Theory of disordered itinerant ferromagnets. II. Metal-insulator transition

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The theory for disordered itinerant ferromagnets developed in a previous paper is used to construct a simple effective-field theory that is capable of describing the quantum phase transition from a ferromagnetic metal to a ferromagnetic insulator. It is shown that this transition is in the same universality class as the one from a paramagnetic metal to a paramagnetic insulator in the presence of an external magnetic field and that strong corrections to scaling exist in this universality class. The experimental consequences of these results are discussed.

I. INTRODUCTION

It is well known that interacting electrons in the presence of quenched disorder at zero temperature form a disordered Fermi-liquid or paramagnetic metal state that shows, with increasing disorder, an instability against the formation of an insulator. This Anderson-Mott transition (PM to PI in the schematic phase diagram shown in Fig. 1) is believed to be the metal-insulator transition observed in doped semiconductors and other disordered electron systems, and it has been studied theoretically in considerable detail.[1,2] Similarly, with increasing exchange interaction, the Fermi-liquid state is unstable against the formation of long-range ferromagnetic order (PM to FM in Fig. 1).

This quantum phase transition has also been studied, both with and without quenched impurities.[4] In the Fermi-liquid phase, the PM-PI transition is preceded by nonanalyticities of various observables (e.g., the conductivity, the tunneling density of states, the spin susceptibility, etc.) as functions of wave number, frequency, or temperature. These nonanalyticities are often referred to as ''weak-localization effects.'' They are caused by soft modes, viz. diffusive particle-hole excitations (''diffusons''), and can be studied in perturbation theory.[5] The diffusons are known to drive the metal-insulator transition, at least near two dimensions, which is the lower critical dimensionality for this transition. The analogous soft-mode effects in the metallic ferromagnetic state have recently been investigated in Ref. 6 (to be referred to as paper I), but the quantum phase transition that must occur from a ferromagnetic metal to a ferromagnetic insulator upon increasing the disorder (FM to FI in Fig. 1) has never been considered.

In this paper we address the latter problem. In particular, we derive and analyze an effective-field theory that is capable of describing the disorder driven transition from a ferromagnetic metal to a ferromagnetic insulator. An interesting theoretical question that arises in this context is the role of the Goldstone modes that occur due to the broken spin rotational symmetry, i.e., the spin waves. Since they constitute soft modes in addition to the diffusons, one would *a priori* expect them to influence the critical behavior. It was shown in paper I that the Goldstone modes, while they contribute to the leading frequency nonanalyticity of $O(\Omega^{(d-2)/2})$ in the conductivity, yield a prefactor that is of $O(1)$, while the diffusons contribute a prefactor that is of $O(1/(d-2))$ and thus diverges as $d \rightarrow 2$. Since it is known that this singularity drives the transition near two dimensions,[7] it follows that the Goldstone modes do not contribute to the asymptotic critical behavior. Largely as a consequence of this, the ferromagnetic-metal–to–ferromagnetic-insulator transition turns out to be in the same universality class as the one from a paramagnetic metal to a paramagnetic insulator in the presence of an external magnetic field. We find strong corrections to scaling for this universality class.[8]

Another important motivation for the present study is the recently observed apparent metal-insulator transition in Si metal-oxide-semiconductor field-effect transistors (MOSFETs) and other two-dimensional (2D) electron systems,[9] which contradicts the orthodox theoretical results that predict an insulating state in $d=2$ even for arbitrarily weak disorder.[10,1,2] Since it is known that magnetic fluctuations have a tendency to increase the conductivity in or close to two dimensions,[2,11] it is conceivable that there might be a ferromagnetic metallic phase at small but nonzero disorder in $d=2$. We find that this is not the case, which rules out a

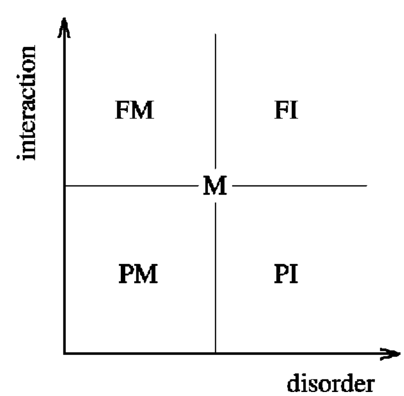

FIG. 1. Schematic phase diagram of disordered, interacting electrons, showing paramagnetic metal (PM), paramagnetic insulator (PI), ferromagnetic metal (FM), and ferromagnetic insulator (FI) phases. *M* denotes a multicritical point.

possible mechanism for a metal-insulator transition in $d=2$.

This paper is organized as follows. In Sec. II we briefly recall the general Q-matrix field theory for itinerant ferromagnets that was developed in paper I. On the basis of this, and using the perturbative results of paper I, we construct an effective theory for the most relevant soft modes in the system. This theory takes the form of a generalized nonlinear σ model. In Sec. III we show that this model describes a ferromagnetic-metal–to–ferromagnetic-insulator transition in $d>2$, and we calculate the critical behavior at this transition in a $d=2+\epsilon$ expansion. In Sec. IV we conclude with a discussion of our results.

II. EFFECTIVE FIELD THEORY FOR DISORDERED ITINERANT FERROMAGNETS

A. Q-matrix theory

In paper I it was shown that disordered itinerant ferromagnets are described by the following action:

$$\mathcal{A}[Q,\tilde{\Lambda}]=\mathcal{A}_{\text{dis}}+\mathcal{A}_{\text{int}}+\frac{1}{2}\,\text{Tr}\ln(G_0^{-1}-i\tilde{\Lambda})+\int d\mathbf{x}\,\text{tr}[\tilde{\Lambda}(\mathbf{x})\,Q(\mathbf{x})]. \tag{2.1a}$$

Here

$$G_0^{-1}=-\partial_\tau+\partial_{\mathbf{x}}^2/2m_{\text{e}}+\mu \tag{2.1b}$$

is the inverse free-electron Green operator, with ∂_τ and $\partial_{\mathbf{x}}$ derivatives with respect to imaginary time and position, respectively, m_{e} is the electron mass, and μ the chemical potential. Q and $\tilde{\Lambda}$ are matrix fields that carry two Matsubara frequency indices n and m, and two replica indices α and β. The matrix elements $Q_{nm}^{\alpha\beta}$ and $\tilde{\Lambda}_{nm}^{\alpha\beta}$ are spin-quaternion valued. They are conveniently expanded in a basis

$$Q_{nm}^{\alpha\beta}(\mathbf{x})=\sum_{r,i=0}^{3}(\tau_r\otimes s_i)\,{}_r^iQ_{nm}^{\alpha\beta}(\mathbf{x}) \tag{2.1c}$$

and analogously for $\tilde{\Lambda}$. Here $\tau_0=s_0=\mathbb{1}_2$ is the 2×2 unit matrix and $\tau_j=-s_j=-i\sigma_j$ $(j=1,2,3)$, with $\sigma_{1,2,3}$ the Pauli matrices. In this basis, $i=0$ and $i=1,2,3$ describe the spin singlet and the spin triplet, respectively. An explicit calculation reveals that $r=0,3$ corresponds to the particle-hole channel while $r=1,2$ describes the particle-particle channel. In Eq. (2.1a), Tr denotes a trace over all degrees of freedom, including the continuous position variable, while tr is a trace over all those discrete indices that are not explicitly shown. For the disorder part of the action one finds[12]

$$\mathcal{A}_{\text{dis}}[Q]=\frac{1}{\pi N_F\tau}\int d\mathbf{x}\,\text{tr}[Q(\mathbf{x})]^2, \tag{2.2}$$

with τ the single-particle scattering or relaxation time.[13] The electron-electron interaction $\mathcal{A}_{\text{int}}$ is conveniently decomposed into four pieces that describe the interaction in the particle-hole and particle-particle spin-singlet and spin-triplet channels.[12] We will need only the particle-hole channel, and thus write

$$\mathcal{A}_{\text{int}}[Q]=\mathcal{A}_{\text{int}}^{(s)}+\mathcal{A}_{\text{int}}^{(t)}, \tag{2.3a}$$

$$\mathcal{A}_{\text{int}}(s)=\frac{T\Gamma^{(s)}}{2}\int d\mathbf{x}\sum_{r=0,3}(-1)^r\sum_{n_1,n_2,m}\sum_{\alpha}\times\{\text{tr}[(\tau_r\otimes s_0)Q_{n_1,n_1+m}^{\alpha\alpha}(\mathbf{x})]\}\times\{\text{tr}[(\tau_r\otimes s_0)Q_{n_2+m,n_2}^{\alpha\alpha}(\mathbf{x})]\}, \tag{2.3b}$$

$$\mathcal{A}_{\text{int}}^{(t)}=\frac{T\Gamma^{(t)}}{2}\int d\mathbf{x}\sum_{r=0,3}(-1)^r\sum_{n_1,n_2,m}\sum_{\alpha}\sum_{i=1}^{3}\times\{\text{tr}[(\tau_r\otimes s_i)Q_{n_1,n_1+m}^{\alpha\alpha}(\mathbf{x})]\}\times\{\text{tr}[(\tau_r\otimes s_i)Q_{n_2+m,n_2}^{\alpha\alpha}(\mathbf{x})]\}. \tag{2.3c}$$

Here $\Gamma^{(s)}>0$ and $\Gamma^{(t)}>0$ are the spin-singlet and spin-triplet interaction amplitudes, respectively. $\Gamma^{(t)}$ is responsible for producing magnetism.

