Nonlinear ac response of an unpinned two-dimensional Wigner crystal

Yuri G. Rubo*
Centro de Investigación en Energía, Universidad Nacional Autónoma de México, Temixco, Morelos 62580, Mexico
(Received 11 September 2000; published 22 January 2001)

We analyze the macroscopic response of an unpinned two-dimensional Wigner crystal (WC) for the case of nonlinear losses with a nonuniform ac excitation in the Corbino geometry. For a WC above a liquid-helium surface, the nonlinearity arises from the Bragg-Cherenkov resonance, which gives saturation of the drift velocity with driving force. We present an analytical theory of the current distribution in the vicinity of the resonance. The theory relates the measured magnetoconductivity of WC to the microscopic characteristics of the resonance, in particular to the low-velocity tail of the frictional force.

DOI: 10.1103/PhysRevB.63.073401 PACS number(s): 73.21.−b, 73.20.Mf, 67.90.+z

I. INTRODUCTION

Low-density free electrons tend to form a periodic structure, a phenomenon predicted by Wigner in 1934. The transition to the Wigner crystal (WC) phase was first discovered experimentally for two-dimensional (2D) electrons above the surface of liquid helium.[1] More recently, evidence was found for 2D Wigner crystals in semiconductor heterostructures.[2] There are substantial differences in the physical properties of the WC in semiconductor structures and on helium. In particular, WC's observed in semiconductors are pinned to the interface defects, so that the electron system is then insulating. In contrast, electrons above the helium surface are not localized by defects. Static disorder comes from the helium atoms in the vapor. Their concentration, however, is negligible at temperatures below the WC melting temperature T_m ($T_m<1$ K for typical electron concentrations $n<10^9$ cm^{-2}). The electron WC can thus move freely on the helium surface and exhibits a finite static conductivity. In the case of ^{4}He, the scattering at low temperatures arises from the losses due to the interaction of the WC with the capillary waves on the helium surface (ripplons).[3]

It was observed[4–8] that the dissipative magnetoconductivity in the WC state is strongly nonlinear. The conductivity increases with driving force or drift velocity[6,7] and then, at some threshold, the conductivity switches from a high to a low conducting state. Below the threshold, the magnetoconductivity is closely proportional to the excitation voltage or electron velocity.[7] Hysteresis near the threshold force[4,5] and fluctuations (instabilities)[8] above the threshold have been reported. The nonlinearity is a consequence of strong coupling to the ripplons with a wave vector close to a reciprocal-lattice vector of the electron crystal (the Bragg ripplons).[1,9] As in Bragg scattering, the interference between the waves emitted by different electrons allows only these ripplons to take away the energy of the moving crystal. The energy conservation law for the ripplon emission implies the same condition as for the Cherenkov effect, leading to the Bragg-Cherenkov scattering.[10] The ripplon dispersion relation is $\omega(q)\propto q^{3/2}$ and the losses increase resonantly with the driving as the Hall velocity approaches the phase velocity $v_1=\omega(G_1)/G_1$ of the ripplons with the first reciprocal-lattice vector $G_1=(8\pi^2 n/\sqrt{3})^{1/2}$. The Hall velocity then saturates to v_1, and the dissipative magnetoconductivity increases proportionally to the driving force.[7]

A particular feature of these Bragg-Cherenkov resonances is their broadening even in an ideal two-dimensional WC.[10] This broadening is a consequence of the lack of true long-range positional order in a 2D crystal. To characterize the resonance, it is convenient to introduce the frictional force $\mathbf{F}$ experienced by an electron in the WC due to the ripplon emission. It has been shown[10] that the frictional force for a perfect WC at low temperatures is described as a function of the WC drift velocity $\mathbf{v}$ by power-shaped tails, $F\propto|v_1-v|^{-(1-\alpha)}$ with $\alpha\ll 1$. For an imperfect WC, e.g., a polycrystal or a finite-size crystal, the frictional force does not diverge at $v=v_1$. Its maximum value F_m, which can be estimated from scaling arguments,[11] defines the switching of the magnetoconductivity. The Bragg-Cherenkov scattering from the classical point of view was considered by Vinen,[12] who, however, gave a Lorentzian shape for the resonance.

