Criterion for crack formation in disordered materials

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Crack formation is conventionally described as a nucleation phenomenon despite the fact that the temperatures necessary to overcome the nucleation barrier are far too high. In this paper we consider the possibility that cracks are created due to the presence of frozen disorder (e.g., heterogeneities or frozen dislocations). In particular we calculate the probability for the occurrence of a critical crack in a quasi-two-dimensional disordered elastic system. It turns out that this probability takes the form of an Arrhenius law (as for thermal nucleation) but with the temperature T replaced by an effective *disorder temperature* T_{eff} which depends on the strength of the disorder. The extension of these results to $d=3$ dimensions is briefly discussed.

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I. INTRODUCTION

Cracks are one of the most important sources for the failure of solids.[1] Despite continuous efforts for more than a century a full understanding of fracture has not yet been reached.[2–4] A simple but very appealing picture for the occurrence of cracks goes back to Griffith.[5] Griffith describes the emergence of cracks as a nucleation phenomenon: To open a crack in a thin plate, atomic bonds have to be broken and two new surfaces have to be created. For a crack of linear size a, this costs an energy of order a. Simultaneously, the potential energy of the plate under external load is reduced due to the stress relaxation around the crack. This decreases the energy by an amount of order a^2. Thus, the total crack energy as a function of a increases for small a linearly and reaches a maximum at $a=a_c$ before it decreases quadratically. Cracks of length $a<a_c$ are stable whereas those with $a>a_c$ are unstable. However, contrary to conventional nucleation phenomena the typical energy barriers for crack propagation in a perfect solid under realistic stresses are much too high to be overcome by thermal fluctuations. Instead, the preexistence of microcracks (or preweakened bonds) on scales $a\lesssim a_c$ is usually tacitly assumed. These will then grow under external load. It seems to be reasonable to consider microcracks as well as other heterogeneities in the material as a kind of frozen disorder only amenable to a statistical treatment.

The *propagation* of supercritical cracks in an inhomogeneous material has been the subject of a number of articles[6,7] which have attempted to explain the roughness of crack fronts found experimentally.[8] Unfortunately a convincing explanation of the experimental data is still lacking. The other aspect—the *occurrence* of a critical crack in the first place—has been, to the best of our knowledge, not yet considered. It is the aim of the present paper to address this point by calculating the probability of the occurrence of a critical crack in systems which includes various types of disorder. In particular, we will consider randomness in the atomic bond strength as well as randomly distributed impurities (or other kinds of heterogeneities) and frozen dislocations. It should, however, be mentioned that our considerations are not restricted to crystalline material. The main ingredient of our theory is isotropic elasticity, which also applies to amorphous materials, supplemented by randomly distributed disorder. The latter can also include mesoscopic heterogeneities and microcracks which occur during fabrication. Although the various sources of disorder conceivable may differ considerably in their local properties, the most important aspect from the statistical point of view (which is adopted in this paper) is the spatial decay of the stress fields they create. It is in this sense that the three types of disorder considered below are generic.

The main body of the paper is related to crack creation in a thin plate of infinite extension. Some results can, however, be easily extended to bulk materials. A detailed investigation of cracks in slabs of finite width as well as those in three-dimensional systems will be presented in forthcoming publications.[9]

II. CRACKS IN AN INFINITE TWO-DIMENSIONAL SAMPLE

In this paper we consider a single planar crack extended parallel to the x direction in a two-dimensional elastically isotropic system of infinite extension. The two-dimensional situation can be realized, e.g., by a thin plate of width h in the so-called plane stress configuration.[10] The Lamé coefficients $\bar{\lambda}$ and $\bar{\mu}$ of the two-dimensional system are then related to the Lamé coefficients λ and μ of the bulk by $\bar{\lambda}=2\lambda\mu h/(\lambda+2\mu)$, $\bar{\mu}=\mu h$. The coordinates of the crack are given by

$$-a\leqslant x\leqslant a, \quad y\equiv 0. \tag{2.1}$$

Such a planar crack appears, for instance, in experiments with preweakened bonds.[11]

In two dimensions only mode I and mode II cracks occur characterized by the only nonzero components $\bar{\sigma}_{yy}^{(\mathrm{e})}=\bar{\sigma}_{(\mathrm{e})}$ or $\bar{\sigma}_{xy}^{(\mathrm{e})}=\bar{\sigma}_{yx}^{(\mathrm{e})}=\bar{\sigma}_{(\mathrm{e})}$, respectively, of the applied stress $\bar{\sigma}_{ij}^{(\mathrm{e})}$. In the further treatment we will apply the dislocation theory of cracks:[12] The crack will be described by virtual lattice planes filling its interior such that there is no free crack surface. The lattice planes terminate in crack dislocations with Burgers vector $\mathbf{b}^{(\mathrm{c})}$. The whole crack is then given by a collection of dislocations (and antidislocations) $\mathbf{b}_\alpha^{(\mathrm{c})}$ at positions $\mathbf{r}_\alpha=(x_\alpha,0)$, see Fig. 1. Crack dislocations interact with the ex-

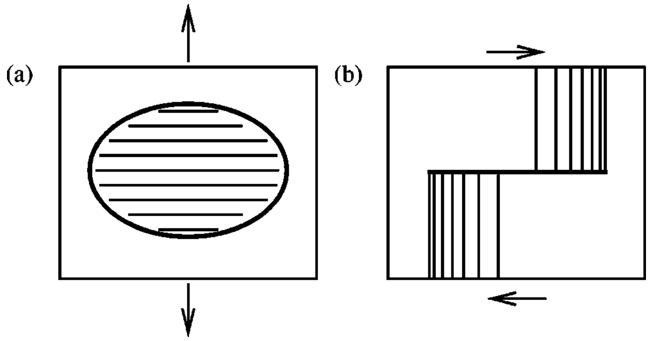

FIG. 1. Lattice planes terminating in crack dislocations in mode I (a) and mode II (b) cracks. The dislocation vectors are perpendicular to the lattice planes. The arrows denote the direction of the external forces $f^{(\mathrm{e})}$. Changing $f^{(\mathrm{e})}$ crack dislocations in mode I and II climb and glide, respectively.

ternal stress $\overline{\sigma}_{ij}^{(\mathrm{e})}$ and the disorder made up of impurities and frozen dislocations. For the further discussion, it turns out to be convenient to introduce a two-dimensional dislocation density $\mathbf{b}(\mathbf{r})=\Sigma_\alpha \mathbf{b}_\alpha \delta(\mathbf{r}\text{-}\mathbf{r}_\alpha)$ .The actual distribution of the crack dislocation will be determined later from a minimum condition for the free energy for given external stresses and crystal imperfections. It should be mentioned, however, that the crack description by dislocations is *not* essential for the final results. We could also have used more traditional elasticity theory combined with the appropriate boundary conditions on the crack surface. In this sense also amorphous materials are included (but there are no frozen dislocations in this case).