As was shown in paper I, the Goldstone modes do not contribute leading singular terms to the conductivity or to any of the two-point vertices in perturbation theory up to one-loop order as $d\to 2$ in the metallic ferromagnetic phase. It was further shown that they do not contribute to the renormalization of the density of states (DOS), and hence they do not contribute to the wave-function renormalization. It follows that both the heat and the charge diffusion constants do not carry any singular renormalizations due to Goldstone modes. If we assume that this signals the absence of Goldstone-mode effects on the critical properties near the metal-insulator transition as well, we can ignore the Goldstone modes for the purpose of constructing an effective-field theory for the soft modes that drive the metal-insulator transition.[14] Furthermore, we can ignore the particle-particle channel, which is massive in a system with a nonvanishing magnetization. Accordingly, we drop both the particle-particle channel $(r=1,2)$ and the transverse spin-triplet channels $(i=1,2)$ from our model definition. For the remaining soft modes,we will now construct an effective theory by generalizing the procedure followed in Ref. 12.

B. Soft and massive modes

Let us briefly recall the basic philosophy behind the derivation of a nonlinear σ model in Ref. 12, which in turn was based on the work by Schäfer and Wegner[15] on noninteracting electrons. First one realizes, by means of a Ward identity, that the soft modes are given by the matrix elements Q_{nm} with $nm<0$, while the Q_{nm} with $nm>0$ are massive. This remains true in the present case except for the Goldstone modes, which we can neglect for our purposes. Next one block-diagonalizes the matrix Q in frequency space. Algebraic arguments show that the most general Q can be written as

$$Q=\mathcal{S}P\mathcal{S}^{-1}. \tag{2.4a}$$

Here $\mathcal{S}$ is a matrix that represents an element of the coset space $\text{USp}(8Nn,\mathcal{C})/\text{USp}(4Nn,\mathcal{C})\times\text{USp}(4Nn,\mathcal{C})$, and P is block diagonal in Matsubara frequency space,

$$P = \begin{pmatrix} P^{>} & 0 \\ 0 & P^{<} \end{pmatrix}, \tag{2.4b}$$

where $P^{>}$ and $P^{<}$ are matrices with elements P_{nm} where $n,m>0$ and $n,m<0$, respectively. It is further convenient to define a transformed field Λ by

$$\Lambda(\mathbf{x}) = \mathcal{S}^{-1}(\mathbf{x})\tilde{\Lambda}(\mathbf{x})\mathcal{S}(\mathbf{x}) \tag{2.5}$$

and to write the action in terms of these variables,

$$\begin{aligned}\mathcal{A}[\mathcal{S},P,\Lambda] = {} & \mathcal{A}_{\rm dis}[P] + \mathcal{A}_{\rm int}[\mathcal{S}P\mathcal{S}^{-1}] \\ & + \frac{1}{2}\,{\rm Tr}\ln(G_0^{-1} - i\mathcal{S}\Lambda\mathcal{S}^{-1}) \\ & + \int d\mathbf{x}\,{\rm tr}[\Lambda(\mathbf{x})P(\mathbf{x})].\end{aligned} \tag{2.6}$$

The next step is to expand $\mathcal{S}$, P, and Λ about their respective saddle-point values, which we denote by $\langle\mathcal{S}\rangle$, $\langle P\rangle$, and $\langle\Lambda\rangle$, respectively. From Sec. II B in paper I we have

$$\langle\mathcal{S}\rangle = \mathbb{1}\otimes\tau_0, \tag{2.7a}$$

$$\langle P\rangle_{12} = \delta_{12}\frac{i}{2V}\sum_{\mathbf{p}} [(\tau_0\otimes s_0)\mathcal{G}_{n_1}(\mathbf{p}) + (\tau_3\otimes s_3)\mathcal{F}_{n_1}(\mathbf{p})], \tag{2.7b}$$

$$\begin{aligned}\langle\Lambda\rangle_{12} = {} & \delta_{12}(\tau_0\otimes s_0)\ \frac{-i}{\pi N_{\rm F}\tau}\frac{1}{V}\sum_{\mathbf{p}} \mathcal{G}_{n_1}(\mathbf{p}) \\ & - \delta_{12}(\tau_0\otimes s_0)2i\Gamma^{(s)}T\sum_m e^{i\omega_m 0}\frac{1}{V}\sum_{\mathbf{p}} \mathcal{G}_m(\mathbf{p}) \\ & + \delta_{12}(\tau_3\otimes s_3)\frac{-i}{\pi N_{\rm F}\tau}\frac{1}{V}\sum_{\mathbf{p}} \mathcal{F}_{n_1}(\mathbf{p}) \\ & + \delta_{12}(\tau_3\otimes s_3)2i\Gamma^{(t)}T\sum_m e^{i\omega_m 0}\frac{1}{V}\sum_{\mathbf{p}} \mathcal{F}_m(\mathbf{p}),\end{aligned} \tag{2.7c}$$

with $\mathcal{G}$ and $\mathcal{F}$ from paper I, Eqs. (2.13). In the popular approximation that replaces the wave-vector sum over a Green function by an integral over $\xi_{\mathbf{p}} = \mathbf{p}^2/2m_e$,[16] we have

$$\frac{1}{V}\sum_{\mathbf{p}} \mathcal{G}_n(\mathbf{p}) \approx \frac{-i\pi}{2}N_{\rm F}\,{\rm sgn}\,\omega_n, \tag{2.8a}$$

$$\frac{1}{V}\sum_{\mathbf{p}} \mathcal{F}_n(\mathbf{p}) \approx 0. \tag{2.8b}$$

In this approximation,[17] we can write Eq. (2.7b) as

$$\langle P\rangle_{12} \approx \pi_{12}, \tag{2.9a}$$

with

$$\pi_{12} = \delta_{12}(\tau_0\otimes s_0){\rm sgn}\ \omega_{n_1}, \tag{2.9b}$$

with $\omega_n = 2\pi T(n+1/2)$ a fermionic Matsubara frequency. For our purposes this approximation will be sufficient for reasons that were explained in detail in Ref. 12.

We now write

$$P = \langle P\rangle + \Delta P, \quad \Lambda = \langle\Lambda\rangle + \Delta\Lambda, \tag{2.10}$$

and expand in powers of ΔP, $\Delta\Lambda$, and derivatives of $\mathcal{S}$. Let us first consider the Tr ln term in Eq. (2.6). Using the cyclic property of the trace, we can write it in the form

$$\begin{aligned}{\rm Tr}\ln(G_0^{-1} - \mathcal{S}i\Lambda\,\mathcal{S}^{-1}) & \\ = {\rm Tr}\ln(\mathcal{S}^{-1}G_0^{-1}\mathcal{S} - i\Lambda) & \\ = {\rm Tr}\ln(G_{\rm sp})^{-1}) + {\rm Tr}\ln\Big[& 1 + G_{\rm sp}\mathcal{S}^{-1}(\partial_\tau\mathcal{S}) \\ & + \frac{1}{m}G_{\rm sp}\mathcal{S}^{-1}(\nabla\mathcal{S})\nabla + \frac{1}{2m}G_{\rm sp}\mathcal{S}^{-1}(\nabla^2\mathcal{S}) \\ & - G_{\rm sp}i(\Delta\Lambda)\Big].\end{aligned} \tag{2.11a}$$

with

$$G_{\rm sp} = (G_0^{-1} - i\langle\Lambda\rangle)^{-1} \tag{2.11b}$$

the saddle-point Green function. This is formally the same expression as in the absence of ferromagnetism,[12] only the saddle-point Green function is more complicated. In particular, the transformation matrix $\mathcal{S}$ appears only in conjunction with some derivative and is therefore soft, while the fluctuations $\Delta\Lambda$ are massive. Expanding the second term on the right-hand side, the simplest contribution is the one involving the time derivative,

$$\begin{aligned}{\rm Tr}[G_{\rm sp}\mathcal{S}^{-1}(\partial_\tau\mathcal{S})] &= \int d\mathbf{x}\,{\rm tr}[i\Omega\mathcal{S}(\mathbf{x})G_{\rm sp}(\mathbf{x}=0)\mathcal{S}^{-1}(\mathbf{x})] \\ &= \frac{\pi N_{\rm F}}{2}\int d\mathbf{x}\,{\rm tr}[\Omega\,\hat{Q}(\mathbf{x})] + O(\Omega^2\hat{Q}).\end{aligned} \tag{2.12}$$

