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Small-amplitude mobile solitons in the two-dimensional ferromagnet

B. A. Ivanov,[1,2] C. E. Zaspel,[3] and I. A. Yastremsky[2]
[1]Institute of Magnetism, National Academy of Sciences of Ukraine, 03127, Kiev, Ukraine
[2]Taras Shevchenko Kiev University, 03127, Kiev, Ukraine
[3]Department of Physics, Montana State University–Bozeman, Bozeman, Montana 59717
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Two-parameter small-amplitude magnetic solitons moving with arbitrary velocity on a two-dimensional easy-axis ferromagnet are obtained from the approximate solution of the dynamics equations for this system. Both radially symmetric solitons and structures with a quasivortex core are considered. These solitons are characterized by precession of the magnetization about the easy axis, and the corresponding integrals of motion, namely, the energy E, the linear momentum P, and the number of bound magnons N are calculated. The radially symmetric solitons are shown to have the lower energy of the two and they are stable, whereas the lower energy solitons are unstable. For the stable solitons it was found that the dispersion relation $E=E(P,N)$ was found to have a minimum at the value of $E_0=11.2JS^2$ on the ellipse of arbitrary size. It is remarkable that this energy is approximately independent of the parameters except for the exchange constant J and the value of the atomic spin S. Also, the value of E_0 is only slightly smaller than the energy of the well-known Belavin-Polyakov soliton, $E_0=0.93E_{\mathrm{BP}}$, where $E_{\mathrm{BP}}=4\pi JS^2$. Finally the soliton dispersion relation is used to calculate the soliton density, and a comparison of the soliton density with the magnon density shows that there is a wide range of temperatures where solitons will give the dominant contributions to thermodynamic quantities. It is expected that these solitons will give an essential contribution to observed dynamical quantities such as the spin-correlation functions.

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I. INTRODUCTION

Nonlinear topologically nontrivial excitations, or solitons, are known to exist in lower dimensional magnetic systems. There is both theoretical and experimental indication of solitons, and in some cases, their effects dominate the thermodynamic behavior of one-dimensional (1D) and two-dimensional (2D) magnetic systems. Soliton effects were possibly first established through the proof of the absence of long-range order in lower dimensional magnets, where from simple entropy arguments it has been shown that this order is not possible in 1D and 2D isotropic systems. Later it was shown that kinks in 1D magnets[1–3] and localized Belavin-Polyakov solitons (BP solitons) in 2D isotropic magnets[4] are responsible for the destruction of the long-range order at a nonzero temperature. Furthermore, the presence of vortices in 2D easy-plane magnets gives rise to the Berezinskii-Kosterlitz-Thouless phase transition.[5,6] Solitons in 1D and 2D magnets have not been directly observed; however, the dynamic soliton effect such as soliton motion and the soliton-magnon interaction results in soliton contributions to the dynamic response functions, which can be studied experimentally. For example, translational motion of vortices leads to the so-called soliton central peak,[7–9] which can be detected through neutron-scattering experiments. The same should be valid for localized excitations such as solitons of the BP type.[10,11]

The first experimental detection of localized solitons in 2D antiferromagnets was done indirectly through measurement of the temperature dependence of the electron paramagnetic resonance (EPR) linewidth.[12–15] The experiments were done using layered manganese compounds in the narrow temperature range immediately above the Neel temperature. For these experimental conditions the magnetic correlations are 2D and Mn(II) is nearly classical with a spin of $\frac{5}{2}$. In this temperature range, an Arrhenius ($e^{E/T}$) behavior of the temperature-dependent linewidth is observed and the excitation energy is identified as the soliton energy. For a review of EPR detection of solitons see Ref. 10.

This previous theoretical work can be explained within the scope of the so-called soliton phenomenology, where the magnetic system can be described as a two-component gas of elementary excitations: solitons and magnons. However, the general behavior of 2D soliton dynamics is not clear at present.

In this article we investigate the structure and dynamics of different types of solitons with finite energy in the easy-axis ferromagnet, and an exact analytical dispersion relation for nontopological solitons is constructed. It is shown that these solitons are characterized by a universal value of the energy, which appear in an Arrhenius law for all thermodynamic quantities of 2D ferromagnets. This value is independent of the anisotropy, it is determined by the ferromagnetic exchange integral, and it is slightly smaller than the energy of topological (BP) solitons in isotropic magnets. We also discuss the dynamical properties of solitons in antiferromagnets and compare the soliton features for these two types of magnets. Finally, the relative magnitudes of the effects arising from solitons or magnons is calculated and it is shown that there is a temperature range where solitons have the dominant influence on thermodynamic quantities.

II. FERROMAGNETIC MODEL, MOTION INTEGRAL, AND SOLITON CLASSIFICATION

In the dissipationless limit the ferromagnetic magnetization may be described by the unit vector **m**, which is expressed in terms of angular variables:

$$m_z = \cos\theta, \quad m_x + i m_y = \sin\theta \sin\varphi.$$

Then, the dynamics of these variables can be described by the following Lagrangian:[3,16]

$$L = \frac{\hbar S}{a^2} \int (1-\cos\theta) \frac{\partial\varphi}{\partial t} d^2 r - W\{\theta,\varphi\}, \tag{1}$$

where S is the atomic spin, a is the lattice constant, and W is the ferromagnetic energy functional

$$W\{\theta,\varphi\} = \frac{S^2}{2} \int d^2 r \{J[(\nabla\theta)^2 + (\nabla\varphi)^2 \sin^2\theta] + K \sin^2\theta\} \tag{2}$$

for the uniaxial magnet. Here J is the exchange integral, K is the anisotropy constant, which is assumed to be small, $(K \ll J)$. Owing to spin symmetry of the Lagrangian (1) and regarding rotational invariance of the energy (2), the z component of the total spin is conserved in the ferromagnet, and the corresponding integral of motion,

$$N = \frac{S}{a^2} \int (1-\cos\theta) d^2 r, \tag{3}$$

can be interpreted as corresponding to the number of bound magnons in the soliton. The Lagrangian (1) also determines the total linear momentum, **P** of the magnetization field:

$$\mathbf{P} = -\frac{\hbar S}{a^2} \int (1-\cos\theta) \nabla\varphi \, d^2 r. \tag{4}$$

For topological solitons the value of **P** is not conserved,[17] but for solitons with zero topological charge, which we are interested in, **P** defined by Eq. (4) is an integral of motion. For nonmobile solitons one has $P=0$, and the soliton dependence of the energy on N (Ref. 18) for this special case was established numerically. However, the question of the nature of the dispersion law for 2D solitons in the ferromagnet, $E = E(N,P)$, is still open. In more recent work this case has been analyzed numerically for isotropic[19] and easy-plane[20] ferromagnets.