The interaction between the external stress $\overline{\sigma}_{ij}^{(\mathrm{e})}$ and a dislocation with Burgers vector $\mathbf{b}$ is given by the Peach-Köhler force[13] $f_i=-\epsilon_{il} b_m \overline{\sigma}_{lm}^{(\mathrm{e})}$, where ϵ_{il} denotes the total antisymmetric unit tensor. With the help of this relation one obtains for the total contribution of $\overline{\sigma}_{ij}^{(\mathrm{e})}$ on the crack dislocation energy

$$E^{(\mathrm{e})}=-\epsilon_{xl}\overline{\sigma}_{lm}^{(\mathrm{e})}\sum_\alpha x_\alpha b_{\alpha,m}^{(\mathrm{c})}=-\overline{\sigma}_{ym}^{(\mathrm{e})}\int_{-a}^{a} dx x b_m^{(\mathrm{c})}(x). \tag{2.2}$$

Thus, in mode I (II) only the $y(x)$ component of $\mathbf{b}^{(\mathrm{c})}(x)$ contributes to $E^{(\mathrm{e})}$. Since $E^{(\mathrm{e})}$ is the only macroscopic term which favors the existence of crack dislocations, it is clear that in mode I (II) only those with $\mathbf{b}^{(\mathrm{c})}(\mathbf{r})$ parallel to the $y(x)$ axis will occur. This will be used in the following.[14]

The stress field $\overline{\sigma}_{ij}$ generated by dislocations is related to the Airy stress function $\chi(\mathbf{r})$ by[15] $\overline{\sigma}_{ij}=\epsilon_{ik}\epsilon_{jl}\partial_k\partial_l\chi(\mathbf{r})$, where χ is a solution of

$$(\boldsymbol{\nabla}^2)^2\chi(\mathbf{r})=\overline{Y}\epsilon_{ji}\partial_j b_i(\mathbf{r})\,. \tag{2.3}$$

Here $\overline{Y}=4\overline{\mu}(\overline{\lambda}+\overline{\mu})/(2\overline{\mu}+\overline{\lambda})$ denotes the Young modulus in two dimensions.

In an infinite system the solution of Eq. (2.3) is given by[13]

$$\chi(\mathbf{r})=\overline{Y}\int d^2\mathbf{r}'\, g(\mathbf{r}-\mathbf{r}')\,\epsilon_{ij}\partial_i' b_j(\mathbf{r}'). \tag{2.4}$$

Here $g(\mathbf{r})=\mathbf{r}^2(\ln|\mathbf{r}|+C)/(8\pi)$, and C is an arbitrary constant. The elastic energy of the dislocations is then given by

$$\begin{aligned} E^{(\mathrm{c})}&=\frac{1}{2}\int d^2\mathbf{r}\overline{\sigma}_{ij}u_{ij}\\ &=-\frac{\overline{Y}}{2}\int d^2\mathbf{r}\int d^2\mathbf{r}'\,\epsilon_{ij}\epsilon_{kl}b_j(\mathbf{r})b_l(\mathbf{r}')\partial_i\partial_k g(\mathbf{r}-\mathbf{r}'), \end{aligned} \tag{2.5}$$

where we used the relation

$$u_{ik}=\frac{1}{2\overline{\mu}}\overline{\sigma}_{ik}-\frac{\overline{\lambda}}{4\overline{\mu}(\overline{\lambda}+\overline{\mu})}\delta_{ik}\overline{\sigma}_{ll}\,. \tag{2.6}$$

The elastic energy of the crack dislocations is then given by

$$E^{(\mathrm{c})}=-\frac{\overline{Y}}{8\pi}\int_{-a}^{a}dx\int_{-a}^{a}dx'\, b^{(\mathrm{c})}(x)b^{(\mathrm{c})}(x')\ln\left|\frac{x-x'}{a_0}\right|. \tag{2.7}$$

$b^{(\mathrm{c})}(x)$ denotes the crack dislocation density along the x axis and a_0 denotes a microscopic cutoff (of the order of the lattice spacing and in general different for mode I and mode II). Equation (2.7) is valid both for mode I and mode II cracks. Note that $E^{(\mathrm{c})}$ is always positive.

So far we did not consider the core contributions of the dislocations. In the present context this is replaced by the two-dimensional energy density $\overline{\gamma}(x)=\overline{\gamma}_0+\overline{\gamma}_1(x)$ of the crack surface:

$$E^{(\mathrm{s})}=2\int_{-a}^{a}dx\,\overline{\gamma}(x)=4\overline{\gamma}_0 a+E_1^{(\mathrm{s})}(a). \tag{2.8}$$

Here $\overline{\gamma}_1(x)$ reflects the randomness in the strength of the bonds broken. For simplicity we assume Gaussian disorder with $\langle\overline{\gamma}_1(x)\rangle=0$ and

$$\langle\overline{\gamma}_1(x)\overline{\gamma}_1(x')\rangle=\overline{\delta\gamma}^2 a_0\delta_{a_0}(x-x'), \tag{2.9}$$

where $\langle\cdots\rangle$ denotes the average over the disorder and $\delta_{a_0}(x)$ a delta function of width a_0. In general the correlation length a_0 of the disorder appearing in Eq. (2.9) is different from the cutoff appearing in Eq. (2.7). Similarly, further correlation lengths could be introduced for the distribution of impurities and frozen dislocations to be considered below. To avoid a too clumsy notation we will, however, use everywhere the length a_0 as a small scale cutoff but keeping in mind this complication. Depending on the type of material under consideration, a_0 may vary between the size of an atom in crystalline and the size of a grain in granular materials, respectively. The precise value of a_0 will be of course important if comparison with experiments is made. Then $\langle E_1^{(\mathrm{s})}(a)\rangle=0$ and we find for the variance of $E_1^{(\mathrm{s})}(a)$

$$\langle [E_1^{(s)}(a) - E_1^{(s)}(a')]^2 \rangle = \Delta_{(s)} |a - a'|, \tag{2.10}$$

where $\Delta_{(s)} = 8\overline{\delta}\gamma^2 a_0$. Clearly Eq. (2.9) is only valid for $|a-a'| \gtrsim a_0$.

