High-energy phonon creation from cold phonons in pulses of different length in He II

A. F. G. Wyatt and M. A. H. Tucker
School of Physics, University of Exeter, Exeter EX4 4QL, United Kingdom

I. N. Adamenko, K. E. Nemchenko, and A. V. Zhukov
Kharkov State University, Svobody Square 4, Kharkov, 310077, Ukraine
(Received 30 December 1999)

We have carried out a theoretical investigation of the experimentally observed phenomenon that long-lived high-energy (h) phonons are generated by a moving cloud of low-energy (l) phonons. The h phonons are created from the l phonons by four phonon processes ($4pp$) and they are lost from the trailing edge of the l phonon cloud, because they have a lower velocity than the l phonons, and form the h phonon cloud. We obtain a set of equations which completely describe these phenomenon. The solution of these equations accounts for the high efficiency of the conversion process: a major part of the energy in the l phonons can be converted to h phonons within the propagation time of the pulse ($<10^{-4}$ s). In short pulses ($<10^{-7}$ s) the h phonons escape as soon they are created, but in long pulses the h phonon density increases within the l cloud. It is shown that in long phonon pulses there can be a suprathermal number of h phonons within the l cloud. The theory describes the cooling of pulses of different length due to energy being transformed into h phonons. It also accounts well for the important characteristics of h phonon generation which is an unusual example of energy transferring from low-energy to high-energy states.

I. INTRODUCTION

It has been known for some time that high-energy phonons can be created in liquid ^{4}He by an interacting phonon cloud from a high input power[1,2] but it has only recently been recognized how efficient this process can be at medium input powers for phonons with $\varepsilon \geqslant 10$ K.[3,4] A recent measurement showing this effect can be seen in Fig. 1(a) in Ref. 6, which is the signal resulting from a single heat pulse. The phonons have propagated ~15 mm in cold liquid ^{4}He at zero pressure. There are clearly two groups of phonons; the first is low (l)-energy phonons with $T \approx 1$ K and the second is high (h)-energy phonons with $\varepsilon \approx 10\,\mathrm{K} (\varepsilon = \hbar\omega/k)$. In this example there are similar energies in the two phonon signals (the signal sizes are quite unequal as the sensitivity of the detector increases with phonon energy[5]). The division of energy between these two groups of phonons can be varied widely and it depends on the input power, pulse duration, and on the pressure of the liquid ^{4}He. In this paper we present a theory which gives a good explanation for the various observations at zero pressure.

We have recently described how high (h)-energy phonons are generated from a short pulse of low (l)-energy phonons propagating in liquid ^{4}He.[6] The basic picture is that h phonons are created within the moving cloud of l phonons and are then lost out of the back of the l cloud due to the slower speed of the h phonons relative to the l phonons. This causes the number of h phonons to drop below the equilibrium value for the l cloud which drives the production of more h phonons by four phonon processes, $4pp$, when two phonons scatter into two others. In a short pulse the h phonons are lost as soon they are created, which explains how a major fraction of the energy in the l phonons can be converted into h phonons. The problem is more complex for long pulses as we must consider the h phonons within the l phonon cloud. This entails understanding the lifetime of the h phonons and the dynamics of their population. We find the interesting result that a suprathermal density of h phonons is created, that is their distribution function is about two orders of magnitude higher than that of a Bose-Einstein distribution at the temperature of the low-energy phonons.

That two phonon groups arise from one input pulse is due to the anomalous dispersion in liquid ^{4}He.[7] At zero pressure the dispersion curve first bends upwards and then at a relatively high energy bends downwards towards the maxon peak.[8] The upward curvature allows three phonon processes $3pp$,[9–11] up to an energy ≈ 8.3 K, so a phonon in this energy range can decay spontaneously into two phonons and can be created from two phonons, with energy and momentum conservation.

Between 8.3 K and $\varepsilon_c = 10$ K,[8,12] $3pp$ are forbidden but a phonon in this range can decay into three or more phonons[11,13] and similarly can be created from many phonons. There can be $4pp$ at all energies but they are generally weaker than $3pp$,[14,15] however, for $\varepsilon > \varepsilon_c$ only $4pp$ are allowed so a phonon in this energy range can be scattered only if another input phonon is available. This means that if such a phonon is isolated in helium at $T=0$, its lifetime is infinite.

