Proposed measurements of the small entropy carried by the superfluid component in liquid helium II

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The microscopic theory of liquid helium II due to Fliessbach [T. Fliessbach, Nuovo Cimento D **13**, 211 (1991)] allows a certain amount of entropy to be carried by the superfluid component. The experimental results tell us that the entropy carried by the superfluid component is smaller than 2% with respect to the total entropy. The nonstandard model of liquid helium II deduced from extended thermodynamics is not in contrast with the theory by Fliessbach. In this work, using this model, we show that accurate measurements of the speeds and the attenuations of the first, second, and fourth sounds in liquid helium II allow the determination of this small superfluid entropy, or, at least, they will establish an upper bound for this quantity.

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I. Introduction. The most known model able to describe the anomalous behavior of liquid helium below the λ point is the two-fluid model.[1,2] This model, inspired by considerations of quantum statistical mechanics,[3] treats liquid helium II as a two-component mixture: a normal viscous component, and a superfluid component, able to flow through very thin capillaries and porous media without viscosity. One of the main assumptions of this model is that the entropy of the superfluid fraction vanishes.

As observed by Puttermann,[4] the statement that the superfluid component does not carry entropy has not been demonstrated theoretically, so that it must be regarded as an additional postulate. The experimental results[5,6] tell us that the entropy carried by the superfluid component is smaller than 2% with respect to the entropy carried by the normal component. In Ref. 7 theoretical (microscopic) motivations in favor of a small superfluid entropy have been advanced; an extension of the two-fluid model, where a small entropy transfer associated with the superfluid component is allowed, has been proposed.[8]

A macroscopic nonstandard model of liquid helium II, which is based on the extended thermodynamics (ET),[9,10] has been formulated.[11,12] The fundamental fields of this model are the density ρ, the velocity v_i, the absolute temperature T, and the heat flux q_i. Also in this model, a small entropy transfer by helium which flows through very thin capillaries or porous media is allowed. A comparison between this model and the two-fluid model with superfluid entropy by Schäfer and Fliessbach[8] has been made in Ref. 13.

In this work, using this extended model, we show that this small entropy can be determined through accurate measurements of the speeds and the attenuations of the first, second, and fourth sounds in liquid helium II.

II. The extended model of liquid helium II. A recent approach to nonequilibrium thermodynamics is extended thermodynamics (ET): this is a macroscopic theory of nonequilibrium processes based on a dissipative differential system of balance laws using dissipative fluxes, beside the traditional variables, as independent fields. This theory is particularly useful for studying the thermodynamics of nonequilibrium steady states and systems with long relaxation times.

The fundamental fields of ET are the density ρ, the velocity $\mathbf{v}=(v_i)$, the temperature T, the heat flux $\mathbf{q}=(q_i)$, and the nonequilibrium stress, which we decompose into its trace $3p_V$, p_V being the viscous pressure and its deviatoric part $p_{\langle ij\rangle}$. The linearized system of field equations of the ET of a fluid is[9]

$$\frac{\partial \rho}{\partial t}+\rho\frac{\partial v_j}{\partial x_j}=0, \tag{2.1a}$$

$$\rho\frac{\partial v_i}{\partial t}+\frac{\partial}{\partial x_j}[(p_0+p_V)\delta_{ij}+p_{\langle ij\rangle}]=0, \tag{2.1b}$$

$$\rho\frac{\partial \epsilon}{\partial t}+\frac{\partial q_j}{\partial x_j}+[(p_0+p_V)\delta_{ij}+p_{\langle ij\rangle}]\frac{\partial v_i}{\partial x_j}=0, \tag{2.1c}$$

$$\frac{\partial q_i}{\partial t}+\zeta\frac{\partial T}{\partial x_i}-\beta' T^2\zeta\frac{\partial p_V}{\partial x_j}-\beta T^2\zeta\frac{\partial p_{\langle ij\rangle}}{\partial x_j}=-\frac{1}{\tau_1}q_i, \tag{2.1d}$$

$$\tau_0\frac{\partial p_V}{\partial t}+\zeta\frac{\partial v_j}{\partial x_j}+\beta' T\xi\frac{\partial q_j}{\partial x_j}=-p_V, \tag{2.1e}$$

$$\tau_2\frac{\partial p_{\langle ik\rangle}}{\partial t}+2\eta\frac{\partial}{\partial x_j}\frac{\partial v_{\langle i}}{\partial x_{k\rangle}}-2\beta T\eta\frac{\partial q_{\langle i}}{\partial x_{k\rangle}}=-p_{\langle ik\rangle}. \tag{2.1f}$$

In these equations, p_0 is the pressure of thermostatics, ϵ is the internal specific energy, ξ and η are the bulk and the shear viscosity, τ_0 and τ_2 are the relaxation times of the nonequilibrium pressure p_V and of the stress deviator $p_{\langle ij\rangle}$, τ_1 is the relaxation time of the heat flux q_i, ζ equals κ/τ_1, κ being the thermal conductivity, β and β' are coefficients which characterize the dissipation of thermal nature.

