Absence of a metallic phase in random-bond Ising models in two dimensions: Applications to disordered superconductors and paired quantum Hall states

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When the two-dimensional random-bond Ising model is represented as a noninteracting fermion problem, it has the same symmetries as an ensemble of random matrices known as class D. A nonlinear σ model analysis of the latter in two dimensions has previously led to the prediction of a metallic phase, in which the fermion eigenstates at zero energy are extended. In this paper we argue that such behavior cannot occur in the random-bond Ising model, by showing that the Ising spin correlations in the metallic phase violate the bound on such correlations that results from the reality of the Ising couplings. Some types of disorder in spinless or spin-polarized *p*-wave superconductors and paired fractional quantum Hall states allow a mapping onto an Ising model with real but correlated bonds, and hence a metallic phase is not possible there either. It is further argued that vortex disorder, which is generic in the fractional quantum Hall applications, destroys the ordered or weak-pairing phase, in which non-Abelian statistics is obtained in the pure case.

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I. INTRODUCTION

Ising models with quenched random bonds have been considered over many years. Negative couplings produce frustration and this is the starting point for the spin glass problem.[1] A large class of models possess a ''Nishimori line'' in their phase diagram, on which the internal energy is analytic,[2] and the correlation functions of the Ising spins obey certain identities.[2,3] In two dimensions, the Ising model can be represented as a noninteracting fermion problem, even when the bonds are random.[4] The problem then reduces to something similar to a two-dimensional (2D) tight-binding Hamiltonian with quenched disorder. Properties of the Ising model are then related to those of the fermion system, in particular to the fermion Green's functions corresponding to the ''Hamiltonian,'' at a fixed ''energy,'' namely zero (this ''energy'' is not directly related to the energy in the sense of the Ising Hamiltonian). Then it is of interest to understand the properties of the fermion eigenstates near this energy, in particular whether they are localized or extended. In this paper, we consider such problems, and in particular argue that a recent proposal[5] that there exists a phase of the Ising model in which the fermion eigenstates at zero ''energy'' are extended (a ''metallic phase'') is ruled out. We also apply the results to paired fermion systems as in superconductors and quantum Hall states, which map onto similar noninteracting fermion problems.

Models of noninteracting fermions can in principle be considered using the methods of localization theory and random matrices. A list of symmetry classes (larger than the standard list due to Dyson) of ensembles of matrices was introduced by Altland and Zirnbauer (AZ).[6] The work of AZ was motivated by problems of disordered superconductors. Within the mean-field approximation, the fermionic quasiparticles of a superconductor are noninteracting, thus can be described using a single-particle formulation. The latter involves a Hamiltonian which in general contains quenched disorder, and could be a tight-binding Hamiltonian in 2D, for example. The energy levels of this Hamiltonian are the excitation energies of the quasiparticles. Once again, we may ask questions about the nature of the fermion eigenfunctions and eigenvalues. For superconductor problems, the natural zero of energy is a special point in the spectrum (unlike the case of a normal metal, for example).[6]

Among the symmetry classes found by AZ, one, denoted class D, describes disordered superconductors with broken time-reversal and spin-rotation symmetries. The symmetries are the same as those of the fermion problem in the two-dimensional (2D) random-bond Ising models (RBIM's), and ''energy'' for the fermions of the Ising model corresponds to excitation energy for the fermions in the superconductor. The nonlinear σ model for class D,[6] which in effect defines this ensemble for dimensions greater than zero, has been shown, in the 2D case, to flow under the renormalization group to weaker values of the coupling constant.[7,5,8,9] The coupling constant is related to the inverse of the thermal conductivity of the superconductor, and this flow implies that there is a phase in which there is a nonzero density of extended fermion eigenstates at zero excitation energy, and a superconductor described by this model would be in a thermal metal phase. We will refer to such a phase simply as a metallic phase. See also Refs. 10 and 11, respectively, for the 1D and 3D cases.

Senthil and Fisher[5] considered possibilities for the application (via the fermion mapping) of results for class D to 2D RBIM's. One scenario they discussed includes a metallic region in the phase diagram, below the Nishimori line, at relatively strong disorder and low temperature. They suggested that such a phase would have vanishing expectation values for both the Ising spin (''order'') and the dual ''disorder'' variables. Another scenario was that the metallic phase should be identified with the zero-temperature spin-glass region of a RBIM.

The preceding statements will be formulated more precisely in the course of this paper. Here we will begin by writing the Ising model Hamiltonian,

$$\beta\mathcal{H}=-\sum_{ij} K_{ij}\sigma_i\sigma_j, \tag{1}$$

where $\beta=1/T$ is the inverse temperature, the Ising spins $\sigma_i=\pm 1$, i, j label sites of the lattice, and $K_{ij}=J_{ij}/T$ is a convenient notation for the Ising couplings (bonds). We will assume that J_{ij} is zero unless i, j are nearest neighbors on (say) the square lattice, and that there is a T-independent probability distibution for J_{ij}, such that the different nearest-neighbor bonds are statistically independent and identically distributed. The statistical assumptions are not crucial and could be relaxed further, but we will see that it is important that the J_{ij} are real, not complex. The partition function is then

$$Z=\sum_{\{\sigma_i\}} \exp(-\beta\mathcal{H}), \tag{2}$$

where the sum is over all spin configurations $\sigma_i=\pm 1$ for all i. We will avoid discussing the boundary conditions on the lattice, or the thermodynamic limit, since we are mainly concerned with averages over the disorder of correlations of operators at separations that can be held fixed and far from the boundaries as the system size is taken to infinity after the disorder average.

We now recall a trivial fact, which will be central to the later arguments: the Ising spin correlation function for a fixed set of bonds J_{ij},

$$\langle\sigma_i\sigma_j\rangle\equiv\sum_{\{\sigma_k\}} \sigma_i\sigma_j \exp(-\beta\mathcal{H})/Z, \tag{3}$$

is bounded above by 1 and below by -1:

$$|\langle\sigma_i\sigma_j\rangle|\leqslant 1. \tag{4}$$

The bound is attained in the zero-temperature limit in pure or unfrustrated models, which include the antiferromagnetic models (all $J_{ij}<0$) on a bipartite lattice, as well as ferromagnetic (all $J_{ij}>0$) models. The bound follows from the Boltzmann-Gibbs probabilities $\exp(-\beta\mathcal{H})/Z$ being positive (and summing to 1), due to the reality of the couplings J_{ij}.

In this paper, we will discuss the statistics of the correlation functions in the order and disorder operators in a RBIM and in the class D nonlinear σ model. Our central result is that in the metallic phase, the moments of either correlation function increase as powers of distance, which for the order (Ising spin) correlations eventually violates the upper bound, Eq. (4). This implies that the metallic phase described by the σ model cannot occur in a RBIM as long as the couplings between the Ising spins are real. Our results apply to both nonzero and zero temperature in the Ising model. We trace the difference between the behaviors to differences in the form of the disorder, and suggest that the metallic phase may not after all occur in spinless or spin-polarized superconductors, or in paired fractional quantum Hall states with disorder.