Although it was realized that h phonons could be created in an interacting phonon cloud of high-energy density by $3pp$ and $4pp$,[1,2] the experiments with superconducting aluminum tunnel junction detectors did not show how much of the pulse energy was converted to h phonons. This is because these dectectors have a low-energy cutoff at 4.47 K and so are insensitive to the majority of the l phonons. Later experiments with superconducting bolometers[3,4] which are sensitive (but not equally so) to all phonons showed that the conversion process can be extremely efficient; even at medium heater powers ~10 mW/mm^2 a majority of the l phonon energy can be converted to h phonons.

The h phonons were also studied by quantum evaporation.[16] Measurements of the energy of the evaporated atoms confirmed that the energy of the h phonon was 10 K. However, these experiments showed the puzzling behavior of the time of flight; it was ~ 5 μs too fast.[17–19] This was eventually explained;[3,4] it is due to the h phonons being created in the liquid helium some millimetres in front of the heater. The faster than expected time of flight of the quantum evaporation signal is due to the faster l phonon propagation from the heater to the point where they are converted to the slower h phonons. This interpretation was confirmed by pressure measurements[3] which showed that h phonons have an energy $\sim \varepsilon_c(P)$ which means that they are not injected by the heater, as this would be independent of pressure, but are created by up scattering in the liquid ^{4}He. Furthermore, measurements with different propagation paths[20] showed that the probability for creating h phonons decreases with distance beyond ~ 1 mm in front of the heater, and it is very small after ~ 5 mm.

A heater immersed in liquid helium always injects phonons into the liquid together with a relatively small number of R^+ rotons.[21] Unless the pulse power is very low, <1 mW/mm^2, the phonons rapidly thermalize within the liquid helium through the fast $3pp$ scattering rate[11,15] and so the phonons that are detected are not the original phonons but are the result of phonon-phonon interactions. The thermalized phonons are predominantly low energy as their temperature is <1 K.[22] The h phonons, $\varepsilon > \varepsilon_c = 10\,\mathrm{K}$, have an energy in the high-energy tail of this distribution. The number of h phonons that are created initially increases with pulse length while it is still a short pulse (the criterion for short will emerge later, in practice it means $<10^{-7}$ s), but then saturates when the pulse is long.

There can be a high density of h phonons in the l cloud and this density governs the loss rate of h phonons from the back of the cloud. One aim of this paper is to calculate the h phonon flux as a function of the duration of the heater pulse.

We shall see that the behavior of the h phonons depends critically on their lifetime within the l phonon cloud. This is a dynamic property because the h phonons only scatter by $4pp$ and so they need a low-energy phonon to interact with. However, because of the restrictions imposed by the conservation of energy and momentum, the momentum of the low-energy phonon must make a large angle with the h phonon in order to scatter the h phonon to a lower energy $\varepsilon < \varepsilon_c$. Interestingly, the h phonon lifetime is always much longer than if it were in an isotropic Bose-Einstein distribution of l phonons. This is a direct consequence of the narrow cone of momenta in the thermalized l phonon cloud.

In this paper we first discuss the thermalization of the l phonons, then consider how the h phonon cloud is created and hence the behavior of short pulses. We then analyze the time and space dependence of the h phonons within the pulse which leads to expressions for the cooling of pulses of different lengths and the corresponding h phonon fluxes. We compare the theoretical results with the available measured data for long and short pulses.

II. INITIAL PHONON THERMALIZATION

A heater injects mainly low-energy phonons into liquid ^{4}He. Phonons injected by the heater are in two groups, the first is phonons in the narrow cone Ω_p perpendicular to the heater surface. These peak phonons create the l pulse which travels to the detector. The second group is phonons emitted into the entire half space, these are the so called background phonons. The existence of the first group was predicted theoretically [23] and found experimentally.[24] The possible existence of background phonons was considered in the papers[25,26] and was discovered independently with experiments on phonon emission from cleaved crystals.[24,27] From measurements we know that the total energy in the background is an order of magnitude larger than the energy in the peak,[28] but because the solid angle of the peak is so small, the energy per unit solid angle of the peak is larger than for the background. The injected phonons will have energies equal to or lower than those in the heater depending on whether they are peak or background phonons, respectively.[29] With a gold film heater and short pulses these peak phonons are found to occupy a cone in momentum space with a solid angle $\Omega_p = 0.115$ steradians.[4]

The distribution function for phonons injected by a heater into a narrow cone of solid angle Ω_p is[30]

$$n = \frac{A}{e^{\varepsilon/T_s} - 1}, \tag{1}$$

where T_s is the temperature of the heater, and $A \ll 1$ is a constant which includes small parameters $\rho c/\rho_s c_s$. Here ρ and ρ_s are the densities and c and c_s the velocities in the liquid helium and heater, respectively. For a gold heater $A = 5.59 \times 10^{-3}$.[30]

The distribution (1) is not an equilibrium function under $3pp$ scattering, so it rapidly decays to an equilibrium Bose-Einstein function with temperature $T_0 = A^{1/4} T_s$; these are then the l phonons. The time for this relaxation is estimated to be 10^{-10} s,[31–33] which is short compared to the heater pulse duration t_p and all other times in the experiment.

