale $G\Omega^2$, the anisotropic pairing and antiferromagnetic correlations $H - H'_0$ become important, and $SU(4)$ breaks to its subgroups. Then different low-temperature phases appear, with the favored phases depending on the competition between $D^\dagger D$ and $\vec{Q} \cdot \vec{Q}$ interactions as a function of doping concentrations.

From the preceding discussion, at zero temperature we expect the $SO(4)$ AF phase to dominate at half filling if $p > 0.5$, because the energy surface is minimized at $\beta = [\frac{1}{4}(1-x)]^{1/2}$, implying that the staggered magnetization is large: $Q = n/2$. On the other hand, at larger doping the $SU(2)$ pairing phase is favored (the pairing energy is optimized and the staggered magnetization minimized) [see Fig. 2(d)]. Finally, the intermediate doping region is described naturally by the $SO(5)$ critical dynamical symmetry that interpolates between SC and AF behavior with doping. Thus, $SU(4)$ symmetry implies the schematic phase diagram of Fig. 3, independent of detailed calculations.

B. Phase transitions and symmetry breaking

In the Hamiltonian (14), the parameter p takes on the values 0, 1, and 1/2 in the three symmetry limits of the theory. For any other allowed value of p the system exhibits $SU(4)$ symmetry but there is no dynamical symmetry (if $p \neq 0, 1, 1/2$, the eigenstates of the system are linear superpositions of eigenstates from the three dynamical symmetry chains). In this case, phase transitions are driven by microscopically determined control parameters that change the expectation value of the terms of the Hamiltonian such that $\langle \vec{Q} \cdot \vec{Q} \rangle$ is dominant in some circumstances while $\langle D^\dagger \cdot D \rangle$ dominates in others. This permits phase transitions among the AF [$SO(4)$ limit], $SO(5)$, and SC phases [$SU(2)$] to be studied using a fixed Hamiltonian. In cuprates, the hole-doping x is an example of such a microscopically determined control parameter. Thus, a Hamiltonian that possesses an approximate $SO(5)$ symmetry [antiferromagnetic perturbed $SO(5)$] can have AF solutions at small hole doping and SC solutions at large hole doping. We shall give an explicit example of a phase transition driven microscopically by changing hole doping in the next section.

VII. THE TRANSITION BETWEEN ANTIFERROMAGNETISM AND SUPERCONDUCTIVITY

Above, we have used the method of generalized coherent states to give an interpretation of the $SO(5)$ subgroup as a critical dynamical symmetry interpolating between AF and

SC order and having the character of a “spin glass” or perhaps leading to stripe phases for a large range of intermediate doping parameters.¹⁶ In this section we address further the relationship between the other two phases of the theory. We show that the microscopic symmetry incorporates a differing dependence on doping for SC and AF order. This implies that the group structure itself controls the transition between the superconducting $SU(2)$ symmetry and the antiferromagnetically ordered $SO(4)$ symmetry. Thus, we conclude that the $SU(4)$ symmetry leads naturally to AF order at half-filling and to d -wave superconductivity as the system is hole-doped away from half filling for a broad range of Hamiltonian parameters.

A. A simplified picture

To simplify the discussion, we drop the common dependence of both phases on the spin and charge generators and consider the competition between the $SO(4)$ stabilization energy coming from a term $\vec{Q} \cdot \vec{Q}$ and the $SU(2)$ stabilization energy coming from a term $D^\dagger D$ in the Hamiltonian. These clearly have fundamentally different behaviors with changing particle number. From Table II and Eq. (16), we may conclude that, in the respective ground states,

$$\begin{aligned} \langle \chi \vec{Q} \cdot \vec{Q} \rangle &= \chi N(N+2), \\ \langle G_0 D^\dagger D \rangle &= G_0 N(\Omega - N + 1), \end{aligned} \quad (27)$$

where the pair number is $N = \frac{1}{2}n$. At half filling ($N = \Omega/2$), if $\chi/G_0 > 1$, the Hamiltonian exhibits an effective $SO(4)$ symmetry because the $SO(4)$ correlation energy $\chi \vec{Q} \cdot \vec{Q}$ dominates. As the particle number decreases (hole doping x increases), the $SO(4)$ correlation energy decreases quadratically but the d -wave pairing $D^\dagger D$ decreases much more slowly (essentially linearly). Therefore, the pairing correlation $D^\dagger D$ will eventually dominate and the Hamiltonian exhibits effective $SU(2)$ symmetry.