In the remainder of this paper, we present our results. In Sec. II, we show that the Kadanoff-Ceva disorder correlation function[12] in a RBIM has moments bounded below by 1, and that its logarithm is symmetrically distributed, whenever the bonds are symmetrically distributed, as in an Edwards-Anderson (EA) spin-glass model. This relatively simple result will serve to illustrate points in the later discussion. In Sec. III, we obtain our central result, that the logarithms of the squared order and disorder correlations in the metallic phase are normally distributed, with mean zero and variance increasing as the logarithm of the distance, and hence the even moments of the correlations increase as powers of distance. Several steps are involved to set this up. An important point that arises along the way is that the distinctions between ensembles D, B, and BD, introduced in Ref. 9, are not important for local properties, such as these correlations. In Sec. IV, we consider another model, the O(1) model, and show that both its order and disorder correlations have properties like those in Sec. II. This model is most likely in the metallic phase. The crucial difference between such a model, and the RBIM, is that (in network model[13] language, discussed in Sec. III) the disorder adds π fluxes or vortices on one sublattice in the RBIM, but on both in the O(1) model; in Ising model language, the O(1) model corresponds to an Ising model with some couplings being complex. We also obtain the exact exponent for the mean order and disorder correlations at the critical point in another network model, the class C, or spin quantum Hall, model of Ref. 14. In Sec. V, we consider applications of our results to spinless or spin-polarized p-wave superconductors or paired fractional quantum Hall effect (FQHE) states. We show that independent insertion of vortices on a single sublattice corresponds to the RBIM situation, and cannot produce a metallic phase, at least at low densities. We argue that such ''vortex disorder'' destroys the Ising low-temperature ordered, or weak-pairing phase. For correlated vortices, the latter phase can occur, and there may be transitions in the universality classes found in the RBIM, rather than an intermediate metallic phase. Section VI is the conclusion.

II. DISORDER CORRELATIONS FOR A SYMMETRIC DISTRIBUTION OF BONDS

Our first result concerns the dual correlations in the EA spin glass case where the mean of J_{ij} is zero. The two-point correlation of the Kadanoff-Ceva disorder variable μ_α is defined in the following way (adapted from the pure case.[12]) The disorder variables are associated with sites α of the (graph-theoretic) dual lattice, that is, plaquettes of the original lattice. Given a choice of two such sites α, β, we take the Hamiltonian (1) and modify it by reversing the sign of the J_{ij}'s on the links of the lattice crossed by a path on the dual lattice that runs from α to β. We can then construct the corresponding modified partition function $Z_{\rm mod}$. Then we define

$$\langle\mu_\alpha\mu_\beta\rangle\equiv Z_{\rm mod}/Z. \tag{5}$$

This definition is independent of the choice of path from α to β, because of $\mathbf{Z}_2$-gauge properties of the Ising model. Note that $\langle \mu_\alpha \mu_\beta \rangle > 0$ when the J_{ij}'s are real.

Now we consider the statistical properties of the disorder correlation function. We denote the average over the random bonds by an overbar, for example $\overline{\langle \mu_\alpha \mu_\beta \rangle}$. We again make use of $\mathbf{Z}_2$ gauge properties, this time of the distribution function for J_{ij}. There is a statistical $\mathbf{Z}_2$ gauge invariance if the distribution is symmetric, $P(J_{ij}) = P(-J_{ij})$ for each i, j. However, such reversed bonds were exactly what was used in the definition of the disorder correlation. The set of bonds used in Z_{mod} occurs with the same probability, or probability density, as those in Z. Also, interchanging the original with the modified bonds exchanges Z_{mod} with Z. Hence $\ln\langle \mu_\alpha \mu_\beta \rangle$ is symmetrically distributed, and

$$\overline{(\ln\langle \mu_\alpha \mu_\beta \rangle)^m} = 0 \tag{6}$$

for m odd, while

$$\overline{(\ln\langle \mu_\alpha \mu_\beta \rangle)^m} \geqslant 0, \tag{7}$$

for m even. For the correlation function itself, we have

$$\overline{\langle \mu_\alpha \mu_\beta \rangle} = \overline{Z_{\mathrm{mod}}/Z} = \frac{1}{2}(\overline{Z_{\mathrm{mod}}/Z} + \overline{Z/Z_{\mathrm{mod}}}) \geqslant 1. \tag{8}$$

The same argument works for any moment of the correlation function,

$$\overline{\langle \mu_\alpha \mu_\beta \rangle^m} \geqslant 1, \tag{9}$$

for any positive or negative integer m. The bounds are attained in the high-temperature limit, where $\langle \mu_\alpha \mu_\beta \rangle = 1$.

We can predict how the disorder correlation function would behave in some well-known phases. In the paramagnetic phase, where $\langle \sigma_i \sigma_j \rangle \to 0$ as r_{ij} (the distance between i and j) goes to infinity, we expect that the mean disorder correlation goes to a constant at large distances, as in the pure case, and as in the high-temperature limit. The constant must be $\geqslant 1$, and it appears that it will increase with decreasing temperature. We also expect that the width of the distribution of the logarithm of the correlation goes to a constant. A finite-temperature spin-glass phase is believed not to occur in 2D, but if it did we would predict that the distribution of $\ln\langle \mu_\alpha \mu_\beta \rangle$ would have a width that goes as $(C_1 r_{\alpha\beta}^\theta + C_2)/T$ at low temperature, where C_1, C_2 are positive constants, and θ is an exponent that characterizes the spin-glass phase[15] as follows. In the spin glass, the insertion of the disorder variables induces a domain wall terminating at α and β. The wall is a fractal object, with a fractal dimension less than 2, and its free energy, which is random and can be positive or negative, scales as $r_{\alpha\beta}^\theta$.[15] This exponent is believed to be the same one that enters the effect of reversing the boundary conditions, from periodic to antiperiodic, in one direction in a finite system of size L; the change in free energy scales as L^θ. The exponent θ must be positive if the spin-glass phase is to be stable at finite T; it is found numerically to be negative, for continuous (e.g., Gaussian) distribution of J_{ij}, indicating that no finite T spin-glass phase exists in 2D.[1] For some special discrete distributions, such as the bimodal $\pm J$ distribution (which has many degenerate ground states, giving an extensive entropy at $T=0$), θ is small and negative, or possibly zero.[16] Finally, for a critical point, r_{ij}^θ in the width should be replaced by $\ln r_{ij}$ to a power $\geqslant 1/2$, but < 1, when certain conditions hold, or most generally a function of r_{ij} that is smaller than $\ln r_{ij}$ as $r_{ij} \to \infty$ (these follow from general results in Ref. 17).

The result Eq. (9) is in stark contrast to the Ising (order) correlations. With a symmetric probability distribution, the odd moments of $\langle \sigma_i \sigma_j \rangle$ vanish, and the even moments are $\leqslant 1$. These opposite inequalities illustrate the extreme *lack* of duality for a symmetric probability distribution. If a metallic phase did occur in the RBIM, it would have to be due to frustration, as recognized in Ref. 5. It would then naively be expected to occur for a symmetric distribution of bonds. We have now shown that the idea of a phase in which the mean disorder correlation tends to zero is untenable in any RBIM with a symmetric distribution of bonds.

What has happened to the duality present in the pure 2D Ising model? Kramers and Wannier showed that the Ising model on the square lattice can be reformulated as a dual model on the dual lattice, with Ising spins $\mu_\alpha = \pm 1$, and dual couplings $\tilde{K}$. If the disorder variables become Ising spins, why does one not again obtain a correlation less than one? In the pure case, of course, one does. But the general relation between the original couplings K_{ij} and the corresponding $\tilde{K}_{\alpha\beta}$ is

$$\exp(-2\tilde{K}_{\alpha\beta}) = \tanh K_{ij}. \tag{10}$$

For $K_{ij} < 0$, $\tilde{K}_{\alpha\beta}$ has an imaginary part $i\pi/2$ (modulo a multiple of $i\pi$). The Boltzmann-Gibbs weights of the dual spin configurations become complex in general. However, the weights for a given nearest-neighbor bond α, β for the two values of $\mu_\alpha \mu_\beta = \pm 1$ differ simply by a sign. The disorder correlation for fixed J_{ij}'s then becomes a weighted average of $\mu_\alpha \mu_\beta$ (for arbitrary α, β) with weights that sum to 1 but can be positive or negative. Hence the disorder correlation can be larger than 1. Put another way, Z may be smaller than Z_{mod}, unlike the pure case.

For more general distributions, including those with a nonzero mean for J_{ij} (which we can take to be positive without loss of generality), we cannot obtain a general result so easily. It is clear that when the disorder is weak (say, the standard deviation is small compared with the mean), there will be a ferromagnetically ordered Ising phase, as in the pure Ising model, and in this the disorder correlation goes to zero at large distances. In order to rule out the existence of a metallic phase in the intermediate region with nonzero mean J_{ij}, another approach is needed.

