Average conductance coefficients in multiterminal chaotic cavities

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We calculate exactly the average conductance coefficients of a ballistic chaotic cavity coupled via tunnel barriers to an arbitrary number of reservoirs. Explicit formulas are derived for several regimes of interest: ideal contacts, strongly overlapping resonances, locally weakly absorbing limit, equivalent channels, and equivalent terminals.

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I. INTRODUCTION

Coherent transport of waves in complex systems has been a subject of much interest both theoretically and experimentally. Application include light propagation in opaque media such as white paint,[1] transmission of radiation through interstellar clouds,[2] microwave scattering by chaotic cavities,[3] and electron transport through disordered and ballistic mesoscopic devices.[4] An important feature of these systems is the irregular behavior of the transport characteristics as a function of certain parameters, such as the energy of the incoming wave or the shape of the scattering potential. In experiments and numerical simulations, it is always possible to smooth out the irregularity by performing averages over a sufficiently large interval of the relevant parameter associated with the fluctuations. Such a procedure usually leads to very robust statistical laws and reveals a great degree of universality (independence of microscopic details). It has been a challenge to the theorists in the field to explain and derive these universal statistical laws from microscopic approaches.

An interesting explanation for this universality can be produced by using arguments put forward in Ref. 5. In the universal continuum limit, the statistical models of the above-mentioned systems flow to the same mathematical structure: Efetov's supersymmetric nonlinear σ model.[6,7] Using scaling arguments, the authors of Ref. 5 have shown that the observables obtained from the one-dimensional version of this model are given by universal scaling functions of dimensionless combinations of the unrenormalized coefficients in the action (a phenomenon known as ''no-scale-factor universality.'')[8,9] The universal scaling functions for the first and second moment of the two-terminal conductance of a thick wire can be found in Ref. 10. More recently, using a stable numerical algorithm for large products of random matrices and finite size scaling, Plerou and Wang[11] obtained the universal scaling functions of several moments of the conductance.

In Ref. 12, we have studied in detail the universal scaling functions of arbitrary linear statistics of a disordered quasi-one-dimensional conductor with broken time-reversal symmetry and coupled ideally to two electron reservoirs via perfectly conducting leads (labeled 1 and 2) with arbitrary numbers of open channels: N_1 and N_2. In particular, we have demonstrated, by explicit calculations, that the first three moments of the two-terminal conductance have the following ''no-scale-factor'' universal form:

$$\langle (G/G_0)^m \rangle = f_m(N_1, N_2, 2L/\xi), \tag{1}$$

where L is the length of the wire, ξ is the localization length and $G_0 \equiv 2e^2/h$ is the conductance quantum. The results obtained in Ref. 10 are recovered by taking the limits $N_1, N_2 \to \infty$. Interestingly, in the limit $L/\xi \to 0$ the system behaves as an open ballistic chaotic cavity (i.e., the classical dynamics of a confined particle that scatters only at its boundary is chaotic) and Eq. (1) can be interpreted as describing the universal quantum dot to disordered wire crossover. Important extensions of this result would involve the inclusion of more terminals (typical experiments use four probes: two for voltage and two for current measurements), the presence of tunnel barriers at the lead-wire contacts and dephasing effects.

In this paper, we shall consider some of these extensions in the somewhat simpler but important case of the limit $L/\xi \to 0$. Specifically, we calculate exactly the average conductance coefficients of a ballistic chaotic cavity coupled via tunnel barriers to an arbitrary number of electron reservoirs.

An experimental realization of this system is a high-mobility semiconductor quantum dot with many voltage/current terminals at very low temperature.[13–22] In such devices, conductance coefficients are usually measured by applying a very low frequency ac current through the system and observing the induced voltage differences between the various probes. The experimental and theoretical importance of these coefficients was established by Büttiker in Refs. 23 and 24, where they were used to explain some observed reciprocity symmetries.[25] A notable feature of the conductance coefficients is the fact that, at zero temperature, they are completely determined by states at the Fermi surface. This property makes it possible to describe low-temperature transport in these systems as a scattering problem. This point of view has been pioneered by Landauer[26,27] and was later generalized by Büttiker.[23,24]

The most important simplifying aspect of the transport regimes amenable to a scattering description is the existence of widely separated time scales. There are two well-developed approaches that take benefit of this fact. The first one (called ''the Heidelberg approach'') has been presented in detail in Refs. 28 and 29, and builds on earlier work on scattering theory of resonant nuclear reactions.[30] It basically assumes that the coupling to the reservoirs leads to the appearance of two well-separated time scales, one associated with direct processes (described by the smooth energy averaged scattering matrix) and the other with long-living resonances (described by the strongly fluctuating part of the S matrix). The S matrix is expressed in terms of the Hamiltonian H_c of the cavity and a fenomenological matrix W describing the coupling of the cavity to a continuum of scattering states. The Hamiltonian H_c is substituted by a member from the Gaussian random-matrix ensemble and energy averages are replaced by ensemble averages, which are calculated using the supersymmetry theory.[6,7] This approach has had remarkable success leading to both exact results[31–34] and useful approximations.[35,36] The second scattering formulation has been put forward in Refs. 37 and 38 (for recent reviews, see Refs. 39 and 40) and also uses the assumption of two widely separated time scales for direct and resonant processes, but unlike the Heidelberg approach, the statistical properties of the S matrix are derived using information theoretical ideas, without ever refering to the underlying Hamiltonian. The equivalence of these two methods was established in Ref. 29 (see also Ref. 41).

