

Effects of dissipation on quantum phase transitions

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We discuss the effect of dissipation on quantum phase transitions. In particular, we concentrate on the superconductor to insulator and quantum Hall to insulator transitions. By invoking a phenomenological parameter α to describe the coupling of the system to a continuum of degrees of freedom representing the dissipative bath, we obtain phase diagrams for the quantum Hall and superconductor-insulator problems. Our main result is that, in two dimensions, the metallic phases observed in finite magnetic fields (possibly also strictly zero field) are adiabatically deformable from one to the other. This is plausible, as there is no broken symmetry that differentiates them.

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I. INTRODUCTION

Quantum phase transitions continue to attract intense theoretical and experimental interest; see, for example, Refs. 1–5. Such transitions—where changing an external parameter in the Hamiltonian induces a transition from one quantum ground state to another, fundamentally different one—have been invoked to explain data from various experiments. Transitions that have been studied include the quantum Hall liquid to insulator transition (QHIT), the quantum Hall liquid to quantum Hall liquid or “plateau” transition (QHPT), the metal to insulator transition (MIT), and the superconductor to insulator transition (SIT). Where the transition is continuous, quantum critical phenomena are expected to give rise to interesting, universal physics, which it is common practice to analyze using a straightforward scaling theory, inherited from the classical theory of finite temperature phase transitions.

Effects of dissipation, that is to say, the coupling of the critical modes to a continuum of other “heat-bath” degrees of freedom, can fundamentally alter the character of the phases and of the transitions between them.^{6–8} While in classical statistical mechanics, the dynamics and thermodynamics are independent of each other, in the quantum case they are intimately related. The dynamical relaxation processes that permit the system to reach equilibrium can be neglected in classical problems, but cannot be ignored in a quantum problem.⁹

Recently, compelling experimental evidence has accumulated of the existence of “metallic” phases, that is to say, phases with finite dissipation in the zero temperature limit. There is as yet no microscopic understanding of these observations. We conjecture that a metallic phase is stabilized by strong enough coupling to a dissipative heat bath, which we characterize by a single phenomenological parameter α , in a manner pioneered in early studies of macroscopic quantum tunneling and coherence.¹⁰ Note that the present phenomenological approach is impervious to such important issues as whether the dissipation is intrinsic or extrinsic. For large enough α , quantum coherence can be suppressed even at zero temperature.¹¹ Thus, the conventional picture of quan-

tum phase transitions is certainly dramatically altered, and “intermediate” metallic phases can appear in the phase diagram.

As a paradigmatic example, consider the magnetic field driven SIT, for which the commonly accepted phase diagram is shown in Fig. 1(a). In the neighborhood of the phase boundary, quantum critical scaling with universal exponents is expected (so long as D and $H \neq 0$); in particular, the correlation length exponent should be $\nu \approx 7/3$, as discussed below. With the introduction of dissipation, the phase diagram is modified in a manner that is not presently well understood. Here, we postulate that, as originally proposed in Ref. 12, the result is the phase diagram shown schematically in Fig. 1(b). Here, when $\alpha > \alpha_m$, a finite resistance metallic state appears between the superconducting and insulating phases. Moreover, to the extent that a crossover from a positive to a negative coefficient of resistance occurs along the dashed line in the figure, classical percolation with $\nu = 4/3$ will describe the physics at high temperatures, as is indeed observed experimentally.¹² However, to reconcile the fact that finite dissipation in general tends to suppress quantum fluctuations and thus pin the superconducting phase, the phase boundary emerging from H_c is slightly tilted toward high fields.

The same considerations have direct implications for quantum Hall systems. Here, classical percolation in the limit of slowly varying disorder¹³ and a crossover to quantum percolation near the transition^{14,15} can be motivated from the microscopic theory. It was previously shown¹⁶ by iteratively applying the Chern-Simons flux attachment transformation,¹⁷ that a global phase diagram for quantum Hall systems can be obtained from considerations of the magnetic field driven SIT. Here, the various QHIT’s and QHPT’s are mapped onto a single SIT. Consequently, if we adopt the phase diagram in Fig. 1(a), the corresponding quantum Hall phase diagram is that shown in Fig. 2(a), with direct quantum transitions between the various quantum Hall phases governed by simple selection rules. However, the existence of the intermediate metallic phase in Fig. 1(b) produces for the quantum Hall system the phase diagram shown in Fig. 2(b), where for large enough α each direct transition point opens up into a metallic regime.

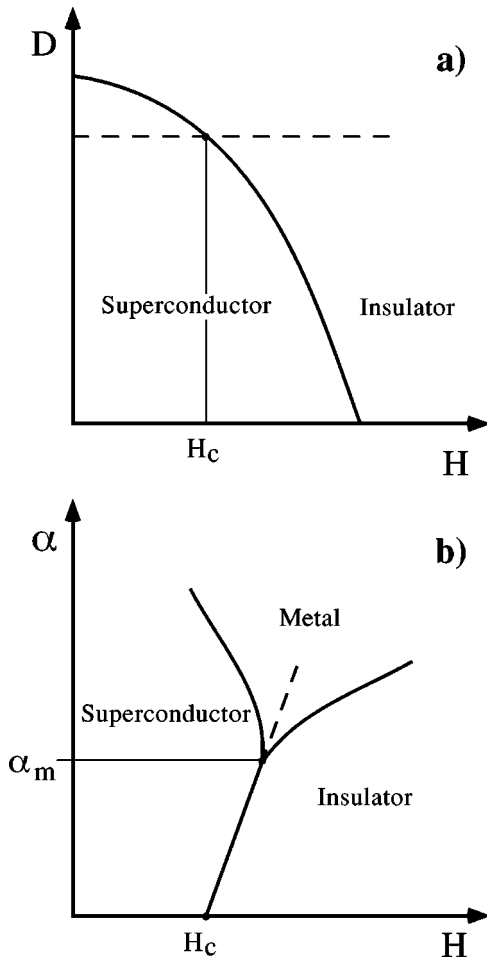


FIG. 1. Phase diagram for the field-tuned SIT. (a) The H - D (D stands for disorder) plane with the dashed line representing a plane at finite disorder, and (b) H - α (α stands for dissipation strength) diagram at finite disorder. H_c marks the SIT critical point, and α_m marks the critical dissipation above which a metallic phase is obtained (see text for details.)