III. SPIN CORRELATIONS IN THE METALLIC PHASE

Now we turn to our second result, which directly concerns the metallic phase in the nonlinear σ model for class D. We ask the question: if such a phase occurs in a random Majorana fermion model, what will be the behavior of the order and disorder correlations? We note immediately that the phase, as discussed in Refs. 5 and 8, is intermediate between two localized phases that would be identified with the paramagnetic and ferromagnetic Ising phases, which are still ap-

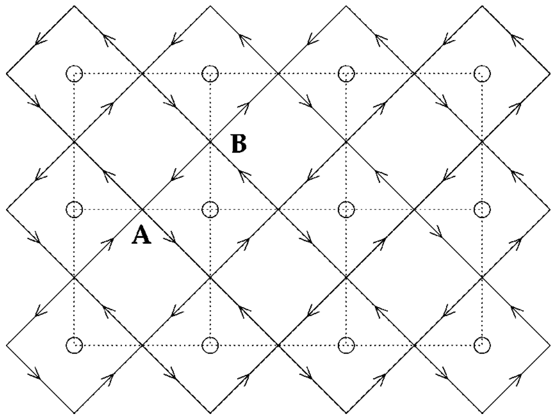

FIG. 1. Relation of the Ising model and the network model. Ising spins are located at the open circles, and bonds are shown dotted. Solid lines with arrows form the ''medial graph,'' on which the network model is defined. Examples of nodes on each of the two sublattices, corresponding to the horizontal and vertical bonds, are labeled *A*, *B*, respectively. Note the form of the edge of the cluster.

proximately dual to each other as in the pure Ising model. Then the intermediate metallic phase maps to itself under duality, and should treat the order and disorder correlations on an equal footing. The asymptotics of the two correlation functions should be similar.

In Sec. III A, we discuss the representation of the Ising model as a lattice free fermion quantum field theory, the relation of this to a network model, and the representation of order and disorder correlations in this language. In Sec. III B, we describe the nonlinear σ model that is used to define the metallic phase. We argue that the distinctions between ensembles D, B, and BD,[9] that differ globally, are not important for local correlations. In Sec. III C, we introduce the ''twist operators'' that represent the order and disorder operators in the nonlinear σ model. Then in Sec. III D, we show that the statistics of the order correlations is incompatible with a RBIM with real bonds, but compatible with other models that violate the latter condition.

A. Fermion representation

The metallic phase in the nonlinear σ model for class D describes Majorana fermions, so we must consider the fermion representation of the Ising model. This can be set up in a variety of ways. The details are not in fact all that important here. The important points are that in fermion language, duality becomes rather self-evident, and both the order and disorder variables are represented as modifying the partition function by inserting an additional $\mathbf{Z}_2$ fluxes or vortices seen by the fermions. A fermion propagating around a vortex picks up a phase factor -1. The difference between the two operators is in the locations on the lattice at which they occur. The duality is most evident if the fermions are considered as moving on the ''medial graph'' of the original square lattice,[18] as shown in Fig. 1 for a simply connected cluster. The medial graph of a given planar graph possesses two sublattices of plaquettes, on one of which each plaquette encloses a site of the original lattice, and on the other of which each plaquette encloses the center of a plaquette of the original lattice (i.e., a site of the dual lattice). The medial graph of the square lattice is another square lattice, as shown, and we consider only square-lattice clusters from here on. Note that we view the corners outside the cluster as nodes, so that the total number of links in the medial graph is a multiple of 4.

The square of the Ising model partition function Z can be represented as a six-vertex model on the medial graph, with free-fermion values of the parameters at each node.[18] (There are also many other ways to represent the Ising model as a noninteracting fermion field theory on a decorated version of the square lattice. One such approach was used in pioneering work by Blackman and Poulter[4] on the RBIM.) This free-fermion system is also equivalent to a (second-quantized representation of) the Chalker-Coddington network model,[13] as has been emphasized recently.[19,20] In other words, the single-particle model underlying the fermion field theory is a network model. We omit a complete description of these models since they have been discussed so frequently in recent years, but an outline of the main points is as follows. The links of the medial graph square lattice are viewed as directed with an arrow on each link; the arrows circulate around the plaquettes, which implies that they circulate in opposite ways for the two sublattices of plaquettes (see Fig. 1). The particle propagates on the links of this medial graph in the direction of the arrows, picking up amplitudes that depend on the original random bonds J_{ij}. The amplitudes for each time step, during which the particle must move ''forward,'' following the arrows, to an adjacent link consistent with the arrows on the network, are elements of a unitary (S) matrix assigned to each node. Thus the time evolution is described by a unitary matrix $\mathcal{U}$, that is real in the present case, and has size a multiple of 4. The sign of the product of amplitudes picked up by the particle propagating once around a plaquette determines whether a $\mathbf{Z}_2$ flux or vortex (we use these terms, or π flux, interchangeably) is present; a vortex is present when the sign is -1. In the pure Ising model, such a vortex (a flux of π) is present on every plaquette. The insertion of negative J_{ij} in the Ising model introduces an additional pair of $\mathbf{Z}_2$ fluxes on the plaquettes of the medial graph that enclose the plaquette centers of the original lattice of the two plaquettes that are adjacent to the bond in question.[4] When we speak of adding vortices or fluxes to plaquettes, the fluxes add mod 2π, since the net phase picked up by the particle is what really counts; the gauge choices involved will not matter. The effect of negative J_{ij}'s in the Ising model is thus to add vortices, but *only on one of the two sublattices of plaquettes* of the medial graph network. By duality, vortices can also be produced in similar pairs on the other sublattice, by adding an imaginary term $i\pi/2$ to K_{ij} (see Sec. II).

The squared partition function of the Ising model, Z^2, is now given by the second-quantized version of the network.[20] The partition function for noninteracting fermions is generally a determinant of the inverse fermion propagator; in the present case, the propagator between two links is a sum over paths, given by the corresponding matrix element of $1+\mathcal{U}+\mathcal{U}^2+\cdots=(1-\mathcal{U})^{-1}$, so we have $Z^2\propto\det(1-\mathcal{U})$. Note that, unlike many other representations of the Ising model as a fermion field theory, in our case the matrix $1-\mathcal{U}$ is not antisymmetric, so we cannot say that Z is the Pfaffian of the same matrix.

Because $\mathcal{U}$ is unitary, its eigenvalues lie on the unit circle and may be written $e^{-i\epsilon}$, where the eigenvalues ϵ of $i\ln\mathcal{U}$ play the role of excitation energy eigenvalues, even though they are defined only mod 2π. It is clear that for the long-time properties, such as the partition function, the important part of the spectrum of ϵ is near $\epsilon=0$. Since $\mathcal{U}$ is real, its complex eigenvalues come in complex conjugate pairs, while 1 and -1 are possible and will usually be nondegenerate. Also, since the network has a two-sublattice property (the particles hop from one type of link to the other alternately), the eigenvalues come in pairs $e^{-i\epsilon}$, $-e^{-i\epsilon}$. This implies that if 1, -1 are present, then so are i, $-i$, since the total number of eigenvalues is a multiple of 4. Thus we could restrict attention to the range $-\pi\leqslant\epsilon\leqslant\pi$, which represents $-\infty$ to ∞ in a continuum model. For the RBIM, the pair 1, -1 does not occur, $\det(1-\mathcal{U})>0$, and the square root can be taken to obtain $Z>0$.[4] The case without the quadruplet 1, -1, i, $-i$ (i.e., when $\det\mathcal{U}=1$) corresponds to random matrices in class D, while the case with that quadruplet, $\det\mathcal{U}=-1$, corresponds to those in what has been termed class B.[9] These random matrix ensembles are of matrices in the Lie algebras of SO($2N$), SO($2N+1$) (for some N), respectively.[6,9] Matrices found in class B possess at least one, and typically only one, exact zero eigenvalue.

