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Transmission through quantum networks

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We use a simple formalism to calculate the conductance of any quantum network made of one-dimensional quantum wires. We apply this method to analyze, for two periodic systems, the modulation of this conductance with respect to the magnetic field. We also study the influence of an elastic disorder on the periodicity of the Aharanov-Bohm oscillations, and we show that a recently proposed localization mechanism induced by the magnetic field resists such a perturbation. Finally, we discuss the relevance of this approach for the understanding of a recent experiment on GaAs/GaAlAs networks.

It is well known that quantum transport exhibits deviations from classical transport, resulting in corrections to the classical addition rules of conductances or resistances. A spectacular example is the Aharonov-Bohm (AB) effect, where the conductance of a ring is a periodic function of the magnetic flux ϕ through its opening, with period $\phi_0 = h/e$. Since the observation of this effect in condensed matter,[1] many papers have been devoted to the study of coherence effects in transport, especially in the ring geometry. One approach uses the Landauer formalism in which the conductance is proportional to the transmission coefficient. In this framework, disorder effects have been considered in single-channel[2] and multichannel rings.[3] On the other hand, the conductance of diffusive systems has been also extensively studied within the Kubo approach, where the weak-localization correction is related to the modulation by the magnetic field of the return probability of a diffusive particule.[4] Although being a transport property, this correction is a spectral quantity, since it is related to the spectrum of the diffusion equation, more precisely to its spectral determinant.[5]

In this paper, we focus on the transmission properties of quantum networks, generalizing the original works of the 1980s.[2,6] This work is motivated by recent conductance measurements of normal metallic networks etched on a two-dimensional (2D) GaAs/GaAlAs electron gas.[7] Remarkably, for the particular $\mathcal{T}_3$ network shown in Fig. 1, the magnetoresistance presents large ϕ_0-periodic oscillations which are barely visible for a more conventional geometry like the square lattice. This is the first observation of strong ϕ_0-periodic oscillations in a macroscopic system where, in principle, ensemble average due to a finite coherence length is expected to destroy them.

The experimental study of the $\mathcal{T}_3$ network has been motivated by the recent prediction of a type of magnetic-field-induced localization. Indeed, it has been shown, in a tight-binding approach, that when the flux ϕ per elementary plaquette equals $\phi_0/2$ (half-flux), the electron motion is completely confined inside the so-called AB cages[8] resulting from a subtle quantum interference effect. This surprising phenomenon has been experimentally observed in superconducting ($\mathcal{T}_3$) networks,[9] where it was found that the critical current almost vanishes at $\phi = \phi_0/2$. The standard mapping between the Ginzburg-Landau theory and the tight-binding problem[10] actually allows one to relate this current to the energy band curvature, predicting a zero critical current at half-flux. However, it is interesting to know whether this localization effect still exists in normal metallic networks and if it could be at the origin of the oscillations discussed above.[7]

The aim of this paper is threefold. First, we describe a simple formalism allowing one to calculate the transmission coefficient of any network made up of one-dimensional wires. Second, we concentrate on two regular structures, the square and the $\mathcal{T}_3$ networks and study the flux dependence of the transmission coefficient which is reminiscent of the butterfly-like structure of the tight-binding spectrum. We then consider the influence of elastic disorder that we model by a distribution of the wire lengths. We show that the $\mathcal{T}_3$ network exhibits ϕ_0-periodic oscillations which are robust with respect to disorder and which are much larger than those observed in the square network. We also discuss the crossover from a ballistic (in the pure case) to a disorder dominated behavior, revealed by the emergence of $\phi_0/2$-periodic oscillations reminiscent of the weak-localization regime. This model gives strong support to the interpretation of the above-mentioned experiment[7] in terms of the AB cages.

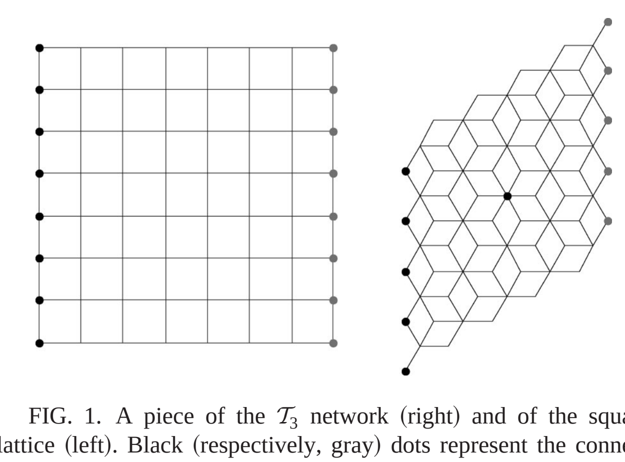

FIG. 1. A piece of the $\mathcal{T}_3$ network (right) and of the square lattice (left). Black (respectively, gray) dots represent the connections to the input (respectively, output) channels. The central black dot is the input channel chosen for the bulk injection.