This has many further consequences.

(1) In quantum Hall systems, in which metallic behavior is observed below an apparent high temperature QHIT, behavior analogous to that observed in the magnetic field driven SIT should be seen. Among other things, this means that in the high temperature scaling regime an apparent exponent $\nu = 4/3$ should be observed, and that at fields considerably larger or smaller than the apparent critical field true low temperature metal-insulator and metal to quantum Hall liquid transitions should be found. Preliminary evidence of the correctness of the first of these predictions is shown in Fig. 3.

(2) In quantum Hall systems, in which a direct QHIT is observed with quantum exponent $\nu \approx 7/3$, a transition to classical percolation behavior with an intermediate phase can be induced by increasing the dissipation in the system. This can in principle be done, using the strategy employed by Rimbarg *et al.*,⁷ by placing a second two-dimensional (2DEG) electron gas that is capacitively coupled²⁰ to the first and tuning the conductivity of the second 2DEG by means of a

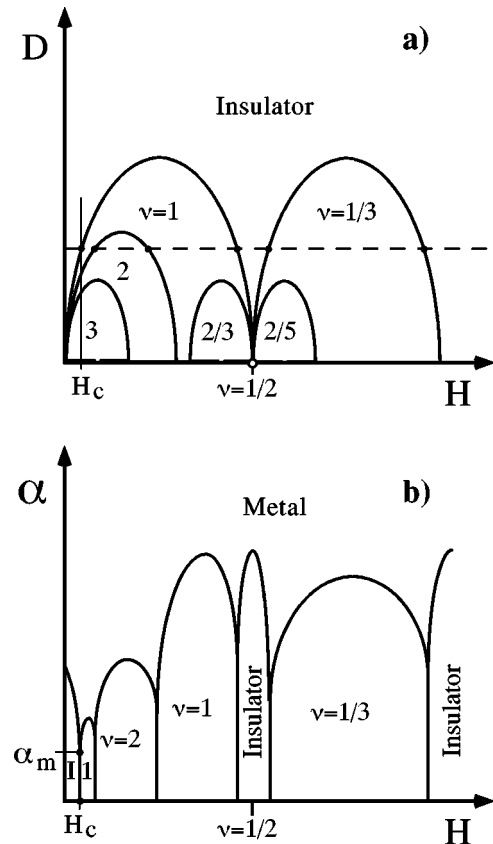


FIG. 2. Similar to Fig. 1 but for the quantum Hall problem. (a) The global phase diagram of Kivelson, Lee, and Zhang (Ref. 16) with dashed line representing a cut at finite disorder. Full circles represent the critical points obtained for a particular realization of disorder. (b) H - α (α stands for dissipation strength) diagram at finite disorder. This figure is obtained by applying flux attachment and the law of corresponding states (see Sec. II D), and it preserves the correct topology of Fig. 1(b) (slopes of the phase boundaries at small α are omitted.) H_c marks the QHIT critical point from $\nu = 1$ to insulator. Other critical points are found in a similar way.

back gate. The critical phenomena associated with the value $\alpha = \alpha_m$ at which the metallic phase first appears should be very interesting.

(3) Conversely, by reducing the amount of dissipation in the system (for instance, by studying Josephson junction arrays or granular films in a field) the intermediate metallic phase observed at low temperatures in experiments on the field driven SIT should be narrowed, and ultimately eliminated. If this can be achieved, rather than classical percolation exponents, values of $\nu \approx 7/3$ are expected.

(4) The flux attachment transformation to relate states at zero H to those at nonzero H is subtle.¹⁶ Even if the average magnetic flux is canceled in this way, the Hamiltonian at zero H respects time-reversal symmetry at the microscopic level whereas the finite field system does not. However, if time-reversal symmetry is spontaneously broken even at $H = 0$, as would occur, for instance, in a spin-glass phase, then the correspondence between zero field and finite H phases might be reliable. In this way, a correspondence between the Hall metal and the zero field metallic phase might occur, with many consequences.^{21,22}

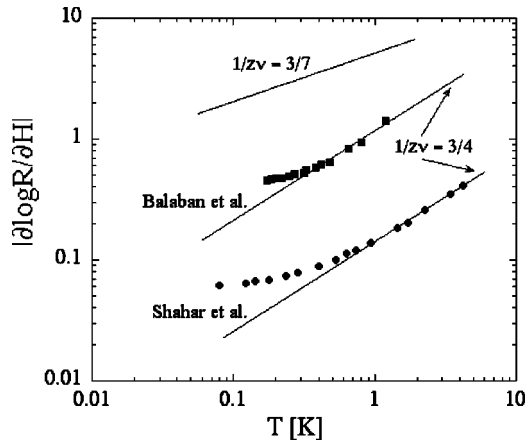


FIG. 3. Data taken from Shahar *et al.* (Ref. 18) and Balaban *et al.* (Ref. 19). The straight lines with slope $1/z\nu = 3/4$ represent the asymptotic trend of the data at “high temperatures” (see text for details.) The straight line with a slope of $1/z\nu = 3/7$ represents the expected behavior for a quantum percolation transition.

(5) All the metallic phases observed in these systems at nonzero H (and possibly even at $H=0$ if time-reversal symmetry is spontaneously broken) are adiabatically connected.

II. INTERMEDIATE “METALLIC” PHASES

The theory of a metallic phase in two dimensions at zero temperature is a matter of intense current debate.^{12,23–29} Indeed, such a phase was thought for many years not to be possible.^{30,31} Recent experiments have shown that metallic phases are not only possible, but exceedingly common whenever interaction effects are strong.² Of course, there is always a question whether the metallic behavior is a finite temperature artifact, as there is no way to prove that the resistivity would not diverge or vanish if the temperature were lowered enough. Below we list some of the most clear-cut cases in which an apparently metallic phase has been observed. Since in each case the experiments access temperatures low compared to all the simple energy scales in the problem, we feel it is reasonable to accept this evidence at face value.