An independent and also very successful approach to transport in quantum dots has been developed by Efetov.[7] It builds on the zero-dimensional limit of the supersymmetric non-linear σ model and its connection with the problem of an electron in the presence of a random-impurity potential. The equivalence with the Heidelberg approach follows from Efetov's proof[6] of the microscopic validity of a random-matrix theory.

In this paper we adopt the Heidelberg approach, mainly because of its appealing physical transparency, which greatly facilitates the interpretation of the final results in terms of concepts from the theory of resonant quantum chaotic scattering. In Sec. II we describe the physical system and give a detailed introduction to the multiterminal version of the Heidelberg approach. An exact formula for the average conductance coefficients is presented in Sec. III. In Sec. IV we derive some useful limits and give a physical interpretation of our central result. A summary and conclusions are presented in Sec. V.

II. THE PHYSICAL SYSTEM AND THE HEIDELBERG APPROACH

In this section we present the Heidelberg approach for the description of a ballistic chaotic cavity of arbitrary geometry coupled to M electron reservoirs via perfectly conducting leads with an arbitrary number of propagating channels. Previous works have been concerned mostly with the particular cases: $M=1$ and $M=2$. As we shall demonstrate, even for these particular situations, the study of the general multiterminal system offers additional insights, which can help improve the understanding of the physical aspects of the problem.

A. Scattering matrix

A central result in the formalism is the scattering matrix formula below, which describes the coupling of the metastable states in the cavity to the asymptotic free propagating states in the leads. A proof of this formula using projection operators can be found in Ref. 30. Simpler derivations have been presented in Refs. 42,43, and 33. The S matrix is given by

$$S=1-2\pi i W^{\dagger}G^{r}W, \tag{2}$$

where G^r is the retarded Green's function defined by

$$G^{r}=(E-H_c-\Sigma^{r})^{-1}, \tag{3}$$

in which H_c is the $N_c\times N_c$ Hamiltonian matrix describing the dynamics of the closed cavity, and

$$\Sigma^{r}=-i\pi WW^{\dagger} \tag{4}$$

is the self-energy function associated with the processes of particles entering into the cavity from the leads and particle emission from the cavity into the leads. The nonideal coupling between the cavity and the leads is described by a nonrandom $N_c\times N$ matrix with the following structure:

$$W_{\mu n}=\begin{cases}(W_1)_{\mu,n} & ; \quad n=1,\ldots,N_1\\ (W_2)_{\mu,n-N_1} & ; \quad n=N_1+1,\ldots,N_1+N_2\\ \vdots & ; \quad \vdots\\ (W_M)_{\mu,n-N+N_M} & ; \quad n=N-N_M+1,\ldots,N.\end{cases}$$

The elements of the matrices W_p not related to W are zero. This implies that the self-energy can be decomposed as

$$\Sigma^{r}=\sum_{p=1}^{M}\Sigma_p^{r},$$

where $\Sigma_p^r=-i\pi W_pW_p^{\dagger}$. It is standard in the Heidelberg approach to assume the absence of direct reactions by imposing the following orthogonality conditions:[28]

$$W_p^{\dagger}W_q=\frac{N_c\Delta}{\pi^2}w_p\delta_{p,q}, \tag{5}$$

where Δ is the mean level spacing and w_p is a diagonal matrix given by

$$w_p=\mathrm{diag}(w_{p,1},\ldots,w_{p,N_p}). \tag{6}$$

B. Landauer-Büttiker Formula

Using a counting argument due to Landauer[26,27] and Büttiker[23,24] one can show that the current in terminal p can be written in terms of the voltage differences in the reservoirs $\Delta V_{pq}\equiv V_p-V_q$ as

$$I_p=\sum_{q=1}^{M}G_{pq}\Delta V_{pq},$$

where

$$G_{pq}=G_0\sum_{n=1}^{N_p}\sum_{m=1}^{N_q}|S_{nm}^{pq}|^2; \quad p\neq q, \tag{7}$$

are the conductance coefficients of the open cavity. Derivations of this multiterminal formula using a standard linear response theory can be found in Refs. 44,45, and 46. More recently,[47,48] the Keldysh technique has been used to establish its limits of applicability.