The initial temperature T_0 of the thermalized l phonons can be obtained by equating the injected energy with the phonon energy density in a volume with the area of the heater and length ct_p, we find

$$gW = \frac{\Omega_p \pi k^4 T_0^4}{120 \hbar^3 c^2}, \tag{2}$$

where g is the fraction of the electrical power per unit area W that goes into the cone in the helium. The pulse duration cancels out if $ct_p \ll$ heater dimensions. For $\Omega_p = 0.115$ steradians, and gW=1 mW/mm^2, Eq. (1) gives $T_0 = 0.9\,\mathrm{K}$.

This phonon thermal equilibrium is rather unusual as it is within a cone of solid angle Ω_p in momentum space, see Fig. 1. Normal thermal equilibrium entails a spherically symmetric distribution of phonons in momentum space. We are here dealing with a slice of that distribution and it is reasonable to consider the phonons in this slice to be in equilibrium because the $3pp$ scattering angles are small and so the scattered phonons remain mostly within the cone, see Fig. 2. Scattering initially broadens the cone but this becomes very slow as the l phonons cool due to both the creation of h phonons and the geometric expansion of the l cloud. This means that any cone spreading occurs within a few mm of the heater.

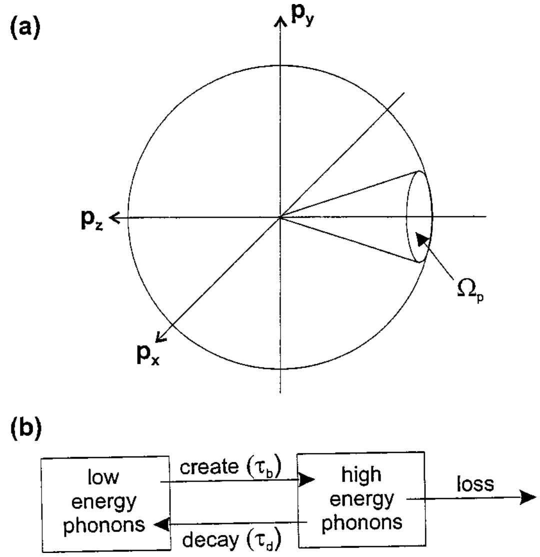

FIG. 1. (a) The low-energy (l) phonons occupy a small solid angle Ω_p in momentum space. (b) A schematic representation of the l and h phonon groups. The arrow indicate the transitions that *create, decay,* and *lose h* phonons.

So we consider the occupied cone in momentum space to be in thermal equilibrium as far as the l phonons are concerned so that they have a Bose-Einstein energy distribution with a characteristic temperature T, see Fig. 3(b) in Ref. 6. We exclude the high-energy phonons from this description because they achieve equilibrium in a different way. We will discuss the dynamic equilibrium of the h phonons in long pulses in Sec. V.

III. CREATION OF HIGH-ENERGY PHONONS

We treat the low- and high-energy phonons as two nearly separate systems. This is possible because the dispersion curve imposes quite different relaxation times on the two groups. The $3pp$ scattering rate[10,11,31] (τ_3^{-1}) is approximately two orders of magnitude faster than that for $4pp$,[14,15] so the low-energy phonons are in thermal equilibrium. We include the phonons with $\varepsilon_{3c}<\varepsilon<\varepsilon_c$ in this group. Although there is no detailed theory of their scattering time it is expected that their relaxation times are more than an order of magnitude smaller than the corresponding times for $4pp$.[10] The liquid ^{4}He in which they propagate is taken to be at zero temperature.

The two groups of phonons are shown schematically in Fig. 2(b). The arrows indicate creation of h phonons from l phonons, and the decay of h phonons to l phonons. Also there is loss of h phonons from the interacting cloud of phonons by dispersion, i.e., we use the words *creation* and *decay* for h phonons within the cloud of l phonons, and *loss* for transmission of h phonons out of the cloud of l phonons. The loss arises because h phonons have a lower group velocity c_h than l phonons so they get left behind as the interacting l phonon cloud moves forward at velocity c.