In the following we add the contributions $E^{(d)} = E^{(fd)} + E^{(i)}$ from randomly frozen dislocations and impurities to the energy. $E^{(d)}$ is given by $\int d^2\mathbf{r}\overline{\sigma}_{ij}^{(d)} u_{ij}^{(c)}$ where $\overline{\sigma}_{ij}^{(d)}$ denotes the stress generated by the disorder and $u_{ij}^{(c)}$ the strain field generated from cracks, respectively.

Using Eq. (2.5), $E^{(d)}$ can be written in the form

$$E^{(d)} = -\frac{\overline{Y}}{4\pi} \int_{-a}^{a} dx b^{(c)}(x) V(x), \tag{2.11}$$

$$V(x) = V^{(fd)}(x) + V^{(i)}(x). \tag{2.12}$$

From Eqs. (2.3), (2.4), and (2.6) one obtains for the potential created from dislocations

$$V^{(fd)}(x) = 4\pi \int d^2\mathbf{r}' \epsilon_{ij} [\partial_k \partial_i g(\mathbf{r} - \mathbf{r}')]_{y=0} b_j^{(fd)}(\mathbf{r}'), \tag{2.13}$$

where $k = x,y$ for mode I, II cracks, respectively. The frozen dislocations are assumed to have both random positions $\mathbf{r}_\alpha$ and directions of their Burgers vectors such that $\langle \mathbf{b}^{(fd)}(\mathbf{r}) \rangle = 0$ and

$$\langle b_i^{(fd)}(\mathbf{r}) b_j^{(fd)}(\mathbf{r}') \rangle = b_{(fd)}^2 c_{(fd)} \delta_{a_0}(\mathbf{r} - \mathbf{r}') \delta_{ij}. \tag{2.14}$$

Here $c_{(fd)}$ and $b_{(fd)}$ denote the concentration and the strength of the dislocation.

Impurities (or more macroscopic inclusions) also generate a long-range elastic displacement field $\mathbf{u}^{(i)}(\mathbf{r})$. Repeating the calculation of Eshelby[16] for $d=2$ dimensions one finds for the strain tensor of an impurity located at the origin

$$u_{ij}^{(i)}(\mathbf{r}) = \frac{\Omega}{2\pi} \frac{\overline{\lambda} + \overline{\mu}}{2\overline{\mu} + \overline{\lambda}} \partial_i \partial_j \ln|\mathbf{r}|. \tag{2.15}$$

Here Ω denotes the two-dimensional volume change due to the impurity which can be of either sign. The interaction energy between the crack dislocations and the impurities of density $\overline{c}_{(i)}(\mathbf{r})$ takes the form

$$E^{(i)} = \frac{1}{2}\Omega \int d^2\mathbf{r} \overline{\sigma}_{ii}^{(c)}(\mathbf{r}) \overline{c}_{(i)}(\mathbf{r}). \tag{2.16}$$

Here we used Eq. (2.15) and

$$\overline{c}_{(i)}(\mathbf{r}) = \sum_\alpha \delta_{a_0}(\mathbf{r} - \mathbf{r}_\alpha) - \overline{c}_{(i)}, \tag{2.17}$$

where the summation is over all impurity sites $\mathbf{r}_\alpha$ and $\overline{c}_{(i)}$ denotes the impurity concentration. With Eqs. (2.3)–(2.6) we find

$$V^{(i)}(x) = \Omega \int d^2\mathbf{r}' c(\mathbf{r}') \left(\partial_k \ln \left| \frac{\mathbf{r} - \mathbf{r}'}{a_0} \right| \right)_{y=0}, \tag{2.18}$$

where again $k = x,y$ for mode I, II cracks, respectively.

The total energy $E = E^{(e)} + E^{(c)} + E^{(s)} + E^{(d)}$ is a functional of the crack dislocation density $b^{(c)}(x)$. Differentiating the saddle point equation $\delta E / \delta b^{(c)}(x) = 0$ with respect to x, we find

$$\frac{4\pi}{\overline{Y}} \overline{\sigma}^{(e)} + \int_{-a}^{a} dx' b^{(c)}(x') \frac{1}{x - x'} + V'(x) = 0. \tag{2.19}$$

Equation (2.19) has the solution[17]

$$\begin{aligned} b^{(c)}(x) &= \int_{-a}^{a} dx' f(x,x';a) \left(\frac{4\pi}{\overline{Y}} \overline{\sigma}^{(e)} + V'(x') \right) \\ &= b_0^{(c)}(x) + b_1^{(c)}(x), \end{aligned} \tag{2.20}$$

where

$$f(x,x';a) = -\frac{1}{\pi^2} \left(\frac{a^2 - x'^2}{a^2 - x^2} \right)^{1/2} \frac{1}{x' - x}. \tag{2.21}$$

The total energy as a function of the crack length follows with the help of Eq. (2.19):

$$\begin{aligned} E(a) = \frac{\overline{Y}}{8\pi} \int_{-a}^{a} dx \int_{-a}^{a} dx' b^{(c)}(x) b^{(c)}(x') \\ \times \ln \left| \frac{x - x'}{a_0} \right| + 2 \int_{-a}^{a} dx \overline{\gamma}(x), \end{aligned} \tag{2.22}$$

where $b^{(c)}(x)$ is given by Eq. (2.20).

For vanishing disorder $b^{(c)}(x) \to b_0^{(c)}(x,a)$ for which we obtain from Eqs. (2.20), (2.21), and (A1)

$$b_0^{(c)}(x,a) = \frac{4\overline{\sigma}^{(e)}}{\overline{Y}} \frac{x}{(a^2 - x^2)^{1/2}}, \tag{2.23}$$

which yields in mode I an elliptic crack of maximal height $2\overline{\sigma}^{(e)} a / \overline{Y}$.