From Eq. (2) we obtain

$$S^{pq}_{nm}=\delta_{pq}\delta_{nm}-2\pi i\sum_{\mu\nu}(W_p^{\dagger})_{n\mu}(G^r)_{\mu\nu}(W_q)_{\nu m}. \qquad (8)$$

Defining $G^a\equiv(G^r)^{\dagger}$ and $\Gamma_p\equiv 2\pi W_pW_p^{\dagger}$ we find, after inserting Eq. (8) into Eq. (7) that

$$G_{pq}=G_0C_{pq}, \quad p\neq q,$$

where

$$C_{pq}=\mathrm{Tr}(\Gamma_pG^r\Gamma_qG^a). \qquad (9)$$

This last formula also applies for $p=q$.

There is an interesting sum rule (Ward identity) satisfied by C_{pq} that can be obtained as follows.[49] Define the spectral function $A\equiv i(G^r-G^a)$, and observe that it can be written as $A=G^r\Gamma G^a=G^a\Gamma G^r$, where $\Gamma=\Sigma_{p=1}^M\Gamma_p$. Summing Eq. (9) over q and using these results we get

$$\sum_{q=1}^{M}C_{pq}=\sum_{q=1}^{M}C_{qp}=\mathrm{Tr}(\Gamma_pA). \qquad (10)$$

This relation is obviously a direct consequence of the unitarity of the S matrix.

C. Random-matrix ensemble

Following the Heidelberg approach, we assume that the Hamiltonian matrix H_c describing the dynamics of a noninteracting particle in the chaotic cavity, can be replaced in the universal regime, by a member from the Gaussian unitary ensemble (GUE), which describes systems with broken time-reversal symmetry. We are thus assuming the presence of a weak magnetic field in the cavity region. The probability distribution is given by

$$P(H_c)=\mathcal{N}\exp\left(-\frac{1}{2\mathcal{C}}\mathrm{Tr}(H_c^2)\right), \qquad (11)$$

where $\mathcal{N}$ is a normalization constant, $\mathcal{C}=\lambda^2/N_c$ and consequently

$$\langle(H_c)_{\mu\nu}\rangle=0 \qquad (12)$$

and

$$\langle(H_c)_{\mu\nu}(H_c)_{\mu'\nu'}\rangle=\mathcal{C}\delta_{\mu\nu'}\delta_{\nu\mu'}. \qquad (13)$$

The random-matrix assumption is in fact a long-standing problem in the field of quantum chaos, that goes under the name of ‘‘the Bohigas conjecture.’’ There is numerical evidence[50] for its validity and microscopic justifications have been put forward in Refs. 51,52, and 53, but a rigorous proof is still missing.

D. Coupling parameters

The large separation of time scales for direct and resonant processes in the universal chaotic regime implies that distribution functions describing the stochastic nature of the resonant response depend parametrically on a certain number of coupling coefficients associated with direct processes. This fact has been formally justified by the information-theoretical model[39] in terms of the analyticity-ergodicity constraint that leads to the Poisson kernel distribution of the S matrix. In the Heidelberg approach, these coupling parameters have been called ‘‘sticking probabilities’’ and are defined as

$$T_{pn}=1-|\langle S^{pp}_{nn}\rangle|^2. \qquad (14)$$

The average S matrix has been calculated in Ref. 28 and is given by

$$\langle S^{pq}_{nm}\rangle=\delta_{pq}\delta_{nm}\tanh(\alpha_{pn}/2),$$

where $\alpha_{pn}\equiv-\ln w_{pn}$, with w_{pn} defined in Eq. (6). For the tunnel probabilities T_{pn}, we find

$$T_{pn}=\mathrm{sech}^2(\alpha_{pn}/2). \qquad (15)$$

These coefficients measure the part of the incomming flux that penetrates into the cavity and participates in the formation of long-living resonant states. Note that when $T_{pn}=1$, the coupling to the leads is ideal and direct processes, such as prompt reflection, is absent. On the other hand, when $T_{pn}=0$ there is no coupling between the scattering channels and the resonances, and thus, the incomming flux is completely backscattered.

We finish this section by remarking that since

$$\langle\mathrm{Tr}(\Gamma_pA)\rangle=-2\,\mathrm{Im}\,\mathrm{Tr}(\Gamma_p\langle G^r\rangle),$$

one can use Eq. (10) to derive the following useful sum rule for the average coefficients $\langle C_{pq}\rangle$

$$\sum_{q=1}^{M}\langle C_{qp}\rangle=\sum_{q=1}^{M}\langle C_{pq}\rangle=\sum_{n=1}^{N_p}\frac{4}{e^{\alpha_{pn}}+1}. \qquad (16)$$

This identity will be used later to simplify some integrals obtained with the supersymmetry technique.

