Dipolar induced magnetic anisotropy and magnetic topological defects in ultrathin films

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A Taylor expansion of the spin field enables us to obtain a local representation of dipole-dipole interactions in the plane. The magnetic anisotropy induced by the lattice symmetry by means of the dipolar interaction is analyzed for a hexagonal lattice. The reduction of the divergence of higher terms of the dipolar Hamiltonian leads to a set of nonlinear equations on the partial derivatives of the spin field. Topological defects are shown to be approximate solutions of these equations and are classified according to their validity and occurrence frequency. Because of dipolar interactions, this result is shown to be general for two-dimensional lattices observed on a large scale.

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I. INTRODUCTION

In recent years, the numerous improvements of surface technology enabled several groups to obtain high quality continuous magnetic ultrathin films with controlled thickness and structure.[1–3] These ultrathin films observed by magneto-optical, magnetoelectronic, or magnetoelastic means exhibit magnetic patterns with different symmetries as a function of temperature and lattice structure.[1–4] The magnetization distribution within such patterns gives evidence for topological defects such as vortices, domains, and walls.[1–4] Of course, because of the theoretically known two-dimensional (2D) instability in the presence of only short-range interactions,[5,6] these experimental results have encouraged several analytical studies of long-range dipole-dipole interactions in order to demonstrate their stabilizing properties.[7–9] These experimental results as well as parallel progress in numerical computation have also stimulated several numerical simulations of magnetic patterns observed in these structures.

Simple Hamiltonians with local anisotropy, short-range exchange, and dipole-dipole interactions were considered in numerical simulations[10–13] in order to understand the way of producing such patterns. Among the main results of both the experimental and theoretical approaches, the existence of intermediate states of definite symmetry at high temperature in ultrathin samples[4] is now clear. It generalizes the previous observations of numerous magnetic phase changes with temperature and external field in thicker samples, i.e., thin films,[14] and looks very attractive. The numerical evidence for magnetic vortices in ultrathin films obtained by different groups[12,13] reactivates the interest for magnetism in 2D with Kosterlitz-Thouless singularities.[15] Such magnetic vortices were already observed in thicker samples.[16–18] The present nonobservation of specific vortices in ultrathin films seems linked with the difficulty of observation of in-plane magnetization at an atomic level, while progress in such an experimental mean is expected to be obtained in the near future.[3] Thus the question arises of local magnetic symmetry in ultrathin films as well as the need for a simple theoretical version of the competition between exchange, anisotropy, and specially dipolar interactions arise.

However, because of their long-range character, dipolar interactions cannot be directly introduced in a Landau treatment for a local interaction. So the reduction of dipolar interactions to a local interaction is the goal of the present paper. And one of the consequences of this analysis is a direct treatment of the magnetic symmetry induced by dipolar interactions. A simultaneous result is the deduction of nonlinear equations satisfied by the ground-state magnetization field.

The starting point of the present work is to translate 2D nonlocal dipolar interactions, i.e., interactions between different sites lying in the same unique infinite layer, into strictly local interactions by means of a Taylor's series expansion of the spin field.[19,20] This expansion provides a Landau-like Hamiltonian working on the spin field and its derivatives taken at the same site. Each term of the so-deduced site Hamiltonian H_i contains spin field terms weighted by two-dimensional lattice sums K_n with

$$H=\sum_i H_i,\qquad H_i=\sum_{n=0}^{\infty} F_{i,n}K_n, \tag{1}$$

where $F_{i,n}$ is a function of the spin field and of its n-order derivatives as it will be shown. Because of the long-ranged character of dipolar interactions, the lattice sums K_n are demonstrated to increase with the size L of the sample when n is larger than 1. And because of inversion symmetry only even values of n lead to non-null lattice sums. Thus for a finite lattice the existence of a magnetic ground state implies the set of inequalities $F_{i,2p}\leqslant 0$ at all orders larger than 1. This defines a set of high-order derivative inequalities, all of them homogeneous in the derivative order. For an infinite lattice, the magnetic ground state is expected to have a finite energy and thus just equalities occur: $F_{i,2p}=0$. However, all these equalities or inequalities do not share the same level of validity since the spin field on a lattice is by nature discrete, thus high-order derivatives of the spin field are expected to be null except if there is a strong singularity in the magnetic pattern. These equalities are naturally classed according to the derivation order. Thus this remark justifies us to focus our attention on the first nontrivial equality or inequality, i.e., on $n=2$.