The basic problem of establishing the dispersion relation lies in the fact that the mobile soliton solutions of the Landau-Lifshitz equation in a ferromagnetic system of dimensionality higher than one are unknown. In the case of a nonmobile soliton one can at least determine the functional form of a solution. In fact, variation of $\delta L/\delta\varphi = 0$ leads to the equation

$$\nabla(\sin^2\theta \nabla\varphi) = -\frac{\hbar}{JSa^2} \frac{\partial\theta}{\partial t} \sin\theta, \tag{5}$$

which is satisfied by a solution of the form

$$\varphi = \omega t + q\chi, \quad \theta = \theta(r), \tag{6}$$

where r and χ are the polar coordinates in the plane of the 2D ferromagnet and q is an integer. Such an ansatz does not contradict the equation obtained from the variation $\delta L/\delta\theta$, which after substitution of Eq. (6) becomes an ordinary differential equation for $\theta(r)$, with the condition $\theta \to 0$ as $r \to \infty$:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left(\frac{q^2}{r^2} + \frac{1}{r_0^2}\right) + \frac{\omega}{\omega_0 r_0^2}\sin\theta = 0. \tag{7}$$

Here $\omega_0 = KS/\hbar$ is minimal frequency of the linear magnons (magnon gap), r_0 is the characteristic length determined by the ratio of the exchange energy to the energy of magnetic anisotropy, $r_0^2 = a^2 J/K$. This equation has solutions only for $0 < \omega < \omega_0$, and its form depends essentially on the value of q. If $q=0$, then we get a soliton like the magnon drop[18] with a radial distribution of magnetization. The boundary conditions $\theta < \pi$, $d\theta/dr = 0$ at $r=0$ correspond to such a soliton. If $q \neq 0$ (in the following we consider the only the case $q = 1$), then the various solitonic states are realized. The solution with $\theta(0) = \pi$ corresponds to a topological soliton with Pontryagin index $Q = q$. Another possible solution that has the topological charge $Q=0$, and the condition $\theta(0)=0$ corresponds to this case with the function $\theta(r)$ reaching a maximum value $\theta_{\max} < \pi$ at nonzero r.

The topological solitons have been reviewed[3,16] in detail in a series of works. In the isotropic case $K=0$ they exist as static objects, where the exact analytical BP solution ($\tan[\theta/2] = R/r$, R is the soliton radius) is known. The energy of the BP soliton does not depend on R and is determined by a general formula,

$$E_{\mathrm{BP}} = 4\pi J S^2, \tag{8}$$

and since in such a soliton $N \propto R^2$, the energy does not depend on N. The transverse motion of this soliton in a ferromagnet is impossible, and in this system the momentum is not an integral of motion.[17] The effective mass of this soliton diverges[21–23] with increasing system size, and non-Newtonian dynamical equations have been found to describe the motion of the magnetic vortex in an easy-plane ferromagnet.[24] Despite these remarks, the question of topological soliton motion in ferromagnets, in particular, the dispersion law of these solitons, is of interest and we proceed with the analysis of nontopological solitons.

III. SMALL AMPLITUDE NONTOPOLOGICAL SOLITONS

For both types of nontopological solitons, with $q=0$ or $q \neq 0$, the linear momentum (4) is an integral of motion and mobile solutions of the form $\mathbf{m} = \mathbf{m}(\mathbf{r},t) = \mathbf{m}(\mathbf{r} - \mathbf{V}t)$ or $\theta = \theta(\tilde{r},\tilde{\chi})$, $\varphi = \omega t + \psi(\tilde{r},\tilde{\chi})$ should exist. Here by means of the tilde the moving coordinate system is indicated,

$$\tilde{r}^2 = (x - Vt)^2 + y^2, \quad \tilde{\chi} = \arctan[y/(x - Vt)],$$

with the soliton velocity assumed to be parallel to the x axis. In the following we will deal only with these coordinates and omit the tilde for simplifications of the equations. However, the basic problem here is that one cannot begin with a simple ansatz like Eq. (6) and reduce the problem to the analysis of an ordinary differential equation. The only exception is the case of small soliton amplitude where $\theta_{\max} \ll 1$. It is easily seen that to the order of $\theta_{\max}$, Eq. (5) becomes $\nabla[\theta^2(\nabla\varphi^{(0)} - \mathbf{V}/V_m r_0)] = 0$, where $V_m = 2\omega_0 r_0$ is the minimal phase velocity of spin waves. Then to the first approximation $\theta \leq \theta_{\max} \ll 1$ the solution exhibiting Galilean invariance is

$$\theta = \theta(r), \quad \varphi = \varphi^{(0)} = \omega t + q\chi - Vx/r_0 V_m, \tag{9}$$

and the equation for $\theta(r)$ takes the following form:

$$r_0^2 \nabla^2 \theta - \sin\theta \cos\theta \left[1 + \left(\frac{V}{V_m} \right)^2 + \frac{2qVr_0 \sin\chi}{rV_m} + \frac{q^2 r_0^2}{r^2} \right] + \sin\theta \left(\frac{\omega}{\omega_0} + \frac{qVr_0 \sin\chi}{rV_m} + \frac{V^2}{\omega_0 r_0 V_m} \right) = 0. \tag{10}$$

It is remarked that for $q=0$ Eq. (10) does not conflict with ansatz (9) because if $q \neq 0$ and θ is small, then the terms with $\sin\chi$ cancel each other, and there is a simple solution: $\theta = \theta(r)$, with φ determined by Eq. (9). This situation is a unique, but in the general case the structure of a moving soliton reduces to establishing stationary solutions of a set of two partial differential equations for functions $\theta(\tilde{r},\tilde{\chi})$ and $\psi(\tilde{r},\tilde{\chi})$.