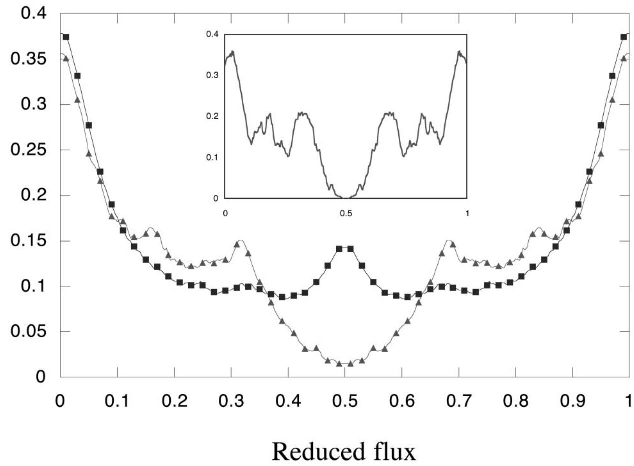

FIG. 2. Averaged transmission coefficient $\langle T(k,f)\rangle_k/N_{in}$ as a function of the reduced flux for a (8×8) square lattice (square) and a piece of the $\mathcal{T}_3$ network (triangle) with 75 sites. Input and output channels are connected as displayed in Fig. 1. Inset: Averaged transmission coefficient for the $\mathcal{T}_3$ network with one input channel at the center of the network.

We consider a graph made up of N nodes and connected to N_{in} wires (also called channels) defining the input reservoir and to N_{out} wires defining the outgoing reservoir (see Fig. 1). In the Landauer approach, the two-terminal conductance is proportional to the total transmission coefficient defined by

$$T=\sum_{i,j}|t_{ij}|^2, \tag{1}$$

where $i\in[1,N_{in}]$ denotes the ith input channel and $j\in[N-N_{out}+1,N]$ denotes the jth output channel. This coefficient is the sum of each individual transmission coefficient obtained by injecting a wavepacket in the ith channel. We emphasize that actually Eq. (1) assumes that there is no phase relationship between the different input channels.[11]

Let us consider an incoming wave function in the ith channel defined by

$$\psi(x)=e^{-ikx}+r_{ii}e^{ikx}, \tag{2}$$

where r_{ii} is the reflexion coefficient in this wire. We need to determine the transmission coefficient t_{ij} giving the probability for the wave packet to outgo into the jth channel. Therefore, we first solve the Schrödinger equation on each bond whose extremities are denoted by α and β. The corresponding eigenfunctions are simply given by

$$\psi_{\alpha\beta}(x)=\frac{e^{-i\gamma_{\alpha x}}}{\sin kl_{\alpha\beta}}[\psi_\alpha\sin k(l_{\alpha\beta}-x)+e^{i\gamma_{\alpha\beta}}\psi_\beta\sin kx], \tag{3}$$

where $\gamma_{\alpha\beta}=(2\pi/\phi_0)\int_\alpha^\beta\mathbf{A}\cdot d\boldsymbol{l}$ is the circulation of the vector potential between α and β, k is the wave vector related to the eigenenergy by $E(k)=\hbar^2k^2/2m$, x is the distance measured from the node α, and $l_{\alpha\beta}$ is the length of the bond (α,β), and $\psi_{\alpha,\beta}=\psi_{\alpha\beta}(x=\alpha,\beta)$. The current conservation on each internal node of the network (not connected to reservoirs) is satisfied if

$$M_{\alpha\alpha}\psi_\alpha+\sum_{\langle\alpha,\beta\rangle}M_{\alpha\beta}\psi_\beta=0, \tag{4}$$

where M is a $(N\times N)$ matrix whose elements are given by

$$M_{\alpha\alpha}=\sum_{\langle\alpha,\beta\rangle}\cot kl_{\alpha\beta},\quad M_{\alpha\beta}=-\frac{e^{i\gamma_{\alpha\beta}}}{\sin kl_{\alpha\beta}}. \tag{5}$$