The effect of optimizing the contribution of the first $n=0$ terms which involve only the spin components is to favor some magnetization directions. The corresponding lattice sums for $n=0$ are calculated or estimated for a simple square lattice and a triangular lattice. This calculation enables us to derive the dipolar induced magnetic anisotropy for these lattice symmetries. Of course this induced anisotropy must be compared to other sources of anisotropy when finally deducing the magnetization direction. The derivation of the inequality corresponding to the case $n=4$ is then considered as an introduction to the general case with arbitrary values of n.

In Sec. II the general framework of a Taylor expansion of the spin field applied to the dipolar Hamiltonian is given, then lattice sums are calculated for different lattices in Sec. III. Section IV is devoted to determine the ground state and to the analysis of different possible solutions for the ground state or weakly excited states such as topological defects. Finally, concluding comparisons with experimental and numerical results are reported in Sec. V.

II. TAYLOR EXPANSION OF THE SPIN FIELD FOR THE DIPOLAR HAMILTONIAN

When introducing the partial derivatives at all orders of the spin field at site i, the spin at site j reads as a function of the local spin field at site i by means of a Taylor expansion:[19]

$$\mathbf{S}_j=\sum_{p,q=0}^{\infty}\frac{x_{ij}^p y_{ij}^q}{p!q!}\left(\frac{\partial^{p+q}\mathbf{S}}{\partial x^p \partial y^q}\right)_i, \tag{2}$$

where the vector $\mathbf{r}_{ij}$ of coordinates (x_{ij},y_{ij}) joins the lattice sites i and j in the plane layer $z=0$. The usual nonlocal version of the dipolar Hamiltonian between spins reads

$$H=\sum_{i,j\neq i}\frac{\mathbf{S}_i\cdot\mathbf{S}_j}{r_{ij}^3}-3\sum_{i,j\neq i}\frac{\mathbf{S}_i\cdot\mathbf{r}_{ij}\mathbf{S}_j\cdot\mathbf{r}_{ij}}{r_{ij}^5}. \tag{3}$$

With the Taylor expansion of the spin field given in Eq. (2) the dipolar Hamiltonian reads as a function of the spin field operators at the same site:

$$H=H_i$$

$$H_i=\sum_{p,q}\frac{\mathbf{S}_i\cdot}{p!q!}\frac{\partial^{p+q}\mathbf{S}_i}{\partial x^p+\partial y^q}\sum_j\frac{x_{ij}^p y_{ij}^q}{(x_{ij}^2+y_{ij}^2)^{3/2}} -3\sum_{p,q,\alpha,\beta}\frac{(\mathbf{S}_i)_\alpha}{p!q!}\frac{\partial^{p+q}(\mathbf{S}_i)_\beta}{\partial x^p\partial y^q}\sum_j\frac{(\mathbf{r}_{ij})_\alpha(\mathbf{r}_{ij})_\beta x_{ij}^p y_{ij}^q}{(x_{ij}^2+y_{ij}^2)^{5/2}}. \tag{3a}$$

This expression is deduced for an infinite lattice with translational invariance. Two kinds of lattice sums appear for such an infinite lattice, namely the isotropic sum $I_{p,q}$ and the anisotropic sum $L_{p,q,\alpha,\beta}$ with

$$I_{p.q}=\sum_j{}'\frac{x_{ij}^p y_{ij}^q}{(x_{ij}^2+y_{ij}^2)^{3/2}},$$
$$L_{p,q,\alpha,\beta}=\sum_j{}'\frac{(r_{ij})_\alpha(r_{ij})_\beta x_{ij}^p y_{ij}^q}{(x_{ij}^2+y_{ij}^2)^{5/2}}, \tag{4}$$

where the site i is an arbitrary 2D lattice site while site j runs over all the other lattice sites. These origin independent sums depend on the lattice symmetry. With the so-defined lattice sums, the local dipolar Hamiltonian reads

$$H=\sum_{i,p,q}I_{p,q}\frac{\mathbf{S}_i\cdot}{p!q!}\frac{\partial^{p+q}\mathbf{S}_i}{\partial x^p y^q} -3\sum_{i,p,q,\alpha,\beta}L_{p,q,\alpha,\beta}\frac{(\mathbf{S}_i)_\alpha}{p!q!}\frac{\partial^{p+q}(\mathbf{S}_i)_\beta}{\partial x^p\partial y^q}. \tag{5}$$

III. LATTICE SUMS AND DIPOLAR HAMILTONIAN FOR 2D LATTICES

In this paper lowest-order sums are calculated for a simple square lattice of lattice parameter a and for an hexagonal lattice of a different lattice parameter a' in order to compare their values for lattices with the same site density and different symmetries and finally to analyze the lattice induced anisotropy originated by dipolar interactions. Assuming the same atomic density in both lattices leads to a lattice parameter $a':a'=2^{1/2}3^{-1/4}a\cong 1.075a$. On the other hand, a general isotropic treatment of the simple square lattice within a continuous approximation is given for all lattice sums.