Here

$$\Omega_{12} = (\tau_0\otimes s_0)\delta_{12}\Omega_{n_1} \tag{2.13a}$$

is a frequency matrix with $\Omega_n = 2\pi Tn$ a bosonic Matsubara frequency, and

$$\hat{Q}(\mathbf{x}) = \mathcal{S}(\mathbf{x})\pi\mathcal{S}^{-1}(\mathbf{x}), \tag{2.13b}$$

with π from Eq. (2.9b). Here we have made use of Eqs. (2.8).[18] This is the same result as the one obtained in the absence of ferromagnetism.[12]

We now turn to the gradient terms. It is convenient to define a matrix-valued d-dimensional vector field

$$\mathbf{s}(\mathbf{x}) = \mathcal{S}^{-1}(\mathbf{x})(\nabla\mathcal{S})(\mathbf{x}), \tag{2.14}$$

and to expand in powers of $\mathbf{s}$. The term linear in $\mathbf{s}$ vanishes for symmetry reasons. To $O(\mathbf{s}^2)$, both the next-to-last term on the right-hand side of Eq. (2.11a) and the square of the preceding term contribute. So far our gradient expansion has been completely general. In order to evaluate the terms of $O(\mathbf{s}^2)$, we now remember that we can neglect the spin waves for the purpose of deriving a soft-mode transport theory; i.e., we have dropped the channels $r=0,3$, $i=1,2$ in the spin-quaternion expansion, Eq. (2.1c). Furthermore, it is well known that the Cooper channel ($r=1,2$) is massive in an external magnetic field,[5,2] and the same is true in a ferromagnetic state. Consequently, the only spin-quaternion degrees of freedom present in $\mathbf{s}_{12}$ are $\tau_{0,3}$ and $s_{0,3}$, and $\mathbf{s}_{12}$ commutes with $\tau_3 \otimes s_3$. This simplifies the evaluation of the gradient terms substantially, and we obtain

$$\mathrm{Tr}\, G_{\mathrm{sp}} \mathcal{S}^{-1}(\nabla^2 \mathcal{S}) - \frac{1}{m}\mathrm{Tr}[G_{\mathrm{sp}} \mathcal{S}^{-1}(\nabla \mathcal{S})\nabla]^2$$

$$= \sum_{12} \sum_{\mathbf{q}} \int d\mathbf{x} \{ \eta^{s}_{12,ij}(\mathbf{q}) \mathrm{tr}[s^{i}_{12}(\mathbf{q}) s^{j}_{21}(-\mathbf{q})] + \eta^{a}_{12,ij}(\mathbf{q}) \mathrm{tr}[(\tau_3 \otimes s_3) s^{i}_{12}(\mathbf{q}) s^{j}_{21}(-\mathbf{q})]\}, \tag{2.15a}$$

where

$$\eta^{s} = (\eta^{+} + \eta^{-})/2, \quad \eta^{a} = (\eta^{+} - \eta^{-})/2, \tag{2.15b}$$

with

$$\eta^{\pm}_{12,ij}(\mathbf{q}) = \delta_{ij}\frac{1}{2}[\mathcal{G}^{\pm}_{n_1}(\mathbf{q}) + \mathcal{G}^{\pm}_{n_2}(\mathbf{q})] + \frac{1}{m} q_i q_j \, \mathcal{G}^{\pm}_{n_1}(\mathbf{q}) \, \mathcal{G}^{\pm}_{n_2}(\mathbf{q}), \tag{2.15c}$$

with $\mathcal{G}^{\pm}_{n}$ the Green functions defined in paper I, Eq. (2.13c). Equations (2.15) are generalizations of the corresponding expressions in the absence of ferromagnetism.[12]

C. Nonlinear σ model

The remaining steps in the derivation of an effective-field theory proceed in analogy to Ref. 12. In particular, we integrate out the massive modes P and Λ in a tree approximation; i.e., we neglect all fluctuations ΔP and $\Delta \Lambda$. $\eta^{\pm}$ can be related to the conductivity in self-consistent Born approximation of a system whose chemical potential has been shifted from its value for nonmagnetic electrons by $\pm \Delta = \pm \Gamma^{(t)} M/\mu_{\mathrm{B}}$. Here M is the magnetization in Stoner approximation, and μ_{B} is the Bohr magneton [see paper I, Eq. (2.15)]. Denoting these conductivities by $\sigma^{\pm}_{0}$ and defining the bare coupling constants

$$1/G = \frac{\pi}{4} m(\sigma^{+}_{0} + \sigma^{-}_{0}), \tag{2.16a}$$

$$1/G_3 = \frac{\pi}{4} m(\sigma^{+}_{0} - \sigma^{-}_{0}), \tag{2.16b}$$

$$H = \pi N_{\mathrm{F}}/8, \tag{2.16c}$$

we obtain for the effective action

$$\tilde{\mathcal{A}} = \frac{-1}{2G}\int d\mathbf{x}\, \mathrm{tr}[\nabla \tilde{Q}(\mathbf{x})]^2 + 2H \int d\mathbf{x}\, \mathrm{tr}[\Omega \tilde{Q}(\mathbf{x})] - \frac{1}{2G_3}\int d\mathbf{x}\, \mathrm{tr}\{(\tau_3 \otimes s_3)[\nabla \tilde{Q}(\mathbf{x})]^2\} + \mathcal{A}_{\mathrm{int}}[\tilde{Q}]. \tag{2.17}$$

Here $\tilde{Q} = \hat{Q} - \pi$ with π the matrix defined in Eq. (2.9b) and $\mathcal{A}_{\mathrm{int}}$ from Eqs. (2.3). We recognize this action as the generalized nonlinear σ model for disordered interacting electrons,[1] augmented by the term with coupling constant $1/G_3$ that is proportional to the magnetization.

It turns out that the bare action, Eq. (2.17), is not sufficient to completely describe the effects of magnetic long-range order, even if one ignores the spin waves as we did. As we will see, under renormalization two additional terms are generated. One is a frequency coupling that is analogous to the second gradient term in Eq. (2.17), and the other is an electron-electron interaction term that is not present in nonmagnetic systems. We therefore need to add these terms to our action. Denoting the respective coupling constants by H_3 and K_3, we obtain our final result for the effective action,

$$\mathcal{A} = \frac{-1}{2G}\int d\mathbf{x}\, \mathrm{tr}[\nabla \tilde{Q}(\mathbf{x})]^2 + 2H \int d\mathbf{x}\, \mathrm{tr}[\Omega \tilde{Q}(\mathbf{x})] - \frac{1}{2G_3}\int d\mathbf{x}\, \mathrm{tr}\{(\tau_3 \otimes s_3)[\nabla \tilde{Q}(\mathbf{x})]^2\} + 2H_3 \int d\mathbf{x}\, \mathrm{tr}[(\tau_3 \otimes s_3)\, \Omega \tilde{Q}(\mathbf{x})] + \mathcal{A}_{\mathrm{int}}[\tilde{Q}]. \tag{2.18a}$$