The symbol $\langle\alpha,\beta\rangle$ indicates that the sums extend to all the nodes β connected to the node α. In addition, the off-diagonal element $M_{\alpha\beta}$ is nonzero only if the nodes α and β are connected by a bond. Consider now the case where the current is injected in the channel $i\in[1,N_{in}]$. The current conservation at this node writes

$$M_{ii}\psi_i+\sum_{\langle i,\beta\rangle}M_{i\beta}\psi_\beta=i(1-r_{ii}). \tag{6}$$

For each node $j\in[N-N_{out}+1,N]$, one also has

$$M_{jj}\psi_j+\sum_{\langle j,\beta\rangle}M_{j\beta}\psi_\beta=-it_{ij}. \tag{7}$$

Finally, for $i\in[1,N_{in}]$ and $j\in[N-N_{out}+1,N]$, the continuity of the wave function reads $\psi_i=1+r_{ii}$ and $\psi_j=t_{ij}$. Equations (4),(6),(7) constitute a $(N\times N)$ linear system[12] from which $t_{ij}=\psi_j(j\in[N-N_{out}+1,N])$ can be calculated. The total transmission coefficient is finally obtained from Eq. (1) by considering the N_{in} input channels.

We now apply this formalism to the case of regular networks where all the bonds have identical length l so that the transmission coefficient $T(k,f)$ is a periodic function of the wave vector k with period $2\pi/l$ and a periodic function of the reduced flux $f=\phi/\phi_0$ with period 1. In principle, the k dependence of the transmission coefficient can be probed experimentally if the wave vector k is well defined, i.e., if the energy of injected electrons is well controlled. Several factors like finite temperature or finite bias contribute to broaden this energy. This can be taken into account by giving a finite width Δk to the Fermi wave vector of the incoming wave packet. For example, in Ref. 14 the conductance of a single ring was measured and it was found that the phase of the AB oscillations could be varied by tuning the gate voltage, and thus the Fermi energy. One may therefore conclude that the width Δk is smaller than the period $2\pi/l$. These oscillations are very well described by a Landauer single-channel formalism, assuming that the ring is assymetric, i.e., the two arms have a different length.[2,14]

For a given k, the flux dependence of $T(k,f)$ has a rich structure which is reminiscent of the complexity of the associated tight-binding spectrum. Here, for simplicity, we have chosen to average the transmission coefficient over a period $k\in[0,2\pi/l]$. The flux dependence of the average transmission $\langle T(k,f)\rangle_k$ is shown in Fig. 2 for the square and $\mathcal{T}_3$ networks. One clearly observes a few peaks in the transmission for particular values of the reduced flux: $f=1/2,1/3$ for the square lattice and $f=1/3,1/6$ for the $\mathcal{T}_3$ lattice. One can simply understand this structure by invoking the extended nature of the corresponding eigenstates that are Bloch waves with a spatial period proportional to the denominator of f.[13] Due to the existence of the AB cages, the transmission coef-

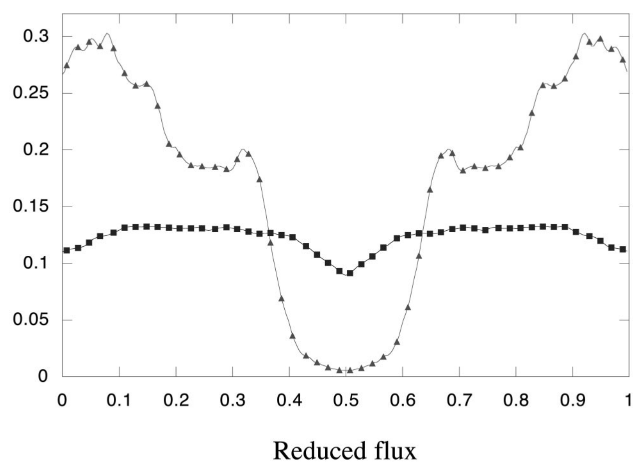

FIG. 3. Transmission coefficient $\langle T(k,f)\rangle_{dis}/N_{in}$ averaged over 50 configurations of disorder for $kl=\pi/3$ and $k\Delta l=1.47$ as a function of the reduced flux.

ficient is minimum at $f=1/2$ for the $\mathcal{T}_3$ network but, surprisingly, it is not exactly zero. This is due to the existence of dispersive edge states[15] that are able to carry current even for $f=1/2$. Therefore, T converges toward a finite value for the $\mathcal{T}_3$ network when the system size (and N_{in}) increases, whereas $T\sim N_{in}$ in the square lattice. However, when one injects current in the bulk of the sample, the transmission completely vanishes for this flux (see the inset of Fig. 2). This study shows that the cage effect, originally predicted in a tight-binding model, also arises in a $\mathcal{T}_3$ network made up of one-dimensional ballistic wires.