A. Simple square lattice

For this lattice of parameter a and square axes x and y, the nonzero isotropic sums $I_{p,q}$ have even values for both indices p and q. And among these $I_{2p,2q}$, the term of lowest order is $I_{0,0}$ with

$$I_{ss\ 0,0}=\frac{4}{a^3}\left[\zeta(3)+\frac{1}{2^{3/2}}+2\sum_{m=2}^{\infty}\frac{1}{(m^2+1)^{3/2}} +\sum_{m=2,n=2}^{\infty,\infty}\frac{1}{(m^2+n^2)^{3/2}}\right], \tag{6}$$

where $\zeta(n)$ is the Riemann zeta function.[21–23] An approximate treatment of $I_{ss\ 0,0}$ was given some years ago by Yafet and Gyorgy.[8] The nonzero values of the anisotropic sums $L_{p,q;\alpha,\alpha}$ are obtained for even values of both p and q, and the lowest-order term of $L_{2p,2q;\alpha,\alpha}$ is $L_{0,0;\alpha,\alpha}$, with

$$L_{ss\ 0,0;\alpha,\alpha}=\frac{4}{a^3}\left[\zeta(3)+\sum_{m=1,n=1}^{\infty,\infty}\frac{m^2}{(m^2+n^2)^{5/2}}\right] \tag{7}$$

while the nonzero values of the anisotropic sums $L_{p,q;\alpha,\beta\neq\alpha}$ are obtained for odd values of both p and q, and the lowest-order term is $L_{1,1;2,1}=L_{1,1;1,2}$, with

$$L_{1,1;1,2}=\frac{4}{a}\left[\sum_{m=1,n=1}^{\infty\infty}\frac{m^2n^2}{(m^2+n^2)^{5/2}}\right]$$
$$=\frac{4}{a}\left[2^{-5/2}\zeta(3)+2\sum_{\substack{m=2,n=1\\n<m}}^{\infty,\infty}\frac{m^2n^2}{(m^2+n^2)^{5/2}}\right]. \quad (8)$$

However, this lattice sum of order 2, as it will be shown later, does not converge for large values of m and n, as expected for a sum of order 2. Finally, the zero order part of the local dipolar Hamiltonian for the simple square lattice reads, with an obvious fourfold symmetry,

$$H_i=I_{ss0,0}S_i^2-3L_{ss0,0,1,1}\left(S_{i,x}^2+S_{i,y}^2\right). \quad (9)$$

Such a fourfold symmetry in the magnetic domain patterns is generally observed for films with a fcc (100) or a bcc (100) surface,[24] or for ultrathin films with a simple square symmetry, experimentally,[4] in analytical calculations,[9] or in numerical simulations[10] in presence of a strong enough uniaxial magnetic anisotropy which enables one to avoid the in-plane magnetization of a pure dipolar film. In such a case, lattice axes x and y are directions of easy magnetization, as observed.

B. Hexagonal lattice

The running point coordinates of the hexagonal lattice being considered are $((ma'/2),(na'\sqrt{3}/2),0)$ where the integers m and n share the same parity. The nonzero values of the isotropic sums $I_{p,q}$ are obtained for even values of both p and q, and the lowest-order term of the form $I_{2p,2q}$ is $I_{0,0}$. Taking advantage of symmetry, it reads

$$I_{h\ 0,0}=\frac{6}{a'^3}\left[\zeta(3)+8\sum_{\substack{m=1,n=1\\m+n\ \text{even}\\n<m}}^{\infty}\frac{1}{(m^2+3n^2)^{3/2}}\right]$$
$$=\frac{2^{-1/2}3^{7/4}}{a^3}\left[\zeta(3)+8\sum_{\substack{m=1,n=1\\m+n\ \text{even}\\n<m}}^{\infty}\frac{1}{(m^2+3n^2)^{3/2}}\right]. \quad (10)$$

