

# Random-bond Ising model in two dimensions: The Nishimori line and supersymmetry

Ilya A. Gruzberg,<sup>1</sup> N. Read,<sup>2</sup> and Andreas W. W. Ludwig<sup>3</sup>

<sup>1</sup>*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030*

<sup>2</sup>*Department of Physics and Department of Applied Physics, Yale University, P.O. Box 208120, New Haven, Connecticut 06520-8120*

<sup>3</sup>*Physics Department, University of California, Santa Barbara, California 93106-9530*

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We consider a classical random-bond Ising model (RBIM) with binary distribution of  $\pm K$  bonds on the square lattice at finite temperature. In the phase diagram of this model there is the so-called Nishimori line which intersects the phase boundary at a multicritical point. It is known that the correlation functions obey many exact identities on this line. We use a supersymmetry method to treat the disorder. In this approach the transfer matrices of the model on the Nishimori line have an enhanced supersymmetry  $\text{osp}(2n+1|2n)$ , in contrast to the rest of the phase diagram, where the symmetry is  $\text{osp}(2n|2n)$  (where  $n$  is an arbitrary positive integer). An anisotropic limit of the model leads to a one-dimensional quantum Hamiltonian describing a chain of interacting superspins, which are irreducible representations of the  $\text{osp}(2n+1|2n)$  superalgebra. By generalizing this superspin chain, we embed it into a wider class of models. These include other models that have been studied previously in one and two dimensions. We suggest that the multicritical behavior in two dimensions of a class of these generalized models (possibly not including the multicritical point in the RBIM itself) may be governed by a single fixed point, at which the supersymmetry is enhanced still further to  $\text{osp}(2n+2|2n)$ . This suggestion is supported by a calculation of the renormalization-group flows for the corresponding nonlinear sigma models at weak coupling.

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## I. INTRODUCTION

For many decades Ising models served as the simplest nontrivial models for the description of magnetically ordered phases and phase transitions between them. This is true both for pure models and for Ising models with randomness. In particular, in the context of the spin-glass problem<sup>1</sup> the relevant Ising models have random bonds of both signs (ferromagnetic and antiferromagnetic). This leads to frustration and the possibility of spin glass order.

In this paper we consider a classical random-bond Ising model (RBIM) of Ising spins  $S_i = \pm 1$  on the two-dimensional (2D) square lattice with the Hamiltonian ( $\beta = 1/T$  is the inverse temperature)

$$\beta\mathcal{H} = - \sum_{\langle ij \rangle} K_{ij} S_i S_j, \quad (1.1)$$

where the bold indices  $\mathbf{i} = (i_x, i_y)$  and  $\mathbf{j} = (j_x, j_y)$  denote 2D vectors of integer coordinates of the sites of the lattice, the summation is over distinct nearest-neighbor bonds (i.e., pairs), and the coupling constants  $K_{ij}$  are independent random variables drawn from the distribution

$$P[K_{ij}] = (1-p)\delta(K_{ij}-K) + p\delta(K_{ij}+K). \quad (1.2)$$

That is, the couplings  $K_{ij}$  are ferromagnetic ( $K > 0$ ) with probability  $1-p$  and antiferromagnetic with probability  $p$ . Notice that  $K$  varies inversely with  $T$ . In what follows we will occasionally also consider Ising models with other distributions of the bond strengths. For simplicity, in most cases, where it cannot lead to confusion, we will simply call the model with the binary distribution (1.2) “the RBIM.”

Later, we will also consider the anisotropic generalization of the model, in which  $K$  takes different values on bonds in the  $x$  and  $y$  directions.

Let us summarize some of what is known about this model. The phase diagram of this model is still somewhat controversial, but is widely believed to be as in Fig. 1.<sup>2-8</sup> First we note that for  $p=1$ , we have a pure antiferromagnetic Ising model, which can be mapped onto the ferromagnetic case by sending  $S_i \rightarrow -S_i$  for  $i$  on one sublattice. More generally, this transformation is equivalent to sending  $p \rightarrow 1-p$ . Hence we need show only the region  $0 \leq p \leq 1/2$ . The solid line is a phase boundary which separates the fer-

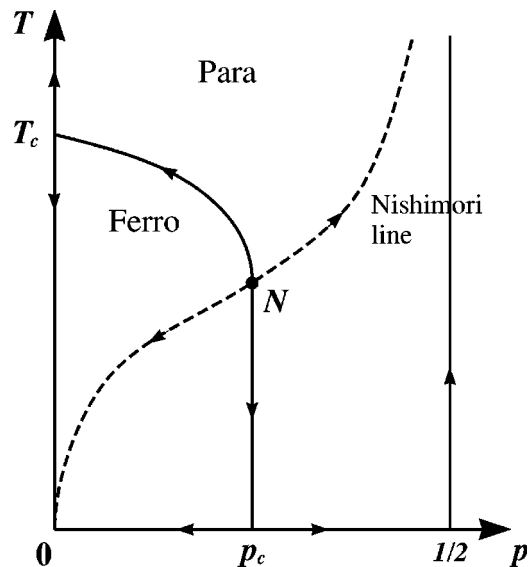


FIG. 1. Phase diagram of the random-bond Ising model of Eq. (1.1), in terms of  $T \propto 1/K$  and  $p$ .

romagnetically ordered from the paramagnetic phase. Figure 1 can also be viewed as a schematic renormalization group (RG) flow diagram, in which the intersection points labeled  $T_c$  (corresponding to  $K_c=0.44\dots$ , the pure Ising transition),  $p_c\approx 0.12$  (the  $T=0$  transition), and another point  $N$  are viewed as RG fixed points that govern the critical behavior for the portions of the phase boundary shown as flowing into these points ( $N$  is an unstable, hence multicritical, point). In 2D, it is generally believed that no spin-glass phase exists at finite temperature. At zero temperature, long-range spin-glass (Edwards–Anderson) order exists trivially when the distribution of bonds is continuous, since in a finite system there is, with probability one, a unique ground state, up to a reversal of all the spins. (Taking the thermodynamic limit in a fixed sample is a very subtle problem; for a very recent discussion, see Ref. 9 and references therein.) However, for the discrete distribution with bonds taking values  $K$ ,  $-K$ , assumed here, the existence of such order in the region  $p > p_c$  is not clear because there will be many degenerate ground states. There is evidence for power-law spin-glass correlations at  $T=0$  in this region.<sup>10,11</sup> In three or more dimensions, there is a spin-glass ordered phase at temperatures below some temperature  $T_{SG}(p)$ , and which extends up to  $p=1/2$ . The three phase boundaries meet in a multicritical point at some  $T < T_c$ ,  $p < 1/2$ . The point  $N$  in 2D is in some sense a remnant of this multicriticality in higher dimensions.

An interesting feature of this phase diagram is the so-called Nishimori line (NL), shown dashed. Such a line can be defined for a broad class of distributions  $P[K_{ij}]$ , and in our special case (1.2) is given by the condition

$$1 - 2p = \tanh K. \quad (1.3)$$

Nishimori<sup>4</sup> found that the model on the NL has a local (“gauge”)  $\mathbb{Z}_2$  symmetry. Using the symmetry he showed that the internal energy of the RBIM is analytic along the NL, and also established a special case of the following identities for correlation functions (proven generally in Ref. 7) which hold on the NL

$$[\langle S_{i_1} \dots S_{i_k} \rangle^{2q-1}] = [\langle S_{i_1} \dots S_{i_k} \rangle^{2q}]. \quad (1.4)$$

Here  $\{S_{i_1}, \dots, S_{i_k}\}$  is any set of the Ising spins, the angular brackets denote the thermal average, and the square brackets the average over the distribution of bonds  $P[K]$ . Georges *et al.*<sup>6</sup> showed that Nishimori’s result concerning the internal energy may be rederived using a supersymmetric formulation. Nishimori later argued, and Kitatani showed,<sup>5</sup> that the ferro-para phase boundary is vertical below the NL. This is supported by the later numerical works among Refs. 2 and 8.

Le Doussal, Georges, and Harris<sup>7</sup> established that in any dimension the NL goes through the multicritical point  $N$  on the ferro-para phase boundary and is one of the exact RG trajectories near this point, the other trajectory being the vertical tangent to the phase boundary at  $N$ . The RG eigenvalues along these trajectories and the corresponding critical exponents were estimated by Singh and Adler<sup>12</sup> from high-temperature expansions of high order. Two of their exponents in 2D are very close to those of classical 2D percolation, as are the results in Ref. 8. We emphasize that,

while Nishimori’s results do not apply to every possible distribution of disorder in the RBIM, the implications for universal critical properties must hold throughout a universality class, and it is believed that this class includes the generic RBIM multicritical point.<sup>7</sup>

More recently, Cho and Fisher<sup>13</sup> proposed a network model similar to the Chalker–Coddington model used to describe transitions between integer quantum Hall plateaus.<sup>14</sup> The Cho–Fisher model was supposed to be in the same universality class as the 2D RBIM. They simulated their network numerically and found a phase diagram with a multicritical point, with numerical critical exponents close to the ones found by Singh and Adler.<sup>12</sup> In her thesis,<sup>15</sup> Cho also simulated another network model which is precisely equivalent to the RBIM, and again found exponents close to some of those in Ref. 12.

Here we will briefly sketch a version of the argument of Ref. 7, since it provided motivation for our work. Let us consider the random-bond Ising model with Gaussian disorder,

$$P[K_{ij}] = \frac{1}{\sqrt{2\pi}\Delta} \exp[-(K_{ij} - K_0)^2 / (2\Delta^2)], \quad (1.5)$$

so the mean of  $K_{ij}$  is  $K_0$ , and the standard deviation is  $\Delta$ . Taking the partition function

$$Z = \sum_{\{S_i^a\}} \exp \sum_{\langle ij \rangle} K_{ij} S_i S_j, \quad (1.6)$$

we replicate and average to obtain

$$[Z^n] = \sum_{\{S_i^a\}} \exp \left[ K_0 \sum_{\langle ij \rangle, a} S_i^a S_j^a + \frac{1}{2} \Delta^2 \sum_{\langle ij \rangle, ab} S_i^a S_j^a S_i^b S_j^b \right], \quad (1.7)$$

where  $a, b = 1, \dots, n$ . For finite  $n$  this has the form of the Ashkin–Teller model, consisting of  $n$  coupled Ising models. Now compare this with the replicated spin-glass model, which is obtained by setting  $K_0 = 0$ :

$$[Z^m] = \sum_{\{S_i^a\}} \exp \left[ \frac{1}{2} \Delta^2 \sum_{\langle ij \rangle, ab} S_i^a S_j^a S_i^b S_j^b \right], \quad (1.8)$$

with  $a, b = 0, \dots, m-1$ . This model has a gauge symmetry: it is invariant under site-dependent transformations  $S_i^a \rightarrow -S_i^a$  for all  $a$  and any set of  $i$ ’s. This local  $\mathbb{Z}_2$  gauge symmetry can be fixed by setting all  $S_i^a = 1$  for all  $i$ , for one value of  $a$ , say  $a=0$ . Then if  $m=n+1$ , we obtain the random-bond partition function with  $K_0 = \Delta^2$ , up to constants. On this line, which is the NL for the Gaussian case, there are still consequences of the underlying gauge symmetry. The  $m=n+1$  replicas are still on an equal footing, and this leads to the identities (1.4) on the NL, as follows. In the gauge-unfixed model, only correlation functions containing an even number of  $S_i^a$ ’s at each site are nonzero. The correlation functions are invariant under permutations of the replicas, and so independent of whether or not  $a=0$  is among the components. Since different replica components represent

distinct thermal averages in the  $n \rightarrow 0$  limit, we obtain the identities (1.4), and others, on gauge fixing. The  $\mathbb{Z}_2$  local gauge symmetry in the replica formalism should not be confused with that of Nishimori, who did not use replicas; it is the enlarged permutational symmetry of the replicas that corresponds to Nishimori's arguments. Off the NL, the identities are lost, but the model can still be written as a gauge-fixed version of a system with  $n+1$  replicas and a local  $\mathbb{Z}_2$  gauge symmetry.

The preceding argument shows that, in the replica formalism, the NL is special because it possesses a larger permutation symmetry  $S_{n+1}$  in place of the usual  $S_n$ . As we saw, even off the NL, an additional "zeroth" replica spin can be introduced into the model, along with a gauge symmetry that can be used to remove the unwanted degrees of freedom, and further on the NL line the zeroth spin is symmetric with the others. Now from work extending back to Onsager, the 2D Ising model can be written in terms of free fermions, which become Majorana (real Dirac) fermions in the continuum limit, and furthermore Ashkin-Teller models can be represented by interacting Majorana fermions, with  $O(n)$  symmetry.<sup>16</sup> Hence we are led to conjecture that, in the replicated fermion representation, it is possible to introduce an additional "zeroth" fermion, together with a local  $\mathbb{Z}_2$  gauge symmetry to remove the unwanted degrees of freedom, and that on the NL, we should find a larger  $O(n+1)$  symmetry. In this paper we demonstrate that this indeed occurs, though we take a different route to do so. We consider the binary distribution above, and we use supersymmetry rather than replicas, so no limit  $n \rightarrow 0$  need be taken. However, the corresponding result for replicas is contained in our results. The network models such as the Cho-Fisher model also depend on the fermion representation of the Ising model, and so we can also consider these models in our framework. We find that the models can be viewed as supersymmetric vertex models, or by using the anisotropic limit as Hamiltonian chains, which act in irreducible representations of the relevant symmetry (super-)group (which is enlarged on the NL), and thus are quantum spin chains, and possibly can also be viewed as the strong-coupling region of a nonlinear sigma model. This greatly enhances the similarity of the problem to the integer quantum Hall effect transition, and to other random fermion problems. However, we find that the NL does not fall into a recent list of nonlinear sigma models that correspond to random matrix ensembles in such problems.<sup>17</sup> Our results also apply to certain one-dimensional fermion problems.

In random fermion problems, including those arising in disordered superconductors, it is usual to attempt classifications, based on symmetries, for generic probability distributions, as in Ref. 17 for random matrices. The ensembles found in Ref. 17 for disordered superconductors differ from the standard ensembles because of the lack of a conserved particle number, and because zero energy is a special point in the spectrum. The second-quantized noninteracting quasiparticle description can in each case be replaced by a "first-quantized" formalism involving a single matrix, which must satisfy certain discrete symmetry and symmetry-like conditions, which distinguish the ensembles. One such class,

termed class D in Ref. 17, is for problems with broken time reversal and no spin-rotation symmetry. This corresponds to the symmetries of the fermion representation of the RBIM. In this class, a nonlinear sigma model analysis in two dimensions indicates that there is a metallic phase in which the fermion eigenstates are extended.<sup>18–21</sup> Senthil and Fisher<sup>19</sup> discussed a scenario in which such a phase occurs in the RBIM, as an intermediate phase between the paramagnetic and ferromagnetic phases, at low  $T$  in the region labeled "Para" in Fig. 1, and with its bordering phase boundaries (one of which is the low  $T$  phase boundary shown) meeting at the multicritical point  $N$  on the NL. (This is the region sometimes claimed to be some kind of spin-glass-like phase in the RBIM literature.) They suggested that the phase would be characterized by the absence of long-range order in the mean of either the ferromagnetic Ising or the dual disorder variable correlations.<sup>19</sup> Presumably such decay would also hold for the mean square (spin-glass) correlations. It is not clear if this is consistent with the  $T \rightarrow 0$  analysis of Ref. 11. An alternative scenario is that a finer analysis of problems with broken time-reversal symmetry is needed, and that the nonlinear sigma model appropriate for class D does not apply to the RBIM. Indeed, a recent paper<sup>21</sup> emphasizes that the target manifold of the class D model is not connected, and that consequently there can be domains of the two components or "phases." It follows that additional parameters are required in order to fully parametrize the systems, independent of those familiar for sigma models with connected targets. This allows for a much richer phase diagram and transitions in this symmetry class. We show in this paper that this is connected with the structure we uncover in the random-bond Ising and network models. In another paper,<sup>22</sup> it is argued that the metallic phase cannot occur in a RBIM with real Ising couplings.

Problems of random noninteracting fermions are among the better understood of disordered systems, and while many results are numerical, in some cases there are even exact results for critical properties in 2D. By making contact between the RBIM and other random fermion problems, and casting them in a common language, we hope to gain understanding of this disordered classical spin problem. At the same time, the RBIM provides an example that may shed light on previously unknown classes or ensembles of random fermions. The analysis presented in this paper does not resolve all aspects of the broken-time-reversal symmetry class, but it does show that the NL is a special subclass.

We now give an outline of the paper. Sections II, III, IV, and VI contain the main technical work. They show how a fermion representation<sup>23</sup> can be used for the RBIM, and how bosons are also introduced to cancel the inverse partition functions, via supersymmetry (SUSY).<sup>24–31</sup> The bosons live in a space with an indefinite metric, a common feature of SUSY methods (see Refs. 26, 31, and compare Refs. 28–30). On the NL, a larger SUSY algebra is found. As an application of this enhanced SUSY, we use it in Appendix A to rederive the equalities (1.4). An anisotropic limit of the model relates it to a Hamiltonian for "superspins" on "split" sites, two for each original site. The Hamiltonian on the NL can be generalized in a natural way, and we introduce

a phase diagram for the generalized models; on one line in this diagram the Hamiltonian possesses a still larger SUSY. Section V discusses a 1D model,<sup>28,32–35</sup> and shows that it possesses a single “Nishimori” point of higher SUSY. In Sec. VII we consider the model of Cho and Fisher<sup>13</sup> and show that it does not correspond precisely to the RBIM, nor does it have the larger SUSY of the NL. In Sec. VIII we argue that much, or possibly all, of the critical surface in the phase diagram in the higher SUSY generalized models is in the universality class of the point with even higher SUSY mentioned above. This is supported by consideration of nonlinear sigma models that correspond to the spin chains, and a weak-coupling calculation for these shows a renormalization-group flow towards the higher SUSY theory. We comment on general network models and nonlinear sigma models with the symmetries of class D. Appendix B gives details of a representation of SUSY that we use. Appendix C contains the details of the calculation of the beta functions at weak coupling in the nonlinear sigma model.

## II. TRANSFER MATRICES AND SUPERSYMMETRY

In this section we will express the Ising model transfer matrices in terms of fermionic replicas, and then introduce bosons to make the system supersymmetric. This allows us to consider (in Sec. III) averages over quenched disorder without taking the replica  $n \rightarrow 0$  limit.

First, we set up some notation. We define the dual coupling  $\tilde{K}$  by

$$e^{-2\tilde{K}_{ij}} = \tanh K_{ij} \quad (2.1)$$

for any sign of  $K_{ij}$ . For positive  $K_{ij} = K > 0$  we denote  $\tilde{K}_{ij} = K^*$ , and for negative  $K_{ij} = -K < 0$ ,  $\tilde{K}_{ij} = K^* + i\pi/2$ . We assume free (not periodic) boundary conditions on the Ising spins in the horizontal ( $x$ ) direction, and periodic in the vertical ( $y$ ) direction. As is well-known (see, for example, Ref. 36), the partition function of the nearest-neighbor Ising model may be written as the trace of a product of row transfer matrices:

$$Z = \text{Tr} \prod_{i_y} T_v(i_y) T_h(i_y). \quad (2.2)$$

Here  $i_y$  is an integer coordinate of a row of sites. The row transfer matrices,  $T_v(i_y)$  for vertical, and  $T_h(i_y)$  for horizontal bonds, do not commute with each other, so the product in Eq. (2.2) must be ordered such that the row coordinate  $i_y$  increases from right to left. The row transfer matrices may in turn be written as products of the transfer matrices for single bonds:

$$T_v(i_y) = \prod_{i_x} T_{vi}, \quad T_h(i_y) = \prod_{i_x} T_{hi}. \quad (2.3)$$

The  $T_{vi}$ 's for different  $i_x$  and the same  $i_y$  commute, and similarly for the  $T_{hi}$ 's. The trace represents the periodic boundary condition in the  $y$  direction.

Following Ref. 23 we write the vertical transfer matrix for a single vertical bond between the Ising spins at points  $i$  and  $i + \hat{y}$  as

$$T_{vi} = \frac{e^{K_{i,i+\hat{y}}}}{\cosh \tilde{K}_{i,i+\hat{y}}} \exp(\tilde{K}_{i,i+\hat{y}} \sigma_i^y), \quad (2.4)$$

where the “light” index  $i$  denotes a site on the 1D lattice corresponding to the vertical row containing the original site  $i$ , and  $\sigma^x$ ,  $\sigma^y$ , and  $\sigma^z$  are Pauli matrices. Similarly, the horizontal transfer matrix for a single horizontal bond between the Ising spins at points  $i$  and  $i + \hat{x}$  is

$$T_{hi} = \exp(K_{i,i+\hat{x}} \sigma_i^z \sigma_{i+1}^z). \quad (2.5)$$

Note that before the averaging over the randomness the transfer matrices explicitly depend on the corresponding bond, and therefore are labeled by the bold 2D indices.