We now consider the calculation of the moments of the two-point functions of the order and disorder variables in the metallic phase of the nonlinear σ model for class D in 2D. In terms of the medial graph or network model, the order and disorder operators are both represented as the ratio of a modified to the unmodified partition function, where order variables are represented by modifying the partition function by inserting vortices on the sublattice of plaquettes that correspond to the sites of the original lattice, and the disorder variables are vortices on the plaquettes that correspond to the plaquettes of the original lattice. Either partition function, when squared, is given by $\det(1-\mathcal{U})$. As a check on the formulas, we can consider the order and disorder correlations in the pure case. An isolated vortex on a site of the original lattice carries a zero eigenvalue $\epsilon=0$ in the high-, but not in the low-temperature phase. For two vortices, the zero modes can mix and split away from zero, by an amount exponential in the separation when the latter is greater than the correlation length. At such large distances, the other eigenvalues tend to nonzero constants, so the behavior of the ratio of products of eigenvalues of $1-\mathcal{U}$ is determined by the eigenvalues that tend to zero. Hence the correlation function tends to zero exponentially with distance in the high-T phase, but goes to a constant in the low-T phase. For the disorder correlation, the situation is reversed.

The average over the disorder of the ratio of determinants is performed by using either the replica method, with $2n$ copies of the system and $n\to 0$, or the supersymmetry method, where $2n$ copies of the system are supplemented by $2n$ copies of the system with a certain kind of boson in place of the fermions, and no $n\to 0$ limit. In the supersymmetry method, the bosons cancel the fermion determinants, as long as they are all unmodified. We will use the replica method, but the same results can easily be obtained with supersymmetry. For technical reasons, it is easiest to consider only the moments with $m=$even of the correlation functions. Then we need to average the ratio of the mth power of the modified partition function to the unmodified partition function. Therefore we will modify the network for m copies of the fermions so that they pick up an additional factor -1 on propagating around either vortex (we can take these on the positions of the original sites, so as to obtain the spin-spin correlation function, but the disorder correlation is similar). The remaining $2n-m$ fermions are unmodified. When $n\to 0$, the partition function of the latter yields the division by Z^m. Thus in the average, the moment of the correlation function is simply the partition function of the replicated system at $n=0$, that is $=1$ when unmodified, but is not when m of the Majorana fermions have been modified. That is

$$\overline{(\langle\sigma_i\sigma_j\rangle)^m}=\overline{(Z_{\rm mod}/Z)^m}=\lim_{n\to 0}\ \mathcal{Z}_{(m)}/\mathcal{Z}, \tag{11}$$

where $\mathcal{Z}$ stands for the partition function of the replicated and averaged system, $\lim_{n\to 0}\mathcal{Z}=1$, and the subscript indicates that m components have been modified.

As we will discuss further below, the nonlinear σ model for the metallic phase in class D requires us to introduce an infrared regulator $\eta>0$ which can be viewed as an imaginary part of the energy (the real part being $=0$) at which we calculate the fermion Green's functions, or as a corresponding shift in the energy eigenvalues. This is necessary in random fermion problems when the mean density of states at $\epsilon=0$ is nonzero, so in general it can be included as a precaution. In the network model, it can be included by replacing $\mathcal{U}$ by $\mathcal{U}e^{-\eta}$. We will need, first, to take moments in a finite size system with $\eta>0$, then take the system size to infinity, then let $\eta\to 0$. Some preliminary investigation suggests that for the moments of the ratio of determinants we consider, with finite separation of i and j (or α and β), this will give the same result as taking $\eta=0$ from the beginning, which is the strict definition for Ising models. This is for a fixed nonzero Ising temperature T. However, if one tries to take the temperature to zero before $\eta\to 0$, problems may arise. The reason is that, in the $T\to 0$ limit, the fermions circulate around the plaquettes of the original Ising lattice, with amplitudes 1 (for $\eta=0$). The eigenvalues of $\mathcal{U}^4$ are then determined by the flux, either 0 or π, on those plaquettes. Hence the eigenvalues ϵ tend to either $4\epsilon=0$ (mod 2π) or $4\epsilon=\pi$ (mod 2π), and when the RBIM has a finite probability for any given plaquette to be frustrated, a finite fraction of eigenvalues ϵ (and also the corresponding eigenvalues of $1-\mathcal{U}$) tend to zero as $T\to 0$.[4] In the modified partition function needed to obtain the spin correlation squared, the number of eigenvalues ϵ that tend to zero is the same as in the unmodified partition function, since otherwise the spin correlation will go to zero or infinity, which is not the case. It is only these eigenvalues that are important in determining the spin correlation in the $T\to 0$ limit. When the partition function is regulated with η, the corresponding eigenvalues of $1-\mathcal{U}e^{-\eta}$ tend to η, independent of ϵ, and the squared spin correlation goes to 1. This is expected in the case of a continuous distribution of random bonds, but definitely not for a bimodal ($\pm J$) distribution, where the $T=0$ spin correlation should be nontrivial. Thus the order of limits $T\rightarrow 0$, $\eta\rightarrow 0$, makes a difference in this case.

B. Nonlinear σ model

The claim about the metallic phase is that, in that phase, the partition function $\mathcal{Z}$, and correlation functions, can be represented at large distances by those of the nonlinear σ model for class D.[7,5,8] In replica language, this model contains a field that takes values in the target manifold $O(2n)/U(n)$. This may be parametrized by a $2n\times 2n$ complex matrix Q, which obeys $Q=Q^{\dagger}$, $Q^2=I_{2n}$, and $Q^t=-\Lambda_x Q\Lambda_x$, where t denotes transpose, and $\Lambda_x=I_n\otimes\tau_x$ (τ_x is a 2×2 Pauli matrix). In terms of $n\times n$ blocks, the top right block V of Q is an $n\times n$ antisymmetric complex matrix. (A different parametrization is used in Ref. 5.) The symmetry operations are $Q\rightarrow OQO^{\dagger}$, where in our basis, a matrix O is in $O(2n)$ if $O^{-1}=O^{\dagger}=\Lambda_x O^t\Lambda_x$ [and in $SO(2n)$ if also $\det O=1$]. Q can be written as $Q=U\Lambda_z U^{-1}$ for U in $O(2n)$, where $\Lambda_z=I_n\otimes\tau_z$ (U should not be confused with $\mathcal{U}$). This represents the coset space $O(2n)/U(n)$ because Q is invariant when $U\rightarrow Ug$, where g is in the $U(n)$ subgroup of $SO(2n)$ parametrized in our basis as $g=\mathrm{diag}(u,u^*)$, where u is a $n\times n$ unitary matrix [thus, in $U(n)$], and u^* is the complex conjugate of u.

In Ref. 9, it was emphasized that $O(2n)/U(n)$ has two disconnected components, corresponding to whether $\det U=\pm 1$. For a zero-dimensional system, the other ensembles, termed classes B and BD, can be obtained by treating the component with $\det U=-1$ differently.[9] These correspond to the existence of a single exact zero mode, $\epsilon=0$, in a finite size system, with probability 1 (for class B) or 1/2 (for class BD). Some network models (still with real $\mathcal{U}$) possess such zero modes, namely whenever $\det\mathcal{U}=-1$, and this can occur, depending on what fluxes are present, and the boundary conditions. We can avoid them by making appropriate choices of the latter. Even when present, they cancel in the regularized ($\eta\neq 0$) ratios of determinants we consider in this paper. The reason is that when we insert two additional π fluxes on the same sublattice, the determinant of $\mathcal{U}$ does not change. (However, an exact zero mode could still affect the other eigenvalues through level repulsion effects, for example.) While the presence or absence of such a zero mode may be important in random matrix ensembles for zero-dimensional systems, or for global properties in higher dimensions, we do not expect it to play a role in *local* properties in more than zero dimensions, such as the correlations we consider here. (This applies to localized, as well as metallic, phases.) Therefore we expect that the distinctions between the nonlinear σ models should not be important, and we will refer to class D/B/BD when this is so.