The nonzero values of anisotropic $L_{hp,q;\alpha,\alpha}$ are obtained for even values of both p and q, and the lowest-order term of the form $L_{h2p,2q;\alpha,\alpha}$ are $L_{h0,0;\alpha,\alpha}$, with

$$L_{h\ 0,0,1,1}=\frac{2}{a'^3}\left[\zeta(3)+16\sum_{\substack{m=1,n=1\\n+m\ \text{even}}}^{\infty,\infty}\frac{m^2}{(m^2+3n^2)^{5/2}}\right]$$
$$=\frac{2^{-1/2}3^{3/4}}{a^3}\left[\zeta(3)+16\sum_{\substack{m=1,n=1\\n+m\ \text{even}}}^{\infty,\infty}\frac{m^2}{(m^2+3n^2)^{5/2}}\right] \quad (11a)$$

and

$$L_{h\ 0,0,2,2}=\frac{2}{a'^3}\left[\frac{\zeta(3)}{3}+48\sum_{\substack{m=1,n=1\\n+m\ \text{even}}}^{\infty,\infty}\frac{n^2}{(m^2+3n^2)^{5/2}}\right]$$
$$=\frac{2^{-1/2}3^{3/4}}{a^3}\left[\frac{\zeta(3)}{3}+48\sum_{\substack{m=1,n=1\\n+m\ \text{even}}}^{\infty,\infty}\frac{n^2}{(m^2+3n^2)^{5/2}}\right], \quad (11b)$$

the asymmetry between these sums indicates the isotropy breakdown due to the lattice symmetry. Next the nonzero values of the anisotropic $L_{hp,q;\alpha,\beta\neq\alpha}$ are obtained for odd values of both p and q, and the lowest nonzero order term is $L_{h;1,1;2,1}=L_{h;1,1;1,2}$ with

$$L_{h;1,1;1,2}=\frac{24}{a'}\left[\sum_{\substack{m=1,n=1\\n+m\ \text{even}}}^{\infty,\infty}\frac{m^2n^2}{(m^2+3n^2)^{5/2}}\right]$$
$$=\frac{2^{5/2}3^{5/4}}{a}\left[\sum_{\substack{m=1,n=1\\n+m\ \text{even}}}^{\infty,\infty}\frac{m^2n^2}{(m^2+3n^2)^{5/2}}\right], \quad (12)$$

but this sum, of order 2, does not converge for large values of m and n for an infinite lattice as expected for this order. Finally the zero order part of the local dipolar Hamiltonian for the hexagonal lattice reads

$$H_i=I_{h\ 0,0}\mathbf{S}_i^2-3(L_{h\ 0,0,1,1}S_{i,x}^2+L_{h\ 0,0,2,2}S_{i,y}^2) \quad (13)$$

or in a more symmetric way,

$$H_i=I_{h\ 0,0}S_{i,z}^2+\left[I_{h\ 0,0}-\frac{3}{2}(L_{h\ 0,0,1,1}+L_{h\ 0,0,2,2})\right](S_{i,x}^2+S_{i,y}^2)$$
$$-\frac{3}{2}(L_{h\ 0,0,1,1}-L_{h\ 0,0,2,2})(S_{i,x}^2-S_{i,y}^2).$$

In the last expression the planar symmetry is obviously not fourfold but sixfold. Such a symmetry is induced by the hexagonal lattice symmetry. Such an induced symmetry appears both in experimental observations on fcc (111) surfaces of thin films[1,6] and in numerical simulations on triangular lattices.[13] Because of the rapid convergence of these lattice sums which was already noticed by Yafet and Gyorgy,[8] the last term which splits the fourfold symmetry of Eq. (13) is easily computed:

$$H_{id,\neq4}=-\frac{3}{2}(L_{h\ 0,0,1,1}-L_{h\ 0,0,2,2})(S_{i,x}^2-S_{i,y}^2)$$
$$=\frac{1.723}{a^3}(S_{i,x}^2-S_{i,y}^2). \quad (13a)$$

Thus the lattice induced anisotropy has directions of easy magnetization in the lattice plane and these directions are normal to the six dense directions of the hexagonal lattice as observed in simulations.[13] In samples where a superimposed uniaxial anisotropy normal to the plane is strong enough to obtain a quasi-Ising spin arrangement, these relative easy orientations drive the domain-wall direction.[1,14]