The transfer matrices act in tensor products of two-dimensional spaces at each horizontal coordinate  $i_x$ . These 2D spaces may be realized as Fock spaces of fermions on a 1D chain of sites. This fermionization is implemented by the Jordan–Wigner transformation relating Pauli matrices to fermionic operators. To use this transformation we first make a canonical transformation

$$\sigma_i^x \rightarrow -\sigma_i^z, \quad \sigma_i^z \rightarrow \sigma_i^x. \quad (2.6)$$

The Jordan–Wigner transformation reads

$$\sigma_i^z = 2c_i^\dagger c_i - 1, \quad (2.7)$$

$$\sigma_i^x \sigma_{i+1}^x = (c_i^\dagger - c_i)(c_{i+1}^\dagger + c_{i+1}), \quad (2.8)$$

where  $c_i^\dagger$  and  $c_i$  are canonical creation and annihilation fermionic operators. In terms of these operators the transfer matrices for individual bonds become

$$T_{vi} = \frac{e^{K_{i,i+\hat{y}}}}{\cosh \tilde{K}_{i,i+\hat{y}}} \exp(-2\tilde{K}_{i,i+\hat{y}}(c_i^\dagger c_i - 1/2)), \quad (2.9)$$

$$T_{hi} = \exp(K_{i,i+\hat{x}}(c_i^\dagger - c_i)(c_{i+1}^\dagger + c_{i+1})). \quad (2.10)$$

Next we replicate the fermions. The number of replicas has to be even, because of the symmetry of the bosons to be introduced below, so we denote it as  $2n$ . We label the replicas by Greek letters. The replicated transfer matrices become

$$T_{vi} = (2 \cosh K)^{2n} \exp(-2\tilde{K}_{i,i+\hat{y}} \hat{N}_{Fi}), \quad (2.11)$$

$$T_{hi} = \exp(2K_{i,i+\hat{x}} \hat{X}_{Fi}), \quad (2.12)$$

where we defined

$$\hat{N}_{Fi} = \sum_{\alpha=1}^{2n} n_{\alpha i}, \quad n_{\alpha i} = c_{\alpha i}^\dagger c_{\alpha i}, \quad (2.13)$$

$$\hat{X}_{Fi} = \sum_{\alpha=1}^{2n} x_{\alpha i}, \quad (2.14)$$



$$x_{\alpha i} = \frac{1}{2}(c_{\alpha i}^\dagger - c_{\alpha i})(c_{\alpha, i+1}^\dagger + c_{\alpha, i+1}). \quad (2.15)$$

The quadratic forms  $\hat{N}_{Fi}$  and  $\hat{X}_{Fi}$  are invariant under the orthogonal transformations mixing the fermions, which becomes especially transparent if we introduce two sets of  $2n$  real fermions per site as

$$\eta_{\alpha i} = \frac{c_{\alpha i}^\dagger - c_{\alpha i}}{\sqrt{2}i}, \quad \xi_{\alpha i} = \frac{c_{\alpha i}^\dagger + c_{\alpha i}}{\sqrt{2}}. \quad (2.16)$$

These fermions satisfy

$$\{\eta_{\alpha i}, \eta_{\beta j}\} = \{\xi_{\alpha i}, \xi_{\beta j}\} = \delta_{ij} \delta_{\alpha\beta}, \quad \{\eta_{\alpha i}, \xi_{\beta j}\} = 0. \quad (2.17)$$

Terminologically, we note here that any set of self-adjoint operators, say  $\psi_a$ ,  $a = 1, \dots, M$ , for some  $M$ , with anticommutation relations  $\{\psi_a, \psi_b\} = \delta_{ab}$ , constitutes a Clifford algebra. For us, the set of  $\xi$ 's, either for one or for many sites, or similarly of  $\eta$ 's, or a combination of these, are all Clifford algebras. A little of the general theory of these algebras will be used later. In terms of these fermions, or Clifford algebra generators, the quadratic forms become

$$\hat{N}_{Fi} = i \eta_{\alpha i} \xi_{\alpha i} + n, \quad \hat{X}_{Fi} = i \eta_{\alpha i} \xi_{\alpha, i+1}, \quad (2.18)$$

where from now on we assume that repeated indices from the beginning of the Greek alphabet ( $\alpha$ ,  $\beta$ , etc.) are summed from 1 to  $2n$ , unless stated otherwise.

The generators of the global symmetry algebra  $\text{so}(2n)$ , in this notation, are  $\sum_i (\eta_{\alpha i} \eta_{\beta i} + \xi_{\alpha i} \xi_{\beta i})$ , for pairs  $\alpha, \beta$ , and because of the anticommutation relations we may take only  $\alpha < \beta$ , corresponding to the antisymmetric  $2n \times 2n$  matrices. These generators commute with  $\hat{N}_{Fi}$  and  $\hat{X}_{Fi}$ , proving that the transfer matrices are invariant under  $\text{so}(2n)$ . The replicated partition function  $Z^{2n}$ , which is now given by a trace in the  $2n$ -component fermion Fock space, is invariant under  $\text{so}(2n)$ . Note that we capitalize the name of the group or supergroup, such as  $\text{SO}(2n)$ , but not the name of the corresponding Lie (super-)algebra, such as  $\text{so}(2n)$ .

The supersymmetric counterpart of the fermionic algebra  $\text{so}(2n)$  is the symplectic algebra  $\text{sp}(2n)$ . This motivates the introduction of bosons with this symplectic symmetry as follows. We start with  $2n$  complex ‘‘symplectic’’ bosonic operators  $s_{\alpha i}$  (and their adjoints  $s_{\alpha i}^\dagger$ ), satisfying

$$[s_{\alpha i}, s_{\beta j}^\dagger] = i \delta_{ij} J_{\alpha\beta}, \quad (2.19)$$

where  $J_{\alpha\beta}$  is a nonsingular real antisymmetric  $2n \times 2n$  matrix. (It is because the number of bosons must be even that the number of fermions must be also.) Without loss of generality (by appropriate change of basis) this matrix may be taken to be (in block form)

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}, \quad (2.20)$$

where  $\mathbb{1}_n$  is the  $n \times n$  identity matrix. We also need to define the vacuum state for our bosons. We will see that the SUSY

requirement makes this choice essentially unique, and the resulting space of states has indefinite metric (some states have negative squared norms).

The bosonic counterparts of the forms  $\hat{N}_{Fi}$  and  $\hat{X}_{Fi}$  are the symplectic forms

$$\hat{N}_{Bi} = i s_{\alpha i}^\dagger J_{\alpha\beta} s_{\beta i}, \quad (2.21)$$

$$\hat{X}_{Bi} = \frac{i}{2} (s_{\alpha i}^\dagger - s_{\alpha i}) J_{\alpha\beta} (s_{\beta, i+1}^\dagger + s_{\beta, i+1}). \quad (2.22)$$

To parallel the fermionic case we also introduce two sets of  $2n$  real bosons per site as

$$q_{\alpha i} = \frac{s_{\alpha i}^\dagger + s_{\alpha i}}{\sqrt{2}}, \quad r_{\alpha i} = \frac{s_{\alpha i}^\dagger - s_{\alpha i}}{\sqrt{2}i}. \quad (2.23)$$

These bosons satisfy

$$[q_{\alpha i}, q_{\beta j}] = [r_{\alpha i}, r_{\beta j}] = i \delta_{ij} J_{\alpha\beta}, \quad [q_{\alpha i}, r_{\beta j}] = 0. \quad (2.24)$$

(These have the form of the commutation relations for canonically conjugate coordinates and momenta.) In terms of the real bosons the forms (2.22) are

$$\hat{N}_{Bi} = -r_{\alpha i} J_{\alpha\beta} q_{\beta i} - n, \quad \hat{X}_{Bi} = -r_{\alpha i} J_{\alpha\beta} q_{\beta, i+1}. \quad (2.25)$$

The generators of the global symplectic symmetry algebra  $\text{sp}(2n)$  are  $\sum_i (q_{\alpha i} q_{\beta i} + r_{\alpha i} r_{\beta i})$ , where because of the commutation relations we may use only  $\alpha \leq \beta$ , corresponding to symmetric matrices. These operators are the generators of global linear canonical transformations on the  $q$ 's and  $r$ 's. The forms  $\hat{N}_{Bi}$  and  $\hat{X}_{Bi}$ , and hence the transfer matrices, are invariant under this algebra.

We now address the question of the bosonic vacuum. We will find it using the requirement that the spectrum of the bosonic form  $\hat{N}_{Bi}$  is the SUSY counterpart of the integer spectrum of the fermionic form  $\hat{N}_{Fi}$ . In this case we will have

$$\text{STr} \exp(-\text{const}(\hat{N}_{Fi} + \hat{N}_{Bi})) = 1. \quad (2.26)$$

This condition is essential in the SUSY approach to ensure that the partition function of the RBIM is unity for any realization of the disorder. Here we used the notation  $\text{STr}$  for the supertrace in the space of states of our problem. We will now discuss how this supertrace is defined.

In general, the supertrace in a supervector space must be defined using the notion of a grading for the states (or vectors). This can be done by choosing a basis and then defining one subset of basis vectors as ‘‘even,’’ and the remainder as ‘‘odd,’’ vectors. The vector space then contains two complementary subspaces of even and odd vectors, respectively; the zero vector, and linear combinations of vectors from both subspaces, are viewed as having no definite grading. The vectors are then said to be  $\mathbb{Z}_2$ -graded. Operators on the vector space can likewise be classified as even or odd, according to whether they preserve or reverse the grading of basis vectors

on which they act; usually, only operators for which this rule gives a consistent answer (those with well-defined grading) are of interest. Thus the grading is usually treated as a superselection rule. The supertrace  $\text{STr } Y$  of an even operator  $Y$  is then defined, like an ordinary trace, as the sum of the diagonal matrix elements of  $Y$  in a basis of even and odd vectors, except that for the supertrace, the diagonal elements in the odd basis vectors are weighted by a minus sign. (Note that the matrix elements  $Y_{IJ}$  of an operator  $Y$  are obtained as the coefficients in the system of equations  $Y|J\rangle = \sum_I Y_{IJ}|I\rangle$ , where  $|I\rangle$ ,  $I=1,2,\dots$ , are the basis vectors, without using an inner product on the vector space.) The supertrace has a number of nice properties, like the ordinary trace in an ordinary vector space; in particular a form of the cyclic property still holds,  $\text{STr } AB = \pm \text{STr } BA$ , with a plus if both  $A$  and  $B$  are even, and a minus if both are odd operators. The grading and the supertrace are needed in connection with supersymmetry algebras, but otherwise do not necessarily have to be considered. We also note here that it is possible to form a graded tensor product of graded vector spaces, in a way that preserves the grading.

In this paper most of our constructions use a Fock space. In a Fock space generated by boson and fermion operators acting on a vacuum, there is a natural grading, defined using an occupation number basis, in which states are even or odd as the total number of fermions (of all types) is even or odd. However, we will not use such a grading to define the supertrace above. The reason is that we have already introduced the ordinary trace in writing the partition function for fermionic replicas; in this trace, all diagonal matrix elements are taken with weight  $+1$ , including those in states with an odd number of fermions. It is of course quite standard to use an ordinary trace even when dealing with fermions, which have a natural grading. The natural grading is used in defining a tensor product, such that fermion operators on different sites (i.e., in different factors in the tensor product) anticommute. These are the tensor products usually used by physicists for second quantized fermion problems. Each time we write a tensor product of spaces, it will be the graded tensor product using the natural grading that we mean. There is nothing wrong with the use of the trace, unless we are concerned about SUSY. The grading that we use in introducing SUSY into our representation is defined by specifying that states with an even (odd) number of bosons are even (odd), and so

$$\text{STr} \dots = \text{Tr}(-1)^{\sum_i \hat{N}_{Bi}} \dots \quad (2.27)$$

For states with no bosons, this reduces to the usual trace.

Now that we have defined the supertrace, we must arrange to satisfy the condition (2.26). The form  $\hat{N}_{Bi}$  may be diagonalized with the transformation to two other sets of  $n$  complex bosons. Namely, we define

$$a_{\mu i} = \frac{s_{\mu i} + i s_{\mu+n,i}}{\sqrt{2}}, \quad \bar{a}_{\mu i} = \frac{s_{\mu i} - i s_{\mu+n,i}}{\sqrt{2}}, \quad (2.28)$$

and the adjoint operators, where the index  $\mu$  (and other indices from the middle of the Greek alphabet, like  $\nu$ , etc.) runs from 1 to  $n$ . These bosons satisfy

$$[a_{\mu i}, a_{\nu j}^\dagger] = \delta_{ij} \delta_{\mu\nu}, \quad [\bar{a}_{\mu i}, \bar{a}_{\nu j}^\dagger] = -\delta_{ij} \delta_{\mu\nu}, \quad (2.29)$$

and the rest of the commutators vanish. In terms of these bosons we have

$$\hat{N}_{Bi} = a_{\mu i}^\dagger a_{\mu i} - \bar{a}_{\mu i}^\dagger \bar{a}_{\mu i}. \quad (2.30)$$

If we introduce the vacuum for  $a$  and  $\bar{a}$  bosons in the usual manner

$$a_{\mu i}|0\rangle = \bar{a}_{\mu i}|0\rangle = 0, \quad (2.31)$$

then the spectrum of  $\hat{N}_{Bi}$  is the set of non-negative integers, which is the SUSY counterpart of the spectrum of  $\hat{N}_{Fi}$ . This ensures that Eq. (2.26) holds. But the price to pay is that the states with an odd number of  $\bar{a}$  bosons have negative norms. By another choice of the vacuum we could avoid negative norms, but then we would not have the supersymmetry. With these definitions, we have now defined a Fock space  $\mathcal{F}$ , which is a tensor product of Fock spaces at each site,  $\mathcal{F} = \otimes_i \mathcal{F}_i$  in an obvious notation. The tensor product of Fock spaces is defined using the natural grading, however, our choice of grading also behaves well in the product; the grading of states is determined by the product of the ‘‘degrees’’ ( $= \pm 1$  for even, odd, respectively) of the states on the sites, because boson numbers add. Note that fermion operators are viewed as even in our grading.

The transfer matrices including fermions and bosons supersymmetrically are now

$$T_{vi} = \exp(-2\tilde{K}_{i,i+y} \hat{N}_{Si}), \quad T_{hi} = \exp(2K_{i,i+\hat{x}} \hat{X}_{Si}), \quad (2.32)$$

where the subscript  $S$  stands for ‘‘supersymmetric,’’

$$\hat{N}_{Si} = \hat{N}_{Fi} + \hat{N}_{Bi}, \quad \hat{X}_{Si} = \hat{X}_{Fi} + \hat{X}_{Bi}. \quad (2.33)$$

The SUSY transfer matrices are invariant under the orthosymplectic superalgebra  $\text{osp}(2n|2n)$ , since the forms  $\hat{N}_{Si}$  and  $\hat{X}_{Si}$  commute with the generators  $\sum_i (\xi_{ai} \xi_{\beta i} + \eta_{ai} \eta_{\beta i})$ ,  $\sum_i (q_{ai} q_{\beta i} + r_{ai} r_{\beta i})$ , and  $\sum_i (\xi_{ai} q_{\beta i} + \eta_{ai} r_{\beta i})$  of  $\text{osp}(2n|2n)$ . The last set of generators are the ‘‘odd’’ (with respect to either grading), fermionic, or supergenerators of the superalgebra, and  $\alpha$  and  $\beta$  can take arbitrary values there. Note that, in a superalgebra, two even operators obey commutation relations, two odd operators anticommute relations, and an even with an odd generator obeys a commutation relation. Thus the definition of the superalgebra structure again involves the grading. The definition of the supertrace also respects supersymmetry.

The condition (2.26) applies when only vertical-bond transfer matrices are present. To prove the supersymmetry of the full problem, namely, that the supersymmetrized partition function  $Z_{\text{SUSY}}$  (the supertrace of the product of supersymmetrized transfer matrices) is unity for any realization of the disorder, we use a graphical representation of  $Z$ . Imagine a high-temperature expansion of the horizontal transfer matrix  $T_{hi}$ , where we expand it in powers of  $K_{i,i+\hat{x}}$ . At each horizontal row the operator

$$2\hat{X}_{Si} = (c_{ai}^\dagger - c_{ai})(c_{\alpha,i+1}^\dagger + c_{\alpha,i+1}) + i(s_{ai}^\dagger - s_{ai})J_{\alpha\beta}(s_{\beta,i+1}^\dagger + s_{\beta,i+1}) \quad (2.34)$$

may create or destroy two particles at neighboring sites, or it may transfer one particle between neighboring sites. The vertical transfer matrices are diagonal, so they only propagate particles in the vertical direction.

We can represent these processes graphically by lines starting and ending at lattice sites (creation and annihilation), joining the horizontal pairs of sites (hopping), and joining vertical pairs of sites (propagation due to  $T_v$ ). When two such lines start at a pair of neighboring sites on a row and end at another pair of neighboring sites on a row, we call this a closed loop (similar to closed loops in high temperature expansion of the pure Ising model). Then the supersymmetrized partition function is equal to 1 plus the sum of contributions of all closed loops. Then we have to prove that for a closed loop the contributions of fermions and bosons cancel each other.

Let us take the smallest possible loop, where two particles are created and destroyed on two adjacent rows. This is represented by two short vertical lines between two neighboring pairs of lattice sites. For a given fermionic replica, say 1, this loop contributes the following term:

$$-\langle 0 | c_{1i} c_{1,i+1} e^{-2\tilde{K}_{i,i+\hat{y}} \hat{N}_{Fi}} e^{-2\tilde{K}_{i+\hat{x},i+\hat{x}+\hat{y}} \hat{N}_{F,i+1}} c_{1i}^\dagger c_{1,i+1}^\dagger | 0 \rangle = e^{-2(\tilde{K}_{i,i+\hat{y}} + \tilde{K}_{i+\hat{x},i+\hat{x}+\hat{y}})}. \quad (2.35)$$

The corresponding bosonic contribution is

$$\langle 0 | i s_{n+1,i} s_{1,i+1} e^{-2\tilde{K}_{i,i+\hat{y}} \hat{N}_{Bi}} \times e^{-2\tilde{K}_{i+\hat{x},i+\hat{x}+\hat{y}} \hat{N}_{B,i+1}} i s_{1i}^\dagger s_{n+1,i+1}^\dagger | 0 \rangle. \quad (2.36)$$

To evaluate this expression we note the following. From the definition of the symplectic bosons it follows that

$$[s_{\alpha j}, \hat{N}_{Bi}] = \delta_{ij} s_{\alpha j}, \quad [s_{\alpha j}^\dagger, \hat{N}_{Bi}] = -\delta_{ij} s_{\alpha j}^\dagger. \quad (2.37)$$

We use these relations to pull the exponentials through to the vacuum on the right in Eq. (2.36), which then becomes

$$-\langle 0 | s_{n+1,i} s_{1,i+1} s_{1i}^\dagger s_{n+1,i+1}^\dagger | 0 \rangle e^{-2(\tilde{K}_{i,i+\hat{y}} + \tilde{K}_{i+\hat{x},i+\hat{x}+\hat{y}})}. \quad (2.38)$$

Next we notice that, as a consequence of Eq. (2.28) and the definition (2.31), the vacuum state  $|0\rangle$  is annihilated by  $s_{ai}$ . Then we commute the operators in the first factor in the last expression, after which it becomes exactly opposite to the fermionic contribution (2.35). This argument is easily generalized to arbitrary loops (including those that wrap around the system, thanks to the definition of the supertrace), and proves that the supersymmetrized partition function is indeed equal to one, for any realization of the disorder.

### III. AVERAGING AND ENHANCED SUPERSYMMETRY ON THE NISHIMORI LINE

In this section we perform the average with respect to the distribution  $P[K]$  and find that on the NL the averaged trans-

TABLE I. Correspondence between states in the spaces  $\mathcal{F}_i$  and  $\mathcal{F}'_i$ .