(1) An apparently metallic phase occurs in superconducting films in a magnetic field for intermediate magnetic field strengths^{12,32–36} and in arrays of Josephson junctions.³⁷ In particular, this behavior is observed at low temperatures in the magnetic field range at which, at higher temperatures, scaling behavior is observed that was formerly interpreted^{12,35,38} as indicative of a SIT.

(2) Analogous behavior has been observed in semiconductor heterojunctions for magnetic fields in the neighborhood of a putative QHIT.^{18,19} Specifically, metallic behavior is observed at low temperatures in the neighborhood of the critical magnetic field at which higher temperature data are indicative of a QHIT.

(3) In quench-condensed films of Ga, Pb, and In at $H=0$, an anomalous metallic phase is seen^{39–41} below the local superconducting transition temperature, in which the resistance decreases strongly (by as much as five decades) with decreasing temperature, roughly as $R = R_0 \exp[T/T_0]$, but ex-

trapolates to a finite value R_0 as $T \rightarrow 0$.

(4) In Si metal-oxide-semiconductor field-effect transistors (MOSFETs), and other high mobility semiconductor devices that access the strongly interacting (large r_s) physics of the two-dimensional electron gas, an unexpected metal-insulator transition at $H=0$ has now been widely and convincingly documented.^{2,42,43} The stability of the metallic phase has also been supported by studies^{44–46} of the quantum Hall effect at small nonzero H : Theoretically,^{47,48} a sequence of transitions resulting from the “floating up” of the extended states is predicted to occur as the expected insulating state is approached as $H \rightarrow 0$, and indeed this is observed in smaller r_s devices.^{49,50} However, at large r_s , the delocalized states (or, more precisely, the critical lines separating different integer quantum Hall states) do not move up in energy as $H \rightarrow 0$, thus allowing for a metallic state at $H=0$.

(5) A related set of experiments^{51,52} on the behavior of high mobility 2DEG’s in the small magnetic field limit show behavior that we interpret as indicative of a metallic phase for a range of weak magnetic fields. Note that these experiments were interpreted somewhat differently by their authors: The data clearly reveal a failure of the delocalized states to “float up” as $H \rightarrow 0$ and a breakdown of the selection rules¹⁶ thought theoretically to govern quantum Hall transitions, including the QHIT. However, the measurements were originally interpreted in terms of a QHIT in which the ground state at H smaller than a (electron density dependent) critical field H_c was said to be insulating. Indeed, in this field range, the resistance increases with decreasing temperature, but only weakly, and it appears to saturate at low temperatures. In the following, we will reinterpret these data in terms of a quantum Hall liquid to metal transition, rather than as a “new universality class of QHIT.”

(6) Metallic phases were also found, under special circumstances, in high quality devices (especially in GaAs heterojunctions) in the presence of a quantizing magnetic field (i.e., H big enough that the cyclotron energy $\hbar\omega_c$ is larger than the temperature, the disorder potential, and even the Coulomb strength). This “Hall metal” has been observed⁵³ and extensively studied⁵⁴ for a range of magnetic fields near filling factor $\nu = 1/2$ (i.e., two magnetic flux quanta per electron) and related even fractions. Precisely at $\nu = 1/2$ there is some indication of an upturn in the resistance at the lowest temperatures,^{55,56} which might indicate that the true ground state is insulating, but overall in this range of fields, the preponderance of experimental evidence supports the existence of a true metallic phase.

(7) In weakly disordered metallic films (with large values of $k_F l$) the phase coherence length, which according to theory controls the low temperature behavior of weakly localized systems, appears⁵⁷ to saturate at low temperatures (the determination of l_ϕ was done using magnetoresistance measurements) rather than diverging as required by theory.⁵⁸

III. EXTREMAL ZERO TEMPERATURE PHASES

In this section we summarize our theoretical understanding of the relevant zero temperature electronic phases in two spatial dimensions in the absence of dissipation ($\alpha=0$).

This discussion is not meant to be exhaustive as there are numerous other phases, for example, phases that spontaneously break time-reversal symmetry,^{22,59} which may be important under some circumstances.

A. The superconducting state

Numerous two-dimensional systems in zero magnetic field exhibit a superconducting or superfluid phase below a nonzero transition temperature. The theoretical understanding⁶⁰ of this state is not in question, as far as we know: at finite temperature, the superfluid phase is characterized by a nonzero superfluid density, but no broken symmetry, although in the zero temperature limit a true broken symmetry state is expected. The zero temperature superconducting state persists in the presence of a nonzero magnetic field H so long as it is small enough, $H < H_c$. In the absence of disorder, this is due to the crystallization of the field induced vortices and pinning at the boundaries (note that with no pinning at the boundaries any finite current will cause the vortex lattice to slide and thus result in dissipation); this vortex crystal (Abrikosov lattice) itself persists to a finite melting temperature, so in this limit superconductivity survives at low, but nonzero temperatures. In the presence of quenched disorder, the vortex crystal at small H is certainly disrupted,⁶¹ but at $T=0$ the dilute vortices are localized by the disorder. However, in this case superconductivity is destroyed at any nonzero temperature due to thermally activated vortex motion.⁶² Thus, in systems of interest, for which disorder is an important feature of the physics, no $T > 0$ transition to the superconducting state is expected; in this phase the conductivity diverges continuously as $T \rightarrow 0$, together with a diverging length that describes the proximity to this zero temperature vortex glass phase. Consequently the I - V characteristics of the system will exhibit nonlinear behavior above some threshold current that itself vanishes as the temperature tends to zero.⁶³

B. The insulating phase

In the limit of large disorder, large H , or strong interactions between particles, all the particles are localized and an insulating ground state occurs. This is independent of whether the constituent particles are taken to be bosons or fermions (or anyons, for that matter). In the absence of disorder, this insulating state has Wigner crystalline long range order, and correspondingly a finite temperature phase transition. However, disorder should couple to the charge order of the Wigner crystal as a random field. Thus, from the Imry-Ma random field arguments, no true finite temperature phase transition to the insulating state occurs. The insulating state is then characterized by a resistivity that diverges continuously as $T \rightarrow 0$.