We now consider the case of disordered networks, the motivation being to see whether the cage phenomenon per-

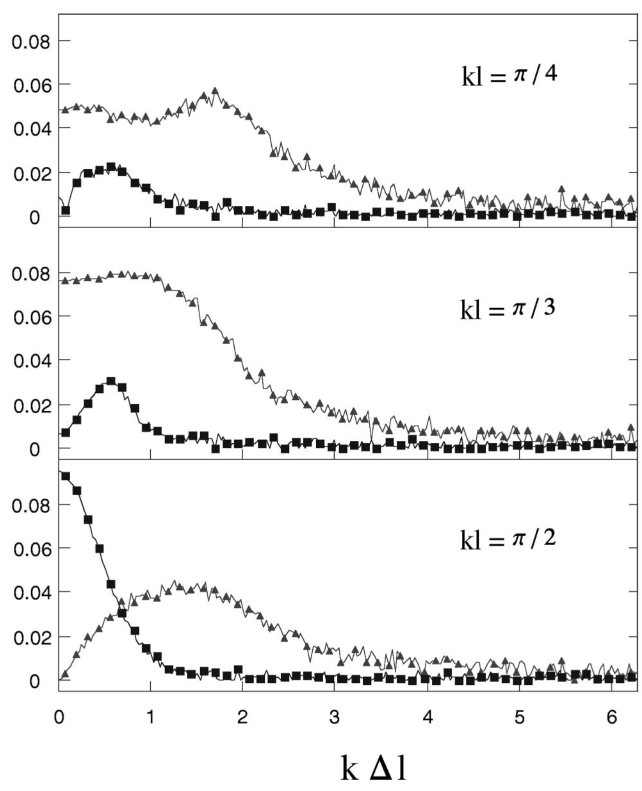

FIG. 4. Variation of $\langle|\tilde{T}(k,1)|\rangle_{dis}/N_{in}$ versus disorder for different values of k after averaging over 50 configurations.

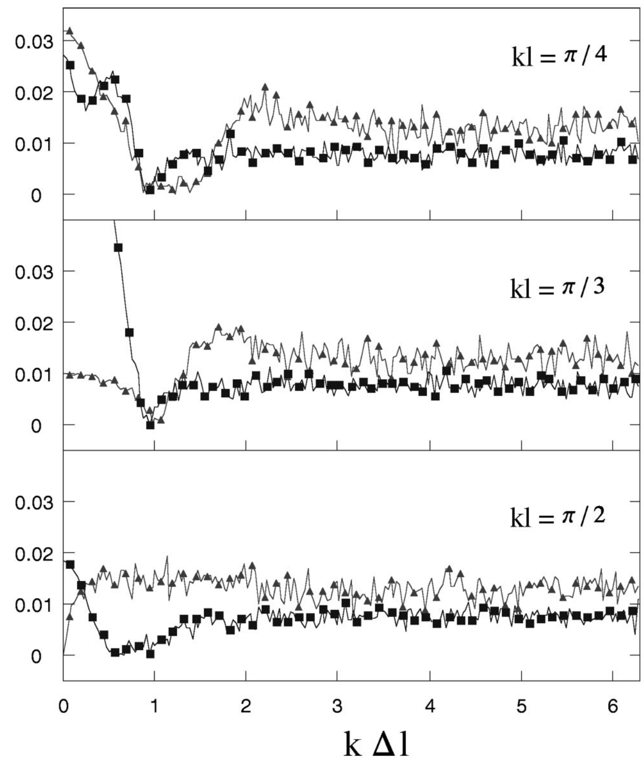

FIG. 5. Amplitude of the second harmonic $\langle|\tilde{T}(k,2)|\rangle_{dis}/N_{in}$ versus disorder for different values of k after averaging over 50 configurations.