$\mathcal{F}_i$	$\mathcal{F}'_i$
$ 0\rangle$	$ 0\rangle$
$c_{1i}^\dagger  0\rangle$	$c_{0i}^\dagger c_{1i}^\dagger  0\rangle$
$c_{2i}^\dagger  0\rangle$	$c_{0i}^\dagger c_{2i}^\dagger  0\rangle$
$c_{1i}^\dagger c_{2i}^\dagger  0\rangle$	$c_{1i}^\dagger c_{2i}^\dagger  0\rangle$
$a_{1i}^\dagger  0\rangle$	$c_{0i}^\dagger a_{1i}^\dagger  0\rangle$
$c_{1i}^\dagger a_{1i}^\dagger  0\rangle$	$c_{1i}^\dagger a_{1i}^\dagger  0\rangle$
$c_{2i}^\dagger a_{1i}^\dagger  0\rangle$	$c_{2i}^\dagger a_{1i}^\dagger  0\rangle$
$c_{1i}^\dagger c_{2i}^\dagger a_{1i}^\dagger  0\rangle$	$c_{0i}^\dagger c_{1i}^\dagger c_{2i}^\dagger a_{1i}^\dagger  0\rangle$
$\vdots$	$\vdots$

fer matrices have enhanced supersymmetry  $\text{osp}(2n+1|2n)$ . In Appendix A we use this enhanced SUSY to rederive the equality (1.4) for the Ising correlators.

Here we need some more notation. We introduce a parameter of the form of the Ising coupling,  $L$ , and its dual  $L^*$ , related to the probability  $p$ :

$$1 - 2p = \tanh L = e^{-2L^*}. \quad (3.1)$$

In terms of  $L$  the equation (1.3) of the NL is  $L = K$ . Below the NL  $L < K$ , and above the NL  $L > K$ .

Since the couplings  $K_{ij}$  are independent, we can average transfer matrices for different bonds separately. For a vertical transfer matrix this gives (recall that the disorder averages are denoted by square brackets)

$$T_{1i} = [T_{vi}] = \exp(-2K^* \hat{N}_{Si})(1 - p + p(-1)^{\hat{N}_{Si}}). \quad (3.2)$$

Note that after the averaging the translational invariance is restored, and this allows us to label the average transfer matrices by 1D (“light”) indices.

The value of the last factor in Eq. (3.2) depends on the value of  $\hat{N}_{Si}$  in the state, on which  $T_{1i}$  acts. For an even  $\hat{N}_{Si}$  it equals 1, for an odd  $\hat{N}_{Si}$  it gives  $1 - 2p = e^{-2L^*}$ . Then we can rewrite the operator (3.2) in a slightly different form. Namely, we introduce an additional (zeroth) fermionic state and operators  $c_{0i}^\dagger$ ,  $c_{0i}$  and consider the subspace  $\mathcal{F}'_i$  given by the following constraint:

$$\hat{N}'_{Si} = \hat{N}_{Si} + n_{0i} = \text{even}, \quad (3.3)$$

where  $n_{0i}$  is defined in analogy with  $n_{ai}$ :

$$n_{0i} = c_{0i}^\dagger c_{0i}. \quad (3.4)$$

That is, the number of fermions plus bosons on each site must be even. There is a one-to-one correspondence between the states in this subspace and the original Fock space  $\mathcal{F}_i$ . This correspondence is illustrated for the case  $n = 1$  in Table I. The grading in the space  $\mathcal{F}' = \otimes_i \mathcal{F}'_i$  is taken to be the same as that in  $\mathcal{F}$ , which was not the natural grading. However, we see that in a Fock space with a constraint of the form of Eq. (3.3), the number of bosons is odd if and only if the number of fermions is odd (this is true for each site and also for the

tensor product). Hence our grading on  $\mathcal{F}'$  is the same as the one obtained from the natural grading on the larger Fock space with  $2n+1$  fermion species, when restricted to the subspace. Now we can replace  $T_{1i}$  by the operator

$$T'_{1i} = \exp(-2K^* \hat{N}'_{Si} - 2L^* n_{0i}), \quad (3.5)$$

which has the same matrix elements in the constrained subspace  $\mathcal{F}'_i$  as  $T_{1i}$  had in the original space  $\mathcal{F}_i$  (between the corresponding states). From now on in this section we will denote transfer operators acting in the constrained spaces  $\mathcal{F}'$  by a prime. For a horizontal transfer matrix the averaging gives

$$T_{2i} = [T_{hi}] = \frac{\cosh(2K \hat{X}'_{Si} + L)}{\cosh L}. \quad (3.6)$$

To find the corresponding operator in  $\mathcal{F}'$  we need to establish some substitution rules for basic operators.

Single creation and annihilation operators like  $c_{1i}^\dagger$ , which are quite legitimate in the space  $\mathcal{F}_i$ , do not act within  $\mathcal{F}'_i$ . Using the correspondence between the states, given in Table I, it is easy to establish that in  $\mathcal{F}'_i$  the operator  $c_{1i}^\dagger$  must be replaced by the operator  $(c_{0i}^\dagger - c_{0i})c_{1i}^\dagger$ . However, this operator is bosonic (i.e., even with respect to the natural grading on  $\mathcal{F}'$ ), so in the tensor product  $\mathcal{F}' = \otimes_i \mathcal{F}'_i$  it will not have the anticommutation properties that  $c_{1i}^\dagger$  had in  $\mathcal{F}$ . To make it fermionic in the total space  $\mathcal{F}'$  we need to attach to it a string:

$$c_{1i}^\dagger \rightarrow (c_{0i}^\dagger - c_{0i})c_{1i}^\dagger \Sigma_{0i}, \quad (3.7)$$

$$\Sigma_{0i} = \prod_{j>i} (-1)^{n_{0j}}. \quad (3.8)$$

Repeating this argument for other operators, we obtain the following rules of substitution:

$$\begin{aligned} c_{ai}^\dagger &\rightarrow (c_{0i}^\dagger - c_{0i})c_{ai}^\dagger \Sigma_{0i}, \\ c_{ai} &\rightarrow (c_{0i}^\dagger - c_{0i})c_{ai} \Sigma_{0i}, \\ a_{\mu i}^\dagger &\rightarrow (c_{0i}^\dagger + c_{0i})a_{\mu i}^\dagger \Sigma_{0i}, \\ a_{\mu i} &\rightarrow (c_{0i}^\dagger + c_{0i})a_{\mu i} \Sigma_{0i}, \\ \bar{a}_{\mu i}^\dagger &\rightarrow (c_{0i}^\dagger + c_{0i})\bar{a}_{\mu i}^\dagger \Sigma_{0i}, \\ \bar{a}_{\mu i} &\rightarrow (c_{0i}^\dagger + c_{0i})\bar{a}_{\mu i} \Sigma_{0i}. \end{aligned} \quad (3.9)$$

As an alternative to looking at all the states, the correspondence can be established by verifying that the right-hand sides of these expressions have the same (anti-)commutators as the left-hand sides. It follows from these rules that any product of an even number of creation and/or annihilation operators in  $\mathcal{F}_i$  remains the same when going to  $\mathcal{F}'_i$ .

Now note what happens with  $x_{ai}$  [see Eq. (2.15)] upon transition from  $\mathcal{F} = \otimes_i \mathcal{F}_i$  to  $\mathcal{F}'$ :

$$\begin{aligned} 2x_{ai} &= (c_{ai}^\dagger - c_{ai})(c_{\alpha, i+1}^\dagger + c_{\alpha, i+1}) \\ &\rightarrow (c_{0i}^\dagger - c_{0i})(c_{ai}^\dagger - c_{ai})(-1)^{n_{0, i+1}}(c_{0, i+1}^\dagger - c_{0, i+1}) \\ &\quad \times (c_{\alpha, i+1}^\dagger + c_{\alpha, i+1}) \\ &= 4x_{0i}x_{ai}, \end{aligned} \quad (3.10)$$

where  $x_{0i}$  is defined similarly to Eq. (2.15):

$$x_{0i} = \frac{1}{2}(c_{0i}^\dagger - c_{0i})(c_{0, i+1}^\dagger + c_{0, i+1}) = i\eta_{0i}\xi_{0, i+1}. \quad (3.11)$$

With these substitution rules established, we can see that in the space  $\mathcal{F}'$  the operator corresponding to  $T_{2i}$  is given by

$$T'_{2i} = \frac{\cosh(2K \hat{X}'_{Si} + 2Lx_{0i})}{\cosh L}, \quad (3.12)$$

which easily follows from the substitution rule (3.10) and the fact that  $x_{ai}^2 = x_{0i}^2 = 1/4$ . From Eqs. (3.5) and (3.12) we see that the transfer matrices commute with the constraint, Eq. (3.3). In the full space of states that includes states of odd, as well as even, fermion plus boson number at each site, there are local  $Z_2$  operations given by  $(-1)^{\hat{N}'_{Si}}$ . This is therefore a gauge symmetry under which the allowed states and transfer operators (and also physical observables) must be invariant.

The forms (3.5) and (3.12) are very convenient for the discussion of the symmetry properties of our model on the NL. Indeed, we see that on the NL, where  $K=L$ , the operators  $T'_i$  become

$$T'_{1i} = \exp(-2K^* \hat{N}'_{Si}), \quad (3.13)$$

$$T'_{2i} = \frac{\cosh(2K \hat{X}'_{Si})}{\cosh K}, \quad (3.14)$$

where

$$\hat{X}'_{Si} = \hat{X}_{Si} + x_{0i} = i\eta_{ai}\xi_{a, i+1} - r_{ai}J_{\alpha\beta}q_{\beta, i+1}, \quad (3.15)$$

and the Latin subscripts from the beginning of the alphabet denote the fermionic indices running from 0 to  $2n$ .

On the NL, the expressions (3.13) and (3.14) have enhanced supersymmetry: they are now invariant under an  $\text{osp}(2n+1|2n)$  algebra. The generators of this algebra have a similar form as before, but involve the  $2n+1$  fermion operators:  $\Sigma_i(\xi_{ai}\xi_{bi} + \eta_{ai}\eta_{bi})$ ,  $\Sigma_i(q_{ai}q_{\beta i} + r_{ai}r_{\beta i})$ , and  $\Sigma_i(\xi_{ai}q_{\beta i} + \eta_{ai}r_{\beta i})$ . The last set of generators are the odd ones, with respect to our grading, or to the natural one on the Fock space of which  $\mathcal{F}'$  is a subspace; we have seen these are equivalent in the constrained subspace.

As anticipated in the Introduction, this enhanced continuous SUSY replaces the gauge symmetry of Nishimori, and the enhanced permutational symmetry of the replica approach, previously known to exist in Ising spin language on the NL.<sup>4,7</sup> The symmetry has many consequences, such as the equalities (1.4) among different correlation functions on the NL. In Appendix A we briefly show how these equalities may be obtained from the enhanced SUSY exhibited in this



section. We have also obtained the local  $\mathbb{Z}_2$  gauge symmetry anticipated in the Introduction.

#### IV. STRUCTURE OF THE SPACE OF STATES AND THE HAMILTONIAN LIMIT

In this section we first analyze (in Sec. IV A) the structure of the space of states of our quantum problem and then take the time continuum limit of our transfer matrices and obtain a quantum Hamiltonian describing our system. This has the form of a spin chain with irreducible representations of the symmetry algebra  $\text{osp}(2n+1|2n)$  at each site. Then in Sec. IV B we consider a more explicit construction of these irreducible representations.

##### A. Superspin chain and Hamiltonian limit

Let us consider the structure of the constrained space  $\mathcal{F}'_i$  [Eq. (3.3)], with its natural grading, under transformations of  $\text{osp}(2n+1|2n)$ . It is easy to see that  $\mathcal{F}'_i$  is not irreducible under this algebra. Rather, it has the structure of the tensor product of two irreducible spinors of  $\text{osp}(2n+1|2n)$ .

Indeed, let us consider first the fermionic replicas only, i.e., the replica approach where  $n \rightarrow 0$  in the end. Then we have the modified transfer matrices which are invariant under the orthogonal algebra  $\text{so}(2n+1)$ , and the subspace they act on is given at each site by the constraint

$$\hat{N}'_{Fi} = \hat{N}_{Fi} + n_{0i} = \text{even}. \quad (4.1)$$

This space has dimension  $2^{2n}$ , and, under  $\text{so}(2n+1)$ , it transforms as the tensor product of two spinors of  $\text{so}(2n+1)$ , each of dimension  $2^n$ . These two spinors can be identified as the spaces on which the two parts  $\xi_{ai}\xi_{bi}$ ,  $\eta_{ai}\eta_{bi}$  of the generators  $\xi_{ai}\xi_{bi} + \eta_{ai}\eta_{bi}$  act. The tensor product decomposes into irreducible representations of  $\text{so}(2n+1)$  corresponding to each even value of  $\hat{N}'_{Fi}$  in the range  $0-2n$  allowed by the constraint (4.1). Similarly, the orthogonal subspace  $\hat{N}'_{Fi} = \text{odd}$  is also a tensor product of spinors and has a similar decomposition.

When the bosons are included as in the SUSY approach, the two parts  $q_{ai}q_{bi}$ ,  $r_{ai}r_{bi}$  of the  $\text{sp}(2n)$  generators at a site  $i$  generate infinite-dimensional spinor representations of  $\text{sp}(2n)$  (sometimes known as metaplectic representations). When the fermions and bosons are combined together with the constraint that  $\hat{N}'_{Si}$  be even, the resulting space is a tensor product of irreducible spinors of  $\text{osp}(2n+1|2n)$ . The fact that a single such tensor product is involved is the nontrivial part of this statement, and is addressed further in Sec. IV B. These spinors comprise one lowest-weight representation of  $\text{osp}(2n+1|2n)$ , which we denote by  $R$ , and one highest-weight representation of  $\text{osp}(2n+1|2n)$ , which we denote by  $\bar{R}$ . Thus we may write  $\mathcal{F}'_i = R_i \otimes \bar{R}_i$ .

This organization of states suggests a picture of our model as a system of ‘‘superspins’’ [spinors  $R$  and  $\bar{R}$  of  $\text{osp}(2n+1|2n)$ ] sitting in pairs on the sites of the 1D lattice. It is convenient to combine the corresponding generators of  $\text{osp}(2n+1|2n)$  into square matrices consistent with reality

properties satisfied by the matrices of  $\text{osp}(2n+1|2n)$  in the defining representation. Namely, the generators of  $\text{osp}(2n+1|2n)$  acting in the representation  $R$  are combined into the superspin

$$G = \begin{pmatrix} \xi_a \xi_b - \frac{1}{2} \delta_{ab} & \xi_a q_\beta \\ iJ_{\alpha\gamma} q_\gamma \xi_b & iJ_{\alpha\gamma} q_\gamma q_\beta - \frac{1}{2} \delta_{\alpha\beta} \end{pmatrix}, \quad (4.2)$$

shown here in block  $[(2n+1)+2n] \times [(2n+1)+2n]$  form, and a similar matrix obtained from the generators of  $\text{osp}(2n+1|2n)$  acting in the representation  $\bar{R}$ :

$$\bar{G} = \begin{pmatrix} \eta_a \eta_b - \frac{1}{2} \delta_{ab} & \eta_a r_\beta \\ iJ_{\alpha\gamma} r_\gamma \eta_b & iJ_{\alpha\gamma} r_\gamma r_\beta - \frac{1}{2} \delta_{\alpha\beta} \end{pmatrix}. \quad (4.3)$$

The generators of the global SUSY of the system are now, in matrix form,  $\Sigma_i(G_i + \bar{G}_i)$ .

Next we take a time-continuum (Hamiltonian) limit of the transfer matrices. To do that we actually have to start with an anisotropic RBIM, where the vertical and horizontal couplings take different values of  $K$ . In that case we can arrange for the situation when in the vertical matrix  $K^*, L^* \ll 1$ , and in the horizontal one  $K, L \ll 1$  (we continue to use these notations, so that  $K^*, L^*$  are no longer related to  $K, L$  by the duality relation). Then we can expand the horizontal transfer matrix (3.12) as

$$\begin{aligned} T'_{2i} &\simeq 1 + 2K^2 \hat{X}_{Si}^2 + 4KL \hat{X}_{Si} x_{0i} \\ &\simeq \exp(2K^2 \hat{X}_{Si}^2 + 4KL \hat{X}_{Si} x_{0i}). \end{aligned} \quad (4.4)$$

Then we can combine all  $T'_{1i}$  and  $T'_{2i}$  into a single ‘‘evolution’’ operator in imaginary time with the Hamiltonian

$$H_S = \sum_i (h(\hat{N}_{Si} + \lambda^* n_{0i}) - k(\hat{X}_{Si}^2 + 2\lambda \hat{X}_{Si} x_{0i})), \quad (4.5)$$

where we introduced

$$h = 2K^*, \quad k = 2K^2, \quad \lambda^* = \frac{L^*}{K^*}, \quad \lambda = \frac{L}{K}. \quad (4.6)$$

The parameters  $\lambda$  and  $\lambda^*$  introduce anisotropy in the couplings among the replicas. The Hamiltonian (4.5) may be rewritten in a very suggestive form using the superspins  $G$  and  $\bar{G}$ :

$$H_S = \sum_i (h(\hat{N}_{Si} + \lambda^* n_{0i}) + k \text{str} \Lambda \bar{G}_i \Lambda G_{i+1}), \quad (4.7)$$

where

$$\Lambda = \text{diag}(\lambda, \mathbb{1}_{4n}) \quad (4.8)$$

is a diagonal matrix representing the anisotropy in the superspin space. The supertrace here, denoted  $\text{str}$ , is over the  $4n+1$ -dimensional space as above, with a plus for diagonal matrix elements in the  $2n+1$ -dimensional block, and a minus for those in the remaining  $2n$ -dimensional block. With this definition, the expressions reduce to the ordinary trace

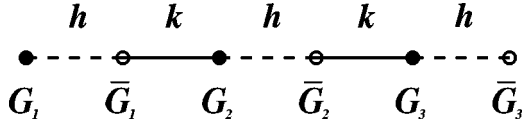


FIG. 2. The graphical representation of the Hamiltonian (4.7) on the split sites. Superspins in the representations  $R$  and  $\bar{R}$  are shown as filled and empty circles. The two types of coupling,  $h$  and  $k$ , are indicated.

for the replica formalism where only the  $\xi$ 's and  $\eta$ 's are kept (and then  $n \rightarrow 0$ ), without an overall change in sign.

Now we should notice that in the anisotropic version of the RBIM, there are in general two couplings,  $K_x, K_y$ , and two parameters for the probabilities,  $L_x, L_y$ . The Nishimori condition becomes two equations,  $K_x = L_x, K_y = L_y$ . Thus the NL is replaced by a two-dimensional surface (or 2-surface) in the four-dimensional space, and so does not divide the phase diagram into two pieces. We will continue to refer to this as the NL. There is presumably a line on this surface at which a transition occurs. The complete phase boundary is three-dimensional, and the multicritical behavior is found on a 2-surface on this 3-surface. The multicritical line on the NL presumably lies in the multicritical 2-surface on the phase boundary. Even though the transfer matrices do not have the larger SUSY everywhere on that 2-surface, we presume by universality that the higher SUSY fixed point theory, to which the multicritical point on the NL flows, controls the entire multicritical 2-surface because anisotropy such as we have introduced usually does not affect the universality class.

In any case, on the NL  $\lambda = \lambda^* = 1$ , and we obtain an  $\text{osp}(2n+1|2n)$ -invariant (or isotropic) Hamiltonian

$$H_S = \sum_i (h \hat{N}_{S_i} + k \text{str } \bar{G}_i G_{i+1}). \quad (4.9)$$

Equation (4.9) has the form of a superspin chain with the alternating lowest- and highest-weight representations  $R$  and  $\bar{R}$ , and corresponding superspin operators  $G$  and  $\bar{G}$ . We can better represent this by splitting the original sites into pairs of split sites. This is shown in Fig. 2. The Hamiltonian (4.9) has two types of couplings on the alternating bonds. Both these couplings are antiferromagnetic in nature. This means that the lowest energy state for a given bond is the singlet of  $\text{osp}(2n+1|2n)$  contained in the decomposition of the tensor product of the representations  $R$  and  $\bar{R}$ .

### B. Unconstrained representation of superspins

The picture of the split sites carrying irreducible representations is very attractive, but does suffer from one difficulty at present. This is that we obtained the representations by introducing an additional zeroth fermion  $c_{0i}, c_{0i}^\dagger$ , together with a constraint which refers to both the split sites that comprise the original site. Here and in Appendix B we show how to avoid this by the use of a different construction. In

Sec. VI we will extend this approach further, introducing a further representation which involves a constraint on each split site.