For dimensions larger than zero, a precise prescription for handling the two components of the target manifold has not been given. One would expect there could be domains of the two ''phases'' (in which the field Q is on one or other of the two components of the target manifold). The domain walls would likely cost some action per unit length, and therefore additional parameters will be needed to specify the model. We would expect that there will then be a regime of parameters in which domain walls are costly and all domains of the ''opposite'' phase are small. Then the Q field would essentially be globally on one component or the other. In that case, calculations can be done without domain walls as in other σ models, but with a sum over the two phases. In fact, all existing proposals for a metallic phase in class D/B/BD (Refs. 7, 5, and 8) neglect domain walls. Alternative phases where domain walls proliferate may exist, but have not been identified, and may not be metallic. In the absence of such proposals, we will consider the system without domain walls as defining the metallic phase we consider here. We note that the results in Ref. 20 give a way to handle, in effect, the different components of the target manifold in a strong-coupling situation in dimensions $\leqslant 2$.

We further argue that the regulator η which we introduce suppresses the second component. In the nonlinear σ model, it introduces a term in the action of the form $-\eta\int d^2r\,\mathrm{tr}_{2n}\Lambda_z Q$, where tr_{2n} denotes a trace over $2n$-dimensional space. This term has to be minimized on each component to find the saddle point(s) about which perturbative fluctuations are expanded. We find that at such Q values for the two components, where $Q=\Lambda_z$, $Q=O\Lambda_z O^{-1}$, respectively, and O represents a reflection in a hyperplane, the second component has relative weight like $e^{-\eta L^2}$ compared with the first, where L^2 is the area of the system. Since we take $L^2\rightarrow\infty$ before $\eta\rightarrow 0$, we find that the second component is suppressed. This does not change the partition function $\mathcal{Z}=1$ at $n=0$ for $\eta=0$, since for $\eta=0$ the functional integral over the first component gives 1, and that over the second component gives 0. (The use of just the first component, which includes $U=I_{2n}$, corresponds strictly to class BD.[9]) Therefore, we drop the second component entirely, and no difference between the metallic phases in classes D, B, and BD will be seen in local correlations. (In the total density of states in the ''ergodic'' regime discussed in Ref. 9, smearing by energy resolution η makes all three classes the same when η is greater than the level spacing, of order L^{-2}, consistent with this conclusion.)

C. Twist operators

From a perturbative point of view, Q arises from bilinears in the underlying Majorana fermions, which naturally leads to antisymmetric matrices. To obtain the correct structure of Q, it is essential that we start from the correct vacuum at weak coupling, represented by $Q=\Lambda_z$, which is invariant under the $U(n)$ subgroup introduced above. The basis in which we gave Q corresponds to the use of n complex Fermi fields ψ in place of the $2n$ Majoranas ξ. The diffusing (Goldstone) modes of the model involve only modes of the form $\psi\psi$ or $\psi^{\dagger}\psi^{\dagger}$ (the indices are suppressed), which correspond to the two off-diagonal blocks. (Goldstone modes corresponding to $2n\times 2n$ real antisymmetric matrices would give class DIII,[6] in which time reversal is unbroken.) This parametrization can also be arrived at using the O(1) network model, which in first-quantized form is a single particle propagating on the medial graph network with fixed nodes of a standard form,[13] and picking up ± 1 factors (with independent probabilities 1/2) on each link. Averaging over the group $O(1) \cong \mathbf{Z}_2$ in a replicated second-quantized representation leads to propagating Goldstone modes, and this model is in class BD, as described in Refs. 9 and 21.

In general, the modified partition function of the model is defined by the presence of a ''twist'' in the Q field. The twist is a boundary condition at the points corresponding to i, j, that is obtained from the fact that m of the Majorana fermion fields pick up a -1. Since m is even, this corresponds to a proper rotation O in $SO(2n)$. We can choose the m components of the fermions that are modified to be the real and imaginary parts of the first $m/2$ of the complex fermions that define our basis for Q. Then O is represented by a matrix in the same $U(n)$ subgroup mentioned above, with $O = \mathrm{diag}(u, u^*)$, and $u = \mathrm{diag}(-1, -1, \ldots, 1, 1 \ldots)$ with -1 appearing $m/2$ times. Hence the modified partition function $\mathcal{Z}_{(m)}$ is defined as the usual one but with the condition on the Q fields at the points i, j, that on making a circuit around these points the Q field is not periodic but changes as $Q \to OQO^\dagger$, using the same O.

D. Result at weak coupling

As mentioned above, the nonlinear σ model for class D/B/BD flows (when $n=0$) to weak coupling. Accordingly, we can compute the spin correlation function in the weak-coupling limit. To leading order, the action can be approximated as Gaussian for small V,

$$S = \frac{1}{2g^2}\int d^2r \ \mathrm{tr}_n \nabla V \nabla V^\dagger, \tag{12}$$

where the trace is over the $n \times n$ matrices V, and g^2 here is the coupling constant squared, proportional to the inverse of the thermal conductivity κ_{xx}.[5,8] We have neglected the topological (θ) term, since it plays no role in the following calculation. We have also omitted the leading nontrivial part of the η term, $\eta \int d^2r \ \mathrm{tr}_n VV^\dagger$ with $\eta > 0$. The limit $\eta \to 0$ is taken after the thermodynamic limit, because massless scalar fields in an infinite 2D system are problematic. In the weak-coupling limit, the twist operators take a simple form, since the operation described by our choice of O acts linearly on V; V transforms as the antisymmetric second-rank tensor representation of $U(n)$. The condition on V on going around the points i, j is that the components corresponding to complex fermions that are both modified or both unmodified are periodic, but those corresponding to one modified and one unmodified fermion pick up -1. Thus $m(2n-m)/4$ distinct (complex) components of V pick up a -1 on going around i or j, and the remainder of the total $n(n-1)/2$ pick up $+1$ (are periodic).

We now need the ratio of the modified to unmodified partition functions for the V field with $n \to 0$. This has the form of a standard problem in conformal field theory[22] (the Gaussian theory is conformal since the coupling g does not get renormalized). A twist operator of a single real massless scalar field, defined as a ratio of partition functions as here, has conformal weight 1/16, and so its left-right symmetric correlation function decays as $r^{-1/4}$. The exponent is doubled for a complex scalar, both of whose components are twisted. Multiplying these for our $m(2n-m)/4$ complex components, we obtain

$$\overline{(\langle \sigma_i \sigma_j \rangle)^m} \sim r_{ij}^{-m(2n-m)/8}, \tag{13}$$

which is the central result of this paper. Note that this result is independent of the coupling g. When $n \to 0$, we obtain $r^{m^2/8}$, a *positive* power of distance. In the full nonlinear σ model, g^2 approaches zero logarithmically with distance when $n=0$.[7,5,8] From standard perturbative renormalization-group (RG) arguments, we expect that the nonconstancy of the coupling produces at worst a factor of the form $\exp[C'(m)(\ln r_{ij})^{\alpha(m)}]$ on the right-hand side, where $\alpha(m) < 1$ is an m-dependent exponent. If the twist operator does not mix with any other operator in the RG, then the factor is only an m-dependent power of $\ln r_{ij}$.