The shape of the Bragg-Cherenkov resonance reflects the structural properties of the WC and it is important to be able to extract the $\mathbf{F}(\mathbf{v})$ dependence from the experimental data. The dynamics of the WC above helium in an applied magnetic field has been studied[4–8] using the capacitative coupling technique of Sommer and Tanner[13] in the circular Corbino geometry (Fig. 1). A set of concentric Corbino disk electrodes is placed below the helium surface. The central electrode is excited with an ac voltage, and the current flow to the surrounding electrode (electrodes) is measured. This method does not provide direct information about the dissipative magnetoconductivity of the electron system $\sigma\equiv\sigma_{xx}$. Nevertheless, in the *linear* conductivity case, it is possible to

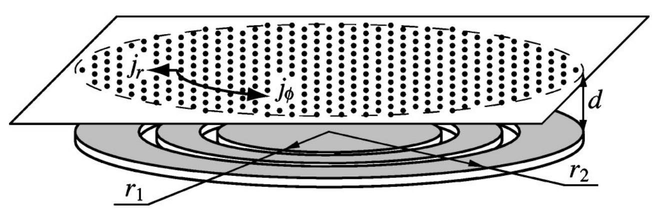

FIG. 1. Showing schematically the electron Wigner crystal above the liquid He surface and the Corbino electrodes below the surface in the typical experimental setup Refs. 5–8. The separation d is much smaller than the electrode radii, but much greater than the interelectron distance.

relate the phase shift of the observed ac current with respect to the excitation voltage to the conductivity.[14,15] The phase shift can be written as $(\pi/2-\phi)$, and, in particular, $\sigma\propto 1/\phi$ for small ϕ. The same relation for the phase shift of the first harmonic of the measured current was used to obtain the WC magnetoconductivity in the nonlinear case. Obviously, its value should be referred to as an *effective* Corbino conductivity. The applied ac electric field is nonuniform over the electron system, and so are the drift velocities of different parts of the WC. The effective magnetoconductivity depends on the geometry of the Corbino electrodes and only qualitatively reflects the nonlinear WC transport.

In this Brief Report, we present an analysis of the macroscopic nonlinear ac response of an unpinned Wigner solid. In the case of resonant behavior of the frictional force, the macroscopic equations for the density of the WC electric current have, in general, bistable solutions, describing domains of the WC moving with small $(v<v_1)$ and high $(v>v_1)$ drift velocities. The analysis of these solutions is complicated since the current can exhibit an azimuthal dependence even under axisymmetric excitation conditions, one has to allow for the edge magnetoplasmons, etc. In the region *before* switching, however, the solution is unique. We develop here a simple analytical approach for the analysis of the current distribution. This approach is particularly useful in the Bragg-Cherenkov limit, when the azimuthal (Hall) velocity becomes saturated to the phase velocity of the Bragg ripplons v_1.

II. GENERAL EQUATIONS

At low excitation frequencies, one can neglect the inductance in the WC response, and the electric current density $\mathbf{j}(\mathbf{r},t)$ can be found from the equations of motion along with the continuity equation,

$$-e(\mathbf{E}+\mathfrak{E})+\frac{1}{cn}(\mathbf{j}\times\mathbf{B})+\mathbf{F}(\mathbf{j})=\mathbf{0}, \tag{1}$$

$$e\frac{\partial\delta n}{\partial t}-\boldsymbol{\nabla}\cdot\mathbf{j}=\mathbf{0}. \tag{2}$$

Here $\mathfrak{E}(\mathbf{r},t)$ is the applied nonuniform ac electric field in the WC plane, $\mathbf{E}(\mathbf{r},t)$ is the ''internal'' electric field due to the change in the electron density $\delta n(\mathbf{r},t)$, and the magnetic field $\mathbf{B}$ is applied perpendicular to the electron layer. The frictional force $\mathbf{F}$ is assumed to be a local function of the current density. We neglect the shear forces in the Wigner solid (see the end of Sec. III).