The representations  $R$  and  $\bar{R}$  can be constructed using complex fermionic and bosonic operators without constraints, that is essentially in  $\mathcal{F}_i$ . For the simplest case of the  $\text{osp}(3|2)$  algebra this is done in Appendix B. Here we note that the complex fermions and bosons used in the construction are related to the real ones (apart from the zeroth fermion) on the split sites introduced so far in the following manner:

$$f_{\mu i} = \frac{\xi_{\mu i} + i \xi_{\mu+n, i}}{\sqrt{2}}, \quad \bar{f}_{\mu i} = \frac{i \eta_{\mu i} + \eta_{\mu+n, i}}{\sqrt{2}}, \quad (4.10)$$

$$b_{\mu i} = \frac{q_{\mu i} + i q_{\mu+n, i}}{\sqrt{2}}, \quad \bar{b}_{\mu i} = \frac{i r_{\mu i} + r_{\mu+n, i}}{\sqrt{2}}. \quad (4.11)$$

All of these operators are canonical, except for the  $\bar{b}$  bosons, which are ‘‘negative norm:’’

$$[\bar{b}_{\mu i}, \bar{b}_{\nu j}^\dagger] = -\delta_{ij} \delta_{\mu\nu}. \quad (4.12)$$

In terms of these complex bosons and fermions the quadratic forms appearing in the transfer matrices (2.32) look especially uniform:

$$\begin{aligned} \hat{X}_{S_i} &= f_{\mu, i+1}^\dagger \bar{f}_{\mu i}^\dagger + \bar{f}_{\mu i} f_{\mu, i+1} + b_{\mu, i+1}^\dagger \bar{b}_{\mu i}^\dagger + \bar{b}_{\mu i} b_{\mu, i+1}, \\ \hat{N}_{S_i} &= f_{\mu i}^\dagger \bar{f}_{\mu i}^\dagger + \bar{f}_{\mu i} f_{\mu i} + b_{\mu i}^\dagger \bar{b}_{\mu i}^\dagger + \bar{b}_{\mu i} b_{\mu i}. \end{aligned} \quad (4.13)$$

In this form the subalgebra  $\text{osp}(2n|2n)$  on a split site is generated by bilinears as before, now of the form  $f^2, (f^\dagger)^2, f^\dagger f, b^2, \dots, f^\dagger b, \dots$ , for  $R$ . We saw earlier that the expressions for  $\hat{X}_{S_i}$  and  $\hat{N}_{S_i}$  are invariant under this  $\text{osp}(2n|2n)$  algebra. However, the extension to  $\text{osp}(2n+1|2n)$  is modified since we do not use the zeroth fermion  $c$ . Instead, the additional generators include string operators such as  $(-1)^{n_f}$ , see Appendix B, where expressions for the generators of  $\text{osp}(2n+1|2n)$  in the case  $n=1$  are given. It is then clear that the states on the unsplit sites decompose into a single tensor product of irreducibles  $R$  and  $\bar{R}$  as claimed. The string operators again correspond to the difference in grading between that natural in  $\mathcal{F}$  and our choice, which agrees with the natural one in  $\mathcal{F}'$ . With our choice, the (anti-)commutation relations obeyed by the generators of the larger  $\text{osp}(2n+1|2n)$  SUSY are consistent with the stated grading, as discussed in more detail in Appendix B.

An advantage of the unconstrained representation of the states on the split sites is that it makes the pure limit  $p=0$  transparent. The pure Ising problem in the anisotropic time-continuum limit in this representation gives a nearest-neighbor ‘‘hopping’’-type Hamiltonian for fermions and bosons, which is a sum over  $i$  of  $\hat{N}_{S_i}$  and  $\hat{X}_{S_i}$ , with coefficients. This is a lattice version of the Dirac fermion and its SUSY partner, the so-called  $\beta$ - $\gamma$  system of bosonic ghosts. However, in the general disordered case, there are additional terms which we expressed previously using the zeroth fer-

mion. In Appendix B we show how the  $G_i \bar{G}_i$  terms in the Hamiltonian can be expressed in the present language. The other term, which becomes  $\hat{N}'_{Si}$  on the NL, is much more difficult to express in this language, and we return to this problem in Sec. VI.

A slight subtlety involved in the definition of the complex bosons is the following. If we express them in terms of the  $a$  bosons, which diagonalize the form  $\hat{N}_{Bi}$ , Eq. (2.30), we obtain a singular Bogoliubov rotation:

$$b_{\mu i} = \frac{a_{\mu i} + \bar{a}_{\mu i}^\dagger}{\sqrt{2}}, \quad \bar{b}_{\mu i} = \frac{a_{\mu i}^\dagger - \bar{a}_{\mu i}}{\sqrt{2}}. \quad (4.14)$$

The singularity of this transformation is seen in the fact that the formal expression for the  $b, \bar{b}$  vacuum (on a single unsplit site  $i$ ), defined by  $b|\bar{0}\rangle = \bar{b}|\bar{0}\rangle = 0$ , is

$$|\bar{0}\rangle \propto \exp\left(-\sum_{\mu} a_{\mu i}^\dagger \bar{a}_{\mu i}^\dagger\right) |0\rangle, \quad (4.15)$$

where  $|0\rangle$  is the  $a, \bar{a}$  vacuum defined in Eq. (2.31) and leads to a series for the squared norm  $\langle \bar{0}|\bar{0}\rangle$  which is not convergent. A way out of this problem is to regularize the Bogoliubov rotation (4.14) as follows:

$$\begin{aligned} b_{\mu i} &= \cos \phi a_{\mu i} + \sin \phi \bar{a}_{\mu i}^\dagger, \\ \bar{b}_{\mu i} &= \sin \phi a_{\mu i}^\dagger - \cos \phi \bar{a}_{\mu i}, \end{aligned} \quad (4.16)$$

where  $\phi = \pi/4 - \omega/2$  with  $0 < \omega \ll 1$ . With such regularized transformation, the  $b, \bar{b}$  vacuum

$$|\bar{0}\rangle = \frac{1}{\cos^n \phi} \exp\left(-\tan \phi \sum_{\mu} a_{\mu i}^\dagger \bar{a}_{\mu i}^\dagger\right) |0\rangle \quad (4.17)$$

is well-defined and normalized to 1.

If we now use the regularized relations (4.16), the expression for  $\hat{N}_{Bi}$  becomes (to first order in  $\omega$ )

$$\hat{N}_{Bi} = b_{\mu i}^\dagger \bar{b}_{\mu i}^\dagger + \bar{b}_{\mu i} b_{\mu i} + \omega(n_{bi} + n_{\bar{b}i}) - n, \quad (4.18)$$

with bosonic number operators defined as

$$n_{bi} = b_{\mu i}^\dagger b_{\mu i}, \quad n_{\bar{b}i} = -\bar{b}_{\mu i}^\dagger \bar{b}_{\mu i}. \quad (4.19)$$

The fermionic sector in our formulation is finite dimensional, and there are no similar problems with the fermions. However, to maintain the exact cancellations between fermions and bosons, we will modify the definition of  $f$  and  $\bar{f}$  similarly to that for the bosons. One effect of this is that the supersymmetric analog of Eq. (4.17) lacks the factor  $1/\cos^n \phi$ . Then the  $\text{osp}(2n|2n)$ -invariant combination (4.13) takes the form

$$\begin{aligned} \hat{N}_{Si} &= f_{\mu i}^\dagger \bar{f}_{\mu i}^\dagger + \bar{f}_{\mu i} f_{\mu i} + b_{\mu i}^\dagger \bar{b}_{\mu i}^\dagger + \bar{b}_{\mu i} b_{\mu i} \\ &+ \omega(n_{fi} + n_{\bar{f}i} + n_{bi} + n_{\bar{b}i}), \end{aligned} \quad (4.20)$$

where the fermionic number operators are defined in a natural way:

$$n_{fi} = f_{\mu i}^\dagger f_{\mu i}, \quad n_{\bar{f}i} = \bar{f}_{\mu i}^\dagger \bar{f}_{\mu i}. \quad (4.21)$$

The term first-order in  $\omega$  breaks the SUSY down to  $\text{gl}(n|n)$ , which is still enough SUSY to ensure cancellation of fermions and bosons.

We can make this term appear more natural by the following considerations. It is a regularizer which suppresses contributions to the partition function from high fermion and especially boson numbers on any site. We can introduce it in a more symmetric way by inserting  $\exp \sum_i \omega(n_{fi} + n_{\bar{f}i} + n_{bi} + n_{\bar{b}i})$  between all the  $T_{1i}$ 's and  $T_{2i}$ 's in the partition function; to first order in  $\omega$ , the effect is the same. Such an insertion is a precaution similar to that often used in network models and nonlinear sigma models of localization. The  $\omega$  term represents a nonzero imaginary part of the frequency in those problems, and as in the present case breaks the symmetry to a subgroup. In the superspin chain language, the operator which  $\omega$  multiplies is one component of the staggered magnetization, the order parameter for the chain. The term, with  $\omega \rightarrow 0$ , is used just in case this develops a spontaneous expectation value, since it picks a direction for the ordering in superspin space and cuts off infrared divergences. Note that the state with each site in the vacuum state for the  $f$ 's,  $\bar{f}$ 's,  $b$ 's, and  $\bar{b}$ 's is the Néel state corresponding to such order, and is invariant under the subalgebra  $\text{gl}(n|n)$ . The symmetry-breaking term will be important in Sec. V.

## V. DIMERIZED LIMIT AND THE ONE-DIMENSIONAL CASE

This section lies somewhat outside of the main line of our development; the latter continues in Sec. VI. Here we consider our model in the vicinity of the NL deep in the low-temperature phase. In terms of the superspin chain with the Hamiltonian (4.7), in this phase we have  $h \ll k$ . Then in the zeroth approximation we may neglect the  $h$  couplings completely. Then the chain (4.7) is broken into disconnected pairs of superspins. The Hamiltonian for one such pair is

$$H_k = 4 \text{ str } \Lambda \bar{G} \Lambda G = 4 \text{ str } \Lambda G \Lambda \bar{G}, \quad (5.1)$$

where the coupling constant  $k$  (overall energy scale) was taken to be equal to 4 for later convenience, and we used the cyclic property of the supertrace. We can try to solve this Hamiltonian and hope to infer some information about the low temperature phase of our original model. However, we wish to sound a note of caution: we are considering a certain double limit of the original lattice Ising model, first the anisotropic limit, then the ‘‘low  $T$ ’’ limit,  $h \ll k$ . It is not entirely clear that this really represents the low  $T$  limit of the nearly isotropic Ising model, where we pass to low  $T$  close to the NL, and perhaps then go to the anisotropic limit.

We will make use of the realization of the representations  $R$  and  $\bar{R}$  in Fock spaces of unconstrained fermions and bosons. For simplicity we will work out the details for  $\text{osp}(3|2)$  only. In this case the necessary construction of  $R$

and  $\bar{R}$  and the invariant products of superspins is given in Appendix B. From it we obtain

$$H_k = \lambda J - J^2, \quad (5.2)$$

with

$$J = f^\dagger \bar{f}^\dagger + \bar{f} f + b^\dagger \bar{b}^\dagger + \bar{b} b. \quad (5.3)$$

We anticipate that the eigenstates of the Hamiltonian  $H_k$  may have arbitrarily large bosonic occupation numbers, and we may encounter convergence problems typical in such cases. These are avoided, however, if we remember the  $\omega$  term, discussed in Sec. IV B. As explained there, it plays the role of a symmetry-breaking regulator that picks a direction for ordering, similar to  $S_1^z - S_2^z$  for the problem of two anti-ferromagnetically coupled  $su(2)$  spins. Thus we add to our Hamiltonian the term

$$H_\omega = \omega(n_f + n_b + n_{\bar{f}} + n_{\bar{b}}). \quad (5.4)$$

The resulting Hamiltonian

$$H = H_k + H_\omega \quad (5.5)$$

is identical to the one studied by Balents and Fisher in a one-dimensional localization problem [see Eq. (3.31) in Ref. 28]. This is the problem of spinless fermions on a 1D lattice with random hopping amplitudes described by the Hamiltonian

$$\mathcal{H} = - \sum_n t_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n). \quad (5.6)$$

The continuum limit of this model gives left and right moving spinless Dirac fermions with random mixing between them:

$$\mathcal{H}_c = \int dx \Psi^\dagger (-i\sigma^z \partial_x + V(x)\sigma^y) \Psi, \quad (5.7)$$

where  $\Psi(x)$  is a two-component spinor field,  $\sigma^i$  are Pauli matrices, and the random potential is Gaussian with nonzero mean and variance:

$$[V(x)] = V_0,$$

$$[(V(x) - V_0)(V(x') - V_0)] = 2D\delta(x - x'). \quad (5.8)$$

The generating functional for the Green's functions of this Hamiltonian at a given energy  $\epsilon + i\eta$  may be supersymmetrized in the standard way. After disorder averaging the  $x$  coordinate may be interpreted as imaginary time, and the two components of the fermion can be viewed as labeling two sites, which correspond to our split sites. This leads to an effective quantum Hamiltonian, which is exactly given by Eq. (5.5) with

$$\lambda = V_0/D, \quad \omega = \eta - i\epsilon. \quad (5.9)$$

The 1D model with the Hamiltonian (5.7), and related models, have a long history, and most of the relevant work is concisely summarized in Ref. 32. In particular, the density of

states for this problem was found for  $\lambda=0$  in 1953 by Dyson,<sup>33</sup> and for arbitrary  $\lambda$  by many authors.<sup>34</sup> The mathematically equivalent problem of diffusion in a 1D random medium was studied by Bouchaud *et al.*<sup>35</sup> The density of states  $\rho(\epsilon)$  behaves at small energies as

$$\rho(\epsilon) \propto \frac{1}{\epsilon |\ln^3 \epsilon|}, \quad \lambda=0, \quad (5.10)$$

$$\rho(\epsilon) \propto \epsilon^{\lambda-1}, \quad \lambda>0. \quad (5.11)$$

In the superspin language the density of states  $\rho(\epsilon)$  is related to the expectation value of some operator in the ground state of the Hamiltonian (5.5).<sup>28</sup> Namely, it is proportional to the staggered component  $h_2 - \bar{h}_2$  of the superspin (see Appendix B)

$$\rho(\omega) \propto \langle 1 - n_f - n_{\bar{f}} \rangle \propto \omega^{\lambda-1}. \quad (5.12)$$

This quantity measures the amount of the symmetry breaking in the ground state of two superspins. From the last equation it follows that the symmetry is spontaneously broken on the NL (which is a point in the 1D model) and below it. Moreover, below the NL, where  $\lambda < 1$ , the density of states (the order parameter of the spin chain) diverges as  $\omega \rightarrow 0$ . On the NL it is constant, and above the NL it vanishes as a power of  $\omega$ .

Because the SUSY representations are the same, we have in fact shown that in the 1D off-diagonal disorder problem, *there is a larger SUSY  $osp(2n+1|2n)$  at the point  $\lambda=1$ .* This has not been noticed previously to our knowledge. This suggests that such Nishimori points, lines, etc., may be common in some classes of random fermion problems. We also note here that in the 1D classical RBIM, which of course has no finite  $T$  phase transition, there is a Nishimori point at which the correlation identities Eqs. (1.4) hold. That problem can be represented using fermions on one unsplit site with the  $T_{1i}$  transfer matrices only, which are of the  $h$ -coupling type, in contrast to the model considered here, and is easily solved in this language.

## VI. FINAL REPRESENTATION AND THE GENERALIZED MODEL

In this section we continue the general consideration of the RBIM problem. Here we focus our attention on the NL, that is, we consider the  $osp(2n+1|2n)$ -invariant Hamiltonian (4.9). First we analyze and solve the problem of finding a way to describe the term  $\hat{N}_{Fi}$  in the spaces  $R, \bar{R}$  on the split sites. The problem is solved by using another representation in a space  $\mathcal{F}'$ , and the spaces can be viewed as representations of a larger SUSY algebra,  $osp(2n+2|2n)$ . Using only terms of the form of the two couplings we have already seen, we then introduce a more general nearest-neighbor superspin chain, and discuss its phase diagram for reference in the following sections.

First let us note that Eq. (4.9) is somewhat schematic. Let us again consider the fermionic replica formalism with  $n \rightarrow 0$  instead of SUSY. The term  $\hat{N}_{Si}'$  is then replaced by



$\hat{N}'_{Fi}$ . According to our general discussion of how to map operators in  $\mathcal{F}$  into  $\mathcal{F}'$  (see Sec. III),

$$\hat{N}'_{Fi} = i\eta_{ai}\xi_{ai} + n + \frac{1}{2} \quad (6.1)$$

is correct as it stands in  $\mathcal{F}'$ . Even though the operator  $\hat{N}'_{Fi}$  is perfectly legitimate, it does not admit any simple expression in terms of the  $\text{so}(2n+1)$  generators  $G_i$  and  $\bar{G}_i$ . We would like to write it as a sum of products of operators in the spinor representations  $R, \bar{R}$  on the split sites. Of course, individual fermion operators  $\xi_{ai}, \eta_{ai}$  do not commute with the constraint, and cannot be used. Instead they must be replaced by  $\mathbb{Z}_2$ -invariant operators. As explained in Sec. III, we can find operators in  $\mathcal{F}'$  with the anticommutation properties of the fermions in  $\mathcal{F}$  for the components *other than the zeroth*. These are fermion bilinears times a string; see Eq. (3.9). A general proof that it is impossible to find a set of operators with the anticommutation relations of the *full* set of real fermion operators  $\eta_{ai}$  and  $\xi_{ai}$  in the space  $\mathcal{F}'$  is to notice that they should form a Clifford algebra with  $2N(2n+1)$  generators, where  $N$  is the number of unsplit sites in the chain. This Clifford algebra has a single non-trivial representation of dimension  $2^{N(2n+1)}$ . This space is the same as an unconstrained Fock space for  $2n+1$  complex fermion operators at each unsplit site, i.e.,  $\mathcal{F}'$  but without the constraints. The total number of states in  $\mathcal{F}'$  is only  $2^{2Nn}$  because of the  $N$  constraints. So the operators we require cannot have the anticommutation relations of free fermions for all the sites. Indeed, in our grading on  $\mathcal{F}$ , single fermion operators are even, and so would be expected to obey commutation relations from a SUSY point of view. In  $\mathcal{F}'$  there are corresponding fermion bilinears, like Eq. (3.9) but without strings, and these do commute on different sites (as mentioned already in Sec. III).

We can also try the unconstrained representation. Then again, we can represent each of the  $2n$  real fermions  $\eta_{ai}$  and  $\xi_{ai}$  on each split site using Eqs. (4.10), and the resulting Clifford algebra for  $2N$  split sites yields the correct number of states. This description carries over easily to the SUSY version. But the above proof shows that no matter what strings or other factors we introduce into a construction of operators, we cannot produce the anticommutation relations for  $2n+1$  real fermions at each split site, and we are no nearer writing  $\hat{N}'_{Si}$  as a product of simple expression in  $R$  and  $\bar{R}$ . What we would have to do is map the problematic part  $n_{0i}$  of the operator back from  $\mathcal{F}'$  to  $\mathcal{F}$ . Because of the constraint in the former space, the resulting operator (still in the fermionic replica formalism) must equal 1 when  $\hat{N}_{Fi}$  is odd, 0 when  $\hat{N}_{Fi}$  is even, or similarly for  $\hat{N}_{Si}$  in the SUSY formalism. It is not clear how we would write this as a coupling of the two split sites at  $i$ .

There is nonetheless a way out of this problem, motivated by the following observation. If we consider a single spinor representation  $R$  of  $\text{so}(2n+1)$  (thus in the fermionic replica formalism once more), then it is in fact possible to find operators with the anticommutation relations of the real fermi-

ons  $\xi_a$ . These are the generators of a Clifford algebra with an odd number  $2n+1$  of generators, which has an irreducible representation of dimension  $2^n$  (the familiar  $2 \times 2$  Pauli matrices are the case  $n=1$ ). The commutators  $i/2[\xi_a, \xi_b]$  of these operators are the generators of  $\text{so}(2n+1)$ , as we have already seen. The operators  $\xi_a$  transform as a vector of  $\text{so}(2n+1)$ . If we now consider these  $\text{so}(2n+1)$  generators together with the  $\xi_a$  (divided by  $\sqrt{2}$ ), then we may use the fact that the *commutators* of these operators (or matrices) together obey the relations of the generators of  $\text{so}(2n+2)$ , and the spinor  $R$  can be identified with one of the two distinct irreducible spinor representations of dimension  $2^n$  of  $\text{so}(2n+2)$ . This construction can also be applied to the representations  $\bar{R}$  [which for  $\text{so}(2n+1)$ , though not for  $\text{osp}(2n+1|2n)$ , is isomorphic to  $R$ ]. This construction also extends easily to the many-site problem, *by taking the operators now replacing  $\xi_a$  to commute on different sites*. We may therefore write down our term  $\hat{N}'_{Fi}$  as a sum of products of bilinears of these operators, and this can also be extended to the SUSY construction, using operators with the relations of  $q_\alpha, r_\alpha$ , on each site, but which anticommute on different sites.

Thus we have learned that our spaces of states  $R, \bar{R}$  at alternate sites can be viewed as irreducible spinor representations of  $\text{osp}(2n+2|2n)$  (but note that this algebra is not a symmetry of our Hamiltonian or transfer matrices so far). There are two inequivalent lowest-weight spinor representations  $R_e, R_o$  (for ‘even’ and ‘odd’) of  $\text{osp}(2n+2|2n)$ , in which all states can be assigned positive norm-squares. We can identify  $R$  with, say,  $R_e$ . Similarly,  $\bar{R}$  can be identified with a highest-weight spinor  $\bar{R}_e$ , which is dual to  $R_e$ , and in which the inner product is indefinite, since as we have seen states with an odd number of  $\bar{b}$  bosons have negative squared norms.