In general at a random critical point, the logarithm of any correlation function is expected to have mean and variance depending logarithmically on the distance; the coefficients of these logarithmic dependences are universal. This arises because each extra factor of (say) 2 in distance is expected to contribute identically distributed, essentially independent factors to the correlation function. The central limit theorem then applies to the distribution as $r_{ij} \to \infty$. [Here we assumed the moments exist. If the distribution of the logarithm of the factors in the correlation function is too broadly distributed for this to hold, then there is still a limiting distribution with universal properties, in particular the mean (or center) of the distribution varies as $\ln r_{ij}$, and the width increases as a universal power, between 1/2 and 1, of $\ln r_{ij}$, both with universal coefficients. For a more general discussion of the scaling forms, not assuming the product ansatz, see Ref. 17.] In our weakly-coupled Gaussian field theory, the moments in Eq. (13) have the form we would obtain by assuming the log of the squared correlation function is Gaussian-distributed; the mean and variance we would obtain are

$$\overline{\ln(\langle \sigma_i \sigma_j \rangle)^2} = \mathcal{O}([\ln r_{ij}]^{\alpha'}),$$

$$\overline{[\ln(\langle \sigma_i \sigma_j \rangle)^2]^2} = \ln r_{ij} + \mathcal{O}([\ln r_{ij}]^{\alpha''}), \tag{14}$$

where α', α'' (both <1) are again some exponents. Thus these resemble the results for a random critical point, if we ignore the possible subleading corrections. Although it is well known that the log-normal distribution is not uniquely defined by its moments, it is plausible that in the present problem the distribution is indeed asymptotically log-normal. Some consideration of diagrams for directly disorder-averaging powers of the logarithm of the ratio of determinants in some models, using the self-consistent Born approximation to obtain weak coupling, also suggests that this is correct (the normal distribution is uniquely defined by its moments). Note that strictly we considered the limit $g^2 \to 0$ (or $r_{ij} \to \infty$) for each m; this suffices to obtain ''weak convergence'' of the distribution. At fixed g^2 or r_{ij}, high moments, or the tails of the distribution, may not conform to the (log-) normal form.

The fact that Eq. (13) eventually exceeds 1 implies that *this behavior is impossible in any RBIM with positive Boltzmann-Gibbs weights*. The metallic phase in class D/B/BD cannot occur in such a model. Instead, there can presumably be only gapped or localized phases and critical points between them (and possibly critical phases, meaning regions with scale invariance but described by a nontrivial fixed-point field theory, not a weakly coupled nonlinear σ model)—unless some other, so far unknown, stable metallic phase with the symmetries of the RBIM exists, that avoids the contradiction found here. This applies to the zero, as well as the nonzero, temperature region. As we saw, the regulated spin correlation goes to one as $T\rightarrow 0$, for any distribution of bonds. Even though this is not the same as the correct, $\eta = 0$, correlation for certain bond distributions, it is still in disagreement with the metallic phase.

We emphasize that results of a similar form can be obtained for the moments of the twist correlations in a variety of other metallic regimes in different ensembles, since these are by definition regions of diffusive behavior that can be described by a nonlinear σ model at weak coupling. This is true even in systems that do not renormalize towards weak coupling, on length scales shorter than that for the crossover to strong coupling. Two other cases, class DIII and the symplectic (e.g., spin-orbit scattering) case of the Wigner-Dyson ensembles, both of which possess Kramers degeneracy due to time-reversal symmetry, flow to weak coupling in 2D like the class D/B/BD case considered here. However the physical significance of the Ising order correlation is less clear in these systems. Another case of interest is a family of nonlinear σ models with target space $SO(2n+1)/U(n)$, which with $n\rightarrow 0$ arose in connection with the Nishimori line.[20] The m=even moments of $\langle\sigma_i\sigma_j\rangle$ can be considered in this case also. The twist operator has the same form, but the total number of Goldstone modes is different: because $SO(2n+1)/U(n)$ is the same, as a manifold, as $SO(2n+2)/U(n+1)$, it is as above but with $n\rightarrow 1$, not 0. The family of models with $SO(2n+1)$ symmetry has two coupling constants in place of g^2,[20] but these do not enter the twist correlation at the Gaussian level. Thus the above result (13) can be used with $n\rightarrow 1$, and the moments go as $\sim r_{ij}^{m(m-2)/8}$. For $m>2$, these increase with r_{ij}, eventually exceeding 1, requiring that $\langle\sigma_i\sigma_j\rangle^2>1$ with nonzero probability. Thus the weak-coupling region of this family of σ models is inaccessible in a RBIM with real couplings. There might in principle be metallic regimes in other weakly coupled nonlinear σ models in which the original Ising correlations are represented by a different sort of twist operator that gives a different result, but we are unaware of any at present.

According to recent work, certain network models are believed to possess a metallic phase.[21] The model of Cho and Fisher[23] is equivalent to an Ising model with couplings $\pm K$ on horizontal links (see Fig. 1), and K, $K+i\pi/2$ on vertical links,[20] with independent probabilities $1-p$, p (K is positive, and p was denoted W in Ref. 23). From our remarks in Sec. III A, this can also be rephrased by saying that in the Cho-Fisher model, π fluxes are added randomly in pairs, one above the other in Fig. 1, on *both* sublattices of plaquettes.[23] On the $p=1/2$ line, the Cho-Fisher model is equivalent by a gauge transformation to the O(1) model described in Sec. III C above.[21] [The equivalence holds in the bulk but breaks down when we consider the boundary conditions; for certain boundary conditions, the Cho-Fisher model has the symmetries strictly of class D ($1-\mathcal{U}$ has no exact zero eigenvalues).] In Ref. 21, the Cho-Fisher model was reexamined numerically, and metallic behavior was found in a region including the whole of the $p=1/2$ line. We expect therefore that that model flows to the weak-coupling regime of the class D/B/BD σ model, and then the above result applies. Hence we see that this does not contradict our claim that no metallic phase can occur in RBIM's with positive Boltzmann-Gibbs weights. The result for the O(1) model is not really surprising, in view of the behavior seen above in the dual of the RBIM, in which the Kramers-Wannier spins μ_α have couplings $\tilde{K}_{\alpha\beta}$ with imaginary parts, and the moments of their correlations can be larger than 1.

IV. CORRELATIONS IN THE O(1) MODEL

Our final result is for the O(1) model, already introduced in Sec. III C. We will argue that it is never in the ordered or disordered phases of the Ising model, by showing that the moments of the squared order and disorder correlations are both bounded below by 1. We also point out that the latter behavior is found in the other network models, in classes A, C. In the class C (spin quantum Hall) case, we find the exact exponent for the mean order and disorder correlations at criticality.

In the O(1) model, or the Cho-Fisher model at $p=1/2$, each plaquette of the network model (medial graph of the Ising model square lattice) encloses a flux of either 0 or π with independent probabilities 1/2 (up to some boundary effects). We consider the order or disorder correlation functions, defined as before in fermion language. Like the disorder correlations in the RBIM with symmetric distribution of J_{ij}'s, *the logarithm of either squared correlation is symmetrically distributed, and the even moments of the correlations are bounded below by 1*. Note that this behavior is consistent with our results in Eq. (14), if the error term in $\overline{\ln(\langle\sigma_i\sigma_j\rangle)^2}$ is zero. This means that if the O(1) model really does flow to the metallic phase, then these universal correction terms, and all higher-order analogs, in the nonlinear σ model, must be zero.

Now we compare this with the behavior expected in the localized phases. Such phases occur at weak disorder (small p) in the RBIM and Cho-Fisher models. Like the two phases of the pure Ising (massive Majorana field theory) model, one or other mean (and mean square) correlation is supposed to decay to zero, and the other to go to a constant, at large distance. Hence the O(1) model is definitely not in either such phase. It is tempting to conclude that it must therefore be in the metallic phase, though this is not really proved; some other localized phase may not be ruled out. As mentioned above, numerical work[21] led to the hypothesis of metallic behavior everywhere in the O(1) model.