There are no general methods to solve such problems, and an analysis can be done numerically with the use of different variational methods.[19,20,25] However, as we will present below, namely for solitons with small amplitude, it is of interest to analyze low-temperature thermodynamics of ferromagnets; therefore, in the following only these solitons are considered. As in the case immobile solitons, the structure of a soliton is determined by solution of Eq. (10) when $\sin\theta$ and $\cos\theta$ are expanded to third order in powers of θ:

$$r_0^2 \nabla^2 \theta - \left[1 - \frac{\omega}{\omega_0} - \left(\frac{V}{V_m} \right)^2 \right] \theta + \left[1 + \left(\frac{V}{V_m} \right)^2 \right] \frac{\theta^3}{2} = 0.$$

The soliton amplitude is small if the coefficient of θ is small; this coefficient, ε, is the small parameter of the theory that is

$$\varepsilon^2 = 1 - \frac{\omega}{\omega_0} - \left(\frac{V}{V_m} \right)^2 \ll 1. \tag{11}$$

The equation with $\varepsilon = 0$ gives the dispersion relation of linear magnons, where the quantity V_m is the minimal phase velocity of magnons, $V_m = \min\{\omega(k)/k\}$. Hence, as in the 1D case, the condition $\varepsilon = 0$ corresponds to the transition of solitons with small amplitude to linear magnons. As will be shown below, a specific feature of the 2D case is the appearance of a typical macroscopic energy of a 2D soliton E_0 that is comparable with energy of the BP soliton E_{BP}.

Using condition (11) and retaining cubic terms in θ, then in this approximation the quantity, $\theta_0(r)$ for is given by the expressions

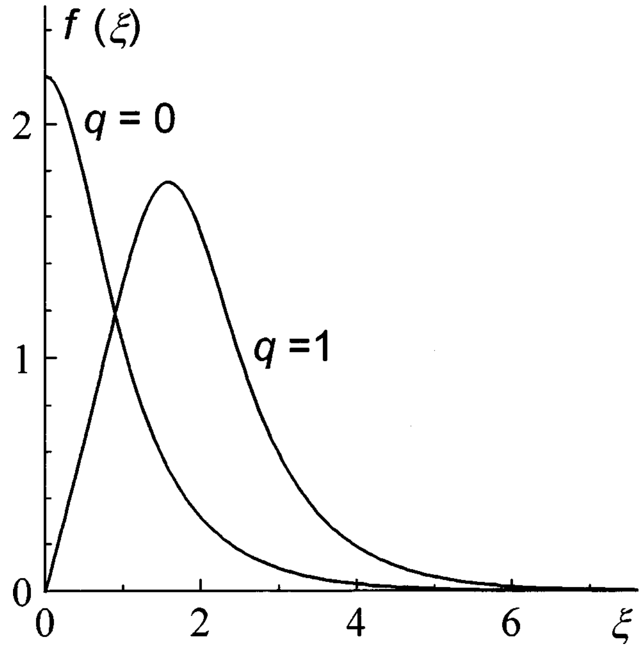

FIG. 1. The solutions of Eq. (13), universal functions $f_q(x)$, for $q=0$ and $q=1$.

$$\theta_0(r) = \frac{\sqrt{2}\varepsilon}{\sqrt{1+(V/V_m)^2}} f(\xi), \quad \xi = \varepsilon \frac{r}{r_0}. \tag{12}$$

Functions $f(\xi)$ for different q are solutions of the equation

$$\frac{d^2 f}{d\xi^2} + \frac{1}{\xi} \frac{df}{d\xi} - f\left(1 + \frac{q^2}{\xi^2} \right) + f^3 = 0, \tag{13}$$

which is similar to the 2D nonlinear Schrödinger (NLS) equation with a cubic nonlinearity. In particular, for $q=0$ it was used for the description of autofocusing of an optical beam.[26] The nontopological soliton solutions of this equation for $q=0$, 1 are found numerically by use of the shooting method are plotted in Fig. 1.

For the description of solitons, at least in this first approximation one can use the NLS equation. However, this approximation, which is adequate in 1D, is not adequate to describe actual properties of solitons, as for example the binding energy of magnons in a soliton. Let us demonstrate this fact by consideration of the example of a soliton with $q=0$. Within the lowest approximation to the energy including terms with θ^2 and $\theta^2(V/V_m)^2$ only, but not θ^4 or $(d\theta/dr)^2$,

$$E^{(\mathrm{zero})} = JS^2 \frac{r_0^2}{2a^2} \int \theta_0^2(r) \left(1 + \frac{V^2}{V_m^2} \right) d^2 r = 2\pi JS^2 \frac{r_0^2}{a^2} \int_0^\infty f^2(\xi) d\xi.$$

Taking into account that $2\pi \int f^2 \xi \, d\xi \approx 11.7$ (Refs. 26 and 18) we get that in this approximation the soliton energy depends neither on the number of bound magnons nor on the total momentum. In this case $E^{(\text{zero})}$ is

$$E^{(\text{zero})}=\hbar\omega_0 N_0,$$

where

$$N_0=2\pi S(r_0/a)^2\int f^2\xi\, d\xi\approx 11.7S(r_0/a)^2.$$

Here the value of N_0 is proportional to the large parameter $(r_0/a)^2=J/K$, and depends strongly on the magnetic anisotropy constant K, but the value of the magnon gap frequency is proportional to K, and the soliton energy in this approximation has the universal value E_0:

$$E_0=2\pi JS^2\int f^2\xi\, d\xi\cong 11.7JS^2. \tag{14}$$

It is remarkable that this is only slightly less that the BP energy $E_{\text{BP}}=4\pi JS^2$, or $E_0\cong 0.93E_{\text{BP}}$. Within this approximation values of N and P can be obtained as integrals over the solution of Eq. (13):

$$N=\frac{2\pi S r_0^2}{a^2}\frac{1}{1+V^2/V_m^2}\int_0^\infty f^2(\xi)\xi\, d\xi=\frac{N_0}{1+V^2/V_m^2},$$

$$P=\frac{2\pi S r_0\hbar}{a^2}\frac{V/V_m}{1+V^2/V_m^2}\int_0^\infty f^2(\xi)\xi\, d\xi=\frac{\hbar N_0}{r_0}\frac{V/V_m}{1+V^2/V_m^2}.$$