III. GENERAL FORMULA FOR THE AVERAGE CONDUCTANCE COEFFICIENTS

In this section we use the supersymmetry method to calculate the average conductance coefficients, $\langle G_{pq}\rangle=G_0\langle C_{pq}\rangle$, for $p\neq q$. For a pedagogical introduction to this method and to the basic definitions of superalgebra, such as the superdeterminant (Sdet) and the supertrace (Str), we refer to the recent review by Zuk.[54]

From Eq. (9) one can see that

$$\langle C_{pq}\rangle=\frac{\partial^2}{\partial h_{p1B}\partial h_{q2B}}\langle\mathcal{Z}(h)\rangle\bigg|_{h=0}, \qquad (17)$$

where

$$\mathcal{Z}(h) = \mathrm{Sdet}^{-1}[1 + J(h)G].$$

We have introduced the following block diagonal matrix of retarded and advanced Green's functions:

$$G = \mathrm{diag}(G^r, G^r, G^a, G^a).$$

The source field is represented by the matrix

$$J(h) = \sum_{q=1}^{M} \sum_{a=1}^{2} \sum_{\sigma=B,F} h_{qa\sigma} \Gamma_q \otimes F_{a\sigma},$$

where

$$F_{1\sigma} = \begin{pmatrix} 0 & 0 \\ k_\sigma & 0 \end{pmatrix},$$

and

$$F_{2\sigma} = \begin{pmatrix} 0 & k_\sigma \\ 0 & 0 \end{pmatrix}.$$

The submatrices k_σ are defined as

$$k_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$k_F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Following the standard procedure we represent $\mathcal{Z}(h)$ as a Gaussian superintegral

$$\mathcal{Z}(h) = \int D\varphi D\overline{\varphi} \exp[i\overline{\varphi}(\mathcal{A} - \mathcal{B})\varphi],$$

where φ is a supervector with components $\varphi_{a\mu}^{\alpha}$ ($\alpha = 0,1$, $a = 1,2$, and $\mu = 1 \ldots N_c$) and $(\overline{\varphi})_{a\mu}^{\alpha} = (-1)^{(1-\alpha)(1+a)}(\varphi_{a\mu}^{\alpha})^*$. The matrices $\mathcal{A}$ and $\mathcal{B}$ are defined as

$$\mathcal{A} = \frac{i}{2}\Gamma \otimes \Lambda + J(h)$$

and

$$\mathcal{B} = H_c \otimes 1_4.$$

We have also defined the following auxiliary matrix $\Lambda = \mathrm{diag}(1,1,-1,-1)$. The ensemble average is performed with the help of the identity

$$\langle \exp(-i\overline{\varphi}\mathcal{B}\varphi)\rangle = \exp\left[-\frac{\mathcal{C}}{2}\mathrm{Str}(\mathcal{S}^2)\right], \tag{18}$$

where

$$\mathcal{S}_{ab}^{\alpha\beta} = \sum_{\mu=1}^{N_c} \varphi_{a\mu}^{\alpha}(\overline{\varphi})_{b\mu}^{\beta}.$$

Inserting Eq. (18) into $\langle \mathcal{Z}(h)\rangle$, linearizing the quadratic depence on $\mathcal{S}$ via a Hubbard-Stratonovitch transformation and performing the integral over the vector variables φ and $\overline{\varphi}$, we get the following exact representation

$$\langle \mathcal{Z}(h)\rangle = \int d\sigma e^{i\mathcal{L}(\sigma,h)}$$

where

$$i\mathcal{L}(\sigma,h) = -\frac{N_c}{2\lambda^2}\mathrm{Str}(\sigma^2) - \mathrm{Str}\ln(\mathcal{A} - \sigma \otimes 1_{N_c}).$$

In the regime $N_c \gg M$, this integral can be further simplified by using the saddle-point method. We end up with an integral over Efetov's coset space $U(1,1|2)/[U(1|1) \otimes U(1|1)]$, whose points are parametrized by supermatrices Q satisfying the constraint $Q^2 = 1$,

$$\langle \mathcal{Z}(h)\rangle = \int dQ \prod_{p=1}^{M} \prod_{n=1}^{N_p} \mathrm{Sdet}^{-1}(1 + w_{pn}QK_p), \tag{19}$$

in which we have introduced the following supermatrix:

$$K_p = \Lambda - 2i\sum_{a\sigma} h_{pa\sigma}F_{a\sigma}.$$

We can calculate $\langle C_{pq}\rangle$ by inserting Eq. (19) into Eq. (17). We find

$$\langle C_{pq}\rangle = -4\sum_{n=1}^{N_p}\sum_{m=1}^{N_q} \langle \mathrm{Str}(\Theta_{pn}^{1B})\mathrm{Str}(\Theta_{qm}^{2B})\rangle_Q - 4\delta_{pq}\sum_{n=1}^{N_p} \langle \mathrm{Str}(\Theta_{pn}^{1B}\Theta_{pn}^{2B})\rangle_Q, \tag{20}$$

where

$$\Theta_{pn}^{a\sigma} = w_{pn}(w_{pn} + \Lambda Q)^{-1}\Lambda F_{a\sigma}$$

and the average $\langle \ldots \rangle_Q$ is defined as

$$\langle f(Q)\rangle_Q = \int dQ f(Q) \prod_{p=1}^{M}\prod_{n=1}^{N_p} \mathrm{Sdet}^{-1}(1 + w_{pn}Q\Lambda).$$