The reason for caution about the last point is obtained by considering other network models for other ensembles. The two models in question are defined similarly to the O(1) model, but in the first, the particles pick up independent, uniformly distributed U(1) phases on the links, and in the second they pick up SU(2) matrices instead (the latter requires two-component wave functions for the particles). These are, respectively, the Chalker-Coddington model for the integer quantum Hall transition (class A),[13] and the Kagalovsky *et al.* model for the spin quantum Hall transition (class C).[14] Both models possess localized phases away from their critical points. We now consider twist (''order'' or ''disorder'') correlations, defined as the ratio of modified to unmodified partition functions (fermion determinants) as before. First we note that for the class A model, $\mathcal{U}$ is a $4N \times 4N$ unitary matrix, its eigenvalues come in pairs $e^{-i\epsilon}$, $-e^{-i\epsilon}$, and $\det(1-\mathcal{U})$ is in general complex. For class C, $\mathcal{U}$ is an $8N\times 8N$ symplectic matrix, and its eigenvalues come in quadruplets, $e^{-i\epsilon}$, $-e^{-i\epsilon}$, $e^{i\epsilon}$, $-e^{i\epsilon}$, similar to class D; hence $\det(1-\mathcal{U})$ is real and positive. This applies to both the modified and unmodified partition functions, and we see that in the U(1) (class A) case we should consider the modulus square correlations, while for the SU(2) (class C) case we can consider the correlations themselves, which are real and positive. The uniform distributions imply independent uniform distributions for the flux [in U(1) or SU(2), respectively] through each plaquette in these models. Multiplying these fluxes (as group elements) by -1 (for the twist insertion operation) leaves the distributions unchanged, and hence again the logarithm of the [modulus squared in the U(1) case] order or disorder correlations in these models are symmetrically distributed, and the moments of the [mod-squared for U(1)] correlations are bounded below by 1, *even in the localized phases*.

Thus it appears that these localized phases behave differently from those in the RBIM and Cho-Fisher models, and cannot be distinguished by Ising order or disorder variables. This may be connected with the continuous distributions for the flux in the plaquettes in these models, as opposed to the discrete distributions for the flux (which was either 0 or π) in the O(1) model. However, we may also point out that the localized phases in the RBIM and Cho-Fisher models, like the localized phases of a Majorana Fermi field with a weakly random mass, are expected to have a vanishing density of states at $\epsilon=0$, at least at weak disorder. In contrast, there is a possibility of a localized phase with the statistics of class D/B/BD, and the mean *local* density of states (which is independent of system size) near $\epsilon=0$ would be expected to be nonvanishing [as in the localized phase in the U(1) network model, which is in the unitary (class A) ensemble, though not the SU(2) model which is in class C] and smoothly varying. Further, it would probably have an ϵ dependence like that for class D in Ref. 9, including a peak at $\epsilon=0$, but with the energy scale for such structure proportional to the inverse-square localization length. Although the existence of such a localized phase was predicted in, for example, Ref. 8, it has not so far been demonstrated to occur in practice in any model. In fact, given the symmetries of the problem, it is not clear why such behavior does not occur in the localized phases of the RBIM, or for the Majorana fermion with random mass. Perhaps such a phase would be consistent with the above form of probability distribution, and have neither order nor disorder correlations decaying to zero. If so, it may, like the metallic phase, be inconsistent with the Ising correlations in a RBIM.

Since we have been discussing the class C (spin quantum Hall) network model,[14] we also include here a result for the critical correlations of the Ising order and disorder operators in that model. We can obtain a result only for the mean values, $\overline{\langle\sigma_i\sigma_j\rangle}$ and $\overline{\langle\mu_\alpha\mu_\beta\rangle}$. Each of these is defined by a twist of each of the two components of the wave function in a single copy of the system. It will be necessary here to use the supersymmetry method, in which the division by the unmodified partition function for a single copy is represented by a single two-component boson field.[24,25] The partition function of the unmodified supersymmetrized system has supersymmetry $\mathrm{osp}(2|2)\cong \mathrm{sl}(2|1)$,[24] and is equal to 1. The mean of either correlation is represented by a modified partition function, in which two twist operators have been inserted. The partition function, like the unmodified one, has a graphical expansion as a sum over coverings by nonintersecting loops on the network model (medial) graph, with certain weights. States of the fermions and bosons flow around the loops; there are only three possible states, which can be labeled by the number of fermions they contain, either 0, 1, or 2.[24,25] In the unmodified partition function, each loop is weighted with a factor 1, because the singly occupied state contributes -1, and the other two states $+1$ each. (In Ref. 24, this mapping was constructed and used to show that several exponents for the spin quantum Hall transition are given by percolation, which has the identical loop expansion.) Because the original twist weights a fermion of either component that propagates once around the twist with a factor of -1, the singly occupied state picks up a -1 and the others are unchanged. That is, a loop that encircles one of the twist insertions and not the other is now weighted by a factor 3, not 1; the other factors which occur at the nodes[24] are unchanged. It follows that either mean correlation is greater than 1, as we proved by another method already. Since the maximum number, and the typical number, of such loops will increase with the separation, without limit in the critical case, we expect that either correlation increases as $\sim r^{-2x}$, where $x<0$ is the scaling dimension of the twist operator. In the loop model description, the twist operator has exactly the form recently considered by Cardy[26] for percolation and other problems. Making use of his Eq. (1), $x=(\chi'^2-\chi^2)/(2g)$, with $g=2/3$, $\chi=1/3$ for percolation, and $2\cos\pi\chi'=3$ for our twist, we obtain

$$x=-\frac{1}{12}-\frac{3\ln^2[(3+\sqrt{5})/2]}{4\pi^2}\simeq-0.154. \qquad (15)$$

It is implicit in this result that we chose the branch for the logarithm such that x is real. With this choice, Cardy's general formulas imply that x is negative whenever the factor for loops that enclose exactly one twist operator is larger than that, $2\cos\pi\chi$, for the unmodified loops.

V. APPLICATIONS TO SUPERCONDUCTORS AND PAIRED FQHE STATES

Paired states of fermions with complex (time-reversal violating) pairing of spinless or spin-polarized particles, or systems with broken time-reversal symmetry and spin-orbit scattering, have the same symmetries as class D/B/BD.[5,8] We will consider only one-component systems such as spinless or spin-polarized fermions with *p*-wave pairing, which are the simplest, and begin with the pure case. We will then argue that their phase diagrams are more like that of the RBIM than has previously been recognized. For a RBIM in which frustrated plaquettes (of the Ising model lattice) are introduced independently, with some density, we argue that the Ising ordered phase is destroyed at $T>0$ for an arbitrarily small density of frustrated plaquettes (vortices). In the FQHE, this implies that the weak-pairing (non-Abelian statistics) phase is destroyed by weak disorder.

It was important in Ref. 8 for the discussion of non-Abelian statistics that vortices carry a Majorana fermion zero mode when they occur in the so-called weak-pairing phase, but not in the strong-pairing phase. These phases occur on the two sides of the transition at which the mass of the Majorana fermions changes sign; the weak-pairing phase corresponds to the Ising ordered (low-T) phase. The notion of a dual (in the Ising sense) vortex with the opposite properties—i.e., carrying a fermion zero mode only in the strong pairing phase—was implicitly discarded. The two types of vortices correspond in the network model to the two sublattices of plaquettes on which vortices (or fluxes) may be added to those already present in the pure model. The first type corresponds to adding a vortex on the network model plaquettes that correspond to plaquettes of the Ising model. In a continuum model, only the first type of vortex was considered because it was argued that in the physical situation a vortex should effectively have a region of strong-pairing phase, or vacuum (which was argued to be effectively the same thing in a ''topological'' sense), at its core.