C. Continuous treatment

All these isotropic and anisotropic lattice sums can be calculated approximately as integrals over large and finite ringlike domains situated between two circles centered at the origin with respective radii a and L. With such a treatment, a perfect rotational invariance of the layer of lattice parameter a is assumed. Thus the lattice induced anisotropy which was considered just before, is completely neglected in this part. For the sake of simplicity, a square lattice is considered. Then the first isotropic sums $I_{p,q}$ read

$$I_{p,q}=\sum_j \frac{x_{ij}^p y_{ij}^q}{(x_{ij}^2+y_{ij}^2)^{3/2}}=\int_a^L r^{p+q-2}dr\int_0^{2\pi}\cos^p\theta\sin^q\theta d\theta =K_{p,q}N_{p,q}, \tag{14}$$

where the radial integrals $K_{p,q}=K_{p+q}$ are defined by

$$K_{p,q}=\int_a^L \frac{r^{p+q}-2}{a^2}dr=\frac{L^{p+q-1}-a^{p+q-1}}{a^2(p+q-1)}=K_{p+q} \tag{14a}$$

of which the dimensional analysis is easily derived: $p+q=n$. It agrees with the previous comments on the divergence of the radial sums K_n for $n>1$ when the sample becomes large. More precisely the isotropic integrals $I_{p,q}$ with non-null p and q values are dominated by long-range contributions which increase with increasing values of p and q, and only the integral $I_{0,0}$ is dominated by short-range contributions with a resulting $1/a$ contribution, i.e., a short-range divergence term which, however, remains finite for a discrete lattice as considered here.

The angular integrals $N_{p,q}$ are

$$N_{p,q}=\int_0^{2\pi}\cos^p\theta\sin^q\theta d\theta. \tag{14b}$$

After angular integration the only nonzero factors $N_{p,q}$ are obtained for even values of p and q with

$$\text{if}\quad p\leq q\quad N_{p,q}=\frac{\pi}{i^q 2^{p+q-1}}(-1)^{(p+q)/2}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\times\binom{q}{\frac{p+q}{2}-k}, \tag{15}$$

$$\text{if}\quad p>q\quad N_{p,q}=\frac{\pi}{i^q 2^{p+q-1}}(-1)^{(p+q)/2}\sum_{k=(p-q)/2}^{(p+q)/2}(-1)^k\binom{p}{k}\times\binom{q}{\frac{p+q}{2}-k}.$$

The non-null values of these angular integrals are for the lowest-order terms:

$$N_{0,0}=2\pi,$$

$$N_{2,0}=N_{0,2}=\pi,$$

$$N_{2,2}=\pi/4;\quad N_{4,0}=N_{0,4}=3\pi/4, \tag{16}$$

$$N_{6,0}=N_{0,6}=5\pi/8;\quad N_{4,2}=N_{2,4}=\pi/8,$$

$$N_{8,0}=N_{0,8}=35\pi/4;\quad N_{6,2}=N_{2,6}=5\pi/64;\quad N_{4,4}=3\pi/64.$$

Using the gamma function,[23] asymptotic formulas for the non-null values of the $N_{p,q}$'s are easily derived with the basic result:

$$N_{p,q}\approx 2\left(\frac{2}{p}\right)^{(q+1)/2}\Gamma[(q+1)/2],$$

$$N_{p,0}\approx 2\sqrt{\frac{2\pi}{p}}. \tag{16a}$$

The anisotropic lattice integrals $L_{p,q,\alpha,\beta}$ are also deduced from the radial and angular integrals, with

$$L_{p,q,\alpha,\beta}=K_{p+q}N_{p',q'}, \tag{17}$$

where the indices α and β are both lower than three for non-null values of these integrals since all spins belong to the same layer $z=0$. The explicit link between the anisotropic $L_{p,q,\alpha,\beta}$ and the radial K_{p+q} and angular $N_{p,q}$ integrals is given by the following rules for defining p' and q' from p and q:

$$\alpha=1,\ \beta=1\quad\text{leads to}\quad p'=p+2,\ q'=q,$$

$$\alpha=1,\ \beta=2\quad\text{leads to}\quad p'=p+1,\ q'=q+1,$$

$$\alpha=2,\ \beta=1\quad\text{leads to}\quad p'=p+1,\ q'=q+1,$$

$$\alpha=2,\ \beta=2\quad\text{leads to}\quad p'=p,\ q'=q+2.$$

Because of these symmetry rules, all the Hamiltonian terms which can be nonzero are factors of K_{2n} with $n\in N$ as already introduced in Eq. (1). Thus an obvious classification of the Hamiltonian terms according to the index n occurs since the strength of the divergence of K_{2n} increases with the value of n as seen in Eq. (14a).