Extended thermodynamics furnishes the following expressions for the entropy flux $\mathbf{J}^s$, the entropy production σ^s, and the Gibbs equation:[9]

$$J_i^s=\rho s v_i+\frac{1}{T}q_i+\beta' p_V q_i+\beta p_{\langle ij\rangle}q_j, \tag{2.2}$$

$$\sigma^s=\frac{1}{\tau_1}\frac{1}{T^2\zeta}\mathbf{q}\cdot\mathbf{q}+\frac{1}{\xi T}p_V p_V+\frac{1}{2\eta T}p_{\langle ij\rangle}p_{\langle ij\rangle}, \tag{2.3}$$

$$T ds=d\epsilon-\frac{p_0}{\rho^2}d\rho-\frac{1}{\rho\zeta T}\mathbf{q}\cdot d\mathbf{q}. \tag{2.4}$$

In Ref. 9 the link between the coefficients ζ, β, and β' and the moments of the fluctuations has been investigated. It is

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found that these coefficients are linked to the moments of the fluctuations of the fluxes by the relations:

$$\zeta=\frac{\kappa}{\tau_1}=\frac{V}{k_B T^2}\langle \delta q_1 \delta q_1\rangle, \tag{2.5}$$

$$\beta=-k_B\frac{\langle C_2 \delta q_1 \delta p_{12}\rangle}{\langle \delta q_1 \delta q_1\rangle\langle \delta p_{12}\delta p_{12}\rangle}, \tag{2.6}$$

$$\beta'=-k_B\frac{\langle C_1 \delta q_1 \delta p_V\rangle}{\langle \delta q_1 \delta q_1\rangle\langle \delta p_V\delta p_V\rangle}. \tag{2.7}$$

In these relations V is the volume, $C_i=c_i-v_i$ the molecular velocity relative to the mean motion, and k_B the Boltzmann constant.

As is well known, in liquid helium II the relaxation time τ_1 of the heat flux is extremely high, while the bulk and the shear viscosity and the relaxation times τ_0 and τ_2 of the nonequilibrium part of the stress are extremely small. In a first approximation, we can neglect the time evolution of the viscous stress, setting zero τ_0 and τ_2 in Eqs. (2.1e) and (2.1f); we obtain

$$p_V=-\xi\left[\frac{\partial v_j}{\partial x_j}-\beta' T\frac{\partial q_j}{\partial x_j}\right], \tag{2.8}$$

$$p_{\langle ij\rangle}=-2\eta\left[\frac{\partial v_{\langle j}}{\partial x_{i\rangle}}-\beta T\frac{\partial q_{\langle j}}{\partial x_{i\rangle}}\right]. \tag{2.9}$$

These relations can be interpreted as constitutive relations for p_V and $p_{\langle ij\rangle}$ in an extended model of liquid helium II in the presence of dissipative phenomena, in which only ρ, T, v_i, and q_i are considered as independent fields. Substituting Eqs. (2.8) and (2.9) in Eqs. (2.1a)–(2.1d) we obtain the following dissipative system of field equations for helium II:

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}+\rho\frac{\partial v_j}{\partial x_j}=0,\\
&\frac{\partial v_i}{\partial t}+\frac{1}{\rho}\frac{\partial p_0}{\partial x_i}-\frac{\xi}{\rho}\frac{\partial}{\partial x_i}\left[\frac{\partial v_j}{\partial x_j}-\beta' T\frac{\partial q_j}{\partial x_j}\right]\\
&\quad-2\frac{\eta}{\rho}\frac{\partial}{\partial x_j}\left[\frac{\partial v_{\langle j}}{\partial x_{i\rangle}}-\beta T\frac{\partial q_{\langle j}}{\partial x_{i\rangle}}\right]=0,\\
&\frac{\partial T}{\partial t}+\frac{T p_T}{\rho c_V}\frac{\partial v_j}{\partial x_j}+\frac{1}{\rho c_V}\frac{\partial q_j}{\partial x_j}=0,\\
&\frac{\partial q_i}{\partial t}+\zeta\frac{\partial T}{\partial x_i}+\xi\beta' T^2\zeta\frac{\partial}{\partial x_i}\left[\frac{\partial v_j}{\partial x_j}-\beta' T\frac{\partial q_j}{\partial x_j}\right]\\
&\quad+2\eta\beta T^2\zeta\frac{\partial}{\partial x_j}\left[\frac{\partial v_{\langle j}}{\partial x_{i\rangle}}-\beta T\frac{\partial q_{\langle j}}{\partial x_{i\rangle}}\right]=-\frac{1}{\tau_1}q_i .
\end{aligned} \tag{2.10}$$

In Eqs. (2.10) c_V is the constant volume specific heat and $p_T=\partial p_0/\partial T$.