Viewing the spaces of states in this way, we can give yet another explicit construction, with which we can finally write the operators  $\hat{N}'_{Fi}$  and  $\hat{N}'_{Si}$  in a simple way. It is convenient to keep much of the notation the same as before. We introduce additional complex fermions  $f_{0i}, \bar{f}_{0i}$  to the set  $f_{\mu i}$  of Eq. (4.10), and define a space  $\mathcal{F}'' = \otimes_i \mathcal{F}''_i$  consisting of the states on the split sites with an even number of fermions plus bosons:

$$n_{bi} + n_{fi} + f_{0i}^\dagger f_{0i} = \text{even}, \quad (6.2)$$

$$n_{\bar{b}i} + n_{\bar{f}i} + \bar{f}_{0i}^\dagger \bar{f}_{0i} = \text{even}. \quad (6.3)$$

It is clear that such states are in one-one correspondence with those in the unconstrained representation. The construction of the correspondence is similar to that for the states in the spaces  $\mathcal{F}$  and  $\mathcal{F}'$  in Sec. III. All the states can be obtained from the vacuum, which is the lowest- (highest-) weight state in  $R_e$  ( $\bar{R}_e$ ), by the action of the bilinears in the creation operators. Then in addition to Eq. (4.10) we also define

$$f_{0i} = \frac{\xi_{0i} + i\xi'_{0i}}{\sqrt{2}}, \quad \bar{f}_{0i} = \frac{i\eta_{0i} + \eta'_{0i}}{\sqrt{2}}. \quad (6.4)$$

Now the  $\text{so}(2n+2)$  generators on a single site are replaced by

$$\frac{i}{2}[\xi_{ai}, \xi_{bi}], \quad \frac{i}{2}[\xi'_{0i}, \xi_{ai}], \quad (6.5)$$

where the first set, again, spans the subalgebra  $\text{so}(2n+1)$ , and the second set transforms as a vector under this subalgebra. There are similar expressions for  $\bar{R}_e$ . We emphasize that the operators  $\xi_{ai}$ ,  $\xi'_{0i}$ ,  $\eta_{ai}$ ,  $\eta'_{0i}$  obey canonical anticommutation relations, while  $q_{ai}$ ,  $r_{ai}$  obey canonical commutation relations, of the same form as in Eqs. (2.17) and (2.24) [with negative norm states appearing in connection with the  $r$ 's, see Eqs. (4.11) and following]. Our choice of grading is again equivalent in the constrained subspaces  $\mathcal{F}'_i$  to their natural grading as subspaces of Fock spaces. Finally, in the representation in  $\mathcal{F}'$ , the operator  $\hat{N}'_{Si}$  undergoes the replacement

$$\begin{aligned} \hat{N}'_{Si} &= i \eta_{ai} \xi_{ai} - r_{ai} J_{\alpha\beta} q_{\beta i} + \frac{1}{2} \\ &\mapsto 2 \eta'_{0i} \eta_{ai} \xi_{ai} \xi'_{0i} + 2i \eta'_{0i} r_{ai} J_{\alpha\beta} q_{\beta i} \xi'_{0i} + \frac{1}{2}. \end{aligned} \quad (6.6)$$

These results may also be established by passing directly from the (averaged) unconstrained representation to the final representation by using a substitution similar to Eq. (3.9) but applied here to the split sites.

We can now organize the generators of  $\text{osp}(2n+2|2n)$  in superspins, similar to Eqs. (4.2) and (4.3), and including the additional odd generators:

$$G' = \begin{pmatrix} 0 & \xi'_0 \xi_b & \xi'_0 q_\beta \\ \xi_a \xi'_0 & \xi_a \xi_b - \frac{1}{2} \delta_{ab} & \xi_a q_\beta \\ iJ_{\alpha\gamma} q_\gamma \xi'_0 & iJ_{\alpha\gamma} q_\gamma \xi_b & iJ_{\alpha\gamma} q_\gamma q_\beta - \frac{1}{2} \delta_{\alpha\beta} \end{pmatrix}, \quad (6.7)$$

$$\bar{G}' = \begin{pmatrix} 0 & \eta'_0 \eta_b & \eta'_0 r_\beta \\ \eta_a \eta'_0 & \eta_a \eta_b - \frac{1}{2} \delta_{ab} & \eta_a r_\beta \\ iJ_{\alpha\gamma} r_\gamma \eta'_0 & iJ_{\alpha\gamma} r_\gamma \eta_b & iJ_{\alpha\gamma} r_\gamma r_\beta - \frac{1}{2} \delta_{\alpha\beta} \end{pmatrix}. \quad (6.8)$$

Note that these  $\text{osp}(2n+2|2n)$  superspins contain the original  $\text{osp}(2n+1|2n)$  superspins  $G$  and  $\bar{G}$  as submatrices. The odd generators are those containing an odd number of fermion operator factors, or equivalently an odd number of boson operator factors. With the help of the  $\text{osp}(2n+2|2n)$  superspins, both terms in the Hamiltonian (4.9) may be written in a unified way:

$$h \hat{N}'_{Si} = \lim_{k \rightarrow 0} \left( \text{str} C G'_i C \bar{G}'_i + \frac{h}{2} \right), \quad (6.9)$$

$$k \text{str} \bar{G}'_i G'_{i+1} = \lim_{h \rightarrow 0} \left( \text{str} C \bar{G}'_i C G'_{i+1} + \frac{h}{2} \right), \quad (6.10)$$

where we have introduced a  $4n+2$ -dimensional diagonal matrices of coupling constants

$$C \equiv C(h, k) = \text{diag}(hk^{-1/2}, k^{1/2} \mathbb{1}_{4n+1}), \quad (6.11)$$

and the supertrace  $\text{str}$  in this space is defined in the same way as the previous  $\text{str}$ . These two terms represent two different  $\text{osp}(2n+1|2n)$ -invariant products of two subsets of the  $\text{osp}(2n+2|2n)$  generators. It should be clear that the representation in  $\mathcal{F}'$  can also be used off the NL, by giving certain terms different coefficients.

It is now natural to consider a generalized Hamiltonian

$$\begin{aligned} H &= \sum_i (\text{str} C(h_A, k_A) G'_i C(h_A, k_A) \bar{G}'_i \\ &+ \text{str} C(h_B, k_B) \bar{G}'_i C(h_B, k_B) G'_{i+1}) + \frac{h_A + h_B}{2} N, \end{aligned} \quad (6.12)$$

parametrized by four coupling constants. In such a Hamiltonian both types of  $\text{osp}(2n+1|2n)$ -invariant couplings appear on every bond between the split sites. In addition, they are staggered between the two sublattices of bonds  $A$  and  $B$  of our chain. Our NL Hamiltonian (4.9) is a particular extreme limit, where on alternate bonds one or the other coupling is zero. It is obtained from Eq. (6.12) for the special values of the parameters

$$h_A = h, \quad k_A = 0, \quad h_B = 0, \quad k_B = k. \quad (6.13)$$

We believe it may be helpful to consider these more general models, since they are so closely related to that for the RBIM, and use only couplings that appear anyway in the RBIM case; however, we emphasize that it may not be possible to obtain these models as anisotropic limits of random fermion or network models. When

$$h_A = k_A, \quad h_B = k_B, \quad (6.14)$$

the model is invariant under the whole of  $\text{osp}(2n+2|2n)$ . We should note that in principle we can also consider this generalization in the discrete imaginary time model, and also off the NL, where, however, the breaking of the symmetries would lead to twice as many parameters. The additional parameters would generalize  $\lambda$ ,  $\lambda^*$  in Sec. IV A, and there would be one for each of  $k_A$ ,  $k_B$ ,  $h_A$ ,  $h_B$ , a total of eight parameters in the Hamiltonian. In particular, another model due to Cho and Fisher<sup>13</sup> fits into this general description, as we will see in Sec. VII.

The Hamiltonian (6.12) contains four parameters, but since the overall energy scale is unimportant, the phase diagram can be plotted in terms of the three independent ratios of parameters. We will consider only positive values of all couplings, though negative values may also give well-defined models. The phase diagram can then be drawn in a symmetrical manner in a three-dimensional tetrahedron, as a portion of projective space (see Fig. 3). Each face of the tetrahedron is defined by the vanishing of one of the four parameters. The opposite vertex is where that parameter goes to infinity, or equivalently the other three all go to zero.

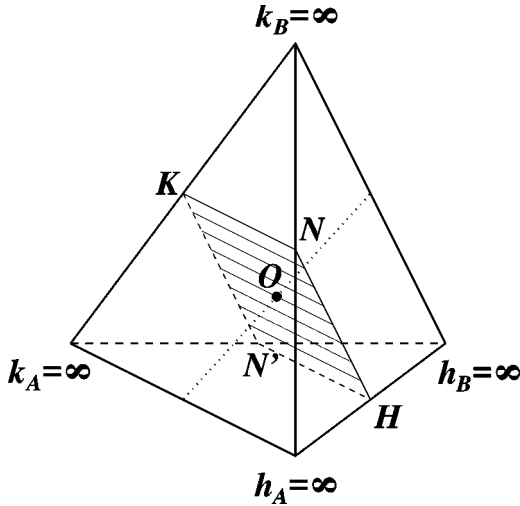


FIG. 3. A possible phase diagram of the generalized Hamiltonian (6.12), discussed in detail in the text.

The edges of the tetrahedron correspond to models with two vanishing couplings. For example, the vertical edge connecting the vertices  $h_A = \infty$  and  $k_B = \infty$  represents the Hamiltonian (4.9) for the RBIM on the NL [to avoid confusion, recall that the whole discussion is a generalization of the NL, since all the models have the larger  $\text{osp}(2n+1|2n)$  SUSY]. There is another such line represented by the horizontal edge connecting the vertices  $h_B = \infty$  and  $k_A = \infty$ . These two Hamiltonians are related by a reflection through a lattice site. Such an operation is thus a symmetry of the whole diagram, which interchanges  $A$  with  $B$ . The line  $k_A = k_B$ ,  $h_A = h_B$  is invariant under this operation, and the operation acts as a  $180^\circ$  rotation about this line. On each NL, there is a multicritical point  $N$  and its image  $N'$ .

The edges where both nonzero couplings are on the same sublattice of bonds (e.g.,  $A$ ) represent the two extreme cases of fully dimerized chains, which have a gap in their energy spectrum. By analogy with other antiferromagnetic (super-)spin chain models, we expect that the regions adjacent to these lines are also gapped phases. There must be at least one phase transition between these two extremes. One way to see this is to consider a chain with open ends, and an even number of split sites. In one phase the dimers extend all the way to the ends of the chain, in the other a single superspin is left unbonded with a neighbor at each end. This corresponds to a chiral edge degree of freedom in the 2D lattice model. A phase transition must occur to change the number of such boundary spins or edge channels, assuming these survive off the edges of the tetrahedron. We will assume that, as expected on the NL in the RBIM, there is a single transition between the two phases. Then there must be a phase-boundary surface between those two edges, indicated schematically (since its exact position is unknown) by the shaded surface in Fig. 3. The points  $N$  and  $N'$  are two vertices of this rhomboidal surface, which also contains the line of reflection symmetry. However, we note that an intermediate phase, in place of some portion of the critical surface, is also possible, though this is not expected on the NL in the RBIM.

The two other edges of the tetrahedron are where either

only the  $k$ 's are nonzero, or only the  $h$ 's, and we denote these models “ $k$ -only” and “ $h$ -only.” They intersect the phase boundary (if there is a unique transition on these edges) at points labeled  $K$  and  $H$  in Fig. 3 (no confusion should result from this notation). In these models, the reflection symmetry and the assumption of a single transition implies that  $K$  is  $k_A = k_B$ , and  $H$  is  $h_A = h_B$  (and other parameters are zero).

The tetrahedral phase diagram also contains the line given by Eq. (6.14) where the generalized Hamiltonian (6.12) has the  $\text{osp}(2n+2|2n)$  symmetry. This line, shown dotted in Fig. 3, intersects the critical surface at a critical point  $O$  (black dot), where all four couplings are equal. Again, this point is unique if we assume there is a single transition on this line; it is  $k_A = k_B = h_A = h_B$ .

## VII. CHO-FISHER, $k$ -ONLY, AND $h$ -ONLY MODELS

In this section we consider a model studied numerically by Cho and Fisher in Ref. 13. This is a network model, similar to the Chalker–Coddington network<sup>14</sup> describing the integer quantum Hall transition, but with only real matrices, and was intended to represent the RBIM problem. We show that the Ising model can be represented exactly as a network model, and that the Cho–Fisher model does not represent the RBIM. Instead, it can be mapped to some of the generalized Hamiltonians without enhanced SUSY, introduced in Sec. VI.

The Cho–Fisher model is a network model, intended to capture the universal aspects of the point  $N$  in the RBIM, which can be viewed as a generalization of the 1D model discussed in Sec. V. It was constructed as a generalization of the model whose action is given in Eq. (5.7), in which the two components of the fermion are replaced by any number of sites in a 1D chain, with random nearest-neighbor hopping that generalizes the  $\sigma^y$  term in Eq. (5.7), and, in general, different parameter values  $V_0$  and  $D$  on alternate bonds in the chain. Then use of replicas or SUSY to perform the disorder average leads to a generalization of the quantum (super-)spin Hamiltonian  $H_k + H_\omega$ , Eq. (5.2) in Sec. V (we will disregard here the regularizing term  $H_\omega$ ), in which the same form of coupling appears for each pair of nearest neighbors, but with the coefficient of  $H_k$  taking two values  $k$ ,  $k'$  on alternate bonds, and similarly for  $\lambda$ ,  $\lambda'$ . We emphasize that at this stage we are using the unconstrained representation of the space of states of the chain. Cho and Fisher specialized to the case  $k = k'$  (i.e., node independent disorder strength), and went back from their time-continuum model to a discrete-time (network) model, similar to that in Ref. 14, in order to perform numerical calculations. They claimed that their network model has a multicritical point in its phase diagram with critical properties remarkably similar to those of the multicritical point on the NL. In particular, the critical exponents along the two scaling axes near the multicritical point were found to be fairly close numerically to the ones known for the point  $N$  in the RBIM from the high-temperature expansion of Singh and Adler.<sup>12</sup> Also, simulation in Cho's thesis<sup>13</sup> of a network model that corresponds precisely to the RBIM (as we will explain) gave similar values. Now we

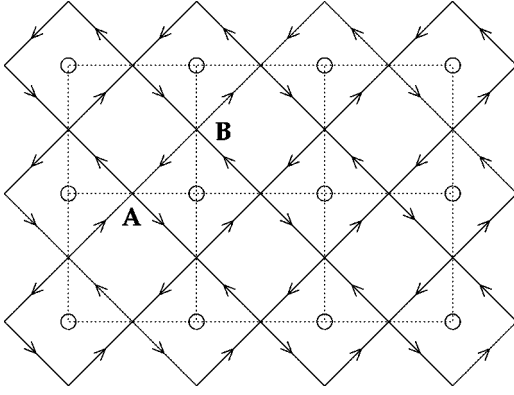


FIG. 4. Relation of the Ising model and the network model. Ising spins are located at the open circles, and bonds are shown dotted. Solid lines with arrows form the “medial graph,” on which the network model is defined. Examples of nodes on each of the two sublattices, corresponding to the horizontal and vertical bonds, are labeled *A*, *B*, respectively. In this paper, we consider periodic boundary conditions in the vertical direction, and free in the horizontal, as shown here.

wish to point out that it is possible to relate network models more directly to the transfer-matrix formulation for fermions, as in the Ising model. First we describe the network models. A portion of the network is shown in Fig. 4, where the solid lines with arrows are where the particles propagate. The particles propagate in discrete time, at each time step moving to the next link in the “forward” direction shown by the arrows, and therefore turning either left or right at each time. The evolution is described by a unitary S-matrix which gives the amplitudes for turning either right or left at each node.<sup>14</sup> This can be replaced by a one-particle transfer matrix, which adds one row of nodes to the system, evolving the wave function of the particle upwards in the figure. In the Cho–Fisher model, the one-particle transfer matrix for one node has the form

$$M = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}. \quad (7.1)$$

The parameter  $\theta$  for each node is random, taking values  $\pm|\theta|$  with probabilities  $1-p$ ,  $p$ , independently. Also, the magnitudes of  $\theta$  can be staggered, taking different values  $|\theta_A|$ ,  $|\theta_B|$  on the two sublattices of nodes labeled *A*, *B* in Fig. 4. The so-called isotropic case is where  $\sinh|\theta_A| \sinh|\theta_B| = 1$ . This leaves a one-parameter family of models; in the original network model<sup>14</sup> a transition occurred when  $|\theta_A| = |\theta_B|$ , the “self-dual” point.

To exhibit a relation with the Ising model transfer matrices we use a second-quantized formulation of the network as a noninteracting fermion field theory. The evolution in the imaginary-time (vertical) direction is described by a transfer matrix constructed from the one-particle one. We can write this by drawing on earlier work.<sup>37</sup> Though the latter was on a different model, the basic Eq. (2) in that work is applicable for any transfer matrix, with matrix elements

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (7.2)$$

Thus in our case  $\alpha = \delta = \cosh \theta$  and  $\beta = \gamma = \sinh \theta$ . Using only one species of fermions, and dropping the bosons in Eq. (2) of Ref. 37 since we will not be averaging here, we replace  $f_1$  in Ref. 37 by  $f$ ,  $f_2$  by  $\bar{f}'$  (where  $f$  and  $\bar{f}'$  obey canonical anticommutation relations) and obtain

$$V = : \exp[\tanh \theta (f^\dagger \bar{f}' + \bar{f}'^\dagger f)] : (\cosh \theta)^{n_f + n_{\bar{f}'}}. \quad (7.3)$$

Here the colons  $:\dots:$  indicate normal ordering with destruction operators to the right. Then, after making the particle–hole transformation  $\bar{f}'^\dagger = \bar{f}'$ , we can prove the identity

$$V = e^{\theta(f^\dagger \bar{f}' + \bar{f} f)} \quad (7.4)$$

by verifying that all matrix elements of the two expressions are equal. But this now has the form of the fermionic representation of the squared Ising model (i.e.,  $n = 1$ ), as in Eqs. (2.32) and (4.13) (dropping the bosons in the latter), up to constant factors in the vertical case. This means that the split sites on a single row correspond to one row of links of the network. The relation of the original Ising lattice and the network model is as shown in Fig. 4; in particular, the two sublattices of nodes *A* and *B* correspond to horizontal and vertical bonds, respectively. The relationship of second-quantized transfer matrices holds true for arbitrary values of  $\theta$  at each node, and also remains true when bosonic partners are introduced in preparation for averaging.

The important corollary to this is that for the transfer matrices of the sort appropriate for the horizontal bonds (labeled *A* in Fig. 4), we have<sup>15</sup>  $2K = \theta$ . The Cho–Fisher network model takes the parameter  $\theta$  at the nodes to have independent random signs. Hence it is precisely equivalent to the use of transfer matrices (2.9) and (2.10), with the binary distribution of the type (1.2) for both the horizontal couplings  $K_{i,i+\hat{x}}$  and the dual  $\tilde{K}_{i,i+\hat{y}}$  to the vertical couplings. Note that the Cho–Fisher model is not in fact isotropic, even when the (magnitudes of the real parts of the)  $K$ ’s and the probabilities  $p$  on the horizontal and vertical links are the same, which is what we termed isotropic above; this is because of the way the random signs are introduced. Another popular parametrization for the network models uses the S-matrix at each node,<sup>14</sup> where the S-matrix is a real orthogonal matrix in the present case, with one of its off-diagonal matrix elements (say, the amplitude for turning right) denoted  $t = \sin \phi$ . In this case the equivalence to the Ising model squared is  $\tanh 2K = \sin \phi$  for the horizontal couplings.