Here $\mathcal{A}_{\mathrm{int}}$ is given by Eqs. (2.3) plus the extra term. Introducing new interaction amplitudes $K_s = -2\pi\Gamma^{(s)}$ and $K_t = 2\pi\Gamma^{(t)}$ to comform with notation used earlier,[2] we write

$$\mathcal{A}_{\mathrm{int}}[Q] = \mathcal{A}^{(s)}_{\mathrm{int}}[Q] + \mathcal{A}^{(t)}_{\mathrm{int}}[Q] + \mathcal{A}^{(3)}_{\mathrm{int}}[Q], \tag{2.18b}$$

$$\mathcal{A}^{(s)}_{\mathrm{int}}[Q] = \frac{-\pi T}{4} K_s \int d\mathbf{x} \sum_{1234} \delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\delta_{1-2,4-3} \times \sum_{r} (-)^r \mathrm{tr}[(\tau_r \otimes s_0) Q_{12}(\mathbf{x})] \times \mathrm{tr}[(\tau_r \otimes s_0) Q_{34}(\mathbf{x})], \tag{2.18c}$$

$$\mathcal{A}^{(t)}_{\mathrm{int}}[Q] = \frac{\pi T}{4} K_t \int d\mathbf{x} \sum_{1234} \delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\delta_{1-2,4-3} \times \sum_{r} (-)^r \mathrm{tr}[(\tau_r \otimes s_3) Q_{12}(\mathbf{x})] \times \mathrm{tr}[(\tau_r \otimes s_3) Q_{34}(\mathbf{x})], \tag{2.18d}$$

$$\mathcal{A}^{(3)}_{\mathrm{int}}[Q] = -4\pi T K_3 \int d\mathbf{x} \sum_{1234} \delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\delta_{1-2,4-3} \times \sum_{rs} \sum_{ij} m_{rs,ij}\, {}^{i}_{r}Q_{12}(\mathbf{x})\, {}^{j}_{s}Q_{34}(\mathbf{x}), \tag{2.18e}$$

where

$$m_{rs,ij} = \frac{1}{4}\mathrm{tr}(\tau_3 \tau_r \tau^{\dagger}_{s}) \mathrm{tr}(s_3 s_i s^{\dagger}_{j}). \tag{2.18f}$$

(This is the matrix that was denoted by m^{03} in paper I.) Finally, we note that $\hat{Q}$ as defined in Eq. (2.13b) obeys

$$\hat{Q}^2(\mathbf{x})\equiv 1, \quad \hat{Q}^\dagger=\hat{Q}, \quad \mathrm{tr}\,\hat{Q}(\mathbf{x})\equiv 0. \tag{2.19}$$

Equations (2.18) and (2.19) represent the analog for itinerant ferromagnets of the nonlinear σ model[1] for paramagnetic electron systems.

D. Metallic fixed point

Before we proceed to use the σ model to study the quantum phase transition from a ferromagnetic metal to a ferromagnetic insulator, let us ascertain that the model, with some correction terms, actually describes a metallic ferromagnetic phase in some parts of parameter space. Since we will want to approach the transition from this phase, its existence within the model is obviously a necessary condition for our program to be viable.

This task is very simple, since it proceeds in exact analogy to the demonstration in Ref. 12 that the model with $1/G_3=H_3=K_3=0$ has a stable Fermi-liquid fixed point. This is because the power-counting procedure used to prove the existence of a stable fixed point does not depend on structural details like the presence of extra τ and s matrices in the various terms of the action, while such details are the only difference between the current model and the one considered in Ref. 12. Accordingly, we parametrize $\hat{Q}$ in terms of a matrix q with elements q_{nm} whose frequency labels are restricted to $n\geqslant 0$, $m<0$,

$$\hat{Q}=\begin{pmatrix} \sqrt{1-qq^\dagger} & q \\ q^\dagger & -\sqrt{1-q^\dagger q} \end{pmatrix}, \tag{2.20}$$

and expand $\mathcal{S}$ in powers of q,

$$\mathcal{S}=\mathbb{1}\otimes\tau_0+\frac{1}{2}\begin{pmatrix} 0 & -q \\ q^\dagger & 0 \end{pmatrix}+O(q^2). \tag{2.21}$$

As in Ref. 12, we assign scale dimensions to $q(\mathbf{x})$,

$$[q(\mathbf{x})]=(d-2)/2, \tag{2.22a}$$

and to the fluctuations of the fields P and Λ,

$$[\Delta P(\mathbf{x})]=[\Delta\Lambda(\mathbf{x})]=d/2. \tag{2.22b}$$

Here the scale dimensions $[\cdots]$ are defined such that the scale dimension of a length L is $[L]=-1$. The fixed point action then consists of the nonlinear σ model action, Eq. (2.18a), expanded to $O(q^2)$, plus the corrections bilinear in ΔP and $\Delta\Lambda$ that arise from Eq. (2.6). All other terms are irrelevant by power counting. The arguments showing this are exactly the same as the ones given in Ref. 12 and need not be repeated here. The correlation functions for this Gaussian action are simply related to the Gaussian propagators of Sec. III in paper I. We will explicitly determine them in Sec. III below. This will show that the fixed point action really describes a disordered itinerant ferromagnet.

In contrast to Ref. 12, however, we cannot discuss the leading corrections to scaling near the stable metallic fixed point within our current framework. The reason is our having neglected the transverse spin-triplet channel that contains the Goldstone modes. While the latter are not expected to influence the leading scaling behavior at the critical fixed point for the reasons pointed out above, they do contribute to the corrections to scaling near the metallic fixed point, as indicated by their contribution to the leading nonanalytic frequency dependence of the conductivity that was studied in paper I.

III. METAL-INSULATOR TRANSITION ON THE BACKGROUND OF FERROMAGNETISM

In this section we perform a one-loop renormalization of the nonlinear σ model, Eqs. (2.18). We first do this for general parameter values, which leads to rather complicated flow equations. They contain a fixed point that corresponds to the known critical fixed point for nonmagnetic electrons in an external magnetic field.[8] We then linearize about this fixed point and show that it is perturbatively stable with respect to the additional terms in the action that represent the presence of a nonzero magnetization.

A. Parametrization and Gaussian order

In order to set up a loop expansion we use the parametrization for the matrix $\hat{Q}$ that is given by Eq. (2.20). Note that this parametrization builds in the constraints given in Eq. (2.19). It is the matrix analog of the usual $(\sigma,\vec{\pi})$ parametrization of the $O(N)$ vector nonlinear σ model.[19] The loop expansion now proceeds as an expansion in powers of q. To Gaussian order, we obtain

$$\mathcal{A}^{(0)}=\frac{-4}{V}\sum_{\mathbf{p}}\sum_{1234}{}^i_r q_{12}(\mathbf{p})\,{}^{ij}_{rs}M_{12,34}(\mathbf{p})\,{}^j_s q_{34}(-\mathbf{p}), \tag{3.1a}$$

where the Gaussian vertex is given by

$$ {}^{ij}_{rs}M_{12,34}(\mathbf{p})=\delta_{13}\delta_{24}\,{}^{ij}_{rs}M^{(0)}_{12}(\mathbf{p}) + \delta_{1-2,3-4}\delta_{\alpha_1\alpha_2}\,\delta_{\alpha_1\alpha_3}2\pi T\,K_{rs,ij}, \tag{3.1b}$$

with

$$ {}^{ij}_{rs}M^{(0)}_{12}(\mathbf{p})=\delta_{rs}\delta_{ij}\frac{1}{G}(\mathbf{p}^2+GH\Omega_{n_1-n_2}) + m_{rs,ij}\frac{1}{G_3}(\mathbf{p}^2+G_3H_3\Omega_{n_1-n_2}), \tag{3.1c}$$

and

$$K_{rs,ij}=\delta_{rs}\delta_{ij}(\delta_{i0}K_s+\delta_{i3}K_t)+m_{rs,ij}K_3, \tag{3.1d}$$

with $m_{rs,ij}$ from Eq. (2.18f).

The Gaussian propagator can be determined by the same methods that were employed in Sec. III of paper I. We find

$$\langle {}^i_r q_{12}(\mathbf{k})\,{}^j_s q_{34}(\mathbf{p})\rangle^{(0)}=\frac{1}{8}\,\delta(\mathbf{k}+\mathbf{p})\,{}^{ij}_{rs}M^{-1}_{12,34}(\mathbf{p}), \tag{3.2a}$$

where M^{-1} has the structure

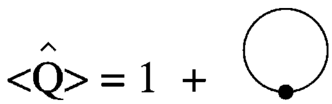

FIG. 2. Perturbation theory for $\langle \hat{Q}\rangle$ to one-loop order.

$$ {}^{ij}_{rs}M^{-1}_{12,34}(\mathbf{p}) = \delta_{13}\delta_{34}[\delta_{rs}\delta_{ij}A_{n_1-n_2}(\mathbf{p}) + m_{rs,ij}B_{n_1-n_2}(\mathbf{p})] + \delta_{1-2,3-4}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}[\delta_{rs}\delta_{ij}C^i_{n_1-n_2}(\mathbf{p}) + m_{rs,ij}D_{n_1-n_2}(\mathbf{p})]. \quad (3.2b)$$

To specify the propagators A, B, $C^0\equiv C^s$, $C^{1,2,3}\equiv C^t$, and D, we define

$$a \equiv a_n(\mathbf{p}) = (\mathbf{p}^2 + GH\Omega_n)/G, \quad (3.3a)$$

$$b \equiv b_n(\mathbf{p}) = (\mathbf{p}^2 + G_3H_3\Omega_n)/G_3, \quad (3.3b)$$

and

$$N\equiv N_n(\mathbf{p}) = (a^2-b^2)[a^2-b^2-2bK_3\Omega_n + a(K_s+K_t)\Omega_n - K_3^2\Omega_n^2 + K_sK_t\Omega_n^2]. \quad (3.3c)$$