Equation (20) can be made explicit by means of Efetov's coordinates, defined by the parametrization

$$Q = U^{-1}\begin{pmatrix} \cos\hat{\theta} & i\sin\hat{\theta} \\ -i\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix} U; \quad U = \begin{pmatrix} e^{i\hat{\varphi}}v_1 & 0 \\ 0 & v_2 \end{pmatrix},$$

where $\hat{\theta} \equiv \mathrm{diag}(i\theta_1, \theta_0)$; $\theta_1 > 0$, $0 < \theta_0 < \pi$, $\hat{\varphi} \equiv \mathrm{diag}(\varphi_1, \varphi_0)$; $0 < \varphi_1, \varphi_0 < 2\pi$ and v_1, v_2 are given respectively by

$$v_1 = \exp\begin{pmatrix} 0 & -\eta_1^* \\ \eta_1 & 0 \end{pmatrix}; \quad v_2 = \exp\begin{pmatrix} 0 & -i\eta_2^* \\ i\eta_2 & 0 \end{pmatrix},$$

where η_1, η_2, η_1^*, and η_2^* are complex anticommuting variables. In these coordinates the invariant integration measure reads

$$dQ=\frac{d\lambda_1 d\lambda_0}{(\lambda_1-\lambda_0)^2}d\varphi_1 d\varphi_0 d\eta_1^* d\eta_2^* d\eta_1 d\eta_2,$$

where $\lambda_1=\cosh\theta_1$ and $\lambda_0=\cos\theta_0$.

To simplify the presentation of the final expressions, let us define the following functions:

$$\mathcal{P}(\{\gamma\};\{\lambda\})=\prod_{p=1}^{M}\prod_{n=1}^{N_p}\frac{\lambda_0+\gamma_{pn}}{\lambda_1+\gamma_{pn}},$$

where $\gamma_{pn}=\cosh\alpha_{pn}$,

$$\mathcal{Q}_{nm}^{pq}(\{\lambda\})=(\lambda_0+\lambda_1)(1+\gamma_{pn}\gamma_{qm}) + (1+\lambda_0\lambda_1)(\gamma_{pn}+\gamma_{qm}),$$

and

$$\mathcal{R}_{nm}^{pq}(\{\lambda\})=\prod_{r=0}^{1}(\lambda_r+\gamma_{pn})(\lambda_r+\gamma_{qm}).$$

Using Efetov's coordinates, together with Zirnbauer's integral theorem[55] to account for the contribution of boundary terms, we obtain from Eq. (20) the following result:

$$\langle C_{pq}\rangle=\sum_{n=1}^{N_p}\sum_{m=1}^{N_q}\int_{\{\lambda\}}\frac{\mathcal{P}(\{\gamma\};\{\lambda\})}{(\lambda_1-\lambda_0)}\frac{\mathcal{Q}_{nm}^{pq}(\{\lambda\})}{\mathcal{R}_{nm}^{pq}(\{\lambda\})} + \delta_{pq}\sum_{n=1}^{N_p}\int_{\{\lambda\}}\frac{(\gamma_{pn}^2-1)\mathcal{P}(\{\gamma\};\{\lambda\})}{\mathcal{R}_{nn}^{pp}(\{\lambda\})} + \delta_{pq}\sum_{n=1}^{N_p}\frac{4}{(e^{\alpha_{pn}}+1)^2}, \tag{21}$$

where

$$\int_{\{\lambda\}}\equiv\int_1^\infty d\lambda_1\int_{-1}^{1}d\lambda_0.$$

The average conductance coefficients can be obtained from the above expression by using the condition $p\neq q$. The final formula is the central result of this paper

$$\langle G_{pq}/G_0\rangle=\sum_{n=1}^{N_p}\sum_{m=1}^{N_q}\int_{\{\lambda\}}\frac{\mathcal{P}(\{\gamma\};\{\lambda\})}{(\lambda_1-\lambda_0)}\frac{\mathcal{Q}_{nm}^{pq}(\{\lambda\})}{\mathcal{R}_{nm}^{pq}(\{\lambda\})}. \tag{22}$$

IV. PHYSICAL ANALYSIS AND USEFUL RESULTS

We can obtain considerable understanding of the physical meaning of Eq. (22) by studying some important particular regimes and situations.