Since negative dual couplings  $\tilde{K}$  correspond to complex Ising couplings  $K$ , the Cho–Fisher model does not faithfully reproduce the RBIM with  $\pm K$  couplings. Instead, in replicated fermion language, both types of bond are represented after averaging by transfer matrices of the horizontal type. One might imagine that this is the  $k$ -only model, with parameters  $\lambda_A$ ,  $\lambda_B$  included, so that the  $\text{osp}(2n+1|2n)$  SUSY is present when  $\lambda = \lambda' = 1$ . But in fact, carefully following our mapping leads to different forms for the two bonds, *A*, *B*. It is necessary once again to use the final ( $\mathcal{F}'$ ) representation,



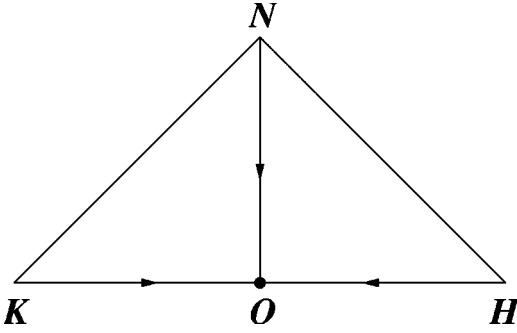


FIG. 5. The critical surface of the generalized Hamiltonian after reduction using symmetry. The arrows indicate our suggested RG flows to a fixed-point theory with the same larger SUSY as the model at  $O$ .

and to pass to it directly from the unconstrained representation. We find that, while the bonds corresponding to the horizontal bonds in the Ising model involve the real fermions  $\xi_{0i}$ ,  $\eta_{0i}$ , in the  $k$ -coupling terms as in Eq. (6.10), for the vertical bonds those fermion operators are replaced by  $\xi'_{0i}$ ,  $\eta'_{0i}$ . For  $\lambda$  or  $\lambda' = 1$ , these terms are invariant under an  $\text{osp}(2n+1|2n)$  SUSY, but these are distinct  $\text{osp}(2n+1|2n)$  sub-superalgebras of  $\text{osp}(2n+2|2n)$  in the two cases, and so the Hamiltonian does not possess a global  $\text{osp}(2n+1|2n)$  SUSY [though there is of course still  $\text{osp}(2n|2n)$ ]. These models therefore lie elsewhere in our space of fully generalized Hamiltonians with (in general) only  $\text{osp}(2n|2n)$  SUSY.

We note here that by rotating the Cho–Fisher network by  $90^\circ$  we obtain after averaging (using the  $\mathcal{F}''$  representation) a model which resembles the  $h$ -only model, but again has different  $\text{osp}(2n+1|2n)$  SUSYs for the two types of bond. Because taking the anisotropic limit usually does not affect the universality class of the critical phenomena, the resulting spin chain model should have the same critical phenomena as the one described above for the Cho–Fisher model.

### VIII. FIXED POINTS AND NONLINEAR SIGMA MODELS

In this section we first speculate that a single fixed point or universality class controls much of the phase boundary in the tetrahedral phase diagram. Then we discuss the nonlinear sigma models that are related to our spin chains, and calculate, at weak coupling, the RG beta functions for the coupling constants. The results support the hypothesis of a flow towards the higher SUSY as at point  $O$  in the phase diagram. Finally, we discuss the nonlinear sigma (and related) models for the more general, lower SUSY [ $\text{osp}(2n|2n)$ ], or class D, random fermion problems.

For the generalized Hamiltonian with  $\text{osp}(2n+1|2n)$  SUSY we argued (on the assumption that there is a single transition surface) that the phase boundary is a rhombus, and further there is a reflection symmetry in the superspin chain, which, on this surface, acts as a reflection. Then the phase boundary is a triangle, with points  $K$ ,  $N$ ,  $H$  at the vertices, as shown in Fig. 5. The point  $O$ , at which the model has the larger SUSY algebra  $\text{osp}(2n+2|2n)$ , is now at the middle of one side of the triangle.

Under the RG, the  $\text{osp}(2n+2|2n)$ -invariant model must flow to a critical quantum field theory (presumably, a conformal field theory) which also has the larger SUSY. Other models, represented by other points in Fig. 5, such as  $N$ ,  $K$ ,  $H$  may flow to some other fixed point theories of lower [ $\text{osp}(2n+1|2n)$ ] SUSY, and it is of course the fate of  $N$  that concerns us in the RBIM model problem. If we must make a guess as to the structure of the flows and fixed points, the simplest guess is the one that involves the fewest fixed points. Since there must be a fixed point theory with  $\text{osp}(2n+2|2n)$  SUSY, the simplest guess is then that the whole critical surface flows to this fixed point. This is schematically illustrated in Fig. 5 by the arrows, which are intended to indicate that all models flow to the fixed point corresponding to  $O$  (note that the models at  $N$ ,  $K$ ,  $H$ , and  $O$  are not themselves fixed points of the RG). If correct, this would imply that the critical exponents are the same at all points in the critical surface shown in Figs. 3 or 5. In particular,  $N$ ,  $K$ , and  $H$  would have the same exponents. We can also imagine other scenarios in which  $N$ ,  $K$ , and  $H$  flow to a common fixed point, or to different ones, that do not have  $\text{osp}(2n+2|2n)$  SUSY. It is certainly possible that (one or both) perturbations away from  $O$  on the surface shown are relevant; we are suggesting that they are both irrelevant. In the absence of any understanding of the conformal field theory of the  $\text{osp}(2n+2|2n)$  or other fixed points in this system, we cannot prove or disprove our suggestion. There are, however, other systems in which an analogous effect occurs, as we will discuss below.

Now we introduce the nonlinear sigma models that should correspond to the superspin chains, and should have transitions in the same universality class or classes. First we utilize a standard relation between antiferromagnetic spin chains and nonlinear sigma models (see Ref. 38 for a fairly general discussion). We define an antiferromagnetic (super-)spin chain as having an irreducible representation at each site, alternating between some (say) lowest-weight representation  $R$  and its dual, say  $\bar{R}$ . The Hamiltonian should be something close to the Heisenberg form which is the invariant bilinear form in the generators of the symmetry algebra, with the antiferromagnetic coupling that for a single pair of spins leads to the singlet ground state (possible because we chose the dual representations). Then the correspondence states that there is a nonlinear sigma model with a certain target (super-)manifold, which can be obtained from the representation  $R$ . The manifold is the same coset space that appears as the coadjoint orbit of  $R$ , or in coherent-state path integral constructions of  $R$ . Put simply, this is the manifold swept out by acting on either a lowest- or highest-weight of  $R$  with all possible (super-)group elements. The long-wavelength action of the nonlinear sigma model in  $1+1$  dimensions contains only terms allowed by symmetry with two derivatives. These comprise the usual “kinetic” type terms, and also possible “ $\theta$ -terms” (this is not the same parameter  $\theta$  we used in Sec. VII). The derivation is controlled by considering a sequence of representations  $R$  with the lowest weight going to infinity in the weight space, like the size of the spin in  $\text{SU}(2)$  going to infinity. Then the reciprocals of the kinetic couplings have magnitudes proportional to the lowest weight, so the non-

linear sigma model is weakly coupled and meaningful in the semiclassical limit. Also, in the absence of staggering of the couplings in the spin chain,  $\theta$  is proportional to the lowest weight with a coefficient  $\pi$  in a suitable normalization (which is such that the bulk physics is periodic when  $\theta \rightarrow \theta + 2\pi$ ). A  $\theta$  term exists and is nontrivial whenever the second homotopy group  $\pi_2$  of the target manifold is nontrivial. More generally, a  $\theta$ -term involves a two-form on the target manifold (i.e., a magnetic field for a charged particle moving on the manifold) that is invariant under the symmetry, and this always exists in this construction because it is part of the coherent-state construction of the representation  $R$  also. Non-compact factors in the manifold are topologically trivial, but the term described always produces boundary effects related to a boundary spin or edge states.<sup>39</sup> The nonlinear sigma model that results from this correspondence in many cases has a phase transition at  $\theta = \pi$ , when the target manifold has nontrivial  $\pi_2$ .

In our case, the target manifold for our general models on the NL would be, for fermionic replicas,  $SO(2n+1)/U(n)$ , or in the SUSY formalism,  $O\text{Sp}(2n+1|2n)/U(n|n)$ . The precise meanings of these coset spaces should be defined as the orbits of our spinors  $R$ . The group in the denominator arises in each case as the invariance group of the lowest weight state. In the SUSY case,  $U(n|n)$  corresponds to the superalgebra  $\mathfrak{gl}(n|n)$  which leaves the vacuum at each split site invariant, and the notation indicates that the ordinary group it contains is the compact form,  $U(n) \times U(n)$ . The manifold underlying the supermanifold is thus  $SO(2n+1)/U(n) \times \text{Sp}(2n, \mathbf{R})/U(n)$ , where the latter factor is non-compact. These (super-) spaces are homogeneous spaces, but not symmetric (super-)spaces. This implies that, in a general nonlinear sigma model for these target manifolds, there is more than one coupling in the kinetic terms,<sup>40</sup> in fact, there are two (see Appendix C). The kinetic term is constructed from the metric on the target manifold, and this metric is not unique, up to a constant factor, unless the manifold is a symmetric space. Otherwise it is a sum of two or more pieces. However, as manifolds (without a choice of metric), these spaces are the same as  $SO(2n+2)/U(n+1)$  [respectively  $O\text{Sp}(2n+2|2n)/U(n+1|n)$ ] (for the former, this is discussed in Ref. 41). This is connected with the fact that, as graded vector spaces,  $R$  is the same as the representation  $R_e$ . The latter manifolds are symmetric (super-)spaces, and there is a unique kinetic coupling. The two couplings in the former point of view are related to the  $k$  and  $h$  terms. When  $k$  and  $h$  on each nearest-neighbor bond are equal, the higher symmetry implies that the two coupling constants in the kinetic terms are such that the  $\text{osp}(2n+2|2n)$  invariant kinetic term is obtained. At the same time,  $\theta$  can be varied by staggering the couplings. Counting parameters, there are four in the spin chain as we have seen, one of which is the overall energy scale which can be ignored, and also the magnitude of the lowest weight. The latter controls the magnitude of the kinetic coupling constant. Ignoring that, the nonlinear sigma model has in general one ratio of kinetic couplings, and  $\theta$ , one parameter less than in the spin chain. However, so far in the nonlinear sigma model we have assumed Lorentz invariance, whereas in the continuum limit of the spin chain this

need not hold. In fact, in the sigma model, the same symmetry considerations imply that each of the two pieces of the kinetic term can be further divided into two terms, which are the second derivatives in the two orthogonal directions in the 2D space, and if Lorentz invariance is not required, all the terms can have different coefficients. That is, a different velocity can apply to the two kinds of spin waves in the spin chain;<sup>42</sup> the two kinds of spin waves correspond to the decomposition of the small fluctuations around the perturbative vacuum into two irreducible representations of the symmetry group  $U(n)$  [or  $U(n|n)$ ] (see Appendix C). Since the overall energy scale, or one velocity, is a redundant parameter, this leaves one additional parameter, the ratio of velocities, as required. Note, however, that if we instead consider the continuum limit of an isotropic network model, such as the RBIM, only one velocity can occur. Therefore we would expect that the universality classes of the transitions would be isotropic and have a unique velocity.

Next, it is natural to raise the question of the RG flow of the two couplings in the  $\text{osp}(2n+1|2n)$ -invariant nonlinear sigma models. At least in perturbation theory the corresponding beta functions that describe such a flow are the same as those for the models with ordinary target manifolds  $SO(2n+1)/U(n)$ , with  $n \rightarrow 0$ . In Appendix C we have computed the beta functions to one-loop order, neglecting the possibility of more than one velocity (in some analogous situations, the ratio of velocities has been shown to renormalize to one,<sup>42</sup> and we expect the same to occur here, as also argued in the last paragraph). The metric on the target manifold is parametrized by two parameters, say  $\eta_1$  and  $\eta_2$ , which appear linearly in the metric, and the inverses of these parameters are the couplings which are small at weak coupling, where perturbation theory is valid. The perturbation expansion has  $1/\eta_1$ ,  $1/\eta_2$  for each propagator,  $\eta_1$  or  $\eta_2$  for each interaction vertex, so naively one ends up with a perturbation expansion in, say,  $1/\eta_1$ , with each term containing powers, positive or negative, of the ratio of  $\eta_1$  to  $\eta_2$ . In fact, only non-negative powers of this ratio appear in the beta functions; see Appendix C. The net power (counted with signs) of  $\eta_1^{-1}$ 's and  $\eta_2^{-1}$ 's corresponds to the number of loops in the Feynman diagrams, as usual. For convenience we will use the parameters

$$\eta_1 = \frac{1}{g}, \quad \eta_2 = \frac{1}{2xg}. \quad (8.1)$$

In this parametrization,  $x=1$  is the point with  $\text{osp}(2n+2|2n)$  SUSY [or  $\text{so}(2n+2)$  in the replica version]. Because of the higher symmetry at  $x=1$ , that line should flow onto itself under RG. Then our one-loop result for the RG flows is ( $l$  is the logarithm of the length scale, as usual)

$$\frac{dg}{dl} = 2g^2(x^2 - 1) + \mathcal{O}(g^3), \quad (8.2)$$

$$\frac{dx}{dl} = -2gx(x-1) + \mathcal{O}(g^2). \quad (8.3)$$

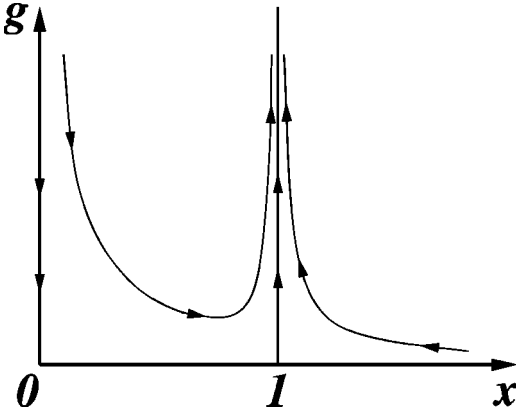


FIG. 6. Sketch of perturbative renormalization group flows for the couplings  $g$ ,  $x$ , including two-loop effects, for the nonlinear sigma model with target space  $\text{OSp}(2n+1|2n)/\text{U}(n|n)$ , including a typical flow line for  $g \neq 0$ ,  $x \neq 0$ , 1.

At  $x=1$ , the one-loop result is zero, so this line is a line of fixed points, to this order. This agrees with the one-loop result for the  $\text{osp}(2n+2|2n)$ -invariant model; the beta function to two-loop order, obtained as the  $n \rightarrow 1$  limit of that for the  $\text{SO}(2n)/\text{U}(n)$  model [see Eq. (C10)], is

$$\frac{dg}{dl} = 4g^3 + \mathcal{O}(g^4). \quad (8.4)$$

Thus it vanishes to one-loop order, but not to two-loop order. At two-loop order,  $g$  flows towards strong coupling (see Fig. 6). The one-loop flows for  $x \neq 1$  take  $x$  closer to 1, and (except for  $x=0$ ) the flows starting at  $g \neq 0$  never reach  $g=0$ . Instead they flow to the region  $x \approx 1$  where the one-loop terms vanish, and the two-loop term cannot be neglected. Since the two-loop term in  $dg/dl$  at  $x=1$  is positive, all flows from weak coupling eventually go towards large  $g$ , with  $x$  approaching 1, except when  $x=0$ . On the latter line,  $g$  flows towards weak coupling. As discussed in Appendix C (but here in SUSY language), on this line the larger SUSY has spontaneously broken, and the system is described by the  $\text{OSp}(2n|2n)/\text{U}(n|n)$  nonlinear sigma model [there is an additional global degree of freedom, described by a point on  $\text{OSp}(2n+1|2n)/\text{OSp}(2n|2n)$ , a “supersphere,” on which the larger SUSY algebra acts]. Flows that begin at small, nonzero  $x$  eventually go to strong coupling. This generates very large crossover lengths, due to the very slow flows near  $x=1$ ,  $g=0$ , where the first nonzero term is at two-loop order; the length scale at which  $g$  becomes of order one is of order  $\exp[1/(2g_0x_0) + 1/(8g_0^2x_0^2)]$ , in units of a short distance cutoff, where  $g_0$  and  $x_0$  are the bare values of  $g$  and  $x$ , and it was assumed that  $x_0$  and  $g_0x_0$  are both small (see Appendix C).

As usual, the perturbative results do not depend on  $\theta$ , but such dependence can be expected nonperturbatively. For the sigma model with  $\text{osp}(2n+2|2n)$  SUSY, and for  $x \neq 0$ , the flows go towards strong coupling, and it is highly plausible, based on our experience with transitions (such as the integer quantum Hall transition) with such behavior of the couplings, that there is a unique fixed point at strong coupling  $g$

and at  $\theta = \pi \pmod{2\pi}$ . Hence we expect that the spin chain at point  $O$  has the same critical theory as the  $\text{osp}(2n+2|2n)$ -invariant sigma model. It is also quite plausible, based on the behavior of the flows, that at least models that map onto the nonlinear sigma models with  $\text{osp}(2n+1|2n)$  SUSY also flow to the same critical theory, when  $\theta = \pi \pmod{2\pi}$ . It is still possible that in our generalized spin chains, points other than  $O$  do not flow to the  $\text{osp}(2n+2|2n)$  critical theory, but it is plausible that there is a nontrivial neighborhood of  $O$  on the critical surface that does. This possibility may seem more plausible if we point out that in some other cases (without SUSY), a similar phenomenon is believed to occur.<sup>42</sup> It is of course less clear that distant points such as  $N$ ,  $K$ , and  $H$  flow to the same theory. Since the spin chain models typically start at bare couplings of order 1, we can almost rule out any flow to the weak-coupling regime of the  $\text{osp}(2n+1|2n)$  or  $\text{osp}(2n+2|2n)$ -invariant nonlinear sigma models (analogous to that in the lower-SUSY  $\text{osp}(2n|2n)$  nonlinear sigma model<sup>18–21</sup>) because that regime is not stable under the RG. A flow to weak coupling is only possible by tuning a parameter, corresponding to putting  $x=0$  or  $g=0$  in the weak-coupling analysis. A natural guess is that the  $h$ -only models might satisfy one of these conditions. This might even occur for a range of values of the staggering, corresponding to changing  $\theta$  away from  $\pi \pmod{2\pi}$  in the nonlinear sigma model, since the value of  $\theta$  is irrelevant (or formally, exactly marginal) at weak coupling. We have been unable to see why any of these models should satisfy such a condition exactly. However, it may be that one of them lies close to  $x=0$ . In that case, the RG flows take them close to  $g=0$ , and since the flows to strong coupling pass near  $x=1$ ,  $g=0$ , where the first nonzero term is at two loops, the crossover length could be very large. That is, very large systems would be needed to see the true asymptotic critical behavior. Another possibility is that these models lie at bare values  $x > 1$ ,  $g$  very small, which again yields a large crossover length. Since we have a two-parameter space of critical models, we may expect to be able to tune  $g$  or  $x$  small somewhere in this space. However, arguments presented elsewhere<sup>22</sup> show that the RBIM, and hence the point  $N$ , cannot flow to the weak coupling region.

The leading alternative scenario seems to be that although most points in the phase diagram flow to the  $\text{osp}(2n+2|2n)$ -invariant fixed point, the multicritical point  $N$  on the true NL may be a distinct universality class, and the perturbation off this point in the phase diagram may be a relevant one that causes a flow to the  $\text{osp}(2n+2|2n)$ -invariant fixed point. Clearly, we cannot answer here the question of which of these scenarios is correct. But the self-duality apparent in the  $\text{osp}(2n+2|2n)$ -invariant model at its critical point, as manifested by the reflection symmetry about a split site in the chain, and the significant lack of it in the RBIM, which instead has the special “symmetry” property that the Ising couplings are all real,<sup>22</sup> suggests that this alternative scenario may be correct.

There is one further point to make about the spin chains and nonlinear sigma models that applies off the NL. In that case the SUSY is broken to  $\text{osp}(2n|2n)$ . It will be convenient here to make use of the replica formalism, in which the



language and notation are simpler and standard, but the ideas extend also to the supergroups, since the additional bosons and  $\text{Sp}(2n, \mathbf{R})$  symmetry, and the odd generators, do not change the form of the argument. In the higher SUSY Hamiltonians, including that for the NL, the global symmetry group can be seen to be  $\text{SO}(2n+1)$ , and making any of the  $\lambda$  parameters  $\neq 1$  reduces the symmetry group to  $\text{O}(2n)$ , not just  $\text{SO}(2n)$  [there seems to be no accepted notation in the supergroup case for the distinction analogous to that between  $\text{O}(N)$  and  $\text{SO}(N)$ , nor for that between  $\text{SO}(N)$  and its covering group  $\text{Spin}(N)$ , which we ignore here]. Furthermore, the representations  $R, \bar{R}$  are reducible under  $\text{SO}(2n)$ ; they split into two nonisomorphic irreducible spinors, each of dimension  $2^{n-1}$ . These two spinors correspond to even and odd numbers of fermions in the unconstrained representation (see Appendix B). However, under  $\text{O}(2n)$ ,  $R, \bar{R}$  do not split;  $\text{O}(2n)$  has irreducible representations of dimension  $2^n$  [this is related to the fact that  $\text{O}(2n)$  is not a direct product of  $\text{SO}(2n)$  with  $\mathbf{Z}_2$ , unlike the case of  $\text{O}(2n+1)$ ]. Thus we will still call the models spin chains, since they involve irreducible representations of their symmetry group,  $\text{O}(2n)$ .