In the absence of a magnetic field, a further distinction may, in principle, distinguish various insulating phases based on their spin (magnetic) structure.^{64,65} Electrons in the extreme low density limit in the absence of disorder form a Wigner crystalline state in which the electron spins are ferromagnetically ordered.⁶⁶ Although even weak disorder eliminates long ranged Wigner crystalline order, presumably

the ferromagnetism would survive up to a critical disorder strength. At higher densities, but still in the Wigner crystalline state, antiferromagnetic exchange processes begin to dominate, so a frustrated antiferromagnet, possibly with some form of spin pairing and a spin gap, may occur.^{64,67,68} Such an insulating state with a spin gap could also occur as the result of a localization transition of a system of Cooper pairs, as would be expected, for instance, in the Coulomb blockade limit of a granular superconductor.⁶⁹ If so, some form of spin pseudogap is likely to survive the introduction of weak disorder. At large disorder, a zero temperature spin-glass phase is also conceivable. At nonzero Zeeman coupling to an external magnetic field, most or all of the magnetic distinctions between various possible insulating phases are removed.

A distinction between insulating states in a magnetic field has also been discussed based⁷⁰ on the asymptotic behavior of the Hall resistance ρ_{xy} as $T \rightarrow 0$. A Wigner crystal, and presumably also an Anderson insulator,⁷¹ have a Hall resistance that diverges in this limit. Conversely, it has been proposed¹⁶ that a ‘‘Hall insulator’’ phase exists that has a vanishing conductivity tensor, but a Hall resistance that approaches a finite value, presumably of order H/nec , as $T \rightarrow 0$. A still more exotic ‘‘quantized Hall insulator’’ phase has also been proposed⁷² in which, despite the vanishing of the conductivity tensor, the Hall resistance approaches a quantized value, for instance h/e , as $T \rightarrow 0$. It is likely, although not proven, that a Hall insulator is a distinct state of matter, and must be separated by a phase transition from an ordinary insulator. This is certainly the case for a quantized Hall insulator,¹⁶ as there must be a first magnetic field at which the Hall resistance ceases to be quantized.

In the present paper, we will always assume that the effects of disorder are important, so we will not explicitly discuss the Wigner crystal or Abrikosov lattice. For simplicity, we will also neglect the various magnetic and transport distinctions between insulating phases, as well.

C. The quantum Hall liquid

The quantum Hall liquid has a vanishing longitudinal conductivity and a quantized Hall conductance $\sigma_{xy} = (e^2/h)s_{xy}$ where s_{xy} is an integer or one of a particular set of rational fractions for the integer and fractional Hall effects, respectively. There is no true broken symmetry, and so no finite temperature phase transition to this state. Quantum Hall liquid states are partially characterized by the quantized value of the Hall resistance. A quantum Hall state is typically most stable at a ‘‘magic’’ value of the magnetic field, $H_{magic} = s_{xy}^{-1} \phi_0 \bar{\rho}$, where $\phi_0 = hc/e$ is the quantum of flux and $\bar{\rho}$ is the mean electron density. However, like the superconducting state, the quantum Hall state at $T=0$ is stable for a finite range of disorder and magnetic field.

Quasiparticles are generated in the ground state when the magnetic field differs from the magic value, or can be nucleated by disorder. They play a role analogous to that of vortices in a superconductor,¹⁷ and when they are not localized they destroy the quantum Hall state. In the absence of disorder, a low density of quasiparticles will crystallize, produc-

ing a quantum Hall state which is analogous to the Abrikosov lattice. In the presence of disorder, the dilute quasiparticles are localized. However, whereas the vortices in a superconductor are bosonic,⁷³ the quasiparticles in a quantum Hall state have statistics, fermionic in the integer case and anyonic in the fractional case, that are related to s_{xy} .

D. Flux attachment and the law of corresponding states

There is a formal transformation,¹⁷ often called flux attachment, that maps a system of interacting particles in a magnetic field with Bose or Fermi statistics to another problem of *transformed* Chern-Simons particles with the same mass, same interactions, but modified statistics, and which interact with a fluctuating statistical gauge field a_μ , whose dynamics are governed by a Chern-Simons term, rather than the usual Maxwell action. The Chern-Simons action has the effect of attaching θ (related to the Chern-Simons coupling constant) quanta of statistical flux to each particle. Consequently, the *net* magnetic flux experienced by the transformed particles is modified according to

$$H^{eff} = H - \theta \phi_0 \bar{\rho}, \quad (1)$$

and the phase (statistical) angle associated with the interchange of two particles is transformed according to

$$\phi_{exchange} = (n - \theta) \pi, \quad (2)$$

where, depending on whether the original particles are bosons or fermions, $n=0$ or $n=1$, respectively. Manifestly, for θ an odd integer, this transformation maps bosons into fermions and vice versa, while, for θ even, it maps bosons into transformed bosons, and fermions into transformed fermions.

The effect of this transformation is a formal relation between seemingly different states of matter. For instance, if for some choice of θ the resulting Chern-Simons bosons are condensed into a superconducting state, the original particles are necessarily in a quantum Hall liquid state, with conductivity tensor

$$\sigma_{xy} = (q^2/h)(1/\theta) \quad \text{and} \quad \sigma_{xx} = 0, \quad (3)$$

where q is the particle charge (i.e., the coupling to the external electromagnetic gauge field). Conversely, if the Chern-Simons bosons are localized in an insulating state, $\sigma_{ij} = 0$. In similar ways, it is possible to relate fermions in an integer quantum Hall state to electrons in a fractional quantum Hall state,^{74,75} etc.