Direct calculations in the continuous approximation lead us to write the first term $H_{0,0}$ of the local dipolar Hamiltonian:

$$H_{0,0}=\pi K_0(-S_{i,x}^2-S_{i,y}^2+2S_{i,z}^2). \tag{18}$$

In agreement with our previous calculations on square and hexagonal lattices in Eqs. (9) and (13), $H_{0,0}$ is finite. Since the value of K_0 is positive, Eq. (18) just shows that the anisotropy induced by dipolar interaction favors an in-plane magnetization in this 2D-rotationally invariant lattice. Such a situation is also expected to occur from classical magnetostatic arguments.[1]

The next term H_2 of the expansion of the site dipolar Hamiltonian H_i contains K_2 as a factor. Using Eqs. (4), (5), (14), (16), and (17), this term reads

$$H_2=\frac{\pi}{8}K_2\left[\begin{array}{c} 4\mathbf{S}\cdot\left(\frac{\partial^2\mathbf{S}}{\partial x^2}+\frac{\partial^2\mathbf{S}}{\partial y^2}\right)-6S_x\frac{\partial^2 S_y}{\partial x\partial y}-6S_y\frac{\partial^2 S_x}{\partial x\partial y} \\ -9S_x\frac{\partial^2 S_x}{\partial x^2}-3S_x\frac{\partial^2 S_x}{\partial y^2}-3S_y\frac{\partial^2 S_y}{\partial x^2}-9S_y\frac{\partial^2 S_y}{\partial x^2}\end{array}\right]. \tag{19}$$

Since the factor K_2 would be infinite for an infinite sample, the ground-state solution must satisfy the nonlinear equation in partial derivatives of the spin field in order to yield a finite site energy:

$$4\mathbf{S}\cdot\left(\frac{\partial^2\mathbf{S}}{\partial x^2}+\frac{\partial^2\mathbf{S}}{\partial y^2}\right)-6S_x\frac{\partial^2 S_y}{\partial x\partial y}-6S_y\frac{\partial^2 S_x}{\partial x\partial y}-9S_x\frac{\partial^2 S_x}{\partial x^2} -3S_x\frac{\partial^2 S_x}{\partial y^2}-3S_y\frac{\partial^2 S_y}{\partial x^2}-9S_y\frac{\partial^2 S_y}{\partial x^2}=0. \tag{20}$$

Of course for a finite sample, Eq. (20) becomes the inequality for a metastable state:

$$4\mathbf{S}\cdot\left(\frac{\partial^2\mathbf{S}}{\partial x^2}+\frac{\partial^2\mathbf{S}}{\partial y^2}\right)-6S_x\frac{\partial^2 S_y}{\partial x\partial y}-6S_y\frac{\partial^2 S_x}{\partial x\partial y}-9S_x\frac{\partial^2 S_x}{\partial x^2} -3S_x\frac{\partial^2 S_x}{\partial y^2}-3S_y\frac{\partial^2 S_y}{\partial x^2}-9S_y\frac{\partial^2 S_y}{\partial x^2}\leqslant 0. \tag{20a}$$

The next term H_4 of the expansion of the site dipolar Hamiltonian contains K_4 as a factor. Using Eqs. (4), (5), (14), (16), and (17), this term enables us to write

$$\begin{aligned}\frac{64H_4}{\pi K_4}=&2S_z\frac{\partial^4 S_z}{\partial x^4}-3S_x\frac{\partial^4 S_x}{\partial x^4}+S_y\frac{\partial^4 S_y}{\partial x^4}+2S_z\frac{\partial^4 S_z}{\partial y^4}+S_x\frac{\partial^4 S_x}{\partial y^4}\\ &-3S_y\frac{\partial^4 S_y}{\partial y^4}+4S_z\frac{\partial^4 S_z}{\partial x^2\partial y^2}-2S_x\frac{\partial^4 S_x}{\partial x^2\partial y^2}-2S_y\frac{\partial^4 S_y}{\partial x^2\partial y^2}\\ &-4S_x\frac{\partial^4 S_y}{\partial x\partial y^3}-4S_x\frac{\partial^4 S_y}{\partial x^3\partial y}-4S_y\frac{\partial^4 S_x}{\partial x\partial y^3}-4S_y\frac{\partial^4 S_x}{\partial x^3\partial y}.\end{aligned} \tag{21}$$