These results are now combined to get the soliton dispersion law $E^{(\text{zero})}(P,N)$, which is the same as for N noninteracting magnons with momentum $\mathbf{p}=\mathbf{P}/N=\mathbf{k}\hbar$ on each:

$$E^{(\text{zero})}(P,N)=E_0=N[\hbar\omega_0+JSa^2(P/\hbar N)^2].$$

This equation can be expressed in the form

$$\left(\frac{N}{N_0}-\frac{1}{2}\right)^2+\left(\frac{P}{P_0}\right)^2=\frac{1}{4}, \tag{15}$$

where $P_0=\hbar N_0/r_0$. This shows that the dispersion relation in the limit of small-amplitude solitons corresponds to an ellipse on the $\{N,P\}$ plane, as can be seen in Fig. 2. The values of N_0 and P_0 are macroscopically large, for example, P_0 is much larger that the characteristic value of momentum for 1D solitons, $P_{1\text{D}}\sim\hbar/r_0$ (P_0 is even larger that the size of the Brillouin zone $P_B\sim\hbar/a$). On the other hand, the momentum per magnon is small, $p=P/N\sim\hbar/r_0$, and the applicability of the macroscopic approximation is upheld.

The same features are present for nontopological solitons with $q=1$, only in this case the numbers E_0 or N_0 are different; for example, when $q=1$ the value of $N_0\cong 48.3S(r_0/a)^2$ and E_0 is larger that E_{BP}. Thus, in the lowest approximation to the soliton amplitude nontopological solitons in 2D ferromagnets have the fixed value of energy E_0. In this approximation soliton states are strongly degenerate and their energies do not depend on the number of magnons or momentum, but only on the combination given by Eq. (15), which is characteristic of noninteracting magnons. Thus, the minimal energy is reached not at the point $P=0$, but on the curve in the $\{N,P\}$ plane.

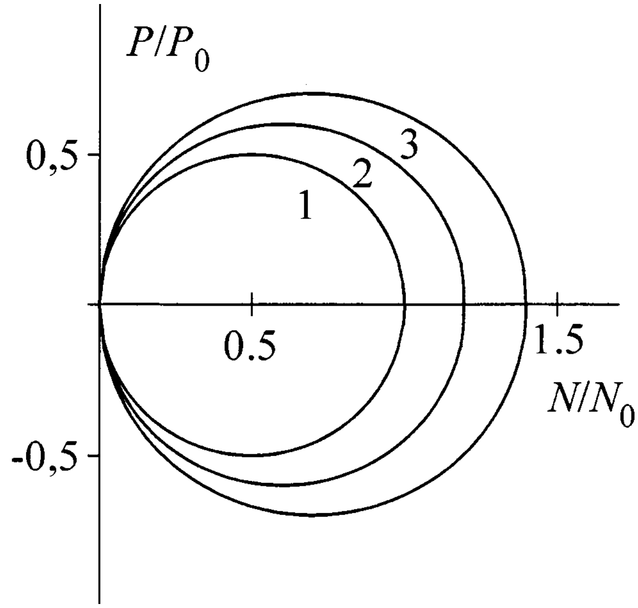

FIG. 2. The curves in the plane (N, P) for given values of the soliton energy E. Curves 1, 2, and 3 correspond to $E=E_0(\varepsilon=0)$, $E=1.2E_0$, and $E=1.4E_0$, respectively.

To describe the binding energy, calculate the dispersion law, and investigate the soliton stability, it is necessary to go beyond the lowest approximation based on the NLS equation. In doing so one has to use not only more exact expressions for P, N, and E (for instance, terms like θ_0^4 or $[d\theta_0/dr)^2]$, but it is also necessary to take into account the higher-order corrections to θ_0 and $\varphi^{(0)}$:

$$\theta(r,\chi)=\theta_0(r)+\vartheta(r,\chi),\quad \varphi(r,\chi)=\varphi^{(0)}+\psi(r,\chi),$$

where the small quantities ϑ, ψ are proportional to higher powers of the small parameter ε compared with θ_0 or $\varphi^{(0)}$, for instance, $\vartheta\sim\varepsilon^3$. The functions $\vartheta(r,\chi)$ and $\psi(r,\chi)$ can be determined from the Landau-Lifshitz equations linearized about θ_0 and $\varphi^{(0)}$. These corrections make contributions to all quantities of interest: energy, momentum, and magnon number. When one tries to do analytical calculations of these quantities in the actual approximation on ε, there arises the problem that the corrections ϑ, ψ are determined by nonhomogeneous differential equations with coefficients depending on the functions $f(\xi)$ known from numerical data only. But this problem can be simplified by means of a method based on the following.

The Landau-Lifshitz equations can be taken from the condition of minimization of the Lagrangian (1) and (2). Due to general properties of variational methods, if the solutions of such equations are known with an accuracy to $\delta\ll 1$, the value of Lagrangian $L=L(V,\omega)$, calculated on this approximate solution, gives the exact value of $L(V,\omega)$ with the accuracy δ^2. Thus, we can find the function $L(V,\omega)$, and then restore the quantities of interest by use of the general relations[16]

$$N=\partial L/\hbar\partial\omega,\quad P=\partial L/\partial V,\quad E(P,N)=\hbar\omega N+PV-L(V,\omega). \tag{16}$$

The concrete analysis will be done for the cases $q=0$ and $q=1$ in the next two sections, showing that the use of this trick can simplify the calculation of the dispersion law $E=E(P,N)$. For example, when $q=0$, use of a definite form for ϑ becomes unimportant, and only the correction ψ must be calculated. The determination of ψ can be done without extra numerical work, and the function $E(P,N)$ can be expressed through two integrals of the universal function $f(\xi)$.