As follows from Eq. (2.20) the total energy (2.22) can be divided into contributions E_n which are proportional to $(\overline{\sigma}_{(e)})^{2-n}$ with $n = 0,1,2$, respectively. Here E_1 and E_2 depend on the disorder. The disorder-independent contributions to the energy are given by the Griffith expression

$$E_0(a) = 4\overline{\gamma}_0 a - \frac{\pi a^2 \overline{\sigma}_{(e)}^2}{\overline{Y}} = 4\overline{\gamma}_0 a \left(1 - \frac{a}{2a_c} \right), \tag{2.24}$$

which shows a maximum at $a = a_c = 2\overline{\gamma}_0 \overline{Y} / (\pi \overline{\sigma}_{(e)}^2)$ corresponding to an energy barrier $E_0(a_c) = 2\overline{\gamma}_0 a_c$.

The contributions E_1 and E_2 depend on the frozen disorder and can be characterized by their mean value and variance. E_1 can be rewritten using partial integration and Eqs. (A1) and (A2) as

$$E_1=\frac{\overline{Y}}{4\pi}\int_{-a}^{a}\int_{-a}^{a}dx\,dx'\,b_0^{(c)}(x)\,b_1^{(c)}(x')\ln\left|\frac{x-x'}{a_0}\right|$$
$$=-\frac{\overline{Y}}{4\pi}\int_{-a}^{a}dx\,V(x)\,b_0^{(c)}(x). \tag{2.25}$$

For impurities we obtain from Eqs. (2.17) and (2.18)

$$\langle V^{(i)}(x)V^{(i)}(x')\rangle=\Omega^2\overline{c_{(i)}}\pi\ln g\frac{R}{|x-x'|}, \tag{2.26}$$

where R is a cutoff of the order of the system size which has to be send to infinity. This gives with Eqs. (A1) and (A2) for the impurity contribution $E_1^{(i)}$ to the variance of E_1

$$\langle[E_1^{(i)}(a)-E_1^{(i)}(a')]^2\rangle=\Delta_{(i)}|a^2-a'^2|. \tag{2.27}$$

Here $\Delta_{(i)}=(\pi/2)\overline{c_{(i)}}(\Omega\overline{\sigma_{(e)}})^2=\overline{c_{(i)}}\Omega^2\overline{\gamma_0}\,\overline{Y}/a_c$.

For frozen dislocations we get from Eqs. (2.13) and (2.14)

$$\langle V^{(fd)}(x)V^{(fd)}(x')\rangle=\frac{6}{\pi}c_{(fd)}b_{(fd)}^2\left(\frac{(\pi R)^2}{3}-(x-x')^2\right), \tag{2.28}$$

which gives with Eq. (2.25) for the dislocation contribution $E_1^{(fd)}$ to the variance of E_1

$$\langle[E_1^{(fd)}(a)-E_1^{(fd)}(a')]^2\rangle=\Delta_{(fd)}(a^2-a'^2)^2, \tag{2.29}$$

with $\Delta_{(fd)}=(3/\pi)c_{(fd)}(b_{(fd)}\overline{\sigma^{(e)}})^2=6c_{(fd)}b_{(fd)}^2\overline{\gamma_0}\,\overline{Y}/(\pi^2a_c)$. It is easy to see from Eq. (2.11) and the condition that the crack be closed, i.e., $\int_{-a}^{a}dx\,b^{(c)}(x)=0$, that there the dependence on R vanishes in Eqs. (2.27) and (2.29). These equations are clearly valid only for $|a-a'|$ larger than a_0. For $|a-a'|$ smaller than the mean distance between the impurities or dislocations, respectively, the statistics of $E_1(a)$ is no longer Gaussian, but Eqs. (2.27) and (2.29) still give the correct order of magnitude of the fluctuations of $E_1(a)$.

The mean values as well as the fluctuations of E_2 are proportional to $\overline{c_{(i)}}$ and $c_{(fd)}$ and hence small if the disorder is weak as we will assume in the following. Then the average energy of the crack is given by the Griffith expression $E_0(a)$. The energy barrier $E_0(a_c)=2\overline{\gamma_0}a_c$ is typically large and cannot be overcome by thermal fluctuations. Indeed, for crystalline solids with[5] $\overline{\gamma_0}\lesssim\overline{Y}a_0$ one finds

$$E_0(a_c)\lesssim k_BT_m(\overline{Y}/\overline{\sigma})^2, \quad T_m=\overline{\gamma_0}a_0/k_B, \tag{2.30}$$

where T_m is a characteristic temperature comparable to, but typically bigger than, the solid's melting temperature.[18] For relatively large strain $\overline{Y}/\overline{\sigma}$ is of order 10 such that the nucleation rate for a supercritical crack is of order $\omega_0\exp(-100T_m/T)$. Here ω_0 is a microscopic attempt frequency of order 10^{13} s^{-1}. In the further discussion we will therefore mostly neglect thermal fluctuations.

Let us denote the probability that a crack of length a has a negative energy by $W_{E<0}(a)$. The smallest crack one can think of has a length of the order a_0. Thus a crack can only appear if this smallest crack has a negative energy, $E(a_0)<0$. This occurs with the probability $W_{E<0}(a_0)$. (On this smallest scale even thermal fluctuations may help to create a crack as we will see below.) The crack can then only propagate further if for a given disorder configuration the force on the crack tip $f(a)=-\partial E/\partial a$ is positive for *all* $a\geqslant a_0$ (we neglect effects of inertia).

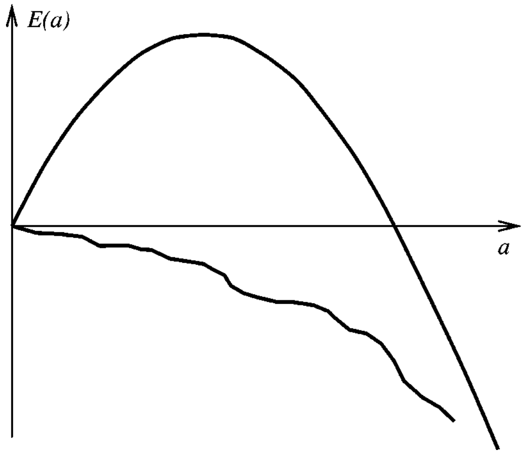

FIG. 2. Schematic plot of the crack energy as a function of a in the pure and random system.