These features imply immediately that, if $\chi/G_0 > 1$, antiferromagnetism will tend to dominate at half filling but, as the system is doped away from half filling with holes, eventually pairing will dominate and the system will become superconducting. The transition point will depend on the relative strengths of the $D^\dagger D$ and $\vec{Q} \cdot \vec{Q}$ terms in the effective Hamiltonian, but the AF ground state at half-filling and the tendency to superconductivity as the system is doped away from half filling is a direct consequence of the group structure, independent of detailed parameter choices.

B. Analogies in nuclear structure

This competition between antiferromagnetism and superconductivity bears many strong resemblances to the competition in nuclear physics between spherical and deformed structure for nuclei. There, it has been shown that the transition from spherical nuclei, which dominate the beginnings and endings of shells, to deformed nuclei, which often dominate the middle of shells, is controlled by the microscopic competition between long-range quadrupole-quadrupole interactions favoring deformation and short-range monopole

pairing interactions favoring spherical vibrational structure. This competition in nuclear structure may be expressed algebraically as a competition between a dynamical symmetry that favors pairing and a dynamical symmetry that favors multipole (particle-hole) interactions.¹

The essential physics of the spherical-deformed transition in nuclear structure physics has been shown to be determined by the differing particle number dependence of the dynamical symmetries: nuclear pairing energy increases linearly with particle number but the quadrupole deformation energy is quadratic in particle number [that is, essentially the same relationship as for Eq. (27)]. Thus, the group structure dictates that spherical vibrational nuclei (favored by pairing energy) dominate the ends of shells and deformed nuclei (favored by the deformation energy) dominate the middle of shells.²⁰ This behavior is a close analog of the competition between antiferromagnetism dominating the half-filled lattice and superconductivity dominating the hole-doped lattice that we have discussed in this paper.

This analogy between condensed-matter and nuclear physics might not be accidental. The structure of heavy nuclei and of high-temperature superconductors both correspond to complex many-fermion problems in which strong particle-hole and pairing interactions involving d pairs play pivotal roles. It has been demonstrated that dynamical symmetries in the microscopic fermion degrees of freedom provide a simple but powerful unifying principle for the nuclear structure problem.¹ Recent results, such as those presented here and in Refs. 4 and 21, suggest that related symmetries may play a similar role for condensed matter and that these problems from rather different fields of physics may have a common dynamical algebraic structure.

VIII. SU(4) OR SO(5)?

Let us now discuss more precisely the relationship between our SU(4) model and the Zhang's SO(5) model.⁴ Although the methodologies used to obtain the two models are rather different, the Zhang SO(5) group is a subgroup of our SU(4) group and both model Hamiltonians possess antiferromagnetic perturbed SO(5) symmetry, implying that the two models are closely related to each other. Our SU(4) model and Zhang's SO(5) model have the same building blocks [the operator set (4), but see Ref. 22]. The essential difference is that we implement the full quantum dynamics (commutator algebra) of these operators exactly, while in Ref. 4, the dynamics is implemented in an approximate manner: a subset of ten of the operators acts as a rotation on the remaining five operators $\{D^\dagger, D, \vec{Q}\}$, which are treated phenomenologically as five independent components of an order-parameter vector (superspin \vec{n}). Thus only 10 of the 15 generators of our SU(4) are treated fully dynamically in the Zhang SO(5). If the full quantum dynamics (full commutator algebra) of the 15 operators is taken into account, the symmetry is SU(4), not SO(5).

The embedding of SO(5) in our larger algebra has various physical consequences that do not appear if the SO(5) subgroup is considered in isolation. We list four:

(1) As we have seen, a phase transition from antiferro-

magnetism to superconductivity at zero temperature that is controlled by the doping emerges naturally and microscopically, whereas in the SO(5) model a symmetry-breaking term proportional to a chemical potential has to be introduced by hand. This occurs because our SO(5) is embedded in a larger group that contains generators breaking SO(5) but preserving SU(4).

(2) The present results suggest that the SO(5) subgroup is only one of the symmetries relevant to the cuprate problem. It is a transitional symmetry that links AF to SC behavior, suggesting that it is most useful for the underdoped region. But the AF phases at half filling and the optimally doped superconductors are more economically described by our SO(4) and SU(2)_p symmetries, respectively. All are unified in the SU(4) highest symmetry.

(3) As we discuss in a separate publication,¹⁸ the present SU(4) theory leads naturally to the appearance of pseudogap behavior,²³ which occurs above the SC transition temperature T_c . The embedding of SO(5) in SU(4) is central to this behavior.