After the heater has injected a pulse of energy into the liquid the l phonon cloud rapidly equilibriates and occupies a cone in momentum space with $\theta_p=11°$ half angle.[4] The cloud has length $L=ct_p$ and is composed almost entirely of l phonons because any injected h phonons will be attenuated by the background excitations. Although the l cloud is in thermal equilibrium for energies $\varepsilon<\varepsilon_c$, the cloud is not in equilibrium for $\varepsilon\geqslant\varepsilon_c$ as initially there are essentially no phonons in this energy range; the distribution has lost its tail. As the system attempts to equilibrate, the l phonons give their energy to the h phonons on a time scale $t_b(T)$.

The sequence of events for a short pulse is shown schematically in Fig. 2(b) in Ref. 6. At time t_1 the cloud has moved a distance ct_1 and has created some h phonons, a fraction of which are trailing behind the cloud with velocity c_h. We call this the h cloud. This loss of h phonons maintains the deficiency of h phonons in the l cloud which again causes more h phonons to be created by $4pp$. At t_2 the h cloud has lengthened because of the continued creation of h phonons and their subsequent loss from the l phonon cloud. As the l phonon cloud loses h phonons it cools which causes the creation rate of h phonons to drop. At t_3 the two clouds appear to be separate because at some time between t_2 and t_3 the l phonon cloud has cooled so much that it has essentially stopped producing h phonons. The two clouds are now independent and move with velocities $\mathbf{c}_h$ and $\mathbf{c}$ to the detector.

Let us now derive a set of equations which completely describe this phenomenon. The kinetic equation for the distribution function n_h of h phonons in the frame moving with the l phonons is

$$\frac{\partial n_h}{\partial t}+\mathbf{u}_h\cdot\nabla n_h=\frac{n_h^{(0)}}{t_b}-\frac{n_h}{t_d}, \tag{3}$$

where $\mathbf{u}_h=\mathbf{c}_h-\mathbf{c}$ is the relative velocity of h and l phonons,

$$n_h^{(0)}=\frac{1}{e^{\varepsilon_1/T}-1} \tag{4}$$

is the Bose-Einstein distribution function for h phonons (phonon 1) with energy ε_1, ($\varepsilon_1>\varepsilon_c$), with the temperature T of the l phonons,

$$t_b^{-1}=\int_{\Omega_{3,4}\leqslant\Omega_p}W(\mathbf{p}_1,\mathbf{p}_2|\mathbf{p}_3,\mathbf{p}_4)n_2^{(0)}(1+n_3^{(0)})\times(1+n_4^{(0)})d\Gamma_2 d\Gamma_3\Gamma_4 \tag{5}$$

is the $4pp$ scattering time for the creation of h phonons within the cloud, and

$$t_d^{-1}=\int_{\Omega_2\leqslant\Omega_p}W(\mathbf{p}_3,\mathbf{p}_4|\mathbf{p}_1,\mathbf{p}_2)n_2^{(0)}(1+n_3^{(0)})\times(1+n_4^{(0)})d\Gamma_2 d\Gamma_3 d\Gamma_4 \tag{6}$$

is the decay time for h phonons within the cloud. In Eq. (5) we take into account $n_3^{(0)}n_4^{(0)}(1+n_1^{(0)})(1+n_2^{(0)})=n_1^{(0)}n_2^{(0)}(1+n_3^{(0)})(1+n_4^{(0)})$ and that $n_1\ll 1$ and $n_1^{(0)}\ll 1$. Also W is the probability function for the $4pp$ and $n_i^{(0)}=n^{(0)}(\varepsilon_i)$ with $i=2,3,4$, is the Bose-Einstein distribution function for the l phonons, $d\Gamma=p^2dpd\Omega/(2\pi\hbar)^3$ and $\Omega/2\pi=1-\cos\theta$. In Eqs. (3), (5), and (6) we consider that the l phonons are in thermal equilibrium through the fast $3pp$ interactions. According to Eq. (5) l phonons 3 and 4 are within the cloud and scatter to create phonons 1 and 2 so the