Because of its mathematical simplicity, we consider here first a *necessary* condition to be fulfilled which is $E(a)<0$ for *all* a. Clearly, if $E(a)>0$, an (essentially) macroscopic energy barrier exists and the crack cannot propagate. The probability $W_{E<0}(a)$ is given by

$$W_{E<0}(a)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\phi(a)}dx\,e^{-x^2/2}, \tag{2.31}$$

with

$$\phi(a)=\frac{4\overline{\gamma_0}a[1-a/(2a_c)]}{(\Delta_{(s)}a+\Delta_{(i)}a^2+\Delta_{(fd)}a^4)^{1/2}}. \tag{2.32}$$

Here we used the fact that $E_1^{(s)}+E_1^{(i)}+E_1^{(fd)}$ is Gaussian distributed with variance $\Delta_{(s)}a+\Delta_{(i)}a^2+\Delta_{(fd)}a^4$. Note that this expression for the variance makes sense only for a much larger than a_0. Among the cracks of various lengths a there is one length $\tilde{a}_c$ for which the probability $W_{E<0}$ is minimal. This minimum appears at the maximum of $\phi(a)$. One finds $\tilde{a}_c$ given by the solution of

$$0=\left(1-\frac{3a}{2a_c}\right)\Delta_{(s)}-\frac{a^2}{a_c}\Delta_{(i)}-2a^3\Delta_{(fd)}. \tag{2.33}$$

It is tempting to consider $W_{E<0}(\tilde{a}_c)\equiv\tilde{W}_{E<0}$ [compare Eqs. (2.31) and (2.33)] as the probability for the occurrence of the crack. This conclusion will be elucidated further below. It is instructive to consider the three different sources of disorder separately.

(i) If only the surface energy is random, then $\phi_{(s)}(a)$ vanishes at $a=0$ and $a=2a_c$ and has an maximum at $\tilde{a}_c$

$=(2/3)a_c$. For weak disorder $\phi^{(s)}(\tilde{a}_c)\equiv\tilde{\phi}^{(s)}$ is large and since, for $\phi(a)\gg 1$, $W_{E<0}(a)\approx -\exp(-\phi^2/2)/(\sqrt{2\pi}\phi)$, we get

$$\tilde{W}^{(s)}_{E<0}\approx\frac{1}{\sqrt{2\pi}\phi^{(s)}(\tilde{a}_c)}\exp\left(-\frac{16\overline{\gamma_0^2}a_c}{27\overline{\delta\gamma^2}a_0}\right). \tag{2.34}$$

(ii) In the case of randomly distributed impurities $\phi_{(i)}(a)$ has a maximum for vanishing a. For weak disorder and $a_c \gg a_0$ we get

$$\tilde{W}^{(i)}_{E<0}\approx\frac{1}{\sqrt{2\pi}\phi^{(i)}(\tilde{a}_c)}\exp\left(-\frac{8\overline{\gamma_0}a_c}{\Omega^2\overline{c_{(i)}}\overline{Y}}\right). \tag{2.35}$$

(iii) Finally, for frozen dislocations $\phi_{(fd)}(a) = 4\overline{\gamma_0}[1/a\text{-}1/(2a_c)]/\sqrt{\Delta_{(fd)}}$. Hence $\phi_{(fd)}(a)$ takes its maximum again at $a\rightarrow 0$ where $\phi(a)$ diverges. In this case a has to be replaced by the minimal length scale a_0. This gives

$$\tilde{W}^{(fd)}_{E<0}\approx\frac{1}{\sqrt{2\pi}\phi^{(fd)}(\tilde{a}_c)}\exp\left(-\frac{4\pi^2\overline{\gamma_0}a_c}{3b^2_{(fd)}c_{(fd)}a_0^2\overline{Y}}\right). \tag{2.36}$$

It may be worthwhile to mention that thermal fluctuations could be at least partially incorporated into the present treatment by relaxing the condition $E(a)<0$ to $E(a) < k_B T\ln(\omega_0 t)$ since barriers of order E are overcome on time scales of order $t\approx\omega_0\exp(E/T)$. This replacement changes the numerator of $\phi(a)$ in Eq. (2.32) from $4\overline{\gamma_0}a[1-a/(2a_c)]$ to $4\overline{\gamma_0}a[1-a/(2a_c)]-k_B T\ln(\omega_0 t)$. Since barriers on scales $a\lesssim a_T$ with

$$a_T\approx a_0\frac{T}{4T_m}\ln(\omega_0 t) \tag{2.37}$$

disappear, a_0 has to be replaced by $\max(a_0,a_T)$ in the expressions for $\tilde{W}_{E<0}$ (we still assume $a_c\gg a_T$). This replacement effects essentially only the result for the frozen dislocations, Eq. (2.36). We emphasize, however, that in general the three types of disorder will work in parallel.

Next we consider the probability to fulfill the *sufficient* condition $\partial E/\partial a<0$ for $0<a<\infty$, i.e., that the force $f(a)$ acting on the crack tip is always positive and the crack can propagate forever (see Fig. 2). We decompose f into a deterministic and a stochastic contribution:

$$f(a)=-\frac{\partial E(a)}{\partial a}=f_0(a)+f_1(a), \tag{2.38}$$

$$f_0(a)=4\overline{\gamma_0}\left(1-\frac{a}{a_c}\right), \tag{2.39}$$

$$f_1(a)=f_1^{(s)}(a)+f_1^{(i)}(a)+f_1^{(fd)}(a). \tag{2.40}$$

Since $f_1(a)=-\partial E/\partial a$ is Gaussian distributed, the joint probability distribution of $f_1(a)$ for $0<a<\infty$ is also Gaussian but in general *nonlocal*. Its form can be reconstructed from the second moments

$$\langle f_1^{(s)}(a)f_1^{(s)}(a')\rangle=\Delta_{(s)}\delta_{a_0}(a-a'),$$

$$\langle f_1^{(i)}(a)f_1^{(i)}(a')\rangle=2a\Delta_{(i)}\delta_{a_0}(a-a'),$$

$$\langle f_1^{(fd)}(a)f_1^{(fd)}(a')\rangle=4aa'\Delta_{(fd)}.$$

In the case of random surface tension and randomly distributed impurities correlations are local and the joint probability distribution of the f's factorizes. In this case the probability that the force $f(a)$ on the tip of a crack of length a is positive is given by

$$W_{f>0}(a)=\int_{-\infty}^{-f_0(a)/\langle f_1^2(a)\rangle^{1/2}}\mathrm{d}x\,\frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \tag{2.41}$$