(4) The different methodology of the SU(4) dynamical symmetry approach suggests a different physical interpretation of the SO(5) subgroup symmetry. We suggest that it should be viewed as an effective symmetry that operates in a severely truncated space. As we shall elaborate below, this implies that its microscopic validity is a question of the physical correctness of the matrix elements evaluated in that truncated model space, not of whether a particular Hamiltonian with some relevance for the full space (Hubbard, for example) possesses such a symmetry.

Thus, both the Zhang SO(5) theory and the present SU(4) theory imply the existence of an approximate SO(5) symmetry in the Hamiltonian of high-temperature superconductors. However, the SU(4) theory encompasses a broader range of physics that may be relevant for cuprate superconductor. Further, as we have argued, a unified quantum mechanical treatment of the generators and order parameter vector of the Zhang SO(5) implies that the full quantum mechanical symmetry is SU(4) and not SO(5).

IX. DISCUSSION

The $U(4) \supset U(1) \times SU(4)$ symmetry represents a fully microscopic fermion system in which SC and AF modes enter on an equal footing. At this "unification" level, there is in a sense no distinction among these degrees of freedom, just as in the Standard Electroweak Theory of elementary particle physics the electromagnetic and weak interactions are unified above the intermediate vector boson mass scale. We may expect this symmetry to hold while the temperature of the system is sufficiently high that anisotropic quantum fluctuations in the directions associated with these collective degrees of freedom can be neglected [recall that the SU(4) scale is set by H'_0 and the anisotropic scale by $G\Omega^2$], but not so high that thermal fluctuations destroy the integrity of the collective modes corresponding to the SU(4) symmetry. This SU(4) region then corresponds to the pseudogap regime.¹⁸

Generalizing language already introduced by Zhang,⁴ in this regime we may view the system as having condensed

into $SU(4)$ pairs, which fixes the length of the state vectors [through the $SU(4)$ Casimir expectation value] but not their direction in the state-vector space. Physically, this means that the system is paired, with the pair structure exhibiting $SU(4)$ symmetry, but is neither SC nor AF because fluctuations in those directions in the $SU(4)$ space are small relative to the scale set by the temperature. Stated in another way, the $SU(4)$ pairs are of collective strength, but are not condensed into a state with long-range order. Stated in yet another way, neither the AF nor SC order parameters have finite expectation values in this regime, but a sum of their squares [the $SU(4)$ Casimir] does. This constraint implies an intimate connection between superconductivity and antiferromagnetism in the $SU(4)$ model. They are, in a sense, different projections of the same fundamental vector in an abstract algebraic space.

Compared to the Hubbard or t - J models, the dynamical symmetry approach applied here is just a different way to simplify a strongly correlated electron system. In the Hubbard or t - J models, approximations are made to simplify the Hamiltonian but no specific truncation is assumed for the configuration space, although in practical calculations a truncation is typically necessary. In contrast, we make no approximation to the Hamiltonian. The only approximation is the (severe) space truncation. The symmetry dictated Hamiltonian includes all possible interactions in the truncated space. In principle, if all the degrees of freedom of the system are included, this approach constitutes an exact microscopic theory. The validity of this approach depends entirely on the validity of the choice of truncated space and its effective interactions, which may be tested by comparison with the data.

Thus, we suggest that effective low-energy theories of the type discussed here need not have much direct relation to the Hamiltonian or wave functions of the Hubbard or t - J models, because both the Hamiltonian and the wave functions in different model spaces could be very different. What is observable quantum mechanically is the matrix elements, not operators or wave functions separately. As the $SU(4)$ theory makes clear, we may view these dynamical symmetries as operating in a severely truncated collective subspace in which the truncation has been implemented primarily by symmetry considerations and only secondarily by energy criteria. Thus, it is possible that the matrix elements of the $SU(4)$ theory and a Hubbard or t - J model calculation might be comparable, even if the Hamiltonians and wave functions are separately quite different.

The advantage of the dynamical symmetry approach is its cleanness and simplicity. It is clean because the *only* approximation is the selection of the truncated model space. Thus, a failure of the method is a strong signal that one has chosen a poor model space. It is simple because the method supplies analytical solutions for various dynamical symmetry limits as a starting point. These symmetry-limit solutions provide an immediate handle on the physics and permit an initial judgment of the model's validity without large-scale numerical calculations. Beyond the symmetry limits, numerical calculations are necessary. However, because of the low dimensionality of the models spaces and the power of group

theory, such numerical calculations are much simpler than those in, say, a Hubbard or t - J model.