There is another transformation, which is a version⁷³ of a standard duality transformation, that relates the physics of different phases, and which can be implemented^{16,76} straightforwardly in the bosonic Chern-Simons field theory. Formally, this transformation relates the particle three-current J_μ to a dual gauge field a_μ^{dual} ,

$$J_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda^{dual}, \quad (4)$$

and, conversely, a dual particle (vortex) current to the original gauge field a_μ ,

$$J_\mu^{dual} = \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda, \quad (5)$$

where $\epsilon_{\mu,\nu,\lambda}$ is the Levi-Civita symbol. The remarkable feature of this transformation in 2+1 dimensions is that the action has the same form in terms of the original and transformed variables but the statistical angle is transformed according to

$$\theta^{dual} = -1/\theta. \quad (6)$$

Moreover, when considering the coupling to an external electromagnetic gauge field the particle charge q transforms according to

$$q^{dual} = -q/\theta \quad (7)$$

and the conductivity tensor (computed from the Kubo formula) for the original variables is related to that computed in the dual theory according to

$$\sigma_{ij} = \sigma_{ij}^{dual} + (q^2/h)(1/\theta) \epsilon_{i,j}. \quad (8)$$

(This expression appears to differ from, but is actually consistent with, a more familiar relation that involves the one-gauge-field irreducible response functions of the Chern-Simons particles, i.e., the particle response to the combined external and statistical gauge fields. In terms of these irreducible response functions, the conductivity of the original particles is related to the resistivity of the dual particles.)

Comparing the expressions in Eq. (3) and Eq. (8), it is clear that if a set of Chern-Simons bosons are condensed in a superconducting state the dual bosons must be localized, and conversely. In the context of the SIT, this allows one to view the insulating state as a condensed state of vortices.⁷³ In the case of the integer quantum Hall effect with $s_{xy} = 1$ (and hence $q = e$ and $\theta = 1$), the quantum Hall liquid state can be viewed as a condensed state of charge e bosons, with vortex excitations (quasiholes) with charge $q^{dual} = -e$ and fermionic statistics ($\theta^{dual} = -1$), while the proximate insulating state can be viewed as a vortex condensate, and the original electrons become the vortices in the dual theory.

These two transformations form the basis for a law of corresponding states,^{16,77} which relates seemingly different states of matter. However, it should be recalled that, because the fluctuations of the statistical gauge field induce additional interactions between particles, at a microscopic level the correspondence may be quite complicated. However, if only the topology of the phase diagram and the nature of the phases are of principal interest, the law of corresponding states can be adopted without caveat.

Within the context of the dirty boson model, it is thought that there are only two zero temperature phases⁷⁸ in the presence of disorder, a superconducting phase and an insulating phase, as shown in Fig. 1(a). Assuming that this is the case for the Chern-Simons bosons in the quantum Hall effect, a global phase diagram for the quantum Hall systems can be constructed.¹⁶ The result is summarized in Fig. 2(a).

IV. CRITICAL PHENOMENA

A. Quantum phase transitions

It is clear that an understanding of the phases is a necessary precursor to an understanding of the quantum critical phenomena associated with the transitions between them. However, unlike the analogous classical problem, it is possible for the universality class of a quantum transition to depend on the nature of the dynamics as well as on the character of the two phases.

The simplest possibility, about which the most is known theoretically, is that some or all of the observed transitions are equivalent to a SIT in a system of disordered, interacting fundamental bosons, the so-called dirty boson problem. Because the electrons form Cooper pairs in the superconducting state, and because it is ultimately fluctuations in the phase of the order parameter that are expected to destroy superfluidity (so that the electrons can plausibly be viewed as paired even on the disordered side of the transition), it is an appealing view that the dirty boson problem captures the universal features of the SIT in actual superconducting films. One of the most striking predictions that has been made on the basis of this model⁷⁹ is that the conductivity tensor at the critical point takes on a universal value in units of $(e^*)^2/h$ (where $e^*=2e$ for Cooper pairs and $e^*=e$ for electrons in the quantum Hall effect). To the best of our knowledge, there are no convincing calculations, either numerical or analytic, for the value of the critical conductance or the critical exponents at this transition. An argument can be made on the basis of particle-hole symmetry that the Hall conductance at the transition is 0. Independent arguments, based on the $1/r$ form of the Coulomb interaction, lead to the expectation that the dynamic exponent $z=1$.

The most obvious, potentially dangerous piece of physics that is omitted in this approach is the effect of low energy quasiparticle excitations in such systems. Presumably, in a granular superconductor, where a relatively clean, large gap in the quasiparticle spectrum is expected and, indeed, observed,^{40,80} the neglect of quasiparticles is a safe bet. However, to date, nothing resembling the expected SIT has been observed in granular films—what is observed is best described as a crossover (an actual transition point is difficult to identify in the data) from a superconductor to a strange metal to an insulator. In disordered films, even in the absence of a magnetic field, one would expect on theoretical grounds that in the neighborhood of the critical film thickness the superconducting gap would be filled in with a large density of gapless quasiparticle excitations, and this expectation is apparently borne out in experiment. In the presence of a magnetic field, where gapless quasiparticle excitations are expected in the cores of vortices, this expectation is even stronger, especially when the spacing between vortices is not large compared to their radius. In both cases dissipation due to gapless quasiparticle excitations can potentially alter the critical phenomena.