In terms of these quantities, we have

$$A_n(\mathbf{p}) = a/(a^2-b^2), \quad (3.4a)$$

$$B_n(\mathbf{p}) = -b/(a^2+b^2), \quad (3.4b)$$

$$C^s_n(\mathbf{p}) = \frac{-2\pi T}{N}[a^2K_s + b^2K_t + a(-2bK_3 - K_3^2\Omega_n + K_sK_t\Omega_n)], \quad (3.4c)$$

$$C^t_n(\mathbf{p}) = \frac{-2\pi T}{N}[a^2K_t + b^2K_s + a(-2bK_3 - K_3^2\Omega_n + K_sK_t\Omega_n)], \quad (3.4d)$$

$$D_n(\mathbf{p}) = \frac{2\pi T}{N}[-a^2K_3 + ab(K_s+K_t) - b(bK_3 + K_3^2\Omega_n - K_sK_t\Omega_n)]. \quad (3.4e)$$

B. Perturbation theory to one-loop order

We now proceed to perform a one-loop renormalization of the theory. We do this by renormalizing the two-point vertex ${}^{ij}_{rs}M_{12,34}$, Eqs. (3.1). This procedures proves the renormalizability of the theory to one-loop order; i.e., it makes sure that no coupling constants in addition to the ones present in the bare theory are generated under renormalization. We also need to determine the wave-function renormalization. This we do by considering the one-point vertex function $\Gamma^{(1)} = \langle\hat{Q}\rangle^{-1}$.

1. One-point vertex

Let us first consider the one-point propagator $\langle\hat{Q}\rangle$. To one-loop order, the only diagram that contributes is shown in Fig. 2.

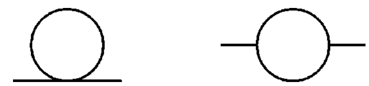

FIG. 3. One-loop contributions to the two-point vertex.

In spin-quaternion space, there are two nonvanishing matrix elements of $\langle\hat{Q}\rangle$, viz., $\langle{}^0_0Q\rangle$ and $\langle{}^3_3Q\rangle$. These expectation values are diagonal in both frequency and replica space. Their inverses constitute one-point vertex functions that we denote by $\Gamma^{(1)}_0(\Omega_n)$ and $\Gamma^{(1)}_3(\Omega_n)$, respectively. A simple calculation using the results of Sec. III A yields

$$\Gamma^{(1)}_0(\Omega_n) = 1 + \frac{1}{8}[I^s_1(\Omega_n) + I^t_1(\Omega_n)], \quad (3.5a)$$

$$\Gamma^{(1)}_3(\Omega_n) = 1 + \frac{1}{4}I^3_1(\Omega_n). \quad (3.5b)$$

Here we have defined the integrals

$$I^{s,t}_1(\Omega_n) = \frac{1}{V}\sum_{\mathbf{p}}\sum_{l=n}^{\infty} C^{s,t}_l(\mathbf{p}), \quad (3.6a)$$

$$I^3_1(\Omega_n) = \frac{1}{V}\sum_{\mathbf{p}}\sum_{l=n}^{\infty} D_l(\mathbf{p}), \quad (3.6b)$$

2. Two-point vertex

We now turn to the two-point vertex $\Gamma^{(2)}$, whose Gaussian approximation is given by Eqs. (3.1). To one-loop order, we write

$$ {}^{ij}_{rs}\Gamma^{(2)}_{12,34}(\mathbf{p}) = {}^{ij}_{rs}M_{12,34}(\mathbf{p}) + {}^{ij}_{rs}(\delta M)_{12,34}(\mathbf{p}). \quad (3.7)$$

There are two topologically distinct diagrammatic contributions to δM, which are shown in Fig. 3.

They arise from quartic terms, i.e., terms of $O(q^4)$, and cubic terms, i.e., terms of $O(q^3)$, respectively, in the expansion of the action in powers of q. An evaluation of the diagrams is straightforward but very tedious, since the two topological structures can be dressed in many ways with the various indices carried by the q field. The calculation reveals that the one-loop contributions can be grouped into three distinct classes: (1) Quartic contributions that are logarithmically divergent as $\Omega_n \to 0$ in $d=2$, (2) cubic contributions that have the same degree of divergence, and (3) contributions, both quartic and cubic, which individually diverge more strongly (''superdivergent terms''), but combine to yield again terms that are only logarithmically divergent. Denoting the contributions of these three classes to the one-loop renormalization of $\Gamma^{(2)}$ by $\delta M^{(4)}$, $\delta M^{(3)}$, and $\delta M^{(sd)}$, respectively, we find for the first of these classes

$$ {}^{ij}_{rs}(\delta M)^{(4)}_{12,34}(\mathbf{p}) = \delta_{13}\delta_{24}\Bigg\{ \delta_{rs}\delta_{ij}\frac{1}{8G}(\mathbf{p}^2+GH\Omega_N)[I_1^{(s)}(\Omega_N)+I_1^{(t)}(\Omega_N)] + \delta_{rs}\delta_{ij}\frac{1}{8G_3}(\mathbf{p}^2+G_3H_3\Omega_N)2\,I_1^{(3)}(\Omega_N) $$

$$ + m_{rs,ij}\frac{1}{8G}(\mathbf{p}^2+GH\Omega_N)2I_1^{(3)}(\Omega_N) + m_{rs,ij}\frac{1}{8G_3}(\mathbf{p}^2+G_3H_3\Omega_N)\,[I_1^{(s)}(\Omega_N)+I_1^{(t)}(\Omega_N)]\Bigg\} $$

$$ -\delta_{1-2,3-4}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\frac{\pi}{4}T\{\delta_{rs}\delta_{ij}[(K_s+K_t)\,J(\Omega_N)+2K_3J^{(3)}(\Omega_N)] + m_{rs,ij}[2K_3J(\Omega_N) $$

$$ +(K_s+K_t)\,J^{(3)}(\Omega_N)]\}, \tag{3.8a} $$

with $I_1^{(s,t,3)}$ given by Eqs. (3.6),

$$ J(\Omega_N) = \frac{1}{V}\sum_{\mathbf{p}} A_N(\mathbf{p}), \tag{3.8b} $$

$$ J^{(3)}(\Omega_N) = \frac{1}{V}\sum_{\mathbf{p}} B_N(\mathbf{p}), \tag{3.8c} $$

and N an external frequency (e.g., $N=n_1-n_2$). For the second class we obtain

$$ {}^{ij}_{rs}(\delta M)^{(3)}_{12,34}(\mathbf{p}) = -\delta_{1-2,3-4}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}(\pi T)^2\{\delta_{rs}\delta_{ij}[(\delta_{i0}K_s^2+\delta_{i3}K_sK_t+K_3^2)\,I_4^{(s)}(\Omega_N)+(\delta_{i3}K_t^2+\delta_{i0}K_sK_t+K_3^2)I_4^{(t)}(\Omega_N) $$

$$ +(\delta_{i0}K_sK_3+K_sK_3+\delta_{i3}K_tK_3)I_5^{(s)}(\Omega_N)+(\delta_{i0}K_sK_3+\delta_{i3}K_tK_3+K_tK_3)I_5^{(t)}(\Omega_N)] $$

$$ + m_{rs,ij}[(\delta_{i0}K_s^2+\delta_{j3}K_sK_t+K_3^2)I_5^{(s)}(\Omega_N)+(\delta_{i3}K_t^2+\delta_{i0}K_sK_t+K_3^2)I_5^{(t)}(\Omega_N) $$

$$ +(\delta_{i0}K_sK_3+K_sK_3+\delta_{i3}K_tK_3)I_4^{(s)}(\Omega_N)+(\delta_{i0}K_sK_3+\delta_{i3}K_tK_3+K_tK_3)I_4^{(t)}(\Omega_N)]\}. \tag{3.9a} $$

Here we have defined integrals

$$ I_4^{(s,t)}(\Omega_N) = \frac{1}{V}\sum_{\mathbf{p}}\sum_{l=N}^{\infty}[A_l(\mathbf{p})^2+B_l(\mathbf{p})^2+A_l(\mathbf{p})lC_l^{(s,t)}(\mathbf{p})+B_l(\mathbf{p})lD_l(\mathbf{p})], \tag{3.9b} $$

$$ I_5^{(s,t)}(\Omega_N) = \frac{1}{V}\sum_{\mathbf{p}}\sum_{l=N}^{\infty}[2\,A_l(\mathbf{p})\,B_l(\mathbf{p})+B_l(\mathbf{p})lC_l^{(s,t)}(\mathbf{p})+A_l(\mathbf{p})lD_l(\mathbf{p})]. \tag{3.9c} $$