(a) Ideal contacts.

When the contacts between the cavity and the leads are ideal, the tunnel probabilities have maximum value, i.e., $T_{pn}=1$. Substituting this condition into Eq. (22) we get

$$\langle G_{pq}/G_0\rangle=\int_{\{\lambda\}}\frac{2N_pN_q(1+\lambda_0)^{N-1}}{(\lambda_1-\lambda_0)(1+\lambda_1)^{N+1}}=\frac{N_pN_q}{N}, \tag{23}$$

in agreement with the maximum-entropy formulation of Ref. 39. Physically, it means that as we vary the incident energy, the S matrix explores with equal probability all parts of its available manifold.

For the important particular case of two terminals, i.e., $M=2$ and $N=N_1+N_2$, there are a number of known results in the literature,[7,40] both for the average conductance (and its variance) and for the average density of transmission eigenvalues defined as

$$\rho(\tau)=\langle \mathrm{Tr}\,\delta(\tau-tt^\dagger)\rangle,$$

where t is the transmission matrix. From the Landauer formula, $G=G_0\mathrm{Tr}(tt^\dagger)$, we can see that

$$\langle G/G_0\rangle=\int_0^1 d\tau\rho(\tau)\tau. \tag{24}$$

For ideal coupling, Ref. 56 obtained

$$\rho(\tau)=\tau^r\sum_{n=0}^{s-1}(2n+r+1)\{P_n^{(r,0)}(1-2\tau)\}^2, \tag{25}$$

where $r=|N_1-N_2|$, $s=\min\{N_1,N_2\}$, and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial. Inserting Eq. (25) into Eq. (24) we get $\langle G/G_0\rangle=N_1N_2/N$ in agreement with Eq. (23).

(b) Strongly overlapping resonances.

As analyzed in Ref. 29, this is the regime where semiclassical methods work very well. It is defined by the condition $T_{tot}\gg 1$, where $T_{tot}\equiv\Sigma_{pn}T_{pn}$ is the total tunnel probability for the electron to get from the leads into the cavity. We can obtain the average conductance coefficients in this regime by expanding Eq. (22) in inverse powers of T_{tot}. Using the expansion scheme described in Ref. 57 we find

$$\langle G_{pq}/G_0\rangle\simeq\frac{T_pT_q}{T_{tot}}+T_{tot}^{-3}(T_p^{(3)}T_q+T_pT_q^{(3)}+T_p^{(2)}T_q^{(2)}) - T_{tot}^{-3}(T_p^{(2)}T_q+T_pT_q^{(2)})-T_{tot}^{-3}\mu_3T_pT_q + T_{tot}^{-3}\mu_2(T_pT_q-T_p^{(2)}T_q-T_pT_q^{(2)}), \tag{26}$$

where $T_p\equiv\Sigma_n(T_{pn})$, $T_p^{(k)}\equiv\Sigma_n(T_{pn})^k$ and $\mu_k\equiv T_{tot}^{-1}\Sigma_pT_p^{(k)}$ are parameters of order unity. This expansion is correct up to the order T_{tot}^{-4} and we have checked that $\langle C_{pq}\rangle$ calculated within the same scheme and to the same order satisfies the sum rule of Eq. (16).

There are two interesting features in Eq. (26). The first concerns the physical interpretation of the leading term. Since T_q/T_{tot} is the probability of emission from the cavity into lead q and T_p is the probability of entry into the cavity from lead p, the factorized expression $T_p(T_q/T_{tot})$ (known in nuclear physics as the Hauser-Feshbach formula[58]) indicates the statistical independence of these processes. The second feature is the absence of the term of order T_{tot}^{-2}. This can be understood by considering the single channel limit. After appropriate reinterpretation of channel indices as terminal indices, Refs. 57 and 59 obtained the following result

$$\langle |S^{pq}|^2 \rangle_{GOE} = \frac{T_p T_q}{T_{tot}} - \frac{T_p T_q (T_p + T_q - \mu_2)}{T_{tot}^2} + \mathcal{O}(T_{tot}^{-3}),$$

for a system with time-reversal symmetry. On the other hand, Eq. (26) gives

$$\langle |S^{pq}|^2 \rangle_{GUE} = \frac{T_p T_q}{T_{tot}} + \mathcal{O}(T_{tot}^{-3}). \tag{27}$$

Both expressions are valid only for $p \neq q$ and GOE stands for Gaussian orthogonal ensemble. The term of order T_{tot}^{-2} in $\langle |S^{pq}|^2 \rangle_{GOE}$ represents the weak-localization contribution, which is a decrease in magnitude caused by the coherent backscattering of waves in the cavity. The absence of this contribution in $\langle |S^{pq}|^2 \rangle_{GUE}$ and in Eq. (26) indicates the suppression of this effect by the breaking of time-reversal symmetry. Weak-localization effects have been observed in many experiments in ballistic chaotic cavities[60] and has recently been the subject of detailed theoretical analysis.[34]

We conclude by remarking that for the particular case of two terminals, the leading term of Eq. (26) agrees with the perturbative calculation of Ref. 61.