A. Dispersion law for solitons with $q=0$ in ferromagnets

First, it is necessary to calculate corrections to the Lagrangian. To quadratic order of the small functions the Lagrangian of a ferromagnet takes the form $L=L^{(0)}+L^{(1)}+L^{(2)}$, where $L^{(0)}$ depends on θ_0 and $\varphi^{(0)}$ only, $L^{(1)}$ is linear in ϑ and ψ, and $L^{(2)}$ is quadratic in these small functions. The first term is obtained by substitution of θ_0 and $\varphi^{(0)}$ into the Lagrangian. After expansion in powers of θ this term takes the form

$$\frac{L^{(0)}(V,\omega)}{JS^2}=-\frac{A\varepsilon^2}{1+V^2/V_m^2}-\frac{B\varepsilon^4}{2(1+V^2/V_m^2)^2}, \tag{17}$$

where A, B are the integrals over universal function $f(\xi)$:

$$A=2\pi\int_0^\infty\left[\left(\frac{df}{d\xi}\right)^2+f^2-f^4/2\right]\xi\,d\xi \quad\text{and}$$

$$B=\frac{2\pi}{3}\int_0^\infty(f^6-f^4)\xi\,d\xi.$$

Using the following trick, these integrals can be evaluated with minimal numerical work. Multiplying Eq. (13) for $f(\xi)$ by ξf and $df(\xi)/d\xi$, integrating over ξ from 0 to infinity and combining the results, it is easy to show that A can be expressed in terms of the known integral in Eq. (14), $A=2\pi\int f^2\xi\,d\xi\cong 11.7$. Using the same procedure of multiplying Eq. (13) by $\xi^3 f$, one can show that the value of B is positive, $B=2\pi\int f^2(df/d\xi)^2\xi\,d\xi\cong 15.88$. This is the only new numerical data we need to calculate L for the case of the soliton with $q=0$.

Let us next calculate the contribution to $L^{(1)}+L^{(2)}$, taking into account ψ and ψ^2 only. This contribution to the Lagrangian is

$$L^{(1)}+L^{(2)}=JS^2\int\frac{d^2r}{r_0^2}\left[\frac{1}{\omega_0}\psi\cos\chi V\sin\theta(1-\cos\theta)\times\frac{d\theta}{dr}-\frac{r_0^2}{2}\sin^2\theta(\nabla\psi)^2\right],$$

and the variation of this part of the Lagrangian with respect to ψ gives the differential equation

$$-r_0^2\nabla[\sin^2\theta(\nabla\psi)]=\frac{1}{\omega_0}\cos\chi V\sin\theta(1-\cos\theta)\frac{d\theta}{dr}. \tag{18}$$

Earlier it was mentioned that we needed no new numerical data to obtain the Lagrangian. This becomes obvious from the following: the terms with ψ, ψ^2 are comparable, and the total value of $L^{(1)}+L^{(2)}$ is equal to $(1/2)L^{(1)}$. In order to show this, we need to know the solution of this equation for ψ to the lowest approximation in ε. It is convenient to introduce the new variable $\mu=\psi\sin\theta_0$, because the differential operator in Eq. (18) becomes one of the Schrödinger type. Next it is remarked that the solution can be written as $\mu=\varepsilon(V/V_m)\beta(\xi)\cos\chi$, where the dimensionless variable $\xi=\varepsilon r/r_0$ is used. By use of the asymptotic solution (12) we can write the equation for function $\beta(\xi)$ in the form

$$\frac{d^2\beta}{d\xi^2}+\frac{1}{\xi}\frac{d\beta}{d\xi}-\frac{\beta}{\xi^2}+\beta(f^2-1)=-f^2\frac{df}{d\xi}. \tag{19}$$

Next we notice that the localized solution of this equation is equal to $(1/2)df/d\xi$. To check this, it is sufficient to differentiate Eq. (13) for $f(\xi)$ with respect to ξ. Doing this and comparing the result with Eq. (19), one can see that they coincide if $\beta=(1/2)df/d\xi$. Thus, the integrand in $L^{(1)}$ is proportional to $(Vf)^2(df/d\xi)^2$ and the ψ contribution to Lagrangian can be written in terms of the same integral $B=2\pi\int f^2(df/d\xi)^2\xi\,d\xi$ as was used above for $L^{(0)}$. Combining all these values, $L^{(0)}(V,\omega)$ and $L^{(1)}+L^{(2)}=(1/2)L^{(1)}$, we obtain the following form for the Lagrange function of the soliton:

$$L(V,\omega)=-E_0\frac{\varepsilon^2}{1+V^2/V_m^2}-E_0\frac{b\varepsilon^4(1-V^2/V_m^2)}{2(1+V^2/V_m^2)^3}. \tag{20}$$

Here E_0 is the universal value of soliton energy as given by Eq. (14) and $b=B/A\approx 1.36$.

Using Eqs. (16), after long but simple algebra we can establish the relation between the small parameter ε and the values of the integrals of motion,

$$\frac{N}{N_0}+\frac{N_0}{N}\left(\frac{P}{P_0}\right)^2-1=\varepsilon^2\Psi(\kappa), \tag{21}$$

where

$$\Psi(\kappa)=\frac{1+\kappa^2}{N_0^2[b+(2-b)\kappa^2]}$$

and $\kappa=N_0P/NP_0$. From this, one can see that Eq. (21) is a generalization of Eq. (15) taking into account higher-order corrections, and, as expected, they become the same as $\varepsilon\to 0$. The soliton energy can be written as

$$E(P,N)=E_0\left[\frac{N}{N_0}+\frac{N_0}{N}\left(\frac{P}{P_0}\right)^2-\frac{1}{2}\Psi_{(\kappa)}\left(N+\frac{N_0^2}{N}\frac{P^2}{P_0^2}-N_0\right)^2\right]. \tag{22}$$

The analysis of the role of the terms of type ϑ and ϑ^2 shows that their contribution is proportional to ε^6 and is negligibly small, therefore, Eq. (22) is applicable to the $q=0$ soliton. The first term in Eq. (22) describes the zeroth approximation to the soliton energy derived above. Taking into account $E_0=\hbar\omega_0N_0$, it can be rewritten as the dispersion law for free magnons. Thus, the second term describes the interaction or binding energy of magnons, and the multiplier $\Psi(\kappa)$ before the second bracket, which is the small parameter in the next power, can be interpreted as the effective amplitude of the magnon interaction. As one can see from Eq. (21), this amplitude is negative for all values of the soliton parameters.