The electron-electron interaction for the electrons on He is effectively screened over distances of the electron-electrode separation d. This separation is made usually[5,7] much smaller than the radii of the Corbino electrodes below the helium surface. For an excitation with the typical scale about the size of the WC (low excitation frequencies ω), δn is a smooth function of $\mathbf{r}$ on the screening length, and one can write[15]

$$\mathbf{E}=\frac{e}{C_s}\boldsymbol{\nabla}\,\delta n=\frac{mc_l^2}{en}\boldsymbol{\nabla}\,\delta n, \tag{3}$$

where $C_s=\epsilon_{\mathrm{He}}/(4\pi d)$ is the capacitance of the electron-electrode system per unit area (ϵ_{He} is the dielectric constant of liquid He). In this ''screened'' limit, when there is a local relation between $\mathbf{E}$ and δn, the WC can be characterized by the longitudinal sound velocity c_l in zero magnetic field, as introduced in Eq. (3): $c_l^2=e^2n/mC_s$.

When the voltage $V_0\cos(\omega t)$ is applied to the central Corbino electrode with the radius r_1, the WC is subject to an electric field $\mathfrak{E}$, which is sharply peaked at the circumference r_1. The width of the peak is about d, and it is consistent with the local approximation (3) to replace this peak with a δ-function spike. Then, the radial $\mathfrak{E}_r$ and azimuthal $\mathfrak{E}_\phi$ components of the applied field are

$$\mathfrak{E}_r=V_0\cos(\omega t)\,\delta(r-r_1),\quad \mathfrak{E}_\phi=0. \tag{4}$$

In what follows, we will analyze the axisymmetric solutions for the radial $j_r(r,t)$ and azimuthal $j_\phi(r,t)$ components of the current in classically strong magnetic fields, where $\|j_r\|\ll\|j_\phi\|$ and $F_r\ll F_\phi\approx F(j_\phi)$. Equations (1)–(4) in this case are reduced to

$$c_l^2\hat{\mathfrak{D}}_r j_r=\frac{\partial}{\partial t}\left(\omega_c j_\phi-\frac{e^2n}{m}\mathfrak{E}_r\right), \tag{5}$$

$$j_r=\frac{en}{m\omega_c}F(j_\phi), \tag{6}$$

where

$$\hat{\mathfrak{D}}_r=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2} \tag{7}$$

and $\omega_c=eB/mc$ is the cyclotron frequency. Equations (5) and (6) should be solved with the condition of the absence of the current at the outer boundary of the WC at the radius $r_2>r_1$: $j_r(r_2)=j_\phi(r_2)=0$. As already mentioned, the frictional force F in Eq. (6) is a nonmonotonic function of the current in the vicinity of the Bragg-Cherenkov resonance, and this equation can have multiple solutions with respect to j_ϕ. If we restrict ourselves to the region below the threshold, i.e., assume

$$|j_\phi|<j_1=env_1, \tag{8}$$

it is possible to resolve Eq. (6) and express j_ϕ as

$$j_\phi(r,t)=\Phi[j_r(r,t)]. \tag{9}$$

Equations (9) and (5) give a nonlinear diffusion-type equation for the radial current j_r. We write this current as

$$j_r(r,t)=f(r,t)+g(r,t), \tag{10}$$

where $f(r,t)$ is chosen to cancel the excitation term with $\mathfrak{E}_r$ in Eq. (5). Introducing the dimensionless variables

$$x=r/r_2,\quad x_1=r_1/r_2,\quad \tau=\omega t,\quad \chi=\omega\omega_c(r_2/c_l)^2, \tag{11}$$

we define the Green function $G(x,x')$ for the operator $\hat{\mathfrak{D}}_x$ (7), so that $\hat{\mathfrak{D}}_x G(x,x') = -\delta(x-x')$ with the boundary conditions $G(0,x') = G(1,x') = 0$:

$$G(x,x') = \frac{x_<^2}{2x}(1-x_>^2), \tag{12}$$

where $x_< = \min\{x,x'\}$ and $x_> = \max\{x,x'\}$. Then, for the applied electric field given by Eq. (4),

$$f(x,\tau) = -f_0 G(x,x_1)\sin(\tau) \quad f_0 = \omega r_2 C_s V_0, \tag{13}$$

and

$$\hat{\mathfrak{D}}_x g(x,\tau) = \chi \frac{\partial}{\partial \tau}\Phi[f(x,\tau)+g(x,\tau)]. \tag{14}$$