When we consider the corresponding nonlinear sigma models, via the usual correspondence, we naturally consider the orbit of the lowest weight in  $R$  under  $\text{O}(2n)$ . Due to the disconnected nature of  $\text{O}(2n)$ , as opposed to  $\text{SO}(2n)$ , this orbit  $\text{O}(2n)/\text{U}(n)$  falls into two disconnected pieces, which are both of the form  $\text{SO}(2n)/\text{U}(n)$  as manifolds. Similar statements hold for the supermanifolds in the SUSY formalism.

In a recent paper<sup>21</sup> on the class D of random matrix problems, which is the same symmetry class as the RBIM fermion problem we are considering, it was emphasized that the target manifold of the nonlinear sigma model that describes it is  $\text{O}(2n)/\text{U}(n)$  in the replica formalism, which has two connected components, corresponding to those of the group  $\text{O}(2n)$  [or the corresponding supergroup  $\text{OSp}(2n|2n)$ ]. This opens a possibility not usually considered for nonlinear sigma models, that the configurations include fluctuations (i.e., domains) where the sigma model field is on different components of the target manifold. This implies that additional parameters, beyond the usual couplings like  $g, \theta$  for continuous deformations of the field, must appear in the model to describe the domain walls; for example, a fugacity per unit length of domain wall. When the fugacity is small, there are essentially no domain walls, and the model would reduce to that with target space  $\text{SO}(2n)/\text{U}(n)$ .

In our approach, we have arrived at spaces of states that correspond to both parts of the target manifold, and further the spin chain Hamiltonians contain in general eight parameters. Therefore our spin chain models describe a strong-coupling version of the physics of the nonlinear sigma model with domain walls included. These models include the pure Ising model, and weak-disorder, limits. Note that the latter are not accessible simply as the strong coupling,  $g \rightarrow \infty$ , limit of the  $\text{SO}(2n)/\text{U}(n)$  nonlinear sigma model (compare Ref. 19).

What we have found in this paper is that the states in the SUSY description can be viewed not only as domains of two

“phases,” but that the discrete (Ising-like) degree of freedom, which labels which phase [component of the target manifold, or irreducible spinor of  $\text{SO}(2n)$ ] a point in 2D space is in, can be replaced by additional continuous variables. These continuous degrees of freedom turn the model into a nonlinear sigma model with symmetry  $\text{SO}(2n+1)$  [or SUSY  $\text{osp}(2n+1|2n)$ ] broken by certain terms in the action, or else a strong-coupling version of this, at least near the NL. This may be of future use in uncovering the physics of these general class D problems, not only the RBIM. The replicated spin chains for  $\text{O}(2n)$  at nonzero  $n$  have not been considered previously (except for the  $n=1$  case, the usual XXZ model), and are also of interest in their own right. Note finally that in our earlier discussion of  $\text{SO}(2n+1)$ - and  $\text{SO}(2n+2)$ -invariant models, the representations of the stated groups were irreducible, the corresponding target manifolds were connected, and no analogous domain walls were possible.

## IX. CONCLUSION

In this paper we applied the supersymmetry (SUSY) method to analyze an Ising model with a binary distribution of random bonds (RBIM). The Nishimori line (NL) on the phase diagram of the model is a line with the enhanced SUSY  $\text{osp}(2n+1|2n)$ . On the rest of the phase diagram the model has only  $\text{osp}(2n|2n)$  SUSY. The enhanced SUSY on the Nishimori line allows us to rederive the identities (1.4) among various correlation functions. More generally, we have shown that the transition on the NL has very strong analogies with the integer quantum Hall effect transition, and other random fermion problems in 2D, such as the spin quantum Hall transition, which can also be modeled by (super-)spin chains with alternating dual irreducible representations at the sites, and staggered couplings. The conformal field theories of the critical points are mostly unknown at present. We emphasize that, in view of our results and those of Ref. 7, the fixed-point conformal field theory of the multicritical point in the RBIM with a generic distribution for the bonds (not only those satisfying the Nishimori condition) must have at least  $\text{osp}(2n+1|2n)$  SUSY, and this is a requirement for any future proposal for a conformal field theory of the multicritical point within the SUSY formulation. We have also demonstrated that such higher SUSY points occur in other problems, such as a 1D model, and probably elsewhere. After analyzing the phase diagram of generalized Hamiltonians with the same enhanced SUSY as the NL, we suggested that the transitions in many or all of these more general 2D models are in a universality class with a still larger SUSY,  $\text{osp}(2n+2|2n)$ . This hypothesis is supported to some extent by the weak-coupling RG analysis of the nonlinear sigma models that correspond to the spin chains.

Fitting our results into the framework of random matrix ensembles for such problems is an outstanding challenge. It is interesting that the nonlinear-sigma-model target manifold we obtain on the NL is (except for  $x=0$ ) not in the list of those known to correspond to random matrix ensembles in Ref. 17. Possibly there is another random matrix theory with special symmetries as on the NL.



There are of course a number of other outstanding problems, even for the RBIM. We have hardly touched the region below the NL, which remains mysterious. The fixed point at  $K = \infty$  (zero temperature) and  $p = p_c$  is of particular interest. In this region the system can be viewed as a superspin chain, since it is a chain of irreducible representations of its supergroup,  $\text{OSp}(2n|2n)$ , to which the larger SUSY,  $\text{OSp}(2n+1|2n)$ , is broken by superspin anisotropy terms, similar to the XXZ model.

*Note added:* Another numerical work on the multicritical point of the  $\pm K$  RBIM has appeared very recently.<sup>43</sup>

### ACKNOWLEDGMENTS

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### APPENDIX A: EQUALITIES FOR CORRELATORS

We will show in this appendix that the enhanced supersymmetry present on the Nishimori line in our formulation allows us to reproduce the results of the type of Eq. (1.4).

We use the formulation of the correlators in the Ising model in terms of paths and modified partition functions.<sup>44,45</sup> Namely, for a correlator of two spins  $S_{i_1}$  and  $S_{i_2}$  we join the points  $i_1$  and  $i_2$  by an (arbitrary) path on the lattice, shift all the coupling constants  $K$  by  $i\pi/2$  along the path, and calculate the modified partition function  $Z^{(\text{mod})}$  for the system with the modified couplings. Then the correlator is

$$\langle S_{i_1} S_{i_2} \rangle = (-i)^l \frac{Z^{(\text{mod})}}{Z}, \quad (\text{A1})$$

where  $l$  is the length of the path.

In the quantum formalism the vertical coordinate on the original square lattice plays the role of imaginary time  $\tau$ , and the partition function is given by the supertrace of an imaginary time ordered evolution operator  $U$ , composed of the transfer matrices  $T_{hi}$  and  $T_{vi}$  for all the bonds in the model. Because of the supersymmetry the partition function equals 1 by construction [see Eq. (2.26) and following] for any realization of the random couplings:

$$Z_{\text{SUSY}} = \text{STr} T_{\tau} U = 1. \quad (\text{A2})$$

When calculating the correlator (A1) we have to modify the couplings in the transfer matrices along the path only for one particular replica, say, the first fermionic one. Then the correlator will be

$$\langle S_{i_1} S_{i_2} \rangle = (-i)^l \text{STr} T_{\tau} U^{(\text{mod})}. \quad (\text{A3})$$

Similarly, when calculating  $\langle S_{i_1} S_{i_2} \rangle^{2m-1}$ , we have to modify the couplings for  $2m-1$  different replicas, in which case we must have  $2n > 2m-1$ .

Let us see how the transfer matrices are modified when we shift the couplings by  $i\pi/2$ . Start with the horizontal transfer matrix assuming that the couplings are modified for  $2m-1$  fermionic replicas:

$$\begin{aligned} T_{hi}^{(\text{mod})} &= \exp \left( 2K_{i,i+\hat{x}} \hat{X}_{Si} + i \frac{\pi}{2} \sum_{\alpha=1}^{2m-1} 2x_{\alpha i} \right) \\ &= i^{2m-1} T_{hi} \prod_{\alpha=1}^{2m-1} (2x_{\alpha i}). \end{aligned} \quad (\text{A4})$$

Upon averaging over the randomness this becomes

$$T_{2i}^{(\text{mod})} = i^{2m-1} T_{2i} \prod_{\alpha=1}^{2m-1} (2x_{\alpha i}). \quad (\text{A5})$$

Then for the correlator of two spins in the same horizontal row we have

$$[\langle S_i S_{i+r\hat{x}} \rangle^{2m-1}] = \text{STr} T_{\tau} V \prod_{k=i}^{i+r-1} \prod_{\alpha=1}^{2m-1} (2x_{\alpha k}), \quad (\text{A6})$$

where  $V = [U]$ .

As before for single transfer matrices we can rewrite the last expression in terms of operators, acting in the space  $\mathcal{F}'$ , using the substitution rules obtained above in Sec. III:

$$[\langle S_i S_{i+r\hat{x}} \rangle^{2m-1}] = \text{STr} T_{\tau} V' \prod_{k=i}^{i+r-1} (2x_{0k}) \prod_{\alpha=1}^{2m-1} (2x_{\alpha k}). \quad (\text{A7})$$

Now comes the crucial point. On the Nishimori line the zeroth fermion is supersymmetric with the rest of the replicas, so we can replace all  $x_{0k}$  in the last expression by, say,  $x_{2m,k}$ :

$$[\langle S_i S_{i+r\hat{x}} \rangle^{2m-1}] = \text{STr} T_{\tau} V' \prod_{k=i}^{i+r-1} \prod_{\alpha=1}^{2m} (2x_{\alpha k}). \quad (\text{A8})$$

Then we can safely go back to the original space  $\mathcal{F}$ , in which the last expression is easily identified as

$$[\langle S_i S_{i+r\hat{x}} \rangle^{2m}], \quad (\text{A9})$$

which proves the relation (1.4) for this particular case.

Now see how vertical transfer matrices are modified. When we modify the coupling for the fermionic replica 1 on a vertical bond, the vertical transfer matrix for this replica is modified from

$$\frac{e^{K_{i,i+\hat{y}} e^{\tilde{K}_{i,i+\hat{y}}}}}{\cosh \tilde{K}_{i,i+\hat{y}}} \exp(-2\tilde{K}_{i,i+\hat{y}} n_{1i}) \quad (\text{A10})$$

to

$$\begin{aligned} & i \frac{e^{K_{i,i+\hat{y}} e^{-\tilde{K}_{i,i+\hat{y}}}}}{\cosh \tilde{K}_{i,i+\hat{y}}} \exp(2\tilde{K}_{i,i+\hat{y}} n_{1i}) \\ &= 2i \sinh K_{i,i+\hat{y}} \exp(2\tilde{K}_{i,i+\hat{y}} n_{1i}) \end{aligned} \quad (\text{A11})$$

(since when  $K$  is shifted by  $i\pi/2$ , the dual coupling  $\tilde{K}$  changes sign). Adding the rest of fermionic and bosonic replicas, we obtain

$$T_{vi}^{(\text{mod})} = i(\tanh K_{i,i+y})^{\hat{N}_{Si}-2n_{1i}+1}. \quad (\text{A12})$$

If we modify the coupling for replicas 1 through  $k$ , we get similarly

$$T_{vi}^{(\text{mod})} = i^k(\tanh K_{i,i+y})^{\hat{N}_{Si}-\sum_{\alpha=1}^k (2n_{\alpha i}-1)}. \quad (\text{A13})$$

To average this expression, we have to distinguish the cases of odd and even  $k$ . For an even  $k=2m$  we get

$$\begin{aligned} T_{1i}^{(\text{mod})} &= i^{2m}(\tanh K)^{\hat{N}_{Si}-\sum_{\alpha=1}^{2m} (2n_{\alpha i}-1)}(1-p+p(-1)^{\hat{N}_{Si}}) \\ &= i^{2m}T_{1i} \prod_{\alpha=1}^{2m} y_{\alpha i}, \end{aligned} \quad (\text{A14})$$

where we introduced

$$y_{\alpha i} = e^{2K^*(2n_{\alpha i}-1)}. \quad (\text{A15})$$

The corresponding operator in  $\mathcal{F}'$  is

$$T'_{1i}^{(\text{mod})} = i^{2m}T'_{1i} \prod_{\alpha=1}^{2m} y_{\alpha i}. \quad (\text{A16})$$

Then for the correlator of two spins in the same column we get

$$\begin{aligned} [\langle S_i S_{i+rj} \rangle^{2m}] &= \text{STr } T_\tau V \prod_{\tau=j}^{j+r-1} \prod_{\alpha=1}^{2m} y_{\alpha i}(\tau) \\ &= \text{STr } T_\tau V' \prod_{\tau=j}^{j+r-1} \prod_{\alpha=1}^{2m} y_{\alpha i}(\tau). \end{aligned} \quad (\text{A17})$$

For odd number  $k=2m-1$  we obtain instead upon averaging

$$\begin{aligned} T_{1i}^{(\text{mod})} &= i^{2m-1}(\tanh K)^{\hat{N}_{Si}-\sum_{\alpha=1}^{2m-1} (2n_{\alpha i}-1)} \\ &\quad \times (1-p+p(-1)^{\hat{N}_{Si}+1}). \end{aligned} \quad (\text{A18})$$

The last factor here is different from the similar factor in  $T_{1i}$ . Now it gives 1 when  $\hat{N}_{Si}$  is odd, and  $1-2p=e^{-2L^*}$  when  $\hat{N}_{Si}$  is even. In the space  $\mathcal{F}'_i$  this factor may be written as  $e^{2L^*(n_{0i}-1)}$ , and Eq. (A18) is replaced by

$$T'_{1i}^{(\text{mod})} = i^{2m-1}T'_{1i} y_{0i} \prod_{\alpha=1}^{2m-1} y_{\alpha i}, \quad (\text{A19})$$

where

$$y_{0i} = e^{2L^*(2n_{0i}-1)}. \quad (\text{A20})$$

For the vertical correlator we obtain now

$$[\langle S_i S_{i+rj} \rangle^{2m-1}] = \text{STr } T_\tau V' \prod_{\tau=j}^{j+r-1} y_{0i}(\tau) \prod_{\alpha=1}^{2m-1} y_{\alpha i}(\tau). \quad (\text{A21})$$

On the Nishimori line due to the enhanced supersymmetry (and the fact that  $L^*=K^*$ ) the factor  $y_{0i}$  may be replaced by  $y_{2m,i}$  and we again get the equality of the type of Eq. (1.4).

The structure appearing in the formulation above for the correlators is multiplicative in the bonds which are modified along a path connecting the spins. Then it is straightforward to generalize the arguments of this appendix to the case of arbitrary spin correlators.

## APPENDIX B: REPRESENTATIONS $R$ AND $\bar{R}$

In this appendix we review the construction of the representations  $R$  and  $\bar{R}$  of  $\text{osp}(3|2)$  in terms of unconstrained fermions and bosons (for details see Ref. 25). We also discuss how to form a graded tensor product of such representations and obtain the invariant product of the superspins  $G$  and  $\bar{G}$ .

To construct the representation  $R$  we need only one complex boson  $b$  and one complex fermion  $f$  and their conjugates  $b^\dagger, f^\dagger$ , with usual commutation relations. In terms of these the generators of  $\text{osp}(3|2)$  are constructed as follows. For an orthonormal basis of the Cartan subalgebra we use

$$h_1 = \frac{1}{\sqrt{2}} \left( b^\dagger b + \frac{1}{2} \right), \quad h_2 = \frac{i}{\sqrt{2}} \left( f^\dagger f - \frac{1}{2} \right). \quad (\text{B1})$$

In the distinguished system of simple roots of  $\text{osp}(3|2)$  one root  $\alpha_1$  is odd (“fermionic”), and one root  $\alpha_2$  is even (“bosonic”). The generators corresponding to these roots (and their negatives) are

$$\begin{aligned} e_{\alpha_1} &= b^\dagger f, & e_{-\alpha_1} &= f^\dagger b, \\ e_{\alpha_2} &= (-1)^n f f^\dagger, & e_{-\alpha_2} &= f(-1)^n f. \end{aligned} \quad (\text{B2})$$

The other roots are  $\alpha_3 = \alpha_1 + \alpha_2$ ,  $\alpha_4 = \alpha_1 + 2\alpha_2$  (both odd),  $\alpha_5 = 2\alpha_1 + 2\alpha_2$  (even), and their negatives. The corresponding generators are

$$\begin{aligned} e_{\alpha_3} &= (-1)^n f b^\dagger, & e_{-\alpha_3} &= b(-1)^n f, \\ e_{\alpha_4} &= b^\dagger f^\dagger, & e_{-\alpha_4} &= f b, \\ e_{\alpha_5} &= (b^\dagger)^2, & e_{-\alpha_5} &= b^2. \end{aligned} \quad (\text{B3})$$

Note that the generators corresponding to the roots  $\alpha_2$  and  $\alpha_3$  contain expression  $(-1)^n f$ . This is a “twist” operator for the fermion, which means that it anticommutes with  $f$  and  $f^\dagger$ . It is necessary to ensure that these generators obey the (anti-)commutation relations. In other words, these choices reflect the grading appropriate for  $\text{osp}(3|2)$ , instead of that which is natural in the present Fock space. The vacuum for bosons and fermions  $|0\rangle$ , defined in the usual manner

$$b|0\rangle = f|0\rangle = 0, \quad (\text{B4})$$

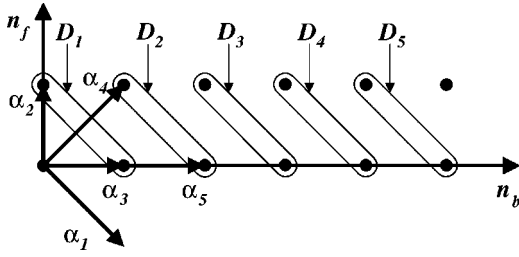


FIG. 7. The weights of states in  $R$ . The action of the positive root generators is shown by arrows. The states are grouped in pairs which are the doublets under  $\mathfrak{gl}(1|1)$  (see text for details).

is the lowest weight state of the  $R$  representation. The remaining states are obtained by the action of the raising generators, and it is easy to see that they span the whole Fock space of  $b$  and  $f$ . The weights of the states in  $R$  in terms of  $n_f$  and  $n_b$  are shown in Fig. 7. We also show in this figure the organization of the states in doublets under the  $\mathfrak{gl}(1|1)$  subalgebra generated by

$$E \equiv n_b + n_f, \quad N \equiv \frac{1}{2}(n_b - n_f),$$

$$F^\dagger \equiv e_{\alpha_1} = b^\dagger f, \quad F \equiv e_{-\alpha_1} = f^\dagger b, \quad (\text{B5})$$

[see Ref. 46, which contains a detailed discussion of the irreducible representations of  $\mathfrak{gl}(1|1)$ .] The doublet of states with  $E = m$  is denoted by  $D_m$ .

From Fig. 7 we can see that the grading of states, consistent with that of the SUSY generators, and such that the vacuum (lowest weight state) is even, is that states are even or odd accordingly as the number of bosons is even or odd. This agrees with the choice we made in Sec. II for other reasons. We may also note that the generators without strings, which are bilinears in the bosons and fermions, generate the  $\mathfrak{osp}(2|2)$  subalgebra. The latter algebra is consistent with the natural grading on the Fock space. This is not in contradiction to the above construction because the Fock space decomposes into two irreducible ‘‘spinor’’ representations of  $\mathfrak{osp}(2|2)$ , which are connected to each other only by the shortest roots  $e_{\pm 2}$ ,  $e_{\pm 3}$  that are not present in  $\mathfrak{osp}(2|2)$ .

The construction of  $\bar{R}$  is similar. The difference is that we start with negative norm bosons  $\bar{b}$  and  $\bar{b}^\dagger$  satisfying

$$[\bar{b}, \bar{b}^\dagger] = -1, \quad (\text{B6})$$

and another pair of the usual fermionic operators  $\bar{f}$  and  $\bar{f}^\dagger$ . One possible choice of the generators of  $\mathfrak{osp}(3|2)$  in the  $\bar{R}$  representation is

$$\bar{h}_1 = \frac{1}{\sqrt{2}} \left( \bar{b} \bar{b}^\dagger + \frac{1}{2} \right), \quad \bar{h}_2 = \frac{i}{\sqrt{2}} \left( \bar{f} \bar{f}^\dagger - \frac{1}{2} \right),$$

$$\bar{e}_{\alpha_1} = \bar{b} \bar{f}^\dagger, \quad \bar{e}_{-\alpha_1} = \bar{f} \bar{b}^\dagger,$$

$$\bar{e}_{\alpha_2} = -(-1)^{n_{\bar{b}}} \bar{f}, \quad \bar{e}_{-\alpha_2} = -\bar{f}^\dagger (-1)^{n_{\bar{b}}},$$

$$\bar{e}_{\alpha_3} = \bar{b} (-1)^{n_{\bar{b}}}, \quad \bar{e}_{-\alpha_3} = (-1)^{n_{\bar{b}}} \bar{b}^\dagger,$$

$$\bar{e}_{\alpha_4} = -\bar{b} \bar{f}, \quad \bar{e}_{-\alpha_4} = -\bar{f}^\dagger \bar{b}^\dagger,$$

$$\bar{e}_{\alpha_5} = -\bar{b}^2, \quad \bar{e}_{-\alpha_5} = -(\bar{b}^\dagger)^2 \quad (\text{B7})$$

Here the number of  $\bar{b}$  bosons is defined as

$$n_{\bar{b}} = -\bar{b}^\dagger \bar{b}. \quad (\text{B8})$$

The minus sign in this expression implies that  $n_{\bar{b}}$  is a non-negative integer with eigenstates  $|n_{\bar{b}}\rangle = (n_{\bar{b}}!)^{-1/2} (\bar{b}^\dagger)^{n_{\bar{b}}} |\bar{0}\rangle$ . Note that now the  $\alpha_2$  and  $\alpha_3$  generators contain a twist operator for the boson  $\bar{b}$ , which also ensures the proper (anti)commutators.