For the investigation of the soliton stability, we can use general relations for two-parameter solitons obtained for different models in Refs. 27–29, 25. In our case the soliton is stable if

$$\Delta \equiv \frac{\partial V}{\partial P}\frac{\partial \omega}{\partial N} - \frac{\partial \omega}{\partial P}\frac{\partial V}{\partial N} < 0.$$

It is convenient to rewrite this in terms of the second derivatives of the energy:

$$\Delta = \frac{\partial^2 E}{\partial P^2}\frac{\partial^2 E}{\partial N^2} - \left(\frac{\partial^2 E}{\partial P\,\partial N}\right)^2 < 0.$$

In lowest approximation on ε Eq. (15a) obviously gives $\Delta = 0$ and nothing can be said about soliton stability. In the next approximation application of Eq. (22) gives

$$\Delta = -\frac{2}{N}\Psi(\kappa)(1+\kappa^2)^2. \tag{23}$$

Thus, the stability criterion is quite simple and transparent: the soliton is stable if the effective amplitude of magnon coupling corresponds to the attraction, $\Psi(\kappa) > 0$. This shows that the small-amplitude soliton with $q=0$ is stable for all possible values of P and N.

B. Dispersion law for solitons with $q=1$ in the ferromagnet

The investigation of the dispersion law for the small-amplitude soliton with $q=1$ in general is the same as for $q=0$, although more complicated, and based more on numerical work. To find the Lagrangian, let us use the same method as for the $q=0$ case. Again the expression for $L^{(0)}(\omega,V)$ can be expanded in powers of θ with the same form as Eq. (17), but in this case the coefficients A and B are defined as follows:

$$A = 2\pi\int_0^\infty\left[\left(\frac{df}{d\xi}\right)^2 + (f/\xi)^2 + f^2 - f^4/2\right]\xi\,d\xi, \tag{24}$$

$$B = \frac{2\pi}{3}\int_0^\infty [f^6 - f^4 - 4(f/\xi)^2]\xi\,d\xi. \tag{25}$$

These integrals are evaluated as before: Multiplying Eq. (13) by $\xi^2\,df/d\xi$ and integrating over ξ from zero to infinity, it is seen that

$$\int_0^\infty f^2\xi\,d\xi = \frac{1}{2}\int_0^\infty f^4\xi\,d\xi;$$

consequently, the expression for A has the form

$$A = \int_0^\infty\left[\left(\frac{df}{d\xi}\right)^2 + \frac{f^2}{\xi^2}\right]\xi\,d\xi. \tag{26}$$

and it is noticed that $A>0$. It is convenient to rewrite the expression for B in following way:

$$B = 2\pi\int_0^\infty f^2\left[\left(\frac{df}{dx}\right)^2 - \frac{f^2}{x^2}\right]. \tag{27}$$

From Eq. (26) we can say nothing about the sign of B, but numerical calculations give the following quantities for the integrals: $A=48.29$ and $B=-6.74$. Notice here that the coefficient B turns out to be negative, which is opposite to the $q=0$ case.

Let us next calculate $L^{(1)}+L^{(2)}$. In contrast to the case with $q=0$, we must take into account both corrections proportional to ψ and ψ^2, and also contributions from terms like ϑ and ϑ^2. The expression for $L^{(1)}+L^{(2)}$ can be written in following form:

$$L^{(1)}+L^{(2)} = JS^2\int\frac{d^2r}{r_0^2}\Biggl\{\Biggl[\frac{1}{2}\vartheta H_\vartheta + \frac{1}{\omega_0}V\sin\theta(1-\cos\theta)\times\left(\frac{\sin\chi}{r} - \frac{\partial\Psi}{\partial x}\right) - r_0^2\frac{\sin 2\theta}{r^2}\frac{\partial\psi}{\partial\chi}\Biggr]\vartheta + \Biggl[\frac{1}{\omega_0}V\sin\theta(1-\cos\theta)\frac{\partial\theta}{\partial x} + \frac{r_0^2}{2}\nabla(\sin^2\theta\nabla\psi)\Biggr]\psi, \tag{28}$$

where H_v is defined in the following way:

$$H_v = r_0^2\nabla^2 - \cos 2\theta\left(1 + \frac{V^2}{V_m^2} + \frac{r_0^2}{r^2}\right) + \frac{1}{\omega_0}\left[\cos\theta\left(\omega + \frac{V^2}{r_0 V_m}\right) + \frac{V\sin\chi}{r}(\cos\theta - \cos 2\theta)\right]. \tag{29}$$

The variation of Lagrangian with respect to ψ and ϑ gives the bounded system of differential equations for ψ and ϑ:

$$H_v v - \frac{1}{\omega_0}V\sin\theta(1-\cos\theta)\frac{\partial\psi}{\partial x} - r_0^2\frac{\sin 2\theta}{r^2}\frac{\partial\psi}{\partial\chi} = -\frac{1}{\omega_0}\frac{V\sin\chi}{r}\sin\theta(1-\cos\theta), \tag{30}$$

$$r_0^2\left[\nabla(\sin^2\theta\nabla\psi) + \frac{\sin 2\theta}{r^2}\frac{\partial v}{\partial\chi}\right] = -\frac{1}{\omega_0}V\sin\theta(1-\cos\theta)\frac{d\theta}{dr}\cos\chi. \tag{31}$$

Taking into account these equations for ψ and ϑ, and integrating the Lagrangian over χ from zero to 2π, we can rewrite the expression for the correction as

$$L^{(1)}+L^{(2)}=2\pi JS^2\int_0^\infty \frac{r\,dr}{r_0^2}\left\{-\frac{1}{2}\vartheta H_\vartheta \vartheta - \frac{r_0^2}{2}\psi\nabla(\sin^2\theta\nabla\psi)\right\}. \tag{32}$$