(c) The locally weak absorption limit.

This regime has been studied in mesoscopic physics in Ref. 62. It corresponds to the case of small transmission coefficients $T_{pn} \ll 1$ and large number of channels, $N \gg 1$. In Eq. (22), it implies the substitutions

$$\mathcal{P}(\{\gamma\};\{\lambda\}) \rightarrow \exp\left[-\frac{1}{2}(\lambda_1 - \lambda_0) T_{tot}\right],$$

and

$$\frac{\mathcal{Q}_{nm}^{pq}(\{\lambda\})}{\mathcal{R}_{nm}^{pq}(\{\lambda\})} \rightarrow \frac{1}{4} T_{pn} T_{qm} (\lambda_1 + \lambda_0).$$

The remaining double integral is straightforward and yields

$$\langle G_{pq}/G_0 \rangle \simeq \frac{T_p T_q}{T_{tot}}, \tag{28}$$

which is again the Hauser-Feshbach fromula and thus the physical interpretation of the statistical independence of the processes of entry into the cavity and emission from it also applies here. This result is in agreement with Eq. (20) of Ref. 62.

(d) Equivalent channels.

In this case, we have $\alpha_{pn} = \alpha$ for all modes in all terminals. From Eq. (22) one gets

$$\langle G_{pq}/G_0 \rangle = \int_{\{\lambda\}} \frac{N_p N_q (\gamma + \lambda_0)^{N-2} \mathcal{Q}(\lambda_0, \lambda_1)}{(\lambda_1 - \lambda_0)(\gamma + \lambda_1)^{N+2}} = N_p N_q F_N(\gamma),$$

where $\mathcal{Q}(\lambda_0, \lambda_1) = (1+\gamma^2)(\lambda_0 + \lambda_1) + 2\gamma(1 + \lambda_0 \lambda_1)$. The double integral defining the function $F_N(\gamma)$, although helpful in numerical estimations, is too unwieldly for direct analytical calculations. Fortunately, we can use the sum rule of Eq. (16) to determine $F_N(\gamma)$ as follows. Using the simplifying assumption $\alpha_{pn} = \alpha$ in Eq. (21) we find

$$\langle C_{pq} \rangle = N_p N_q F_N(\cosh \alpha) + \delta_{pq} \frac{4N_p}{(e^{\alpha}+1)^2} + \delta_{pq} N_p \frac{\tanh^2(\alpha/2) - \tanh^{2N}(\alpha/2)}{N^2 - 1}.$$

From Eq. (16) we get the identity

$$\sum_{q=1}^{M} \langle C_{pq} \rangle = \frac{4N_p}{(e^{\alpha}+1)}.$$

Using this result in the previous equation we obtain

$$F_N(\cosh \alpha) = \frac{N^2 \operatorname{sech}^2(\alpha/2) + \tanh^{2N}(\alpha/2) - 1}{N(N^2-1)}.$$

From this expression and the relation $T = \operatorname{sech}^2(\alpha/2)$ we get the simple formula

$$\langle G_{pq}/G_0 \rangle = \frac{N_p N_q}{N(N^2-1)} [N^2 T + (1-T)^N - 1]. \tag{29}$$

This equation is a new result and represents the generic situation of mildly overlapping resonances. It can be interpreted as describing the smooth crossover between two extreme limits: *strongly overlapping resonances*, where $NT \gg 1$ and the following asymptotic expansion applies

$$\langle G_{pq}/G_0 \rangle = \frac{N_p N_q}{N} \left(T + \frac{T-1}{N^2} + \frac{T-1}{N^4} + [\mathcal{O}(NT)]^{-6} \right),$$

and *isolated resonances*, where $NT \ll 1$ and Eq. (29) yields

$$\langle G_{pq}/G_0 \rangle \simeq \frac{N_p N_q}{N+1} T. \tag{30}$$

Note that the assumption of a large number of probes ($M \gg 1$) is sufficient to recover the Hauser-Feshbach law for equivalent channels $\langle G_{pq}/G_0 \rangle \simeq N_p N_q T/N$, in both limits since $N = \sum_{p=1}^{M} N_p$ would also be large.