sists in such a situation. Disorder can be introduced in several ways (randomly distributed pointlike scatterers, or more generally, random elastic scattering matrix along the bonds). Here, in order to simulate random phase shifts on each bond, we consider a geometrical disorder defined by a random modulation of the wire lengths while keeping the same connectivity. Denoting by Δl the amplitude of the length fluctuations, the relevant dimensionless parameter to characterize the strength of the disorder is the quantity $k\Delta l$ and thus explicitly depends on the energy. Note that the incommensurability between the different lengths breaks the periodicity of T with respect to k. This type of disorder also provides a distribution of areas of width $2l\Delta l$ so that the oscillations are expected to disappear after about $l/\Delta l$ periods. In the following, we will focus on physical situations where $\Delta l/l \ll 1$ and $kl \gg 1$ so that the case $k\Delta l\sim 1$ may be reached without a sizeable dispersion of the areas. Thus, we will not modify the bond lengths in the phase factor $e^{i\gamma_{\alpha\beta}}$ so that the periodicity with respect to kl (for fixed $k\Delta l$) and with respect to the reduced flux will be conserved.

For a given realization of disorder, $T(k,f)$ exhibits a ϕ_0-periodic complex structure which is a signature of the interference pattern through the network. In particular, the transmission extremely sensitive to k. However, experimentally, there is always a finite phase coherence length L_φ. Therefore, a two-dimensional network of typical linear size L must be considered as a set of $(L/L_\varphi)^2$ regions without phase relationship. This provides a natural averaging mechanism over disorder realizations. Thus, we have chosen to study the disorder-averaged transmission coefficient $\langle T(k,f)\rangle_{dis}$ whose variations versus the reduced flux are displayed in

Fig. 3 for fixed k and disorder strength. It is clearly seen that for the square network, the periodicity of $\langle T(k,f)\rangle_{dis}$ with respect to the magnetic flux is no longer ϕ_0 but $\phi_0/2$. The ϕ_0-periodic oscillations have been washed out since they do not have a given phase. By contrast, the $\phi_0/2$-periodic oscillations are still present since they are related to phase-coherent pairs of time-reversed trajectories according to the weak-localization picture. For the $\mathcal{T}_3$ network, the transmission coefficient remains ϕ_0 periodic with a large amplitude. This strongly suggests that the cage effect (which locks the phase of the oscillations) survives for this strength of disorder.

For a finer analysis, it is interesting to compute the discrete Fourier transform of T defined by

$$\tilde{T}(k,\omega)=\frac{1}{n}\sum_{j=0}^{n-1} T(k,j/n)e^{i2\pi\omega j/n}, \quad \omega\in[0,n-1], \quad (8)$$

where n is the number of sampled values of f. Figure 4 displays $\langle|\tilde{T}(k,1)|\rangle_{dis}$ as a function of the disorder strength $k\Delta l$ for different values of k. It shows that, when disorder is increased, $\langle|\tilde{T}(k,1)|\rangle_{dis}$ persists much longer for the $\mathcal{T}_3$ network than for the square network. We are thus led to conclude that the cage effect is robust with respect to disorder. Note that for weak disorder, $\langle|\tilde{T}(k,1)|\rangle_{dis}$ depends on k but this dependence vanishes for $k\Delta l\gtrsim 2$. We strongly believe that this result explains why a ϕ_0-periodic conductance is observed experimentally for the $\mathcal{T}_3$ network while it is not for the square lattice.[7]

The behavior of $\langle|\tilde{T}(k,2)|\rangle_{dis}$ is shown in Fig. 5. It is interesting to see that this harmonic becomes quickly dominant for both networks and remains constant for $k\Delta l\gtrsim 2$. The value of this constant depends on the system size and converges to zero for the infinite lattice. Nevertheless, we leave the precise analysis of this scaling for further studies.

Finally, it should be stressed that, experimentally, $\langle|\tilde{T}(k,2)|\rangle_{dis}$ is further reduced by a factor e^{-2L/L_φ} due to a finite coherence length L_φ, while the ϕ_0 contribution is only reduced by a factor e^{-L/L_φ}, L being the perimeter of an elementary plaquette.[4]

In conclusion, we have used a simple and general formalism to calculate the transmission coefficient of any network made up of single-channel quantum wires. This coefficient can be simply expressed in terms of a connectivitylike matrix. We have used this formalism to study the AB cage phenomenon in the $\mathcal{T}_3$ network and we have shown that this effect is robust to a moderate amount of elastic disorder. As a consequence, the AB oscillations with period ϕ_0 persist in the infinite $\mathcal{T}_3$ networks whereas they vanish in the square lattice.

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[13] Note that since the peak resolution is set by the size of the sample, one would observe higher-order peaks for much bigger systems.

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