Equations (2.10) describe the propagation in liquid helium II of two longitudinal waves: the normal sound wave and the temperature wave. Setting

$$V_1^2=p_\rho, \quad V_2^2=\zeta/\rho c_V, \tag{2.11}$$

the phase speeds $w=\omega/k_r$ of these waves are solutions of the following equation:[10,11]

$$w^4+\left(V_1^2+V_2^2+\frac{T p_T^2}{\rho^2 c_V}\right)w^2+V_1^2V_2^2=0. \tag{2.12}$$

The attenuation coefficients for these sounds have been calculated in Ref. 14. Their expressions contain terms including products of the thermal expansion and of the coefficients ξ, η, and $1/\tau_1$; neglecting these terms, the attenuation coefficients can be written as

$$k_s^{(1)}\simeq\frac{\omega^2}{2\rho w_1^3}\left(\xi+\frac{4}{3}\eta\right), \tag{2.13}$$

$$k_s^{(2)}\simeq\frac{1}{2w_2}\frac{1}{\tau_1}+\frac{\omega^2 T^3\zeta}{2w_2^3}\left(\xi\beta'^2+\frac{4}{3}\eta\beta^2\right). \tag{2.14}$$

This nonstandard model of helium II also explains the experiment known as the (static) fountain effect.[15] Denoting with μ the equilibrium chemical potential, from the Gibbs equation (2.4), in the linear approximation we have obtained $d\mu=(1/\rho)dp-\eta dT$. As is known in the experiment of the fountain effect, the equilibrium is reached as long as $d\mu=0$, so that the following relation, identical to the one obtained by the two-fluid model, is found:[15]

$$(1/\rho)\,dp=s\,dT. \tag{2.15}$$

In Refs. 16–18 this study has been further developed. It was shown that the theory allows the existence of two transverse modes: one of them has a propagation speed near zero, while the other one has a finite propagation speed and a very small penetration depth.[16] In each mode a particular linear combination of the fields v_i and q_i vibrates. In such a way, the two new vector fields

$$u_i^{(n)}=v_i-\beta T q_i \tag{2.16a}$$

and

$$u_i^{(s)}=v_i+(\beta T/P-1)\,q_i \tag{2.16b}$$

with

$$P=1+\rho\beta^2T^3\zeta, \tag{2.16c}$$

having the dimension of a velocity, have been introduced, which can be interpreted as the velocities of the normal and superfluid components of liquid helium II.[16] We have shown that if the theory is reformulated in terms of these fields, no dissipative terms associated with the shear viscosity are present in the field equation for $u_i^{(n)}$, while such terms exist in the field equation for $u_i^{(n)}$. Consequently, these two fields can be interpreted as the velocities of the superfluid and normal components.[16]

In Ref. 17 we also introduced the two scalar fields

$$\rho^{(s)}=(P-1/P)\,\rho, \quad \rho^{(n)}=(1/P)\,\rho, \tag{2.17}$$

associated with $u_i^{(s)}$ and $u_i^{(n)}$, which can be interpreted as the ''densities'' of the superfluid and normal components; furthermore, from Eqs. (2.16a) and (2.16b) we obtained the following expressions for v_i and q_i in terms of $\rho^{(s)}$, $\rho^{(n)}$, $u_i^{(s)}$, and $u_i^{(n)}$:

$$v_i=\frac{\rho^{(s)}}{\rho}u_i^{(s)}+\frac{\rho^{(n)}}{\rho}u_i^{(n)}, \tag{2.18a}$$

$$q_i=\frac{1}{\beta T}\frac{\rho^{(s)}}{\rho}(u_i^{(s)}-u_i^{(n)}). \tag{2.18b}$$

In Ref. 17 we have also shown that, in terms of $\rho^{(s)}$, $\rho^{(s)}$, $u_i^{(s)}$, and $u_i^{(n)}$, the expression for the ''convective'' part of the entropy flux can be written as

$$[J_i^s]_u=\rho^{(s)}\left(s+\frac{1}{\rho\beta T^2}\right)u_i^{(s)}+\rho^{(n)}\left(s-\frac{P-1}{\rho\beta T^2}\right)u_i^{(n)}. \tag{2.19}$$

In particular, the following quantities have been introduced:[17]

$$s_s=s+1/\rho\beta T^2, \tag{2.20a}$$

$$s_n=s-(P-1)/\rho\beta T^2, \tag{2.20b}$$

which can be interpreted as the entropy of the superfluid and normal component.