The total probability $\tilde{W}_{f>0}$ for $f(a)>0$ for $0<a<\infty$ is given by the product of all $W_{f>0}(a)$ factors. Here we take into account that the forces are correlated over a distance of order a_0 such that we can decompose the crack in pieces of the order a_0. It is more convenient to consider the logarithm of $\tilde{W}_{f>0}$:

$$\ln\tilde{W}_{f>0}=\sum_a \ln W_{f>0}(a). \tag{2.42}$$

The sum over $a=na_0$ is here meant over integer numbers n. Since for $a\gg a_c$ the integrals (2.41) are essentially equal to unity, it is sufficient to restrict the summation in Eq. (2.42) to the region $0<a\lesssim a_c$. Moreover, the sum is dominated by the smallest values of $W_{f>0}(a)$ for which we can replace the Gaussian integral (2.41) by the approximate expression used above. This gives

$$\ln\tilde{W}_{f>0}\approx-\int_0^{a_c}\frac{\mathrm{d}a}{2a_0}\left(\frac{f_0^2(a)}{\langle f_1^2(a)\rangle}+\ln\frac{\langle f_1^2(a)\rangle}{2\pi f_0^2(a)}\right). \tag{2.43}$$

In the case of a random surface tension only we obtain

$$\ln\tilde{W}^{(s)}_{f>0}\approx-\frac{a_c}{a_0}\left(\frac{\overline{\gamma_0^2}}{3\overline{\delta\gamma^2}}+1+\frac{1}{2}\ln\frac{\overline{\delta\gamma^2}}{4\pi\overline{\gamma_0^2}}\right). \tag{2.44}$$

Thus $\tilde{W}^{(s)}_{f>0}$ is essentially of the same form as $\tilde{W}_{E<0}$ apart from a replacement of the numerical factor in the exponent ($\frac{16}{27}$ is replaced by $\frac{1}{3}$). A similar calculation for the case of random impurities gives

$$\ln\tilde{W}^{(i)}_{f>0}\approx-\frac{a_c}{a_0}\left[\frac{4\overline{\gamma_0}a_0}{\overline{c_{(i)}}\Omega^2\overline{Y}}\left(\ln\frac{a_c}{a_0}-\frac{2}{3}\right)+1+\frac{1}{2}\ln\frac{\overline{c_{(i)}}\Omega^2\overline{Y}}{32\pi a_0\overline{\gamma_0}}\right], \tag{2.45}$$

which is again the same result for the exponent as for $\tilde{W}_{E<0}$, apart from the logarithmic factor which replaces 2.

The third case of randomly distributed dislocations is more involved. Here we should indeed take into account the long-range correlations of the forces $f_1^{(fd)}$. This requires a more detailed mathematical investigation which we leave for further studies. However, our experience with the two other cases makes it tempting to assume that $\tilde{W}^{(\mathrm{fd})}_{E<0}$ gives essentially the correct expression of the probability for the occurrence of a crack.

III. DISCUSSION AND CONCLUSIONS

It is interesting to remark that in all cases considered above $\tilde{W}$ can be written in the form of an Arrhenius law for thermal nucleation with an effective temperature determined by the strength of the disorder:

$$\tilde{W}\approx\tilde{W}_0\exp\left(-\frac{2\overline{\gamma}_0 a_c}{k_{\mathrm{B}}T_{\mathrm{eff}}}\right). \tag{3.1}$$

Here

$$T^{(\mathrm{s})}_{\mathrm{eff}}=\frac{6\overline{\delta\gamma}^2 a_0}{k_{\mathrm{B}}\overline{\gamma}_0} \tag{3.2}$$

for random surface tension,

$$T^{(\mathrm{i})}_{\mathrm{eff}}=\frac{\Omega^2\overline{c}_{(\mathrm{i})}\overline{Y}}{4k_{\mathrm{B}}} \tag{3.3}$$

for randomly distributed impurities, and

$$T^{(\mathrm{fd})}_{\mathrm{eff}}\approx\frac{3a_0^2 b_{(\mathrm{fd})}^2 c_{(\mathrm{fd})}\overline{Y}}{2\pi^2 k_{\mathrm{B}}} \tag{3.4}$$

for frozen dislocations. The present calculation is not accurate enough to determine the preexponential term $\tilde{W}_0$, which we assume to be of the order 1. Relation (3.1) can be given a very simple meaning: the probability that a crack of minimal length a_0 occurs is given by $\exp[-2\overline{\gamma}_0 a_0/(k_{\mathrm{B}}T_{\mathrm{eff}})]$. Here $2\overline{\gamma}_0 a_0$ denotes the energy of such a crack. The total probability is then the (a_c/a_0)th power of this elementary probability.

Below we want to estimate $\tilde{W}$ for two different materials. We have to keep in mind, that our calculation was strictly two dimensional. To compare the results with real experiments on *thin plates* we have to consider their dependence on the width h of the plate. A necessary condition for the application of the two-dimensional theory is that the critical crack length a_c is *much larger* than h. In the following we will assume that this condition is always fulfilled.

Since $\overline{\gamma}_0=\gamma_0 h$, $\overline{Y}=Yh$ and $\overline{\sigma}_{(\mathrm{e})}=\sigma_{(\mathrm{e})}h$ are proportional to h, the nucleation energy $2\overline{\gamma}_0 a_c=4\overline{\gamma}_0^2\overline{Y}/(\pi\overline{\sigma}_{(\mathrm{e})}{}^2)$ is also proportional to h.

Estimating the h dependence of T_{eff} we have to make sure that all relevant length scales in the xy plane (like a_c) are much larger than the h. For $T^{(\mathrm{i})}_{\mathrm{eff}}$ and $T^{(\mathrm{fd})}_{\mathrm{eff}}$, which were determined by the small scale cutoff a_0, this leads to the severe restriction $h<a_0$. We note, however, that this cutoff will be in general larger than the lattice spacing. It is therefore appropriate to use in these two cases h as the small-scale cutoff. The behavior on even smaller scales is described by three-dimensional physics which we will discuss at the end of the section.

With $\overline{\delta\gamma}^2\approx\delta\gamma^2 h a_0$, where $\delta\gamma$ denotes the fluctuation of the three-dimensional surface tension γ_0, $T^{(\mathrm{s})}_{\mathrm{eff}}$ does not depend on h.