Finally, for the third class we have

$$ {}^{ij}_{rs}(\delta M)^{(sd)}_{12,34}(\mathbf{p}) = \delta_{13}\delta_{24}\frac{1}{8}\{\delta_{rs}\delta_{ij}[I_2^{(s)}(\mathbf{p},\Omega_N)+I_2^{(t)}(\mathbf{p},\Omega_N)] + m_{rs,ij}[I_3^{(s)}(\mathbf{p},\Omega_N)+I_3^{(t)}(\mathbf{p},\Omega_N)]\}, \tag{3.10a} $$

with integrals

$$ I_2^{(s,t)}(\mathbf{p},\Omega_N) = \frac{1}{V}\sum_{\mathbf{k}}\sum_{l=1}^{\infty}\{a_l(\mathbf{k})C_l^{(s,t)}(\mathbf{k})+b_l(\mathbf{k})D_l(\mathbf{k})+2\pi\,T\,K_{s,t}\,A_l(\mathbf{k})+2\pi\,T\,K_3\,B_l(\mathbf{k})-(2\pi\,T)^2K_{s,t}^2\,l\,A_{l+N}(\mathbf{k}+\mathbf{p}) $$

$$ \times[A_l(\mathbf{k})+l\,C_l^{(s,t)}(\mathbf{k})]-(2\pi\,T)^2K_sK_tlB_{l+N}(\mathbf{k}+\mathbf{p})[B_l(\mathbf{k})+l\,D_l(\mathbf{k})]-(2\pi\,T)^2K_3^2lA_{l+N}(\mathbf{k}+\mathbf{p}) $$

$$ \times[A_l(\mathbf{k})+lC_l^{(s,t)}(\mathbf{k})]-(2\pi\,T)^2K_3^2lB_{l+N}(\mathbf{k}+\mathbf{p})[B_l(\mathbf{k})+lD_l(\mathbf{k})]-(2\pi\,T)^22K_3K_{s,t}lA_{l+N}(\mathbf{k}+\mathbf{p}) $$

$$ \times[2B_l(\mathbf{k})+lD_l(\mathbf{k})]-(2\pi\,T)^22K_3K_{s,t}lB_{l+N}(\mathbf{k}+\mathbf{p})lC_l^{(s,t)}(\mathbf{k})\}, \tag{3.10b} $$

$$ I_3^{(s,t)}(\mathbf{p},\Omega_N) = \frac{1}{V}\sum_{\mathbf{k}}\sum_{l=1}^{\infty}\{b_l(\mathbf{k})C_l^{(s,t)}(\mathbf{k})+a_l(\mathbf{k})\,D_l(\mathbf{k})+2\pi TK_{s,t}\,B_l(\mathbf{k})+2\pi TK_3\,A_l(\mathbf{k})-(2\pi\,T)^2K_{s,t}^2lB_{l+N}(\mathbf{k}+\mathbf{p}) $$

$$ \times[A_l(\mathbf{k})+lC_l^{(s,t)}(\mathbf{k})]-(2\pi\,T)^2K_sK_t\,lA_{l+N}(\mathbf{k}+\mathbf{p})[B_l(\mathbf{k})+lD_l(\mathbf{k})]-(2\pi\,T)^2K_3^2\,lB_{l+N}(\mathbf{k}+\mathbf{p}) $$

$$ \times[A_l(\mathbf{k})+lC_l^{(s,t)}(\mathbf{k})]-(2\pi\,T)^2K_3^2\,lA_{l+N}(\mathbf{k}+\mathbf{p})[B_l(\mathbf{k})+lD_l(\mathbf{k})]-(2\pi\,T)^22K_3K_{s,t}\,lA_{l+N}(\mathbf{k}+\mathbf{p}) $$

$$ \times[A_l(\mathbf{k})+lC_l^{(s,t)}(\mathbf{k})]-(2\pi\,T)^22K_3K_{s,t}\,lB_{l+N}(\mathbf{k}+\mathbf{p})[B_l(\mathbf{k})+lD_l(\mathbf{k})]\}. \tag{3.10c} $$

As a check, we consider the superdivergent contributions at zero external frequency and wave number. Although all of the individual terms are linearly divergent, an explicit calculation yields

$$I_2^{(s)}(0,0)+I_2^{(t)}(0,0)=0 \tag{3.11a}$$

and the same for I_3. Therefore,

$${}^{ij}_{rs}(\delta M)^{(sd)}_{12,34}(\mathbf{p})=O(\mathbf{p}^2,\Omega_N), \tag{3.11b}$$

with coefficients of $\mathbf{p}^2$ and Ω_N that are only logarithmically divergent. The cancellation of the superdivergences thus holds as expected (and required by, e.g., particle number conservation and the renormalizability of the theory).

C. Expansion to linear order in the magnetic coupling constants

As is clear from the preceding subsection, the complete one-loop renormalization of our model is rather complicated. While it is certainly possible to determine the renormalization group (RG) flow equations from our perturbative results, it would not be easy to analyze them for fixed points. At this point we therefore take a less general approach that is based on the following physical considerations. We are interested in a phase transition from a metallic magnetic phase to an insulating magnetic phase. Physically, we expect the magnetization to be noncritical at such a transition. The simplest possible scenario is then a fixed point where the renormalized values of the ‘‘magnetic’’ coupling constants $1/G_3$, H_3, and K_3 are all zero. (More complicated possibilities we will come back to in Sec. IV below.) This means that the unversality class of this transition is the same as that for the transition from a paramagnetic metal to a paramagnetic insulator in the presence of an external magnetic field.[8] We can check this scenario by expanding to linear order in the three magnetic coupling constants and investigate the perturbative stability of the nonmagnetic fixed point.

Accordingly, we expand the results of the previous subsection, expressing the result in the form of corrections to the magnetic coupling constants. We use dimensional regularization; i.e., we perform the integrals in $d=2+\epsilon$ to leading order in $1/\epsilon$. We find for the correction to $1/G_3$ to linear order in $1/G_3$, H_3, and K_3,

$$\delta(1/G_3)=\frac{2}{\epsilon}\left[\frac{G}{16G_3}f_{11}(K_s/H,K_t/H)+\frac{H_3}{8H}f_{12}(K_s/H,K_t/H)+\frac{K_3}{4H}f_{13}(K_s/H,K_t/H)\right]. \tag{3.12a}$$

Here we have defined the functions

$$f_{11}(x,y)=g_{11}(x)+g_{11}(y)-2\,\frac{L_x-L_y}{x-y}+2\,\frac{xy}{x-y}[h_{11}(x)-h_{11}(y)], \tag{3.12b}$$

$$f_{12}(x,y)=g_{12}(x)+g_{12}(y)+\frac{L_x-L_y}{x-y}+\frac{xy}{x-y}[h_{12}(x)-h_{12}(y)], \tag{3.12c}$$

$$f_{13}(x,y)=\frac{-1}{x-y}\left(\frac{1}{x}L_x-\frac{1}{y}L_y\right), \tag{3.12d}$$

in terms of

$$g_{11}(x)=\frac{-6}{x}-2+\frac{6}{x}\left(\frac{1}{x}+1\right)\ln(1+x), \tag{3.12e}$$

$$h_{11}(x)=\frac{-2}{x^2}-\frac{2}{x}+\frac{1}{x^2}\left(\frac{2}{x}+3\right)\ln(1+x), \tag{3.12f}$$

$$g_{12}(x)=\frac{1}{x}\left[3-\left(\frac{3}{x}+2\right)\ln(1+x)\right], \tag{3.12g}$$

$$h_{12}(x)=\frac{1}{x^2}\left[2-\left(\frac{2}{x}+1\right)\ln(1+x)\right], \tag{3.12h}$$

and $L_x=\ln(1+x)$, $L_y=\ln(1+y)$.