To calculate $L^{(1)}+L^{(2)}$, it is necessary to first solve the system given by Eqs. (30) and (31). Expanding the functions in this system in the power of ϑ, and neglecting all terms of order more than ε^4, we obtain the following system:

$$\frac{d^2\alpha}{d\xi^2}+\frac{1}{\xi}\frac{d\alpha}{d\xi}-\left(\frac{2}{\xi^2}+1-3f^2\right)\alpha+\frac{2}{\xi^2}\beta=-\frac{f^3}{\xi} \tag{33}$$

$$\frac{d^2\beta}{d\xi^2}+\frac{1}{\xi}\frac{d\beta}{d\xi}-\left(\frac{2}{\xi^2}+1-f^2\right)\beta+\frac{2}{\xi^2}\alpha=-f^2\frac{df}{d\xi}, \tag{34}$$

where the new variables have been introduced:

$$\vartheta=\sqrt{2}V\varepsilon^2\left(1+\frac{V^2}{V_m^2}\right)^{-1}\alpha(\xi)\sin\chi$$

and

$$\psi=\sqrt{2}V\varepsilon^2\left(1+\frac{V^2}{V_m^2}\right)^{-1}\frac{\beta(\xi)}{\sin\theta}\cos\chi.$$

Finally, the Lagrange function $L^{(1)}+L^{(2)}$ is rewritten in terms of α and β to obtain

$$L^{(1)}+L^{(2)}=\frac{JS^2\varepsilon^2V^2}{2V_m^2}\left(1+\frac{V^2}{V_m^2}\right)D, \tag{35}$$

where D is defined in the following way:

$$D=8\pi\int_0^\infty\left[\frac{\beta^2}{\xi^2}+\frac{1}{f^2}\left(f\frac{d\beta}{d\xi}-\beta\frac{df}{d\xi}\right)^2+\alpha\left(\frac{f^3}{\xi}+\frac{2\beta}{\xi^2}\right)\right]\xi\,d\xi. \tag{36}$$

The system given by Eqs. (33) and (34) has been solved numerically by using the shooting method. The results of these calculations for α and β are shown in Fig. 3 giving the numerical value of $D=147.36$. Combining these results, the Lagrangian of the soliton with $q=1$ can be written as follows:

$$L(\omega,V)=E_0\left(1+\frac{V^2}{V_m^2}\right)^{-1}\left[-\varepsilon^2-\frac{\varepsilon^4 b}{2}\left(1+\frac{V^2}{V_m^2}\right)^{-1}+\frac{\varepsilon^4V^2d}{2V_m^2}\left(1+\frac{V^2}{V_m^2}\right)^{-2}\right], \tag{37}$$

where $b=A/B=-0.14$, and $d=D/A=3.05$. The energy of the soliton with $q=1$ is also given by Eq. (22), but in this case the effective amplitude of magnon coupling $\psi(\kappa)$ is

$$\Psi(\kappa)=\frac{1+\kappa^2}{N_0^2(b+(b-d+2)\kappa^2)}. \tag{38}$$

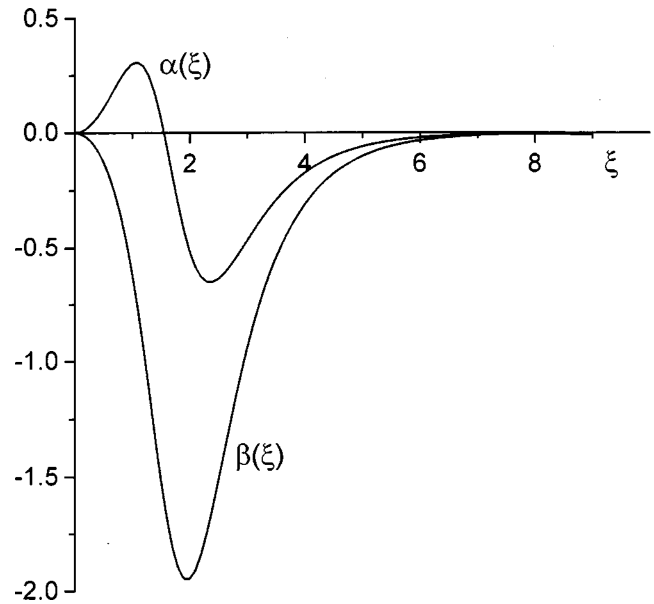

FIG. 3. Functions $\alpha(\xi)$ and $\beta(\xi)$ versus ξ.

In addition Eq. (38) is a general expression for the function $\Psi(\kappa)$. Furthermore, Eq. (40) will coincide with Eq. (21) when we take into account the relation between b and d for the case with $q=0$.

Let us next investigate the soliton stability. In the previous section it was shown that stability criterion is $\Psi(\kappa)>0$. Hence, it is obvious that the small-amplitude soliton with $q=1$ is unstable for all possible values of P and N.

IV. SOLITON THERMODYNAMICS

Now the previous results are used to estimate how the presence of these small-amplitude solitons will affect the thermodynamic quantities in a 2D ferromagnet. Since thermodynamic quantities are affected by both solitons and magnons, we will proceed by finding the relative densities of solitons and magnons. It is then shown that there is a temperature range where the soliton density dominates the magnon density resulting in solitons giving the dominant contributions to thermodynamic quantities. First, the soliton density is estimated. If the dispersion law is isotropic and the values of N are large (macroscopic), then the soliton density is

$$n_{\mathrm{sol}}=\frac{1}{2\pi\hbar^2}\int_0^\infty P\,dP\int_0^\infty dN\exp\left[-\frac{E(P,N)}{T}\right]. \tag{39}$$

Other thermodynamic quantities can be written in the same way, such as the soliton contribution to the mean energy per unit area of the magnet:

$$\langle E\rangle_{\mathrm{sol}}=\frac{1}{2\pi\hbar^2}\int_0^\infty P\,dP\int_0^\infty dN\,E\exp\left[-\frac{E(P,N)}{T}\right]. \tag{40}$$

To calculate these integrals it is more convenient to use the dimensionless variables, $\kappa=PN_0/P_0N$ and $\delta=[N+P^2/N$

$-N_0]/N_0$, where $N=N_0(1+\delta)/(1+\kappa^2)$. After this change of variables the soliton density is the integral

$$n_{\text{sol}}=\frac{P_0^2N_0}{4\pi\hbar^2}\exp\left(-\frac{E_0}{T}\right)\int_0^\infty d\kappa^2\int_0^\infty d\delta\frac{(1+\delta)^2}{(1+\kappa^2)^3}\times\exp\left[-\frac{E_0}{T}\left(\delta-\frac{1}{2}\Psi(\kappa)\delta^2\right)\right]. \tag{41}$$

This integral will be evaluated at the low temperatures, $T \ll E_0$ (we will see below that the temperatures of interest are smaller of E_0) when the lowest approximation on the small parameter δ can be used. In this case integrals like Eq. (41) can be easily calculated and one obtains

$$n_{\text{sol}}=\frac{P_0^2N_0}{8\pi\hbar^2}\frac{T}{E_0}\exp\left(-\frac{E_0}{T}\right), \tag{42}$$

where, as before, $E_0=\varepsilon_0 N_0$ and $\varepsilon_0=\hbar\omega_0=KS$.