In quantum Hall systems a variety of zero temperature continuous phase transitions are expected and, in one form or other, observed. (1) The QHIT, both for the case in which the quantum Hall liquid is an integer Hall state (typically, s_{xy}

$=1$ or, when spin splittings are not resolved, $s_{xy}=2$) or a fractional Hall state (typically $s_{xy}=1/3$). (2) Transitions (plateau transitions) between two distinct quantum Hall liquid states, which again might be integer or fractional. For noninteracting electrons, the critical exponents associated with the QHIT are known to be $\nu \approx 7/3$, and an argument due to Lee and Wang⁸¹ shows that at least short range interactions do not alter this result. Under conditions of particle-hole symmetry, it can be proved⁸² quite generally that the critical Hall conductance at the $s_{xy}=1$ quantum Hall liquid to insulator transition is $\sigma_{xy}^c = (e^2/h)(1/2)$; if the critical conductance is indeed universal then this must be the critical value, even in the absence of (microscopic) particle-hole symmetry. An argument based on particle-vortex duality in the Chern-Simons formulation of the problem leads (among others) to the relation (also called^{83,84,16} the “semi-circle law”) that $\rho_{xx} = (h/e^2)$ in the vicinity of the $s_{xy}=1$ or $s_{xy}=1/3$ to insulator transition. Together, these two results lead to the conclusion that $\sigma_{xx}^c = \sigma_{xy}^c = (e^2/h)(1/2)$ at the $s_{xy}=1$ to insulator critical point, a conclusion that has also been supported by numerical studies of noninteracting electrons.^{85,15}

At least superficially, there is less reason to necessarily expect additional dissipation in quantum Hall systems than in superconducting films. However, in addition to the many possible extrinsic sources of dissipation, there are certainly ways such dissipation could arise intrinsically, even here. For instance, there could be^{86,87} compressible regions of electronic structure associated with the Hall metallic or $\nu=1/2$ state. These might arise naturally, especially in systems with a smoothly varying disorder potential, in the form of “fat” edge states.

A quantum model for dissipation consists of coupling the system variables to a continuum of Ohmic heat-bath degrees of freedom; dissipation is then merely a flow of energy from the system to the heat bath.¹⁰ The integration of the heat-bath degrees of freedom results in an action containing induced temporal long range interactions. The analysis of such long range interactions brings in a new control parameter signifying the coupling to the dissipative heat bath, α , proportional to h/e^2R , where R is the shunt resistance characteristic of the dissipative heat bath. In addition the action will depend on the dimensionality of the space, the symmetry of the order parameter, etc., common in a conventional description of a quantum phase transition.^{1,4}

A quantum phase transition often involves two related phenomena: the formation of a condensate and the associated metastability. For a superconductor or a superfluid, this is widely appreciated. In order to stabilize the superconducting state the phase slip processes must be suppressed and the vortices must be localized. In an incompressible quantum Hall state there is no true broken symmetry, but its metastability hinges upon localizing the quasiparticles over a finite range of disorder and magnetic field (the fermionic quasiparticles playing the role of vortices in a superconductor.) Thus, in the limit of large dissipation, a quantum phase transition can acquire a very special character because dissipation can efficiently reduce quantum fluctuations of the system. As a

result, the competition between the kinetic and potential energies is altered, resulting in a state that is characteristically classical. In other words, the formation of the condensate is tuned by dissipation. Similarly, the metastability of the state can be altered by suppressing quantum tunneling and motions of vortices and quasiparticles, typically by the orthogonality overlap of the continuum degrees of freedom of the heat bath. Such a dissipation tuned transition is well documented in the case of a single Josephson junction, in which, as a function of α , the system undergoes a transition from a quantum state at $\alpha < \alpha_c$, where quantum diffusion of the phase destroys the superconducting state, to a ‘‘classical’’ state at $\alpha > \alpha_c$, where quantum tunneling is suppressed and the junction is truly superconducting at $T \rightarrow 0$. Such a transition can occur in higher dimensional systems, as well, as has indeed been indicated in recent experiments on Josephson junction arrays.⁷

Because the first effect of dissipation is to suppress fluctuation, and hence to stabilize the superconducting phase, we have drawn the superconducting to insulating phase boundary $H_c(\alpha)$ in Fig. 1(b) with a positive slope. However, in the limit of large $\alpha > \alpha_m$, we have indicated a metallic phase, as is suggested, for instance, by the recent work in Ref. 24. The reason for the existence of the metallic phase is that with the suppression of quantum fluctuations the motion of the excitations responding to the external field is diffusive and classical, with characteristics inherited from the heat bath. It is likely that particle exchange with the heat bath, in addition to the usual capacitive coupling, will play an important role in this physics. The existence and the stability of a metallic phase at large α are the central postulates of the present work whose theoretical basis is presently uncertain.

B. Classical percolation

There is another limiting view of any transition between a conducting (or superconducting) and an insulating phase provided by classical percolation; as the conducting regions of the system percolate, the system goes from being globally insulating to globally conducting. In two dimensions, many of the critical properties are well understood theoretically, including the correlation length exponent $\nu_{perc} = 4/3$. The correlation length ξ_{perc} is, roughly speaking, the radius of gyration of the largest typical clusters of the minority phase.⁸⁸ In the case in which the length scale of the disorder is very long, so that quantum tunneling, and more particularly quantum coherence between distinct tunneling events, can be ignored, both the QHIT and the SIT would be expected to be well approximated as percolation transitions. The earliest theories of the QHIT were based on percolation of puddles of Hall liquid.¹³ Likewise, the phenomenologically successful puddle theory of Shimshoni, Auerbach, and Kapitulnik²³ is based on the assumption that the measured transport is dominated by tunneling of vortices (in the case of the SIT) or quantum Hall quasiparticles (for the QHIT) across a characteristic, isolated weak link between puddles.

It is generally believed that at low enough temperature any classical percolation approach must break down. Roughly speaking, if we imagine that, as in weak-