The two parts of the radial current (10) have different meanings. The first term, $f(x,t)$, describes the pure capacitative current in the electron layer. This current is phase-shifted by $\pi/2$ with respect to the applied ac voltage. The second term, $g(x,t)$, contains both an in-phase with voltage component and higher harmonics, providing information about the losses in the electron system. The parameter $\chi \propto \omega$ entering Eq. (14) is typically small,[5,7] $\chi \ll 1$, and $g(x,\tau)$ can be calculated by iterating Eq. (14). When the excitation frequency is small enough and $\|g\| \ll \|f\|$, the solution of Eq. (14) is given by the first iteration

$$g(x,\tau) = \chi \int_0^1 G(x,x') \frac{\partial}{\partial \tau}\Phi[f_0 G(x',x_1)\sin(\tau)]dx'. \tag{15}$$

Before considering the nonlinear response of the WC, we note that for linear conductivity one has $\Phi(j) = (\omega_c \tau_0) j$, where τ_0^{-1} is the relaxation rate. Then Eq. (15) gives the in-phase component of radial current, $g(x,\tau) \propto \cos(\tau)$. Using Eqs. (12), (13), (15), and representing the current (10) as $j_r(x,\tau) = j_r^{\max}\cos[\tau + (\pi/2) - \phi]$, we obtain for the phase shift

$$\phi = \frac{1}{2}\left[\frac{x^2}{1-x^2}\ln(1/x) - \frac{1}{4}(x_1^2 + x^2)\right]\frac{\omega C_s r_2^2}{\sigma}. \tag{16}$$

Here $\sigma = e^2 n / m\omega_c^2 \tau_0$ and $x = r/r_2$ corresponds to the circumference where the current is measured experimentally, $x_1 \leqslant x < 1$. Under the assumptions used, the expression (16) is valid for $\phi \ll 1$. In the case of nonlinear losses, Eq. (16) defines the effective Corbino magnetoconductivity σ from the observed phase shift ϕ.

III. THE BRAGG-CHERENKOV RESONANCE

In this section, we study the current oscillations in the vicinity of the Bragg-Cherenkov saturation, when the azimuthal drift velocity reaches the phase velocity of the resonant ripplons v_1. The frictional force experienced by the electrons in the Wigner crystal resonantly increases at $|j_\phi| \to j_1 = env_1$. This behavior corresponds to the following properties of the function Φ in Eq. (14): (i) $\Phi(-j) = -\Phi(j)$; (ii) $\Phi(j) \approx (\omega_c \tau_0) j$ at small j; and (iii) $\Phi(j)$ approaches j_1 at a large values of the argument. In classically strong magnetic fields ($\omega_c \tau_0 \gg 1$), the azimuthal current (9) reaches the resonant value j_1 at small values $f_0 \sim j_1/(\omega_c \tau_0)$ of the radial current injected in the system (13), and j_ϕ changes between j_1 and $-j_1$ during one cycle. The subsequent increase of the driving only sharpens the switching between these limiting values. As a result, the dissipative component of the radial current $g(x,\tau)$, which is defined by the time derivative of j_ϕ, has a form of the sequence of peaks (Fig. 2).

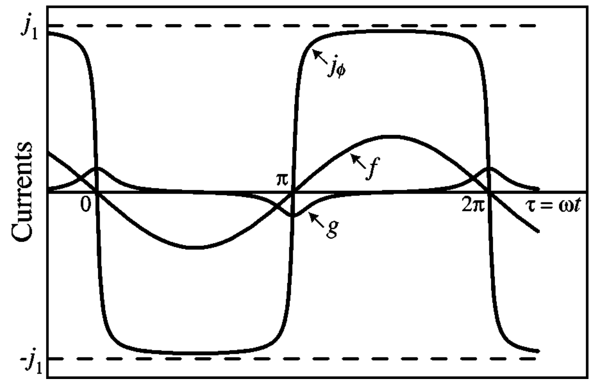

FIG. 2. The oscillations of the currents in the vicinity of the Bragg-Cherenkov resonance (qualitatively). The azimuthal current j_ϕ and two components of the radial current $j_r = f + g$ are shown.