The primary limitation of the dynamical symmetry approach is that the interactions of the model space are necessarily effective interactions that will generally be strongly renormalized by the symmetry-constrained space truncation. Thus, their strengths must be supplied separately from the dynamical symmetry method itself, either phenomenologically (by fitting model-space interaction strengths to data), or by a microscopic study of the relationship between the effective and full-space Hamiltonians.

Finally, let us counter a possible philosophical criticism of the approach taken here. Simple symmetries as a predictor of dynamics has found powerful application in fields such as particle physics or nuclear physics. However, there is a common point of view that the possible ground states in high-temperature superconductors are too complex to permit a simple model like the current one (which operates in a drastically truncated subspace having simple wave functions and highly stylized operators) to have any validity. Although one can make such an argument formally, this ignores the rather obvious point that nature has managed to construct a stable ground state having well-defined, collective properties that change systematically from compound to compound. Extensive experience in many fields of physics suggests that this is the signal that the phenomenon in question is described by a *small effective subspace with renormalized interactions* (that may differ substantially in form and strength from those of the full space) and governed by a symmetry structure of manageable dimensionality. Thus, if an approach like the one proposed here gives correct results for highly nontrivial phenomenology like the doping dependence of observable quantities, one must take seriously the possibility that the corresponding small symmetry-dictated subspace may have relevance to the effective behavior of real physical systems, no matter how complex they may appear to be superficially.

X. SUMMARY AND CONCLUSIONS

In summary, an $SU(4)$ model of high- T_c superconductivity has been proposed that contains three dynamical symmetries: A SC phase identified with the $SU(2)_p$ dynamical symmetry, an AF phase identified with the $SO(4)$ dynamical symmetry, and an $SO(5)$ phase extremely soft against AF and SC fluctuations over a substantial doping fraction that serves as a critical dynamical symmetry interpolating between the other two phases. Realistic systems may mix these subsymmetries while retaining an approximate $SU(4)$ symmetry. Zero-temperature phase transitions are shown to be driven by the competition between the d -wave pairing and the AF $\vec{Q}\cdot\vec{Q}$ interactions, as controlled microscopically by the hole-doping concentration. This model leads naturally to the appearance of pseudogaps in the underdoped regime because it introduces multiple energy scales that permit pairs to form before they condense into states with long-range order.

Thus, we propose that high- T_c behavior of the cuprates results from an $SU(4)$ symmetry realized dynamically, and because this symmetry is microscopic its physical interpreta-

tion is accessible to calculation. This provides a solvable model incorporating many features of cuprate superconductors, a possible understanding of the cuprate phase diagram as arising from dynamically realized symmetries, and substantial insight concerning recent SO(5) theories of d -wave superconductivity.

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⁹One can consider even larger algebras that have the advantage of incorporating a broader range of physics but at the expense of additional parameters. We adopt the point of view that one should first investigate thoroughly the minimal algebra consistent with the degrees of freedom that are likely to be important for the problem at hand.

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¹²The reader should recall that spontaneous symmetry breaking is a useful concept but that the name is a misnomer. A spontaneously broken symmetry corresponds in reality to a conserved symmetry that has been hidden but is still present; it merely manifests itself in a different way. The physical states for a finite system must be constructed by restoring any symmetries broken by approximate solutions, for example through projection techniques. Thus, BCS solutions are in many respects an excellent description of superconductors, but they are approximate and violate particle number conservation. The true physical states of a superconductor must then correspond to a projected combination of BCS states that conserves particle number. Of course, as a practical matter, restoration of the symmetry may not be too

important if the effect of the symmetry violation is small for the physical quantity of interest.

¹³If only part of the system is assumed to constitute the d -wave subspace (for example, in a two-fluid model), the U(1) factor can produce charge density wave excitations in the non-superfluid space. Because of the direct product structure, in the symmetry limits these charge density waves would be independent of any excitations in the d -wave subspace. In the presence of symmetry breaking, the charge density waves could perturb the d -wave subspace and generate inhomogeneous structures like stripe phases, as we discuss in Sec. V D.

¹⁴Groups generally may have more than one Casimir invariant. We shall use the term ‘‘Casimir’’ to refer loosely to the lowest-order of such invariants (which are generally quadratic in the group generators). In the context of the present discussion, quadratic Casimirs are associated with two-body interactions at the microscopic level. Higher-order Casimirs are then generally associated with three-body and higher interactions. The restriction of our Hamiltonians to polynomials of order 2 in the Casimirs is then a physical restriction to consideration of only one-body and two-body interactions.

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