Obviously the right-hand term of Eq. (21), $F_{i,4}$ in the notation of Eq. (1) must also be set equal to zero in order to obtain a finite dipolar energy per site. This leads to a new nonlinear partial derivative equation similar to Eq. (20), $F_{i,4}=0$. This new equation cannot simply be deduced from Eq. (20). This generic process of equating to zero the respective factors of the divergent integrals K_{2n} with $n>0$ generates a full set of nonlinear partial derivative equations at all even order in the case of a long-range interaction such as the dipolar one. Quite obviously equations of order $2n$ involve only derivatives of order $2n$. Taking advantage of the discrete character of the lattice shows that high-order derivatives must be nearly equal to zero in order to avoid the case of functions varying significantly over unoccupied space. Thus Eq. (21) and the following ones $F_{i,2p}=0$ must just be understood as corrective terms to the ground-state determination. which can be neglected in a first approach.

IV. MAGNETIZATION EQUATION OF THE GROUND STATE

The total energy as well as the site energy is minimum in the magnetic ground state. It means that the factor of K_2 in Eq. (19) is negative or zero. Thus the left term of Eq. (20) must be negative or zero as reported in inequality (20a). This inequality leads us to consider the limiting solution of Eq. (20) as the effective condition on the ground-state magnetization of an infinite sample. Such a solution also ensures the finite character of magnetic energy. From the results of extended Monte Carlo simulations with a slow cooling process achieved over finite samples with sizes up to 50 000 spins[13] there is no divergence of site energy and the site energy distribution is rather peaked. This gives a strong evidence of the validity of Eq. (20) in the general case. Then the result of minimization of finite terms as expressed in Eq. (18) for the induced anisotropy is an in-plane magnetization. Without any other source of anisotropy, or if this other anisotropy is small enough, an in-plane magnetization still occurs. Such a planar magnetization is easily described by a polar angle θ with $S_x=\cos\theta$, $S_y=\sin\theta$, $S_z=0$. Taking this into account Eq. (20) can be read as a nonlinear partial derivative equation on the polar angle θ as a function of the lattice coordinates:

$$3\sin 2\theta\left(\frac{\partial^2\theta}{\partial x^2}-\frac{\partial^2\theta}{\partial y^2}\right)-6\cos 2\theta\frac{\partial^2\theta}{\partial x\partial y}+6\sin 2\theta\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial y} +(2+3\cos 2\theta)\left(\frac{\partial\theta}{\partial x}\right)^2+(2-3\cos 2\theta)\left(\frac{\partial\theta}{\partial y}\right)^2=0. \tag{22}$$

The symmetric behavior of this equation leads us to introduce polar coordinates for the lattice ($x=r\cos\varphi$, $y=r\sin\varphi$). Then Eq. (22) reads with usual notations for partial derivatives

$$\begin{aligned}&\theta_{r^2}\sin[2(\theta-\varphi)]-2\frac{\theta_{r\varphi}}{r}\cos[2(\theta-\varphi)]-\frac{\theta_{\varphi^2}}{r^2}\sin[2(\theta-\varphi)]\\ &+2\frac{\theta_\varphi}{r^2}\cos[2(\theta-\varphi)]-\frac{\theta_r}{r}\sin[2(\theta-\varphi)]\\ &+\theta_r^2\left(\frac{2}{3}+\cos[2(\theta-\varphi)]\right)+2\frac{\theta_r\theta_\varphi}{r}\sin[2(\theta-\varphi)]\\ &+\frac{\theta_\varphi^2}{r^2}\left(\frac{2}{3}-\cos[2(\theta-\varphi)]\right)=0.\end{aligned} \tag{23}$$

Of course this equation admits constant values of θ for solutions, i.e., uniform magnetic domains. Such solutions are observed experimentally and numerically in domains between walls or in presence of an external driving field which saturates the sample.[1,13] Moreover, these solutions also satisfy the full set of partial derivative equations $F_{i,2p}=0$ which generalize Eq. (20) at all order since for that solution each spin derivative is equal to zero.