The typical height of the peaks in $g(x,\tau)$ is about $g_m = (\omega_c \tau_0)\chi f_0$. In the case of $(\omega_c \tau_0)\chi \ll 1$ (which, in particular, can be achieved at low excitation frequencies), the dissipative radial current can be calculated using Eq. (15) and written as a Fourier series,

$$g(x,\tau) = a_1(x)\cos(\tau) + a_3(x)\cos(3\tau) + \cdots, \tag{17}$$

where only the odd harmonics are present. When the driving force f_0 increases, the azimuthal current approaches the saturated oscillation shown in Fig. 2 in practically the whole region of x, except for $x \approx 0$ and $x \approx 1$. This also produces a decrease in the width $\Delta t = j_1/\omega \omega_c \tau_0 f_0$ of the peaks in $g(x,\tau)$. The asymptotic form for the low harmonics in Eq. (17), with numbers $i \ll (\omega \Delta t)^{-1}$, can be easily calculated from Eq. (15),

$$a_i(x) \approx a_i^\infty(x) = \frac{4}{3\pi}x(1-x)\chi j_1. \tag{18}$$

Using Eqs. (13), (18), and (11), we obtain the phase shift of the first harmonic of the radial current,

$$\phi_\infty^{-1} = \zeta_p \frac{eV_0}{m\omega_c r_2 v_1}, \quad \zeta_p = \frac{3\pi r_1^2}{8r^2}\left(1 + \frac{r}{r_2}\right). \tag{19}$$

This asymptotic decrease of the phase shift with the driving force depends only on the ripplon velocity v_1, and is not sensitive to the details of the electron-ripplon interaction. Note also that the result (19) is not changed by any renormalization of the electron mass.

More precisely, the amplitude of the first harmonic can be written as $a_1(x) = a_1^\infty(x) - \Delta a_1(x)$, which gives the phase shift $\phi^{-1}=\phi_\infty^{-1}+\Delta\phi$, where $\Delta\phi=(\Delta a_1/a_1^\infty)\phi_\infty^{-1}$. The first correction comes from the deviation of $g(x,\tau)$ from the approximate solution (15) and a small shift of the peaks (to the right in Fig. 2). It can be shown that this gives $\Delta a_1 \propto V_0^{-2}$ and $\Delta\phi \propto V_0^{-1}$. However, the more important contribution to $\Delta a_1(x)$ for large V_0 is due to the incomplete saturation of the azimuthal current to j_1. This correction is determined by the tail of the Bragg-Cherenkov resonance, and its dependence on the driving voltage V_0 reflects a decrease of the frictional force with the detuning (j_1-j_ϕ). We consider a general power-shaped tail of the frictional force $F(j)\propto(j_1-j)^{-1/\nu}$. This corresponds to $\Phi(j)\cong j_1[1-(\gamma/j)^\nu]$ (for large $j>0$), where γ is a constant depending on the electron-ripplon interaction. Then, for $0<\nu<2$,

$$\Delta a_1(x)=\left(\frac{\gamma}{f_0}\right)^\nu \frac{2\chi j_1\Gamma\left(1-\dfrac{\nu}{2}\right)}{\sqrt{\pi}\Gamma\left(\dfrac{3}{2}-\dfrac{\nu}{2}\right)}\int_0^1\frac{G(x,x')}{G(x',x_1)^\nu}dx'. \quad (20)$$

It is seen that the Lorentzian $(\nu=1/2)$ tail of the resonance results in $\Delta a_1(x)\propto V_0^{-1/2}$, so that the correction to the phase shift $\Delta\phi\propto V_0^{1/2}$. In contrast, the tail due to the lack of long-range order in the Wigner crystal ($\nu=1+\alpha$ with $\alpha\ll 1$, Ref. 10) gives $\Delta a_1(x)\propto V_0^{-(1+\alpha)}$ and approximately constant shift $\Delta\phi\propto V^{-\alpha}$.