Let us compare our general results with those of previous works for the particular case of two terminals. In Ref. 56 the density of transmission eigenvalue $\rho(\tau)$, has been calculated as a function of $\gamma = 2/T - 1$. Using this result we find

$$\langle G/G_0 \rangle = \frac{1}{3} T + \frac{1}{6} T^2, \tag{31}$$

for $N_1 = 1 = N_2$ and

$$\langle G/G_0 \rangle = \frac{4}{5} T + \frac{2}{5} T^2 - \frac{4}{15} T^3 + \frac{1}{15} T^4, \tag{32}$$

for $N_1 = 2 = N_2$. Both expressions are in agreement with Eq. (29). Note also that for $T \ll 1$, we get $\langle G/G_0 \rangle \simeq T/3$ and $\langle G/G_0 \rangle \simeq 4T/5$ from Eqs. (31) and (32), respectively, which agree with Eq. (30).

(e) Equivalent terminals.

Let us now consider the case of equivalent probes, i.e., $\alpha_{pn}=\alpha_n$ and $N_p=N_1$ for all p. From Eq. (21) we get

$$\langle C_{pq}\rangle = F(\{\gamma\}) + \delta_{pq}\sum_{n=1}^{N_1}\left(\frac{4}{(e^{\alpha_n}+1)^2} + (\gamma_n^2-1)A_nB_n\right),$$

where

$$A_n = \int_{-1}^{1} dx (x+\gamma_n)^{M-2} \prod_{m(\neq n)} (x+\gamma_m)^M,$$

and

$$B_n = \int_{1}^{\infty} dx (x+\gamma_n)^{-M-2} \prod_{m(\neq n)} (x+\gamma_m)^{-M}.$$

The function $F(\{\gamma\})$ can be evaluated via the sum rule of Eq. (16) with $\alpha_{pn}=\alpha_n$ for all p, we find

$$F(\{\gamma\}) = \frac{1}{M}\sum_{n=1}^{N_1}\left(\frac{4e^{\alpha_n}}{(e^{\alpha_n}+1)^2} + (1-\gamma_n^2)A_nB_n\right).$$

This implies that the average conductance coefficients for the case of equivalent terminals, which from Eq. (22) is given by $\langle G_{pq}/G_0\rangle = F(\{\gamma\})$, read

$$\langle G_{pq}/G_0\rangle = \frac{1}{M}\sum_{n=1}^{N_1}\left(T_n - 4(1-T_n)\frac{A_nB_n}{T_n^2}\right). \qquad (33)$$

The coefficients A_n and B_n can be calculated explicitly but the final form is too cumbersome. In the limit of strongly overlapping resonances, where $T_n \simeq 1$ and $T_{tot} \gg 1$, the contribution of the second term is irrelevant and we recover the Hauser-Feshbach law for nonequivalent channels

$$\langle G_{pq}/G_0\rangle \simeq \frac{\left(\sum_{n=1}^{N_1} T_n\right)\left(\sum_{m=1}^{N_1} T_m\right)}{T_{tot}} = \frac{1}{M}\sum_{n=1}^{N_1} T_n,$$

as expected.

As a consistency check of Eq. (33), let us set $T_n=T$ (equivalent channels), then we get

$$\langle G_{pq}/G_0\rangle = \frac{N_1}{M}\left(T - 4(1-T)\frac{AB}{T^2}\right),$$

where

$$A = \frac{2^{N-1}[1-(1-T)^{N-1}]}{(N-1)T^{N-1}},$$

and

$$B = \frac{T^{N+1}}{2^{N+1}(N+1)}.$$

After substititution, we end up with

$$\langle G_{pq}/G_0\rangle = \frac{N_1^2}{N(N^2-1)}[N^2T + (1-T)^N - 1],$$

in agreement with Eq. (29) for $N_p=N_1=N_q$.

V. SUMMARY AND CONCLUSIONS

In this paper, we have calculated exactly the average conductance coefficients of a ballistic chaotic cavity coupled to continua via tunnel barriers, for an arbitrary number of propagating channels. We have employed the Heidelberg approach, which builds on resonant quantum scattering theory and relates the scattering matrix of the open system to the dynamical Hamiltonian of the closed cavity, which is replaced by a member from the Gaussian unitary ensemble of large random matrices. Our central result, Eq. (22) has been obtained by using the standard mapping of the problem onto the supersymmetric nonlinear σ model. Several important physical limits have been studied in detail, namely: ideal contacts, strongly overlapping resonances, locally weakly absorbing limit, equivalent channels, and equivalent terminals.

From a theoretical point of view, our results constitute a step forward in the complete characterization of the universal scaling functions for the quantum dot to disordered wire crossover problem, since it represents the limit $L/\xi \to 0$. Experimentally, we expect that our explicit expressions may help improve the understanding of the universal features of the transport regimes in low temperature ballistic quantum dots.

We conclude by remarking that there are several extensions of this work that are both physically important and mathematically tractable. These include other ensembles (orthogonal, symplectic, and crossovers), parametric correlations, dephasing effects, and time dependent response. The crucial feature of our approach to these problems is the *geometrization* of its mathematical aspects through the map onto the supersymmetric coset space, or equivalently onto the classical symmetric space of the random-matrix formalism.

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