The presence of the argument $2(\theta-\varphi)$ everywhere in Eq. (23) leads us to consider as other test functions simple fields which correspond to topological defects with constant values of $\theta-\varphi$. Among these functions are the vortex models with two chiralities: $\theta=\varphi+\varepsilon(\pi/2)$ either with anticlockwise vortex for $\varepsilon=1$ or with clockwise vortex for $\varepsilon=-1$. The value of the left-hand term of Eq. (23) for such functions is $-3^{-1}r^{-2}$. Thus these vortices are approximate solutions of the magnetization equation when looking far from the origin. Note that this left-hand term of Eq. (23) is negative and of order L^{-2}, thus it leads to an extra energy $\approx L^{-2}K_2\approx L^{-1}$ per site which tends towards zero in the case of a large sample. On the other hand the local divergence at the origin which appears in this term is obviously avoided in the case of a lattice of finite parameter as it is. Moreover, the site energy of such test functions could be lowered by the introduction of an effective vortex made of several adjacent domains each one with uniform values of θ. And such spin arrangements appears in numerical simulations.[12,13] The origin of the 2D space defines the vortex core. Since it is arbitrary the presence of vortices of any chirality and of coupled vortices is expected to occur in the whole sample, as observed at least in numerical simulations for an effective large sample.[13] Finally, the weak extra energy which is due to such optimized topological defects makes them weakly excited states. Because of the effective boundary conditions in a real space such excited states can be stabilized and they are observed both numerically and experimentally.

The other pair of topological defects to be considered here as a test function for magnetization in Eq. (23) is a source ($\theta=\varphi$) or an antisource ($\theta=\varphi+\pi$) since they both give simple values to the argument $2(\theta-\varphi)$. Sources and antisources are observed experimentally for rather thick samples[24] but numerically such defects are observed just between adjacent vortices of opposite chiralities.[13] And these sources and antisources do not even appear in their full extension. As a matter of fact, these magnetic sources lead to a value of the left term of Eq. (23) equal to $5\times3^{-1}r^{-2}$. This positive extra energy similar to the result calculated for vortices and quite larger than it ensures these sources to be higher excited states. That extra energy explains the infrequent observation by means of numerical simulation of this defect, while its experimental observation in thick samples could be due to the simultaneous presence of a perpendicular component of the magnetization. The fact that common topological defects such as vortices and sources both give rise to a positive extra energy defines them as excited states which can be stabilized in real samples by the result of both the boundary conditions and the slowness of a collective motion.

When the sample is submitted to a strong uniaxial anisotropy which stabilizes a magnetization direction perpendicular to the film, in the walls between adjacent domains with up and down magnetization, in-plane components of the spin field must be accounted for. The previous result on the approximate solutions of Eqs. (20)–(23) are still valid for these in-plane components over limited parts. Such an argument explains the strong topological similarity between the magnetic patterns observed in magnetic films with strong uniaxial anisotropy and the magnetic patterns observed in magnetic films with weak uniaxial anisotropy. In the first case, parallel stripes, chevry domains, and whirled labyrinthine domains occur while in the second case, uniform domains, successive 60° walls and vortices occur with a complete similarity. Locally, the longest side of the Ising domain defines the spin direction of the corresponding *XY* model in accordance with basic magnetostatic considerations of the parallelism between magnetization orientation and border line.[25] This duality is effective because the striped nature of Ising domains ensures the nondegeneracy of the choice of its longest side.

V. CONCLUDING REMARKS

The magnetic anisotropy induced by the lattice symmetry reflected in the dipolar interaction is found for a 2D lattice from a Taylor expansion of the spin field. Moreover, this method enabled us to derive the set of nonlinear equations on spin field derivatives which are satisfied by the ground-state magnetization. These equations are only due to the long-range interaction, here the dipole-dipole interaction, so they are also valid whatever the short-range exchange may be. A typical feature of these nonlinear equations is the absence of any specific distance. This property explains the scale invariance often noticed in 2D observations. Of course, this general property occurs at distances large enough so that the local symmetry can be forgotten, i.e., such as the dipole field due to such a large area can balance exchange and anisotropy fields which are practically local.

Finally, the resulting magnetic structure for a 2D sample is shown to be rather complex because of the possible occurrence of numerous metastable topological defects with very weak extra energy. The overall level of complexity of these stable or metastable states is large since topological defects such as vortices and sources are extended and can overlap. This has been already observed numerically on the optimal states for samples of 40 000 spins[13] and the expected result for the stable or metastable states of a very large sample is thus a mixing of topological defects which satisfies the magnetostatic boundary conditions. The interference between these numerous entangled topological defects gives for the stable or metastable magnetic states a glassy structure of which the evolution is necessarily very slow.

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