In the above analysis, we did not allow for the rigidity of the WC, i.e., we assumed the plastic deformation of the crystal. Shear forces can be incorporated into Eqs. (5) and (6) along the lines of Ref. 15, where the linear conductivity case was considered. This, in particular, removes the local relationship (5) and (9) between the components of the current density and substantially complicates the analysis of the WC response. The effect of the shear modulus, however, can be qualitatively understood by considering the rigid limit, when the azimuthal current $j_\phi(x,\tau)\propto x$. In the Bragg-Cherenkov rigid limit, the azimuthal current reaches the resonant value j_1 at the outer boundary of the WC $(x=1)$. The asymptotic behavior of ϕ^{-1} differs from Eq. (19) by a numerical factor only: ζ_p should be replaced by $\zeta_{\rm rid}=\pi(r_1/r)^2>\zeta_p$. We considered the plastic response in this Brief Report because the experimental increase of the effective magnetoconductivity σ is closer to the dependence given by Eqs. (19) and (16).[16] Comparison of the theory with the experimental data will be given elsewhere.[17]

IV. CONCLUSIONS

In conclusion, we presented an analytical theory of the nonlinear oscillations of the current in an unpinned two-dimensional Wigner crystal under the nonuniform ac excitation in the Corbino geometry. The experimentally measured phase shift of the current, related to the magnetoconductivity of WC, is analyzed in the vicinity of the Bragg-Cherenkov resonance. The asymptotic increase of the magnetoconductivity with the driving is sensitive to the tail of the resonance, and we showed how different microscopic shapes of the resonance can be distinguished experimentally.

ACKNOWLEDGMENTS

I am grateful to M. I. Dykman and M. J. Lea for useful discussions. This work was supported in part by the projects DGAPA-UNAM No. IN104598 and CONACyT No. 34278-E.

*Electronic address: rubo@servidor.unam.mx

[1] C.C. Grimes and G. Adams, Phys. Rev. Lett. **42**, 795 (1979).
[2] M. Shayegan, in *Perspectives in Quantum Hall Effects*, edited by S. Das Sarma and A. Pinczuk (John Wiley & Sons, New York, 1997).
[3] Different properties of the electrons above liquid helium have been reviewed in *Two-Dimensional Electrons Systems on Liquid Helium and Other Substrates*, edited by E.Y. Andrei (Kluwer Academic Press, New York, 1997).
[4] R. Giannetta and L. Wilen, Solid State Commun. **78**, 199 (1991).
[5] K. Shirahama and K. Kono, Phys. Rev. Lett. **74**, 781 (1995).
[6] K. Shirahama and K. Kono, J. Low Temp. Phys. **104**, 237 (1996); K. Kono and K. Shirahama, Surf. Sci. **361-362**, 826 (1996).
[7] A. Kristensen, K. Djerfi, P. Fozooni, M.J. Lea, P.J. Richardson, A. Santrich-Badal, A. Blackburn, R.W. van der Heijden, Phys. Rev. Lett. **77**, 1350 (1996).
[8] K. Djerfi, P. Fozooni, S. Harris, M.J. Lea, P.J. Richardson, A. Santrich-Badal, R.J.F. van Haren, and A. Blackburn, Phys. Rev. Lett. **80**, 806 (1998).
[9] D.S. Fisher, B.I. Halperin, and P.M. Platzman, Phys. Rev. Lett. **42**, 798 (1979).
[10] M.I. Dykman and Y.G. Rubo, Phys. Rev. Lett. **78**, 4813 (1997).
[11] K. Djerfi, V. Dotsenko, P. Fozooni, Y.G. Rubo, and M.J. Lea, Physica B **284-288**, 1918 (2000).
[12] W.F. Vinen, J. Phys.: Condens. Matter **11**, 9709 (1999).
[13] W.T. Sommer and D.J. Tanner, Phys. Rev. Lett. **27**, 1345 (1971).
[14] R. Mehrotra and A.J. Dahm, J. Low Temp. Phys. **67**, 115 (1987).
[15] Y.G. Rubo and M.J. Lea, Phys. Rev. B **58**, 13 197 (1998).
[16] M.J. Lea (private communication).
[17] M.J. Lea, P. Fozooni, A. Kristensen, P.J. Richardson, K. Djerfi, M.I. Dykman, Y.G. Rubo, and A. Blackburn (unpublished).