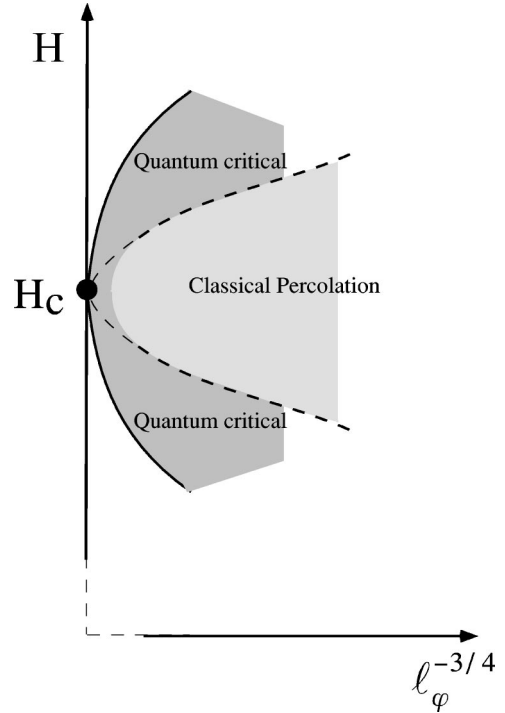


FIG. 4. H - T diagram near the SIT quantum critical point. Here we assume that the classical and quantum critical points coincide. Inside the dashed lines and in the lightly shaded area dissipation is important and the system is classical. However, outside the range of influence of the classical point and between the two solid lines the system (dark shaded area) is quantum critical. Also, as l_φ diverges (e.g., at low enough temperatures), and with a diminishing influence of dissipation, the system becomes quantum critical as well (see text.)

localization theory, there exists some sort of quantum phase coherence length $l_\varphi(T)$ that diverges as the temperature tends to zero, then the classical theory will be valid so long as $\xi_{perc} \gg l_\varphi$, with the true quantum behavior apparent only at such low temperatures that $\xi_{perc} \ll l_\varphi$. In the QHIT, this crossover from classical to quantum percolation has been studied explicitly.⁸¹

In Fig. 4, we illustrate this crossover schematically for the simplest case in which classical percolation and the quantum phase transition occur at the same critical field H_c . (In the case of the $s_{xy} = 1$ QHIT, particle-hole symmetry in the lowest Landau level ensures that this is the case.) Under the assumption that l_φ diverges as $T \rightarrow 0$, the x axis is essentially the temperature. [Specifically, if $l_\varphi(T) \sim T^{-1/p}$, then $x \sim T^{4/3p}$.] The dashed line in the figure denotes $l_\varphi(T) = \xi_{perc}$, while the solid line denotes $l_\varphi(T) = \xi(H)$. In the lightly shaded regime labeled ‘‘classical percolation,’’ where $l_\varphi > \xi_{perc}$, the system can be viewed as a collection of effectively macroscopic droplets of the two phases, and hence some sort of quasiclassical puddle theory applies. In the darker shaded regime, between the dashed line and the solid line, the system is in the quantum critical regime, where $\xi(H) \sim |H - H_c|^{-\nu}$. In ordering these two crossovers, we have assumed that $\nu > \nu_{perc} = 4/3$, which is certainly satisfied if, as discussed, $\nu \approx 7/3$. For very large l_φ , as T approaches

zero, we allow for classical percolation to cease being relevant; hence the darker shaded area covers all the region between the solid lines near H_c . Below the solid line the properties are dominated by dilute thermal excitations above the appropriate ground state phase, a regime that in other contexts^{3,4} is called “renormalized classical.”

Note that Fig. 4 describes a situation in which there is only one critical field for both classical and quantum percolation. For self-dual systems in two dimensions the percolation threshold has to be 50% for either case. However, for general systems the two critical fields will be different. Indeed, where the puddles of the two competing phases are not macroscopic (i.e., when the correlation length for the disorder potential is finite) the notion of classical percolation is not completely well defined, as there is an intrinsic “quantum blurriness” to the edges of the puddles, so there is no precise point at which two puddles can be said to touch.

However, *if* for some reason, perhaps due to strong enough coupling to a heat bath, l_φ does not diverge⁵⁷ at low temperatures, then near enough to the percolation threshold a sort of “metallic” phase, in which the dissipation occurs²³ predominantly at weak links between nearly percolating puddles of the minority phase, will be valid to arbitrarily low temperatures. Moreover, in this case, a true phase transition separates the metallic state in the neighborhood of $H=H_c$ and the extremal quantum states at large values of $|H-H_c|$. At present, we know of no theoretically well understood prototypical systems in which this sort of behavior occurs. Some very promising starts along this line, especially related to the behavior of superconducting grains in a metallic host, are enumerated in the final section. Nonetheless, it is important to realize that analogous behavior is established in certain well understood zero dimensional systems.¹¹ Here, as a function of the strength of coupling to a heat bath, various quantum systems with more than one classical ground state are observed to make a transition from the expected quantum behavior, in which tunneling renders the ground state unique, to zero temperature classical behavior, in which the system is trapped in a single (arbitrary) one of the classical ground states.

C. “Superuniversality”

The Chern-Simons mapping suggests^{16,77} that there should be a correspondence not only between the various stable electronic phases in two dimensions, but also between the quantum phase transitions between them. This idea is referred to as “superuniversality.”⁸⁹ Superuniversality as an approximate statement¹⁶ follows from treating the fluctuations of the statistical gauge field at linear response (one-loop) level. Nonperturbative proofs of superuniversality have been constructed for certain simplified models, without disorder.⁹⁰ Whether it is truly an exact relation asymptotically deep in the critical regime, or whether the effects of higher order fluctuations of the statistical gauge field ultimately destroy superuniversality, is still being debated.⁹¹ If we assume that the correspondences implied by superuniversality hold to a sufficient level of approximation, we can derive a large number of additional results.

The critical exponents ν and z should be the same at all quantum Hall transitions, including the QHIT and the plateau transitions, irrespective of whether the transitions involve integer or fractional quantum Hall states. Assuming that the critical conductivity tensor is indeed universal, one can compute the critical conductance at any given transition from its value at the $s_{xy}=1$ to insulator transition, as discussed in Ref. 16.

The same correspondence implies that the magnetic field driven SIT should have the same critical exponents and a related value of the critical conductance. In particular, if we accept that at the $s_{xy}=1$ to insulator transition $\sigma_{xy}^c = \sigma_{xx}^c = e^2/2h$, then at the SIT we would expect $\sigma_{xy}^c = 0$ and $\sigma_{xx}^c = 2e^2/h$.