Now the vacuum  $|\bar{0}\rangle$  for  $\bar{f}$  and  $\bar{b}$  defined as

$$\bar{f} |\bar{0}\rangle = \bar{b} |\bar{0}\rangle = 0 \quad (\text{B9})$$

is the highest weight state of the  $\bar{R}$  representation, and the remaining states span the whole Fock space of  $\bar{f}$  and  $\bar{b}$ . The states of  $\bar{R}$  are now organized in doublets  $\bar{D}_m$  of the  $\mathfrak{gl}(1|1)$  generated by

$$\bar{E} \equiv -n_{\bar{b}} - n_{\bar{f}}, \quad \bar{N} \equiv \frac{1}{2}(n_{\bar{f}} - n_{\bar{b}}),$$

$$\bar{F}^\dagger \equiv \bar{e}_{\alpha_1} = \bar{b} \bar{f}^\dagger, \quad \bar{F} \equiv \bar{e}_{-\alpha_1} = \bar{f} \bar{b}^\dagger. \quad (\text{B10})$$

Next we have to combine the representations  $R$  and  $\bar{R}$  in the alternating fashion, as in Fig. 2. When we try to do that we immediately realize that the twist operators of individual  $R$  and  $\bar{R}$  representations are not adequate for their job in the tensor product. They should be replaced by ‘‘strings,’’ similar to the ones used in the Jordan–Wigner transformation. One possible convenient choice of these strings is the following. For the representation  $R_i$  (numbering as in Fig. 2) the twist operator  $(-1)^{n_{f_i}}$  is replaced by

$$\Sigma_i = \prod_{k \leq i} (-1)^{n_{f_k} + n_{\bar{f}_k}} \prod_{k > i} (-1)^{n_{b_k} + n_{\bar{b}_k}}. \quad (\text{B11})$$

Similarly, for  $\bar{R}_i$  the operator  $(-1)^{n_{\bar{b}_i}}$  should be replaced by

$$\bar{\Sigma}_i = \prod_{k < i} (-1)^{n_{f_k} + n_{\bar{f}_k}} \prod_{k \geq i} (-1)^{n_{b_k} + n_{\bar{b}_k}}. \quad (\text{B12})$$

Note that in fact  $\Sigma_i = \bar{\Sigma}_{i+1}$ .

For the purposes of Sec. V we need to consider only one pair of the antiferromagnetically coupled superspins. In this case the  $\Sigma$  operator common for both representations is  $\Sigma = (-1)^{n_{f^+} n_{\bar{b}}}$ . First we consider the fully  $\mathfrak{osp}(3|2)$  invariant product appearing in  $H_k$  on the NL. In terms of the root generators this product is given by

$$\text{str } G\bar{G} = -h_1\bar{h}_1 - h_2\bar{h}_2 - \sum_{\alpha>0} K_{-\alpha}^{-1}(e_{\alpha}\bar{e}_{-\alpha} + \bar{e}_{\alpha}e_{-\alpha}), \quad (\text{B13})$$

where  $K_{-\alpha} = K(e_{-\alpha}, e_{\alpha})$  are the values of the Killing form on the pairs  $e_{-\alpha}, e_{\alpha}$ . For  $\text{osp}(3|2)$  they are

$$\begin{aligned} K_{-\alpha_1} &= -2, & K_{-\alpha_2} &= -4, & K_{-\alpha_3} &= -4, \\ K_{-\alpha_4} &= -2, & K_{-\alpha_5} &= -4. \end{aligned} \quad (\text{B14})$$

With the mentioned expression for  $\Sigma$  this gives

$$\begin{aligned} -4 \text{str } G\bar{G} &= (b^{\dagger})^2(\bar{b}^{\dagger})^2 + \bar{b}^2 b^2 + 2(b^{\dagger}\bar{b}^{\dagger} + \bar{b}b)(f^{\dagger}\bar{f}^{\dagger} + \bar{f}f) \\ &\quad - 2n_b n_{\bar{b}} - n_b - n_{\bar{b}} + 2n_f n_{\bar{f}} - n_f - n_{\bar{f}} \\ &\quad - (f^{\dagger}\bar{f}^{\dagger} + \bar{f}f + b^{\dagger}\bar{b}^{\dagger} + \bar{b}b) \\ &= J^2 - J, \end{aligned} \quad (\text{B15})$$

where

$$J = f^{\dagger}\bar{f}^{\dagger} + \bar{f}f + b^{\dagger}\bar{b}^{\dagger} + \bar{b}b. \quad (\text{B16})$$

Note that the term  $J$  in Eq. (B15) comes from the roots  $\pm\alpha_2$  and  $\pm\alpha_3$ , and  $J^2$  comes from all the remaining roots. These remaining roots are exactly the roots of  $\text{osp}(2|2)$ . Therefore the  $J^2$  term is the  $\text{osp}(2|2)$  invariant product. That observation allows us to write the general anisotropic product as

$$4 \text{str } \Lambda G \Lambda \bar{G} = \lambda J - J^2. \quad (\text{B17})$$

### APPENDIX C: PERTURBATIVE BETA FUNCTION FOR $\text{SO}(2N+1)/\text{U}(N)$

In this appendix we derive the perturbative beta function of the weakly coupled nonlinear sigma model on  $\text{SO}(2n+1)/\text{U}(n)$  target space to one-loop order. The underlying ideas for a general sigma model have been discussed extensively by Friedan,<sup>47</sup> and we can be brief (similar calculations can be found in Ref. 42). We then discuss the resulting flows at  $n=0$ .

Consider a general homogeneous space  $G/H$ , where  $H$  is a subgroup of the group  $G$ . The neighborhood of points  $gH \in G/H$  of the ‘‘origin’’  $O \equiv eH$  ( $g$  is an arbitrary element, and  $e$  is the identity, in  $G$ ) may be parametrized in terms of  $\dim G - \dim H$  coordinates  $X^I$  by writing  $g = \exp\{X^I T_I\}$  (repeated indices summed). Here  $T_I$  denotes a basis of the vector space  $\mathcal{G}/\mathcal{H}$  spanned by the generators of  $G$  which are not generators of  $H$  ( $\mathcal{G}$  and  $\mathcal{H}$  denote the Lie algebras). The sigma model on  $G/H$  is then defined by the action

$$S = \frac{1}{2} \int d^2 r \eta_{IJ} \{X(r)\} \partial_{\mu} X^I(r) \partial^{\mu} X^J(r), \quad (\text{C1})$$

where  $r$  is the coordinate of two-dimensional space. The metric  $\eta_{IJ}\{X\}$  on the target space  $G/H$  of the sigma model serves as the coupling constant(s). Every point  $P = gH$  in the

coset space can be reached from the origin by left multiplication, and every element of the tangent space at  $P$  can be similarly obtained from the tangent space  $\mathcal{G}/\mathcal{H}$  at the origin. Therefore the metric at any point  $P$  is uniquely determined by that at the origin  $O$ , where it represents a symmetric bilinear form on the vector space  $\mathcal{G}/\mathcal{H}$ . In order for this bilinear form to represent a metric at the origin it must be invariant under the subgroup  $H$  (which acts by conjugation). Therefore the metrics on the homogeneous space are in 1-to-1 correspondence with  $H$ -invariant symmetric bilinear forms on  $\mathcal{G}/\mathcal{H}$ .

For sigma models on general manifolds (not necessarily homogeneous spaces) the two-loop beta function is<sup>47</sup>

$$\frac{d\eta_{ij}(X)}{dl} = R_{ij}(X) + \frac{1}{2} R_{iklm}(X) R_j^{klm}(X) + \dots, \quad (\text{C2})$$

where  $X^i$  is any system of local coordinates, and  $R^i{}_{klm}(X)$  and  $R_{ij}(X)$  are the Riemann and Ricci tensors, respectively, at the point of the manifold with coordinates  $X$ .

For a homogeneous space  $G/H$  it is enough to compute the beta function for the metric at the origin  $O = eH$  (since all other points can be reached by left multiplication with elements of  $G$  acting as isometries), where it reads

$$\frac{d\eta_{IJ}}{dl} = R_{IJ} + \frac{1}{2} R_{IKLM} R_J^{KLM} + \dots \quad (\text{C3})$$

This form of the beta function is convenient since it does not require reference to any parametrization of the coset space. Rather, the Riemann tensor of the homogeneous space,<sup>47,48</sup> viewed as a Riemannian space, has a simple expression in terms of the structure constants  $f_{IJ}^K, f_{IJ}^a$  of the Lie algebra  $\mathcal{G}$ , and the metric:

$$\begin{aligned} R_{KLIJ} &= -\frac{1}{4} (f_{IML} - f_{MLI} + f_{LIM}) \eta^{MM'} \\ &\quad \times (f_{JKM'} - f_{KM'J} + f_{M'JK}) \\ &\quad + \frac{1}{4} (f_{JML} - f_{MLJ} + f_{LJM}) \eta^{MM'} \\ &\quad \times (f_{IKM'} - f_{KM'I} + f_{M'IK}) \\ &\quad + \frac{1}{2} f_{IJ}^M (f_{MKL} - f_{KLM} + f_{LMK}) + f_{IJ}^a f_{aKL}, \end{aligned} \quad (\text{C4})$$

where indices  $K, L, M$ , and  $M'$  denote generators in  $\mathcal{G}/\mathcal{H}$ , which are lowered and raised by means of the metric  $\eta_{IJ}$  and its inverse  $\eta^{IJ}$ . Indices  $a$  denote generators in  $\mathcal{H}$ . The Ricci tensor is obtained, as usual, by contraction,

$$R_{LJ} = \eta^{KI} R_{KLIJ}. \quad (\text{C5})$$

We now discuss the space of all possible ( $G$ -invariant) metrics on the homogeneous space, that is all  $H$ -invariant symmetric bilinear forms on the vector space  $\mathcal{G}/\mathcal{H}$ . Since the latter transforms in a (real) representation of  $H$  (under conjugation), the bilinear form  $\eta_{IJ} = \eta(T_I, T_J)$  must, by Schur's lemma, be a multiple of the unit matrix on each irreducible



component (assuming, for simplicity, that each such component occurs only once). Consider, for example, the homogeneous spaces  $\text{SO}(N)/\text{SO}(N-1)$ , the familiar  $\text{O}(N)$  vector models. Here  $\mathcal{G}/\mathcal{H} = \text{so}(N)/\text{so}(N-1)$  transforms in the (irreducible) vector representation of  $\text{SO}(N-1)$  and therefore there is only a one-parameter family of metrics. This is the case for all symmetric spaces,<sup>49</sup> whose sigma models have therefore only a single coupling constant (the scale of the metric).

The case of interest in this paper is  $G/H = \text{SO}(2n+1)/\text{U}(n)$ , which is not a symmetric space. It has a two-parameter family of metrics, and the corresponding sigma model has therefore two coupling constants. To see this one notes that the vector space  $\text{so}(2n+1)/\mathfrak{u}(n)$  decomposes (over the real numbers) under the adjoint action of  $\text{U}(n)$  into two irreducible representations. One of them is of dimension  $n(n-1)$ ; the corresponding generators will be denoted  $T_{I_1}$ . The other is of dimension  $2n$ , and the corresponding generators will be denoted  $T_{I_2}$ . These vector spaces may be identified with the cosets of Lie algebras  $\text{so}(2n)/\mathfrak{u}(n)$  and  $\text{so}(2n+1)/\text{so}(2n)$ , respectively. This decomposition corresponds to the chain of subalgebras,  $\mathfrak{u}(n) \subset \text{so}(2n) \subset \text{so}(2n+1)$ . The two metric components can be specified as follows. Consider first the (“standard”) Cartan–Killing metric  $K$  on the entire Lie algebra  $\mathcal{G} = \text{so}(2n+1)$ . We choose the basis of generators  $T_i$  such that  $K(T_i, T_j) \propto \delta_{ij}$  (the structure constants with indices lowered by this metric are then totally antisymmetric). By restriction this is an  $H$ -invariant bilinear form on the subspace  $\mathcal{G}/\mathcal{H}$ , on which it is block-diagonal on the two irreducible representation spaces of  $H$ . The scales of the metric on the two blocks represent the two parameters of the metric, say  $\eta_1 \geq 0$  and  $\eta_2 \geq 0$ , and we can write explicitly

$$\eta_{I,J} = \eta_1 \delta_{I,I_1} \delta_{J,J_1} K(T_{I_1}, T_{J_1}) + \eta_2 \delta_{I,I_2} \delta_{J,J_2} K(T_{I_2}, T_{J_2}). \quad (\text{C6})$$

Note that one may relate the structure constants  $f_{JK}$  of Eq. (C4), with indices lowered with the metric  $\eta_{IJ}$ , to those with indices lowered with the Killing metric  $K(T_I, T_J)$ , which are totally antisymmetric.

The computation of the Ricci (and Riemann) tensor of the homogeneous space  $\text{so}(2n+1)/\mathfrak{u}(n)$  is tedious but straightforward, using Eqs. (C4), (C5), and (C6). In terms of the following parametrization of the metric,

$$\eta_1 = \frac{1}{g}, \quad \eta_2 = \frac{1}{2xg}, \quad (\text{C7})$$

one obtains from Eq. (C3) the one-loop beta functions:

$$\frac{dg}{dl} = 2g^2[x^2 + (n-1)] + \mathcal{O}(g^3), \quad (\text{C8})$$

$$\frac{dx}{dl} = 2(n-1)gx(x-1) \left[ 1 - \frac{n}{n-1}x \right] + \mathcal{O}(g^2). \quad (\text{C9})$$

These equations are valid in the limit  $g \rightarrow 0$  with  $x$  fixed.

The parameter  $x \geq 0$  measures the relative strength of the two metric components. There are two special cases,  $x=0$

and  $x=1$ , which we now discuss in turn. Consider the chain of vector spaces (Lie algebras)  $\mathfrak{u}(n) \subset \text{so}(2n) \subset \text{so}(2n+1)$ . As  $x \rightarrow 0$ , one sees from Eq. (C7) that the stiffness of the fluctuations of the sigma model (C1) associated with the metric component  $\eta_2$ , that is of those in the space  $\text{so}(2n+1)/\text{so}(2n)$ , becomes infinite. At  $x=0$  these fluctuations in the gradients (with respect to  $r$ ) of the sigma model field are forbidden, and the only remaining fluctuations are those associated with the metric component  $\eta_1$ , that is of those in  $\text{so}(2n)/\mathfrak{u}(n)$ , together with a degree of freedom on  $\text{SO}(2n+1)/\text{SO}(2n)$  (a sphere,  $S^{2n}$ ) which is independent of  $r$  and is therefore global. This is related to the structure of  $\text{SO}(2n+1)/\text{U}(n)$ , which [because of the chain of subgroups  $\text{U}(n) \subset \text{SO}(2n) \subset \text{SO}(2n+1)$ ] can be viewed as a fiber bundle with base space  $\text{SO}(2n+1)/\text{SO}(2n) \cong S^{2n}$ , and fiber  $\text{SO}(2n)/\text{U}(n)$ . Thus for each point on the sphere  $S^{2n}$ , there is a copy of the space  $\text{SO}(2n)/\text{U}(n)$  in which the field can fluctuate locally. Because of the global degree of freedom on  $S^{2n}$ , there is still a global  $\text{SO}(2n+1)$  symmetry. In simple terms, the symmetry is spontaneously broken to  $\text{SO}(2n)$ ; this does not violate the Hohenberg–Mermin–Wagner theorem, which applies for integer  $n > 1$ , because the coupling  $1/\eta_2 = 0$ . Neglecting the global degree of freedom, the line  $x=0$  now corresponds to the  $\text{SO}(2n)/\text{U}(n)$  sigma model. (These remarks explain why only non-negative powers of  $x$  appear in the perturbative beta functions.) This line is an invariant of the RG flow, and the beta function (C8) reduces to that of the  $\text{SO}(2n)/\text{U}(n)$  sigma model, which to three-loop order<sup>50</sup> is

$$\begin{aligned} \frac{dg}{dl} = & 2(n-1)g^2 + 2(n^2 - 3n + 4)g^3 \\ & + (3n^3 - 14n^2 + 35n - 28)g^4 + \mathcal{O}(g^5). \end{aligned} \quad (\text{C10})$$

For  $n > 1$ ,  $g$  flows to large values.

At  $x=1$ , on the other hand, one can check by direct calculation that the metric in Eq. (C6) reduces to that of the symmetric space  $\text{SO}(2n+2)/\text{U}(n+1)$  of higher symmetry. Therefore the line  $x=1$  must also be an invariant of the RG flow. On this line, the one-loop beta function in Eq. (C8) reduces to that of the symmetric space  $\text{SO}(2n+2)/\text{U}(n+1)$ , Eq. (C10) with  $n \rightarrow n+1$ , as expected. One sees from Eq. (C9) that for  $n > 1$  both lines  $x=0$  and  $x=1$  are attractive at weak coupling. The line  $x=(n-1)/n$  is the separatrix between these two regimes.

There is also a limit,  $\eta_1 = 0$ , in which fluctuations in  $\text{SO}(2n)/\text{U}(n)$  are “soft” and can be gauged away; compare the discussion in Ref. 42. In this case the model reduces to the nonlinear sigma model with target space  $\text{SO}(2n+1)/\text{SO}(2n) \cong S^{2n}$  mentioned earlier. However, this strong-coupling limit cannot be accessed perturbatively in  $1/\eta_1$ ,  $1/\eta_2$ .

In the replica limit  $n \rightarrow 0$ , of interest in this paper, we obtain Eq. (8.3). Note that now, in contrast to the case  $n > 1$ ,  $x=0$  is repulsive, and that on the line  $x=0$ ,  $g$  flows towards weak coupling. Near  $x=0$ , the one-loop flow lines are hyperbolas,  $xg = \text{constant}$ . In the vicinity of the line  $x=1$  the one-loop flow lines are exponentials:  $g/g^* = \exp$

$\{-2(x-1)\}$ . The one-loop flows are qualitatively different depending on the sign of  $(x-1)$ . When  $x < 1$  the coupling constant  $g$  decreases upon RG flow until it approaches some asymptote  $g = g^*$  at  $x = 1$ , while for  $x > 1$ , on the other hand,  $g$  increases towards  $g^*$  and  $x$  decreases towards 1, as  $l \rightarrow \infty$ .

To two-loop order, as discussed in Sec. VIII, the region around  $x = 1$  flows towards strong coupling. We now consider the behavior of the flows, in particular those which start with bare values near  $x = 0$ . Use of the one-loop equations (8.3) near  $x = 0$ , with bare values  $x_0$  and  $g_0$ , with  $x_0$  and  $g_0 x_0$  assumed small, shows that a value of  $x$  of order 1 is reached when  $l - l_0$  ( $l_0$  is the logarithm of the short distance cutoff, the scale at which  $x_0$ ,  $g_0$  are defined) is  $l - l_0$

$\approx 1/(2g_0 x_0)$ , at which  $g$  is of order  $g_0 x_0$ . Then we use the two-loop flows at  $x = 1$ , which should be sufficiently accurate, starting from these values. Integrating Eq. (8.4) we find finally for the crossover that passes close to the two fixed points at  $x = 0$  and  $x = 1$  that  $g$  becomes of order one when the length scale  $e^l$  is

$$\exp[1/(2g_0 x_0) + 1/(8g_0^2 x_0^2)], \quad (\text{C11})$$

in units of the short distance cutoff,  $e^{l_0}$  (the numerical factors in the exponent should not be taken too literally). The two-loop corrections near  $x = 0$  will generate only a factor of a power of  $g$  in this length scale. Flows that start at  $x > 1$  and  $g$  small give a similar scale,  $\sim \exp[1/(8g_0^2)]$ .

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