In Ref. 17 we have also investigated under what condition in the extended model the superfluid component velocity is curl-free: we have shown that in the linear approximation, the field $u_i^{(s)}$ is curl-free if and only if the superfluid entropy s_s depends on the temperature only.

Finally, in Refs. 14 and 18 the propagation of waves through a very thin capillary (superleak) has been studied: the theory allows the existence of a wave (the fourth sound) in which all the thermodynamic fields vibrate. The speed and the attenuation coefficient of this wave are[13]

$$w_4^2=\left(\frac{\omega}{k_r}\right)_4^2=\frac{P-1}{P}V_1^2+\frac{1}{P}V_2^2(1+\beta T^2 p_T)^2, \tag{2.21}$$

$$k_s^{(4)}=\frac{\xi}{2\rho}\frac{\omega^2}{w_4^3}\left(1-\frac{\beta'}{\beta}\right)^2\frac{P-1}{P}+\frac{1}{2w_4}\frac{1}{\tau_1}\frac{1}{P}. \tag{2.22}$$

In Ref. 17 we have shown that in the linear approximation, the systems of field equations of the extended and the two-fluid model can be identified, if we set

$$\beta=\beta^*:=-1/\rho s T^2. \tag{2.23}$$

In fact, under this hypothesis, the expression (2.18b) of the heat flux q_i is identical with that of the two-fluid model of Tisza[1] and Landau;[2] furthermore, observing that $P-1=\rho^{(s)}/\rho^{(n)}$, the expression of the quantity V_2, characterizing the second sound velocity becomes

$$V_2^{*2}:=\frac{\zeta^*}{\rho c_V} \quad \text{where} \quad \zeta^*:=\rho\frac{\rho^{(s)}}{\rho^{(n)}}Ts^2, \tag{2.24}$$

which is identical to that of the two-fluid model. Finally, we observe that if β equals β^*, the convective part of the entropy flux also reduces just to the expression of the entropy flux postulated by Landau.[2] $[J_i^s]_u=\rho s u_i^{(n)}$. On the contrary, if β is different from β^*, the superfluid component carries a certain amount of entropy.

III. Influence of the superfluid entropy on the sound propagation. In this section we show that the small entropy carried by the superfluid component can be determined through accurate measurements of the speeds and the attenuations of the first, second, and fourth sounds in liquid helium II.

First we observe that as there is experimental evidence that helium II can flow through very small capillaries or porous media for weeks, in order to compare the model just developed with experimental data we can set

$$1/\tau_1=0. \tag{3.1}$$

Now, we write Eq. (2.20a), using Eq. (2.23), as

$$\frac{s_s}{s}=1-\beta^*/\beta. \tag{3.2}$$

From this relation we deduce that the superfluid entropy influences only the quantities depending explicitly on the parameter β. In what follows, we will use an asterisk to denote the quantities referring to the case when the superfluid component does not carry entropy.

First, we study the dependence of V_1 and V_2 on the superfluid entropy s_s. Obviously, V_1 does not depend on s_s, and, because in helium II the thermal dilation is small, from Eq. (2.12) we deduce that in a first approximation the velocity of the first sound can be taken equal to V_1; consequently, neither the velocity nor the attenuation of the first sound depend perceptibly on s_s,

$$w_1\simeq w_1^*, \quad k_s^{(1)}\simeq[k_s^{(1)}]^*. \tag{3.3}$$

On the contrary, the presence of entropy associated with the superfluid component influences both the velocity and the attenuation of the second sound. Indeed, using (2.16c) we can write

$$\zeta=\zeta^*(1-s_s/s)^2 \tag{3.4}$$

so that

$$V_2=V_2^*(1-s_s/s). \tag{3.5}$$

Remembering the expression (2.24) of V_2^* we finally obtain

$$V_2^2=\frac{\rho^{(s)}}{\rho^{(n)}}\frac{T}{c_V}(s-s_s)^2. \tag{3.6}$$

This expression, identical to the one obtained in Ref. 8, allows the determination of the difference $s-s_s$, if the quantities V_2 and $\rho^{(s)}/\rho^{(n)}$ are known.