The argument can be made essentially rigorous by considering a tight-binding Bogoliubov Hamiltonian on a lattice with a finite number N of sites, with an edge, not periodic boundary conditions. Such a Hamiltonian corresponds to a $2N\times 2N$ matrix, which is in the Lie algebra of $\mathrm{SO}(2N)$, and its eigenvalues come in pairs $\epsilon,-\epsilon$, so that it never has an odd number of exact zero eigenvalues. A vortex can be generally defined as an object which, far from its core, approaches a singular gauge transformation that describes the insertion of a flux π into the system without the vortex; this means that both the phase of the gap function, and the vector potential exhibit the winding by π. If we insert one of the postulated dual vortices in the strong-pairing phase with no other vortices present, then it is supposed to carry a zero-energy mode. There are no other zero modes with which it can mix, as we know because far from the vortex, we can use our understanding of the low-energy properties. In particular, there is no chiral spectrum of edge excitations in the strong-pairing phase. Hence it must be an exact zero mode, which is impossible for this Hamiltonian. A similar argument can be given in the weak-pairing phase, where the dual vortex does not carry a zero mode, but induces one on the edge that encloses it. We conclude then that no such dual vortices can exist, and there is only one type of vortex. Clearly we may take a continuum limit and draw the same conclusion. This means that Kramers-Wannier duality does not apply in such Hamiltonian models.

Turning to quenched disorder, it was shown in Ref. 8 that randomly-inserted vortices in independent positions are a relevant perturbation of the pure Ising (Majorana) fixed point theory.[27] It was pointed out that such disorder always occurs in the applications to FQHE systems, where underlying potential disorder can induce vortices, because they are charged. (In the RBIM, it is well known that the vortices are correlated in pairs.) It seemed natural to expect such disorder to cause a flow to the metallic phase in class D/B/BD. Now if we assume that all the vortices are of the first type defined above, then we can construct network models of this situation by adding π fluxes independently, all on the same sublattice, with some density p. This can be described as an Ising model with bond disorder that is not independent for each bond, but the bonds are still real, and of fixed magnitude. This will accurately model the low-energy properties if the vortices are dilute (note that we assume the penetration depth and coherence length of the pure system stay finite at the transition). The results of this paper show that *such a model cannot have the metallic phase in class D/B/BD*.

In fact, such a model may not have a phase transition either. Introducing frustrated plaquettes into the Ising model independently tends to destroy Ising long-range order, though the discussion is complicated by the gauge choices needed in placing the strings of negative bonds that are needed to produce the frustrated plaquettes. We can avoid this difficulty by considering the spin-glass order parameter or correlations instead. Because the bonds are $\pm J$, there will be ground-state degeneracy. Ground states can be represented by lines of frustrated bonds that join the frustrated plaquettes in pairs, chosen so as to minimize the energy. Distinct ways of dividing the frustrated plaquettes into pairs will frequently be exactly degenerate, and reconnecting the lines of frustrated bonds means reversing the spins in some domain. A condition that plausibly is necessary, but may not be sufficient, for the absence of long-range order at $T=0$, and hence of a finite T phase transition, is that, in a ground state in the thermodynamic limit, any given spin lies, with probability 1, in a finite domain that can be flipped with zero energy cost. Heuristic considerations of sufficiently large domains suggest that any spin does lie in at least one such flippable domain (the probability for a domain, formed by reconnections of lines of frustrated bonds between nearby frustrated plaquettes, to have zero energy cost decreases as a power of the length of its boundary, while the number of such domains is exponential in this length). Hence we suspect that there is no ordering even at zero temperature in this model, for any nonzero density of the frustrated plaquettes (however, the zero-temperature state may be critical, as that of the usual $\pm J$ EA model may be also). This means that the model at $T>0$ is presumably in the paramagnetic phase, and all fermion eigenstates are presumably localized. The model can be generalized by introducing a continuous distribution for the magnitudes of the J_{ij}, and will then order at $T=0$, but a similar argument for the free energy at finite T suggests that it will still not order at $T\neq 0$.

A stronger argument can also be given. The mean ground-state energy is a function of the density p of frustrated plaquettes, and is extensive, and varies smoothly with p except at $p=0$. Increasing p slightly means frustrating a small number of previously unfrustrated plaquettes. Thus frustrating one additional plaquette changes the mean total energy by an amount of order 1. Yet this forces a domain wall from the plaquette to the edge of the system, along which the lines of frustrated bonds have been reconnected. The only reasonable conclusion is that the mean energy of the minimum energy domain wall is zero, except for a finite effect from around the added frustrated plaquette. This is the same behavior as in an EA spin-glass model. A similar conclusion holds for two added frustrated plaquettes, and may be compared with the discussion in Sec. II (but note that there the operation also unfrustrated originally frustrated plaquettes, so that for a symmetric distribution of bonds the mean free energy change was exactly zero). We have no information about the width of the distribution of the domain wall energies, but we can expect that, like the usual short-range EA 2D spin-glass models, there will be no finite T transition. (A finite T transition can occur in a 2D spin glass with sufficiently long-range power-law random bonds, but we have no reason to expect this to be realized here.)

Intuitively, the independently inserted vortices appear similar to a random magnetic field, though the relation is not exact. However, it is a field that couples to the dual variables μ_α, and further it has a uniform component. The latter is the most important effect. A uniform magnetic field in a ferromagnetic Ising model destroys the transition, and the resulting phase has correlations like those in the ordered phase for the spin to which the field couples. In the present case, it would lead to the high-temperature paramagnetic phase, in agreement with our conclusion.

The conjecture that vortex disorder destroys the Ising ordered phase has a dramatic consequence for applications to spinless or spin-polarized FQHE paired states. The paramagnetic phase corresponds to the disordered version of the strong-pairing phase; the weak-pairing phase has been destroyed. The weak-pairing or Moore-Read phase was supposed to be the basis for non-Abelian statistics.[8] We are arguing that this behavior, including the chiral Majorana fermion edge modes, is destroyed by any weak vortex (i.e., potential) disorder.

In models, such as the tight-binding Bogoliubov Hamiltonian, in which the vector potential and gap function each has only short-range correlations, vortices will tend to be produced only in pairs, and again can be of only one type. Thus these one-component models appear similar to the RBIM, and may have a similar phase diagram, in which the weak-pairing phase occurs at weak but not at strong disorder. At the transition, the critical behavior may be that of either the pure Ising model (up to logarithms) or the low-T phase boundary in the (frustrated) RBIM. Thus the latter universality class could be realized in one-component superconductors. We conclude (in contrast to Refs. 5 and 8) that, in at least some models of superconductors or FQHE paired states with the symmetries of class D/B/BD, and with only one type of vortex present, there may not be a metallic phase after all, but there may instead be a transition in a distinct universality class from the pure system for some types of disorder.

Clearly, similar possibilities should be explored in connection with other ensembles, which can occur when more components are present, but will not be considered further here. We point out, however, that the case of pairing of spin-1/2 fermions, with spin-orbit scattering and a general random mass, which has the symmetries of class D/B/BD, corresponds to multicomponent models considered in Ref. 9, where it is argued that a metallic phase is produced. Reference 9 also argued that the metallic phase would not occur in the one-component case in the absence of vortex disorder, but did not consider vortex disorder as fully as we have. Also, models of superconductors as disordered grains (each described by a random matrix from class D), coupled by weak hopping, appear similar to multicomponent models, and may have a metallic phase, as a mapping to a weakly coupled nonlinear σ model would suggest.

VI. CONCLUSION

The main point to emerge from this study is that the important difference between the RBIM and the (one-component) models which possess a metallic phase is that in the former the added vortices, in network model language, occur on one sublattice only, but on both in the latter. This appears to be a necessary condition for the existence of the metallic phase. If the vortices are correlated in pairs, sufficiently strong disorder will be required to produce the metallic phase. This result casts some doubt on whether the metallic phase will occur in applications to one-component 2D superconductors and paired FQHE systems, because these possess only one type of vortex, corresponding to those on only one sublattice in the network. On the other hand, if vortices of only one type are present, but are uncorrelated, this may lead to the destruction of one of the phases, and hence of the phase transition. If such vortices are correlated in pairs, the phase diagram may resemble that of the RBIM. It would be interesting to test this numerically, both on models defined by a Hamiltonian, such as tight-binding models, and on network models, and also for other symmetry classes.

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