Next, it is determined whether solitons or magnons give the dominant contribution to the thermodynamic quantities. This is done by comparing the density of free magnons with the soliton density, which is interpreted to be the bound magnon density. The density of bound magnons is given by

$$n_{\text{bound magn}}=\frac{1}{2\pi\hbar^2}\int_0^\infty P\,dP\int_0^\infty N\,dN\exp\left[-\frac{E(P,N)}{T}\right], \tag{43}$$

and the density of free magnons at the temperatures of interest ($T\gg\varepsilon_0$) is

$$n_{\text{free magn}}=\frac{1}{2\pi}\int_0^\infty k\,dk\left[\exp\left(\frac{\varepsilon_0(1+r_0^2k^2)}{T}\right)-1\right]^{-1}\cong\frac{1}{4\pi}\frac{T}{Ja^2S}\ln\left(\frac{T}{\varepsilon_0}\right). \tag{44}$$

When the integral in Eq. (43) is evaluated, we obtain the following expression for the ratios of densities:

$$\frac{n_{\text{bound magn}}}{n_{\text{free magn}}}=\frac{N_0^3}{3\ln(T/\varepsilon_0)}\exp\left(-\frac{E_0}{T}\right), \tag{45}$$

which is accurate to logarithmic accuracy when omitting the numbers and multipliers like $\ln[\ln(J/K)]$. Then the temperature range where the inequality $n_{\text{bound magn}}>n_{\text{free magn}}$ is valid is

$$T>T_0=\frac{E_0}{3\ln(J/K)}\ll E_0.$$

This characteristic temperature, T_0, is high, but it is well below the value of exchange temperature JS^2. Therefore, in the wide temperature range $E_0>T>T_0$ the soliton contribution is dominant. In this temperature interval the exponential temperature dependence of all the soliton contributions $\exp(-E_0/T)$ should be very well pronounced.

V. COMPARISON OF THE DYNAMICS OF FERROMAGNETS AND ANTIFERROMAGNETS

Most of the measurements showing the contribution of solitons to the temperature dependence of the EPR linewidth were done using antiferromagnetic compounds with a layered structure. It is known that antiferromagnets can be described by the nonlinear σ model for the antiferromagnetic vector (normalized sublattice magnetization) **l**; see Refs. 2, 3 and 16. The energy of the antiferromagnet in the static limit has the same form as for the ferromagnet, but with **m** is replaced by **l**. For this reason there is also the BP soliton in the isotropic antiferromagnet case with the energy also given by Eq. (8). However, the dynamic properties of solitons in ferromagnets that are investigated both in the present paper and earlier work do not coincide with the dynamic properties of the antiferromagnet. This is because the equations of the σ model are formally Lorentz invariant, where the characteristic velocity is taken to be the phase velocity of magnons, and the analysis of the soliton motion in antiferromagnets can be carried out by use of the Lorentz transformation applied to the static structures.

In contrast to antiferromagnets and remarked in the Introduction, mobile topological solitons in ferromagnets are not very well understood. However, the role of internal dynamics on the soliton structure is much better understood. Both for topological and for nontopological solitons in ferromagnets, internal dynamics, such as homogeneous precession, can stabilize the soliton. In particular, this work has shown the existence of a small-amplitude nontopological soliton with the internal precession frequency in the range, $0<\omega<\omega_0$, and the energy is slightly smaller than the energy of the topological soliton. For antiferromagnets the situation is different. It is possible to use the ansatz $\varphi=\omega t+q\chi$, but in this case the equation for θ is different (here the θ and φ are the angular variables for the sublattice magnetization vector **l**). For the particular case of the immobile soliton in an antiferromagnet the equation for $\theta(r)$ can be determined from Eq. (7) if we substitute $(\omega/\omega_{\text{ag}})^2\sin\theta\cos\theta$ instead of $(\omega/\omega_0)\sin\theta$, where ω_{ag} is the minimal frequency of magnons in the antiferromagnet (see Refs. 3 and 16). It has been shown[27,30,31] through the use of numerical and qualitative analysis that for the simplest model of uniaxial anisotropy in antiferromagnets there are no soliton solutions for topological solitons ($q\neq 0$) or for nontopological solitons ($q=0$). As was mentioned earlier, the special case that has the structure of the BP soliton with $\tan(\theta/2)=R/r$, and the precession frequency $\omega=\omega_{\text{ag}}$, is possible; however, this case is not physical. There are the soliton solutions with more general anisotropy, namely,

$w_a=\frac{1}{2}K\sin^2\theta-\frac{K}{4}\sin^4\theta$, and with precession frequency

$\omega_{\text{ag}}\left(1-\frac{k}{K}\right)<|\omega|<\omega_{\text{ag}}$ (see Refs. 27, 30, and 31),

which we will not consider here.

In conclusion, our analysis has shown essential differences between the dynamic properties of solitons in ferromagnets and antiferromagnets. In this paper it was shown that there is a nontopological mobile soliton with $q=0$ in 2d ferromagnets. Its energy E_0 is a bit smaller then the energy of the BP soliton, $E_{BP}=4\pi JS^2$, $E_0 \cong 0.93 E_{BP}$. It is expectedthat this soliton will give essential contributions to observed dynamical quantities such as spin-correlation functions. We would like to stress that it is difficult to distinguish the contributions from the nontopological and the topological solitons, owing to the fact that E_0 and E_{BP} are approximately equal, but in antiferromagnets the BP soliton is the only candidate for elementary excitations. For the case of weak anisotropy, BP solitons can exist when systems are approximated by the isotropic antiferromagnet.

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