We now study the dependence of the attenuation of the second sound on s_s. Using Eqs. (3.1), (3.2), (3.4), and (3.5), the expression (2.14) can be written as

$$k_s^{(2)}=\frac{\omega^2\zeta^* T^3}{2V_2^{*3}}\left[\frac{\xi_0\beta'^2}{1-\frac{s_s}{s}}+\frac{4}{3}\frac{\eta\beta^{*2}}{\left(1-\frac{s_s}{s}\right)^3}\right]. \tag{3.7}$$

If, in a simplified analysis, we suppose that the superfluid, although it carries a small amount of entropy, does not produce it (this amounts to setting $\beta=\beta'=\beta^*[1-(s_s/s)]$), we obtain

$$k_s^{(2)}=k_s^{(2)*}(1-s_s/s)^{-3}. \tag{3.8}$$

We conclude that the presence of superfluid entropy has the effect of reducing the velocity of the second sound and of increasing its attenuation. If we suppose that the ratio s_s/s is equal to 2%, an equal variation in percent is obtained in the velocity of the second sound and a percent variation in the attenuation equal to 8%. With the present experimental accuracy these corrections might be observable.

Finally, the expression of the velocity of the fourth sound, in term of s_s, is

$$w_4^2=\frac{P-1}{P}V_1^{*2}+\frac{1}{P}V_2^{*2}\left(1-\frac{s_s}{s}\right)^2\left[1-\frac{p_T}{\rho s}\left(1-\frac{s_s}{s}\right)\right]^2. \tag{3.9}$$

Because $V_1^2 \gg V_2^2$, the first term in Eq. (3.9) is dominant; therefore, the presence of the superfluid entropy s_s has a negligible influence on the velocity on the fourth sound. The attenuation of the fourth sound is zero, if $\beta'=\beta$, independently of s_s. The expression (2.22) of the attenuation coefficient of the fourth sound, if $\beta'\neq\beta\neq\beta^*$, becomes

$$k_s^{(4)}=\frac{\xi}{2\rho}\frac{\omega^2}{w_4^3}\left[1-\frac{\beta'}{\beta^*}\left(1-\frac{s_s}{s}\right)\right]^2\frac{P-1}{P}. \tag{3.10}$$

We conclude that the presence of superfluid entropy produces an increase of the attenuation of the fourth sound.

IV. Determination of the superfluid entropy from sound velocity data. Equation (2.20a) is a simple expression that allows us to evaluate the small amount of entropy carried by the superfluid component of helium if accurate measurements of the velocities of the sounds and of the total entropy s are available.

Starting from the knowledge of p_T, c_V, and of the velocities of the first and second sound, we can determine V_1^2 and V_2^2, observing that, on the basis of Eq. (2.12), they are the solutions of

$$V^4+\left(w_1^2+w_2^2-\frac{Tp_T^2}{\rho^2 c_V}\right)V^2+w_1^2w_2^2=0. \tag{4.1}$$

Then, remembering that $-1/\rho T^2\beta = s-s_s$, from Eq. (2.21), we obtain

$$(s-s_s)^2+\frac{2p_T V_2^2}{\rho(w_4^2-V_2^2)}(s-s_s) -\left[\frac{V_2^2 c_V}{T}\frac{V_1^2-w_4^2}{w_4^2-V_2^2}+\frac{p_T^2V_2^2}{\rho^2(w_4^2-V_2^2)}\right]=0, \tag{4.2}$$

which allows the determination of $s-s_s$ as a function of pressure and temperature: knowing that $s-s_s$ must be positive, we will choose the positive root.

V. Final remarks. The relations obtained in this work, together with direct measurements of the sound speeds and of s, p_T, and c_V, can allow the determination of the small percent of entropy carried by the superfluid component, or, at least, they will establish an upper bound for this quantity. Obviously, in this computation we cannot use, for example, the data reported by Maynard,[19] which are obtained from the velocities of the first, second, and fourth sounds using relations obtained from the two-fluid model: we remark on the fact that Maynard uses Eq. (2.24) in order to determine the entropy s, which is valid only if the superfluid component does not carry entropy.

Finally we consider Fig. 6 of Ref. 19: in this figure both values of entropy, as derived from the two-fluid model and as obtained from the fountain effect, are reported from various values from p and T: we note that at lower temperatures (1.2, 1.3 K), when the superfluid component is more important, the values obtained through the fountain effect are higher than those calculated by Maynard starting from data on the velocities of sounds in helium II, using the two-fluid model. These results agree with the deductions of this work: in fact, as temperature rise, the normal component dominates and the values of entropy obtained through the fountain effect are less different from the values obtained by Maynard, as a consequence of the fact that the amount of the superfluid component is low.

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