V. DISCUSSION

While the classical theory of phase transitions has been extraordinarily successful, there are several reasons to exercise caution when applying this approach to zero temperature, quantum phase transitions: (1) Experiments are always carried out at finite temperature, so that the proper identification of the relevant phases requires an extrapolation to zero temperature. In all the cases cited above, the question has arisen whether a transition between two distinct zero temperature phases has actually been observed, or whether a finite temperature crossover behavior is being misinterpreted as a quantum phase transition. (2) In most experimentally interesting cases, quenched disorder is known to play a central role in the critical physics, and there are increasingly compelling theoretical reasons⁹² to believe that Griffiths singularities, which are typically only of academic interest for classical critical phenomena, can fundamentally complicate the scaling analysis at quantum critical points. In particular they can, under some circumstances, lead to divergent susceptibilities and relaxation times over a finite range of parameters about the quantum critical point, and apparent correlation length exponents that depend on the averaging procedure. (3) Effects of dissipation, that is to say, the coupling of the critical modes to a continuum of other “heat-bath” degrees of freedom, can fundamentally alter the character of the phases and of the transitions between them.^{6,7}

The SIT. On the basis of the dirty boson model, the zero temperature phase diagram of superconducting films is expected to be of the form shown in Fig. 1(a); all along the phase boundary, the SIT is in a single universality class, except at the zero H end point.

How well are these expectations met experimentally? Certainly, at not too low temperatures, a field driven SIT has been apparently observed in a number of superconducting thin-film systems. Moreover, the resistivity (more precisely the I - V curve) as a function of H , T , and I can be successfully collapsed onto a scaling curve, suggestive of quantum critical scaling. However, while the value of $z \approx 1$ extracted from the data is consistent with theoretical expectation, it is always found that $\nu \approx 4/3$, suggestive more of classical than of quantum percolation. Moreover, the critical resistivity is not found to be universal, and is often as much as a factor of 10 smaller than the value $\sigma_{xx}^c = 2e^2/h$ predicted on the basis

of superuniversality. Finally, and most dramatically, it has been emphasized in the recent work of Mason and Kapitulnik (although it is a highly neglected fact that is lurking in many previous data, as well) that at low temperatures the resistivity saturates to a strongly field dependent but temperature independent value. *Thus, in fact no SIT actually takes place in this field range.* Manifestly, there is physics here that is beyond the scope of the dirty boson model.

This observation does not contradict the theoretically solid expectation that superconductivity will occur for a non-zero range of H . Indeed, Mason and Kapitulnik³⁸ have recently found evidence of a low temperature metal-to-superconductor transition in disordered amorphous films in which a field-tuned superconductor-insulator transition is disrupted.¹² This transition is characterized by hysteretic magnetoresistance and discontinuities in the I - V curves. The metallic phase just above the transition is different from the Fermi metal before superconductivity sets in, as is evident from the temperature dependence of the resistance and the I - V characteristics obtained for this phase. Of course, experiments at much higher H to look for the metal to insulator transition predicted by Fig. 1(b) are the next step.

The QHIT and the QHPT. The law of corresponding states applied to the simple phase diagram in Fig. 1(a) leads to the complex phase diagram in Fig. 2(a), and superuniversality would imply that all the phase transitions are in the same universality class.

How well are these expectations met experimentally? The answer is mixed. There is a subset of experiments that are in striking agreement with these expectations. Early experiments on the plateau transitions between various integer quantum Hall states and from the $s_{xy}=2/5$ to $s_{xy}=1/3$ quantum Hall state exhibited good scaling properties characteristic of a quantum phase transition, and apparently universal values of the critical exponent $\nu_z \approx 7/3$. Measurements at finite frequency showed remarkable $\hbar\omega/k_B T$ scaling, again strongly indicative of quantum critical behavior. No sign of saturation of the critical behavior was detected to the lowest temperatures. However, no clear results concerning the critical conductance were obtained in these experiments.

Somewhat later, experiments were carried out^{49,50} on the QHIT, of which some of the clearest made use of purposely low mobility heterojunctions. Transitions from $s_{xy}=2$ to insulator (the factor of 2 is due to the fact that the spin splitting is not resolved in these samples) and $s_{xy}=1/3$ to insulator were studied. Quantum critical scaling is observed on some samples,⁹³ with $\nu_z \approx 7/3$. Moreover, the theoretically predicted universal values of the critical resistance are found to

good approximation. (Ambiguities due to the exact geometry of the current paths make a precise test of this prediction difficult.) Some features of the shape of the global phase diagram in Fig. 2(a) were also confirmed in these experiments, including the existence of a reentrant insulator to quantum Hall liquid to insulator transition as a function of increasing H (also known as the phenomenon of “floating” of the delocalized states^{47,48}). However, other experiments point to the phase diagram of Fig. 2(b) by showing deviations from scaling on both sides of the transition.⁹⁴ It is important to note that deviations from scaling can occur even without noticeable resistance saturation, and it is therefore difficult to determine the conductance states in experiments where scaling is not measured.

A large number of more recent experiments on the QHIT, carried out on high mobility heterojunctions, exhibit behavior that is more like that observed in superconducting films: An apparent transition is observed at high temperatures, which appears to satisfy scaling. However, as pointed out by Mason and Kapitulnik, the apparent value of $\nu_z \approx 4/3$, rather than $7/3$. Moreover, at low temperatures, the resistivity apparently saturates to a finite (metallic) value, rather than diverging on the putative insulating side of the transition or vanishing on the putative quantum Hall side. The one striking difference between these results and the results for superconducting films is that the apparent critical resistance at which the QHIT occurs appears to be, to good approximation, universal (sample independent) and in agreement with the predictions of theory.

Global phase diagram. Mason and Kapitulnik proposed Fig. 1(b), where α is a measure of dissipation. While there are many speculative ideas concerning how this sort of phase diagram could arise, at this stage this proposal must be viewed as phenomenological. In particular, our fundamental postulate that large α stabilizes a metallic phase is supported by recent theoretical work, especially that in Ref. 24, but has not been clearly established. However, once this postulate is accepted, it rationalizes in a simple way the observations on superconducting films. Moreover, on the basis of this idea, as outlined in the introduction, a number of interesting further predictions can be made.

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