Similarly, the correction to H_3 is

$$\delta H_3=\frac{G}{4\epsilon}\left[\frac{GH}{G_3}f_{21}(K_s/H,K_t/H)+H_3\,f_{22}(K_s/H,K_t/H)+K_3\,f_{23}(K_s/H,K_t/H)\right], \tag{3.13a}$$

with

$$f_{21}(x,y)=g_{21}(x)+g_{21}(y)-\frac{1+x+y}{x-y}(L_x-L_y)-\frac{xy}{x-y}[h_{21}(x)-h_{21}(y)], \tag{3.13b}$$

$$f_{22}(x,y)=g_{22}(x)+g_{22}(y)+\frac{L_x-L_y}{x-y}+\frac{xy}{x-y}[h_{21}(x)-h_{21}(y)], \tag{3.13c}$$

$$f_{23}(x,y)=g_{23}(x)+g_{23}(y)+\frac{L_x-L_y}{x-y}+\frac{xy}{x-y}[h_{21}(x)-h_{21}(y)], \tag{3.13d}$$

in terms of

$$g_{21}(x)=1+\frac{x}{2}-\left(\frac{1}{x}+1\right)\ln(1+x), \tag{3.13e}$$

$$h_{21}(x)=\frac{1}{x^2}\ln(1+x), \tag{3.13f}$$

$$g_{22}(x) = -1 + \left(\frac{1}{x} + 1\right)\ln(1+x), \tag{3.13g}$$

$$g_{23}(x) = \frac{-1}{2} + \frac{1}{x}\ln(1+x). \tag{3.13h}$$

Finally, for the correction to K_3 we obtain

$$\delta K_3 = \frac{G}{8\epsilon}\left[\frac{GH}{G_3} f_{31}(K_s/H, K_t/H) + H_3 f_{32}(K_s/H, K_t/H) + K_3 f_{33}(K_s/H, K_t/H)\right], \tag{3.14a}$$

with

$$f_{31}(x,y) = x + y + \frac{x+y}{x-y}[h_{31}(x) - h_{31}(y)], \tag{3.14b}$$

$$f_{32}(x,y) = g_{32}(x) + g_{32}(y) + \frac{y}{x}L_x + \frac{x}{y}L_y + \frac{(x+y)^2}{x-y}[h_{32}(x) - h_{32}(y)], \tag{3.14c}$$

$$f_{33}(x,y) = g_{33}(x) + g_{33}(y) + \frac{y}{x}L_x + \frac{x}{y}L_y + \frac{(x+y)^2}{x-y}[h_{32}(x) - h_{32}(y)], \tag{3.14d}$$

in terms of

$$h_{31}(x) = -2(1+x)\ln(1+x), \tag{3.14e}$$

$$g_{32}(x) = -2x + \ln(1+x), \tag{3.14f}$$

$$h_{32}(x) = \frac{1}{x}\ln(1+x), \tag{3.14g}$$

$$g_{33}(x) = 1 + 3\ln(1+x). \tag{3.14h}$$

Inspection of the integrals in Sec. III B further shows that all corrections to the remaining coupling constants *G*, *H*, and $K_{s,t}$, to the extent that they depend on the magnetic coupling constants, are at least quadratic in the latter and hence can be neglected for our purposes. The ''nonmagnetic'' one-loop corrections are well known,[2] and we do not write them down again.

We also note that $1/G_3 \neq 0$ is sufficient to generate nonzero values of H_3 and K_3 in perturbation theory, even if these coupling constants were not present in the bare action. This is the reason why we have included them in Eq. (2.18a).

D. Renormalization group flow equations

We now perform a RG analysis of our perturbation theory. We define renormalized coupling constants g_3, h_3, and k_3 by

$$G_3 = \kappa^{-\epsilon} Z_{g_3} g_3, \tag{3.15a}$$

$$H_3 = Z_{h_3} h_3, \tag{3.15b}$$

$$K_3 = Z_{k_3} k_3, \tag{3.15c}$$

where the *Z* are renormalization constants, and κ is the arbitrary RG momentum scale.[19] We further define a two-point vertex function $\Gamma_3^{(2)}$ as the ''magnetic piece'' of the general vertex $\Gamma^{(2)}$ defined in Eq. (3.7), i.e., the parts that are proportional to $1/G_3$, H_3, and K_3. From Eqs. (3.1), (3.7), and (3.8)–(3.10) we have

$$\Gamma_3^{(2)}(\mathbf{p},\Omega) = \left(\frac{1}{G_3} + \delta(1/G_3)\right)\mathbf{p}^2 + (H_3 + \delta H_3)\Omega + (K_3 + \delta K_3)\Omega. \tag{3.16}$$

The renormalization constants can then be determined from the renormalization statement

$$\Gamma_{3,\mathrm{R}}^{(2)}(\mathbf{p},\Omega; g_3, h_3, k_3; \kappa) = Z\,\Gamma_3^{(2)}(\mathbf{p},\Omega; G_3, H_3, K_3), \tag{3.17}$$

where $\Gamma_{3,\mathrm{R}}^{(2)}$ is the renormalized counterpart of $\Gamma_3^{(2)}$, and *Z* is the wave-function renormalization. In our notation, we suppress the dependence of the vertex functions on the remaining coupling constants *G*, *H*, $K_{s,t}$, and their renormalized counterparts.

It is *a priori* not clear that a single wave-function renormalization constant will suffice. Indeed, the existence of two distinct one-point vertex functions, Eqs. (3.5), one being related to the density of states and the other to the magnetization, might suggest that one needs at least two. However, as mentioned in Sec. III C above, we do not expect the magnetization to display leading critical behavior at the phase transition we are interested in, despite the fact that the magnetization has nonanalytic contributions in perturbation theory. We therefore expect the only wave-function renormalization to be the one related the vertex $\Gamma_0^{(1)}$,

$$\Gamma_{1,\mathrm{R}}^{(1)}(\Omega; g_3, h_3, k_3; \kappa) = Z\,\Gamma_1^{(1)}(\Omega; G_3, H_3, K_3). \tag{3.18}$$

To linear order in $1/g_3$, h_3, and k_3, *Z* is given by the wave-function renormalization for nonmagnetic electrons in an external magnetic field,[2]

$$Z = 1 - \frac{g}{4\epsilon}(l_s + l_t), \tag{3.19}$$

where $l_{s,t} = \ln(1+\gamma_{s,t})$, with $\gamma_{s,t} \equiv k_{s,t}/h$ the renormalized counterparts of $K_{s,t}/H$. Our perturbative calculation of $\Gamma_3^{(2)}$ is then sufficient to determine the remaining renormalization constants. Using minimal subtraction, we find

$$Z_{g_3} = 1 + \frac{g}{8\epsilon}\left[f_{11}(\gamma_s,\gamma_t) - 2(l_s + l_t) + 2\frac{g_3 h_3}{gh} f_{12}(\gamma_s,\gamma_t) + 4\frac{g_3 k_3}{gh} f_{13}(\gamma_s,\gamma_t)\right], \tag{3.20a}$$

$$Z_{h_3}=1+\frac{g}{4\epsilon}\left[l_s+l_t-f_{22}(\gamma_s,\gamma_t)-\frac{gh}{g_3h_3}f_{21}(\gamma_s,\gamma_t)-\frac{k_3}{h_3}f_{23}(\gamma_s,\gamma_t)\right], \tag{3.20b}$$

$$Z_{k_3}=1+\frac{g}{8\epsilon}\left[2(l_s+l_t)-f_{33}(\gamma_s,\gamma_t)-\frac{h_3}{k_3}f_{32}(\gamma_s,\gamma_t)-\frac{gh}{g_3k_3}f_{13}(\gamma_s,\gamma_t)\right]. \tag{3.20c}$$

From Eqs. (3.20) and (3.15) it is now easy to determine the RG flow equations for the magnetic coupling constants. Our parameter space is spanned by $\mu=(g,h,\gamma_s,\gamma_t,g_3,h_3,k_3)$, and our approximations are valid only in the vicinity of the fixed point (FP) $\mu^*=(g^*,h^*,\gamma_s^*,\gamma_t^*,g_3^*,h_3^*,k_3^*)$, with $1/g_3^*=h_3^*=k_3^*=0$, and g^*, h^*, γ_s^*, and γ_t^* the FP values of these coupling constants for the magnetic-field universality class of nonmagnetic electrons.[2] We therefore immediately linearize about this FP. With $\beta_3\equiv 1/g_3$, and $l\equiv 1/\kappa$ the RG length scale, we find

$$\frac{d\beta_3}{dl}=\left(\epsilon-\frac{g^*}{8}[f_{11}^*-2(l_s^*+l_t^*)]\right)\beta_3-\frac{f_{12}^*}{4h^*}h_3-\frac{f_{13}^*}{2h^*}k_3, \tag{3.21a}$$

$$\frac{dh_3}{dl}=\frac{-1}{4}(g^*)^2h^*f_{21}^*\beta_3+\frac{g^*}{4}[l_s^*+l_t^*-f_{22}^*]-\frac{g^*}{4}f_{23}^*k_3, \tag{3.21b}$$

$$\frac{dk_3}{dl}=\frac{-1}{8}(g^*)^2h^*f_{31}^*\beta_3-\frac{g^*}{8}f_{32}^*h_3+\frac{g^*}{8}[2(l_s^*+l_t^*)-f_{33}^*]k_3, \tag{3.21c}$$

where $f_{11}^*\equiv f_{11}(\gamma_s^*,\gamma_t^*)$, etc.