Next, $\overline{c}_{(\mathrm{i})}=c_{(\mathrm{i})}h$ is a two-dimensional density proportional to h (the three-dimensional density $c_{(\mathrm{i})}$ is independent of h). Here Ω as a two-dimensional cross section of the impurity should be replaced by $\Omega^{3/2}/h$; i.e., we assume essentially spherical impurities. Thus

$$T^{(\mathrm{i})}_{\mathrm{eff}}\to\frac{\Omega^3 c_{(\mathrm{i})}Y}{4k_{\mathrm{B}}}\left(1-\frac{h}{2a_c}\right) \tag{3.5}$$

is essentially independent of $h(\ll a_c)$.

For randomly distributed dislocations $c_{(\mathrm{fd})}$ (as a line density) and $b_{(\mathrm{fd})}$ are unchanged and hence

$$T^{(\mathrm{fd})}_{\mathrm{eff}}\to\frac{h^3 b_{(\mathrm{fd})}^2 c_{(\mathrm{fd})}Y}{4\pi^2 k_{\mathrm{B}}}. \tag{3.6}$$

In Eqs. (3.5) and (3.6) we replaced a_0 by h.

On scales smaller than h the cracks are *three dimensional*. The results of this paper can, however, easily be extended to penny cracks in d dimensions. The Griffith energy then takes the form

$$E_0(a)=\gamma_0 a^{d-1}-Y^{-1}\sigma_{(\mathrm{e})}^2 a^d, \tag{3.7}$$

where we neglect here and in the following all numerical factors which in general depend on the precise crack geometry. $E_0(a)$ has a maximum at $a_c\sim\gamma_0 Y/\sigma_{(\mathrm{e})}^2$; i.e., a_c is essentially unchanged.

If the disorder is taken into account, additional contributions to the energy appear. A randomness in the surface tension leads to a fluctuation of the order $\langle[E_1^{(\mathrm{s})}(a)]^2\rangle\approx\delta\gamma^2(aa_0)^{d-1}$. Following the discussion below Eq. (2.31), the minimum of $W_{E<0}(a)$ follows again for $\tilde{a}_c\sim a_c$. Rewriting the minimum $\tilde{W}_{E<0,d}$ of $W_{E<0}(a)$ in the form

$$\tilde{W}_d\approx\tilde{W}_{d,0}\exp\left(-\frac{\gamma_0 a_c^{d-1}}{k_{\mathrm{B}}T_{\mathrm{eff}}}\right), \tag{3.8}$$

we obtain for the effective disorder the temperature $T^{(\mathrm{s})}_{\mathrm{eff}}(d)\approx\delta\gamma^2 a_0^{d-1}/(k_{\mathrm{B}}\gamma_0)$.

Randomly distributed impurities create an additional contribution $\sigma_{(\mathrm{i})}$ to the stress where

$$|\sigma_{(\mathrm{i})}|=Y(\Omega^d c_{(\mathrm{i})}/a^d)^{1/2}. \tag{3.9}$$

This expression can be understood as follows: An isolated impurity creates in a volume a^d an average stress[13] of the order $Y\Omega^{d/2}/a^d$. With $c_{(\mathrm{i})}a^d$ the number of impurities in this volume [the average stress created by the impurities is assumed to be already incorporated into $E_0(a)$] the fluctuation of the stress created by the impurities is given by Eq. (3.9). Thus we obtain $\langle[E_1^{(\mathrm{i})}(a)]^2\rangle\approx\Omega^d Y\gamma_0 a^d c_{(\mathrm{i})}/a_c$. The minimal probability follows for $d>2$ (contrary to the two-dimensional case) for $\tilde{a}_c \sim a_c$. The disorder temperature is now of the order $T_{\rm eff}^{(\rm i)}(d) = \Omega^d c_{(\rm i)} Y/(4k_{\rm B})$. Clearly, for $a_c \approx h$, $T_{\rm eff}^{(\rm i)}(d)$ agrees with the result for the plate, as it should be.

Similarly, randomly distributed dislocation lines will give a fluctuation contribution $\sigma_{(\rm fd)}$ to the stress of order

$$|\sigma_{(\rm fd)}| = Y b_{(\rm fd)} c_{(\rm fd)}^{1/2}. \tag{3.10}$$

Indeed, a single dislocation (line) creates a stress of the order Yb_0/a. With $c_{(\rm fd)} a^2$ for the number of dislocation lines in the volume a^d which are assumed to have random orientations one obtains Eq. (3.10). The corresponding fluctuation of the energy due to dislocations is therefore given $\langle [E_1^{(\rm fd)}(a)]^2 \rangle \approx a^{2d} \gamma_0 Y b_{(\rm fd)}^2 c_{(\rm fd)}/a_c$. The minimal probability follows as for $d=2$ from small scales $a \approx a_0$, which results in

$$T_{\rm eff}^{(\rm fd)}(d) \approx a_0^2 a_c^{d-2} Y b_{(\rm fd)}^2 c_{(\rm fd)}/k_{\rm B}. \tag{3.11}$$

Clearly, also the case of multiple disorder can be considered as it was done in Eq. (2.31).

Next we have to compare the different probabilities in order to decide which process dominates the formation of cracks in thin plates. For systems with a random surface tension and under the condition $a_c \gg h$, the minimal probability $\tilde{W}$ arises both in two and in three dimensions from cracks of length $\tilde{a}_c \sim a_c$. In this case the probability for crack formation is given by Eqs. (3.1) and (3.2).

For randomly distributed impurities $\tilde{a}_c(d=2) \approx a_0 \approx h$ in two dimensions and $\tilde{a}_c(d=3) \approx a_c$ in three dimensions. Thus crack formation is dominated here by two-dimensional cracks of minimal length $a_0 \approx h$. The corresponding probability is now given by Eqs. (3.1) and (3.5).

Finally, for randomly distributed dislocation, crack formation in both two and three dimensions is controlled by the formation of small cracks of size $\tilde{a}_c \approx a_0$. Comparing the corresponding probabilities in two and in three dimensions (for $d=2$ we have to use $a_0 \approx h$) we find that crack formation is dominated by penny cracks of size a_0. Its probability is given by Eqs. (3.8) and (3.11) for $d=3$.