The fixed point values that enter Eqs. (3.21) depend on whether we consider the long-ranged Coulomb interaction between the electrons, or a short-ranged model interaction. We consider here the former, more realistic, case. Then we have[2]

$$g^*=2\epsilon/(1-\ln 2),\quad \gamma_t^*=-\gamma_s^*=1,\quad l_s^*=-2/\epsilon. \tag{3.22}$$

With this input, we obtain the following eigenvalues for the linearized flow equations, Eqs. (3.21):

$$\lambda_1=-\epsilon/2(1-\ln 2)+O(\epsilon^2)<0, \tag{3.23a}$$

$$\lambda_2=-1/(1-\ln 2)+O(\epsilon)<0, \tag{3.23b}$$

$$\lambda_3=-\frac{3\ln 2-2}{1-\ln 2}\epsilon+O(\epsilon^2)<0. \tag{3.23c}$$

We see that all three eigenvalues are negative, and the fixed point is therefore stable.

E. Critical behavior

As we have seen in the previous subsection, the critical fixed point for the transition under consideration is the same as the one found before for the metal-insulator transition of nonmagnetic electrons in the presence of an external magnetic field.[8] The asymptotic critical behavior is therefore also the same. Choosing the correlation length exponent ν, the critical exponent for the density of states β, and the dynamical critical exponent z as the three independent exponents, we thus have[2]

$$\nu=1/\epsilon+O(1), \tag{3.24a}$$

$$\beta=1/2\epsilon(1-\ln 2), \tag{3.24b}$$

$$z=d. \tag{3.24c}$$

The critical exponent for the conductivity, $s=\nu(d-2)$, is

$$s=1+O(\epsilon). \tag{3.24d}$$

In contrast to the asymptotic critical behavior, the corrections to scaling are different from any previously studied universality class for metal-insulator transitions. The reason for this is the presence of the three irrelevant operators $1/g_3$, h_3, and k_3 in our model. We will not go through a complete analysis of the corrections to scaling here, but only mention that they lead to a nonanalyticity in the magnetization as one crosses the metal-insulator transition, even though the magnetization is not critical. To see this, we recall that the magnetization is proportional to a frequency integral over the inverse of the one-point vertex $\Gamma_3^{(1)}$; see Eq. (3.5b) above and Eq. (2.7c) in paper I. The extra frequency integration makes the integral finite for all $d>0$, and the one-loop contribution to the magnetization is simply proportional to $1/g_3$, h_3, and k_3. Since λ_3 has the smallest absolute value of the three negative eigenvalues given in Eq. (3.23), the magnetization at $T=0$ behaves like

$$M(t,T=0)\propto \text{const}+t^{-\nu\lambda_3}, \tag{3.25a}$$

where t is the dimensionless distance from the critical point. At criticality as a function of temperature we have

$$M(t=0,T)\propto \text{const}+T^{-\lambda_3/z}. \tag{3.25b}$$

Putting $\epsilon=1$ in our one-loop approximation yields $-\lambda_3/z=0.086\ldots$. Our theory thus predicts that the metal-insulator transition is reflected in the magnetization in the form of a very slow temperature dependence.

More generally, the existence of very slow corrections to scaling indicates that it will be very difficult, if not impossible, to observe the true asymptotic critical behavior at the ferromagnetic metal-insulator transition. If we extrapolate our one-loop results to $d=3$ by putting $\epsilon=1$, we have $\nu\lambda_3\approx -0.26$. This mean that in order to obtain critical exponents with an accuracy of 10% one needs to be within about 0.01% of the critical point, $t\lesssim 10^{-4}$. This is not achievable for any metal-insulator transition observed so far.[2] Any observed critical behavior at larger values of t will yield effective exponents that contain contributions from the dominant irrelevant scaling variables. We note in passing that the same conclusion holds for the Anderson-Mott transition of paramagnetic electrons in an external magnetic field.[8]

IV. DISCUSSION

Our chief result is the prediction that the metal-insulator transition from a ferromagnetic metal to a ferromagnetic insulator is in the same universality class as the one from a paramagnetic metal to a paramagnetic insulator in the presence of an external magnetic field. It is important to note that this statement holds independent of what the actual critical exponents, which we can determine only to lowest order in a $2+\epsilon$ expansion, are in three-dimensional systems. It is also independent of the fact that we have considered explicitly only the perturbative stability of the nonmagnetic fixed point. In principle, the full flow equations that follow from the one-loop calculation in Sec. III B could contain other critical fixed points. This is a question that remains to be investigated; here we just mention a possible scenario.

In both the magnetic-field and ferromagnetic material cases a different universality class for the MIT is easy to envisage. First note that the existence of two conductivities $\sigma^{\pm}$ [cf. Eqs. (2.16)] or, equivalently, two diffusivities just reflects the fact that either a magnetic field or a finite spontaneous magnetization leads to a splitting of the energy band. The subband with fewer (more) electrons that have spins aligned in (opposite to) the direction of the magnetic field or spontaneous magnetization is called the minority (majority) subband. If the magnetic energy scale is large compared to other interaction energy scales and comparable to the Fermi energy, then the two subbands are well separated. A polarization scenario for the MIT is that the minority subband carriers become localized first and then act as a static random field for the majority mobile carriers.[20] In this scenario, the MIT occurs when the carriers in the majority band become localized. In this case the MIT is one for spin-polarized, or effectively spinless, electrons. This is mathematically described by the so-called singlet-only or magnetic impurity universality class that was discussed in Ref. 21. One might thus expect a multicritical point separating the magnetic-field universality class, which was discussed above and is relevant for small values of the magnetization or the magnetic field, from the singlet-only or polarization universality class at large values of the magnetization. Experimentally, such a multicritical point could be probed by increasing the magnetic field in the case of an MIT in an external magnetic field or by effectively increasing the triplet interaction for a spontaneously magnetized system. Theoretically, it remains to be seen whether such a behavior is described by our complete flow equations. This point will be investigated in a future publication.

In any event, it would be very interesting to compare experiments on a ferromagnetic metal-to-insulator transition, which has not been studied so far, with the existing results for nonmagnetic systems in a magnetic field.[2] The equivalence of the two universality classes also leads to the conclusion that the existing theory for the nonmagnetic transition in a magnetic field is incomplete since it misses important corrections to scaling.[8]

From a theoretical point of view, this result is *a priori* rather surprising. The critical behavior at the metal-insulator transition, and hence the universality class, is determined by the structure of the soft modes in the system, at least near two dimensions. Since ferromagnetism leads to additional soft modes, namely, the Goldstone modes or spin waves, compared to paramagnetic metals, one would expect the critical behavior to change. The reason why it does not lies in the fact that the Goldstone modes do not lead to a singular correction to the conductivity in $d=2$, in contrast to the diffusive soft modes that are also present in the absence of ferromagnetic long-range order. Since these singular corrections drive the transition in low dimensions, and since the Goldstone modes are the only substantial difference between the soft-mode spectra of ferromagnetic systems and paramagnetic systems in a magnetic field, respectively, the fact that the universality class remains unchanged is at least plausible.

We finally mention again that one of our motivations for the present study had been the observed apparent metal-insulator transition in certain $2-d$ electron systems,[9] which contradicts the results of orthodox theories and is not understood. Since it is known that ferromagnetic fluctuations enhance the conductivity in $d=2$,[2] it was a plausible hypothesis that ferromagnetic long-range order might have an even stronger effect and lead to a metallic phase in $d=2$. Our results rule out this possibility, at least on a perturbative level.

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[17] Improving on Eq. (2.8a) produces terms that turn out to be irrelevant for the purposes of this paper. Improving on Eq. (2.8b) produces terms that are generated in perturbation theory anyway, and which we will include in our final effective model, Eq. (2.18a).

[18] If one does the integration over the Green functions exactly rather than using the approximation, Eqs. (2.8), one finds a real contribution to Eq. (2.8a) that is even in frequency and a nonzero contribution to Eq. (2.8b). The former does not contribute to a theory for the leading effects of soft modes, for reasons that were explained in Ref. 12. The latter is, to lowest order in a frequency expansion, independent of the frequency and thus contributes only an uninteresting constant to the action. However, under renormalization the action acquires a contribution that has the frequency structure of Eq. (2.12) and the spin-quaternion structure of the $\mathcal{F}$ contribution to the saddle-point Green function. See Eq. (2.18a) and Sec. IV below.

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