In a macroscopic sample of linear size L regions of distance greater than $\tilde{a}_c$ can be considered to be essentially independent. The total probability for crack formation is given by $\tilde{W}_d (L/\tilde{a}_c)^d$ where $d=2$ for surface and impurity disorder and $d=3$ for frozen dislocations. Since a_c depends on $\sigma_{(\rm e)}$, the relation $\tilde{W}_d (L/\tilde{a}_c)^d \approx 1$ determines implicitly the critical stress $\sigma_{(\rm e),c}$ for crack formation in dependence on the disorder characteristics and the system size.

TABLE I. Estimates of the effective temperatures $T_{\rm eff}$ for glass and SiC using Eqs. (3.2), (3.5), and (3.6). The corresponding factors A in Eqs. (3.12) and (3.13) are also given. The material constants are taken from Ref. 2.

		Glass	SiC
Y	$[10^9\ \rm Pa]$	70	400
γ_0	$[\rm J\,m^{-2}]$	1.0	4.0
Random surface energy			
Weak disorder: $\delta\gamma/\gamma_0 = 0.1$, $a_0 = 5\times 10^{-10}$ m			
$T_{\rm eff}^{(\rm s)}(d=2)$	[K]	1087	4348
$A^{(\rm s)}$	$[\rm Pa^2\,m^{-1}]$	5.9×10^{30}	1.36×10^{32}
Strong disorder: $\delta\gamma/\gamma_0 = 0.3$, $a_0 = 10^{-6}$ m			
$T_{\rm eff}^{(\rm s)}(d=2)$	[K]	3.9×10^{10}	1.57×10^{11}
$A^{(\rm s)}$	$[\rm Pa^2\,m^{-1}]$	1.65×10^{23}	3.77×10^{24}
Random impurities			
Weak disorder: $\Omega = 2.5\times 10^{-19}\,\rm m^2$, $c = 8\times 10^{24}\,\rm m^{-3}$			
$T_{\rm eff}^{(\rm i)}(d=2)$	[K]	158.5	905.8
$A^{(\rm i)}$	$[\rm Pa^2\,m^{-1}]$	4.07×10^{31}	6.5×10^{32}
Strong disorder: $\Omega = 10^{-15}\,\rm m^2$, $c = 10^{17}\,\rm m^{-3}$			
$T_{\rm eff}^{(\rm i)}(d=2)$	[K]	1.26×10^5	7.24×10^5
$A^{(\rm i)}$	$[\rm Pa^2\,m^{-1}]$	5.09×10^{28}	4.15×10^{29}
Random frozen dislocations			
Weak disorder: $b_{(\rm fd)} = 5\times 10^{-10}$ m, $c_{(\rm fd)} = 10^{14}\,\rm m^{-2}$, $h = 10^{-3}$ m			
$T_{\rm eff}^{(\rm fd)}$(plate)	[K]		1.83×10^{19}
$a_0 = 5\times 10^{-9}$ m			
$A^{(\rm fd)}$	$[\rm Pa^2\,m^{-1}]$		1.71×10^{30}
Strong disorder: $b_{(\rm fd)} = 5\times 10^{-10}$ m, $c_{(\rm fd)} = 10^{16}\,\rm m^{-2}$, $h = 10^{-3}$ m			
$T_{\rm eff}^{(\rm fd)}$(plate)	[K]		1.83×10^{21}
$a_0 = 5\times 10^{-10}$ m			
$A^{(\rm fd)}$	$[\rm Pa^2\,m^{-1}]$		1.71×10^{31}

Next we consider the probability for crack formation in two different materials, one is amorphous (glass) and one crystalline (SiC). The corresponding parameters are summarized in Table I. In the case of random surface tension and randomly distributed impurities we express the probability for crack formation as a function of $h/\sigma_{(e)}^2$ since the probability can be written in the form

$$\ln(\tilde{W}/\tilde{W}_0) = -\frac{4}{\pi} \frac{\gamma_0^2 Y}{k_{\rm B} T_{\rm eff}} \frac{h}{\sigma_{(\rm e)}^2} \equiv -A \frac{h}{\sigma_{(\rm e)}^2}. \tag{3.12}$$

For frozen dislocations we have

$$\ln(\tilde{W}/\tilde{W}_0) \approx -\frac{\gamma_0^2}{a_0^3 b_{(\rm fd)}^2 c_{(\rm fd)}} \frac{a_0}{\sigma_{(\rm e)}^2} \equiv -A^{(\rm fd)} \frac{a_0}{\sigma_{(\rm e)}^2}. \tag{3.13}$$

The corresponding values for $A^{(\rm s)}$, $A^{(\rm i)}$, and $A^{(\rm fd)}$ are also given in Table I.

A number of comments are in order.

(i) The results so far are based on the assumption of short-range correlations of the disorder (surface energy, impurities, frozen dislocations). Experimentally this may not be the most important situation. Long range correlations of the disorder described by a power law decay on the right-hand side of Eqs. (2.9), (2.14), and (2.16) would in general lead to an increase of the probability for the occurrence of a crack.

(ii) In the case of impurities and frozen dislocations the probability $\tilde{W}$ is dominated by the energetics on small length scales. Since the stress on the crack tip diverges in the linear elasticity theory used throughout the paper, nonlinear effects may be of particular importance for small cracks. This could diminish the numerical coefficients in the exponent of $\widetilde{W}$.

(iii) Defects were assumed everywhere to be frozen. In real fatigue experiments often alternating stress is applied which leads to an accumulation of dislocations close to the crack tip, which makes crack propagation easier. This mechanism could also help in a situation where, due to disorder fluctuations, the stress is considerably higher than on average (or the surface energy is lower).

(iv) An interesting question is the relation of our results to crack propagation in quasicrystals,[19] which take an intermediate position between periodic and random media.

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APPENDIX

The relevant integrals are

$$I_1=\int_{-1}^{1} dx'(1-x'^2)^{1/2}\frac{1}{x'-x}=-\pi x \quad \text{for} \quad |x|<1, \tag{A1}$$

$$\begin{aligned} I_2&=\int_{-1}^{1} dx'(1-x'^2)^{-1/2}\frac{x'}{x'-x}\\ &=\begin{cases}\pi & \text{for} \quad |x|<1,\\ \pi-\pi|x|(x^2-1)^{-1/2} & \text{for} \quad |x|>1.\end{cases}\end{aligned} \tag{A2}$$

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