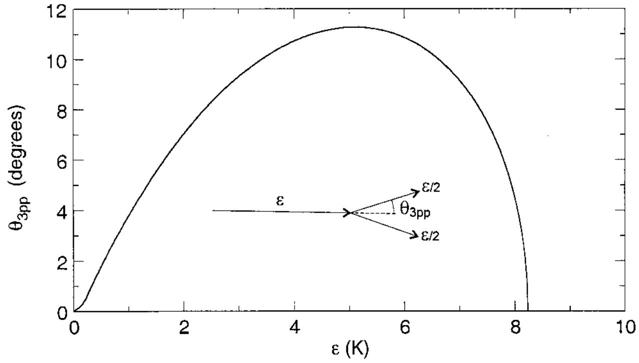

FIG. 2. The variation of the half-angle θ_{3pp} between equal-energy final states in the $3pp$ as a function of initial phonon energy. Note that the angles lie within the l phonon cone.

solid angles Ω_3 and Ω_4 must be less than or equal to the solid angle Ω_p of the pulse. Equation (6) describes the reverse process where h phonon 1 decays because it interacts with 1 phonon 2 from within the cloud, to create phonons 3 and 4. For this case $\Omega_2 \leqslant \Omega_p$. We shall see in Sec. V that the different solid angles in the integrals (5) and (6) leads to the unusual result that t_b may be substantially smaller than t_d in a phonon cloud with a narrow momentum cone with solid angle Ω_p.

Multiplying Eq. (3) by the phonon energy ε and integrating over the phase space occupied by the h phonons we get the equation for the energy density E_h of the h phonons:

$$\frac{\partial E_h}{\partial t} + u_c \frac{\partial E_h}{\partial z} = \frac{E_h^{(0)}}{\tau_b} - \frac{E_h}{\tau_d}, \tag{7}$$

where we make a one-dimensional approximation with the z axis chosen to be antiparallel to the propagation direction, and the average value, u_c ($\approx$50 m/s) is taken to be the relative velocity at $\varepsilon = \varepsilon_c$, because of the exponential factor in the distribution function. The average values of the times for phonons to be born τ_b and to decay τ_d will be considered in Sec. V. Also,

$$E_h^{(0)} = \frac{\Omega_p k^4 T \varepsilon_c^3 e^{-\varepsilon_c/T}}{(2\pi\hbar c_h)^3}, \tag{8}$$

where Ω_p is the occupied solid angle in momentum space, see Fig. 2(a), and we approximate the dependence $\varepsilon(p \geqslant p_c)$ to the expression $\varepsilon = \varepsilon_0 + c_h p$. Neutron-scattering data give $\varepsilon_0 = 2.06$ K and $c_h = 189$ m/s.[8] In Eq. (8), terms of next order in $T/\varepsilon_c \ll 1$ have been neglected.

The energy going to the creation of h phonons comes from the l phonons, so from energy conservation and Eq. (7) we get the rate of change of the energy density of the l phonons:

$$\frac{\partial E_l^{(0)}}{\partial t} = -\frac{E_h^{(0)}}{\tau_b} + \frac{E_h}{\tau_d}, \tag{9}$$

where

$$E_l^{(0)} = \frac{\Omega_p \pi k^4 T^4}{120\hbar^3 c^3} \tag{10}$$

is the energy density of the l phonons assuming linear dispersion.

From Eqs. (9) and (10) we obtain an equation for the cooling of the l phonon cloud,

$$\frac{\partial T}{\partial t} = -\frac{T}{4E_l^{(0)}}\left(\frac{E_h^{(0)}}{\tau_b} - \frac{E_h}{\tau_d}\right). \tag{11}$$

The set of Eqs. (7)–(11) are completed by the initial and boundary conditions. The cloud occupies the space $0<z<L$ and as h phonons move from the point $z=0$, the front of the l cloud, towards $z=L$, the back of the l cloud, so h phonons are absent at $z=0$. Hence the boundary condition is

$$E_h(z=0,t)=0. \tag{12}$$

We take the initial conditions to be

$$T(t=0) \equiv T_0 = \text{const}, \tag{13}$$

$$E_h(z,t=0)=0. \tag{14}$$

The above set of Eqs. (7)–(14) describe the h phonon generation for all pulse lengths. We shall solve them for short and long pulses.

IV. SHORT HEATER PULSES

We will see in Sec. V that if the phonons occupy a narrow cone in momentum space with solid angle $\Omega_p \ll 1$ then τ_d is always longer than τ_b, and the problem can be easily solved for a short pulse when all the created h phonons are lost from the l cloud. This condition is satisfied if the pulse length L is short enough, i.e., $L \ll u_c \tau_d$. In this case we can neglect the term E_h/τ_d in Eq. (9) and for the short pulse we have

$$\frac{\partial E_l^{(0)}}{\partial t} = -\frac{E_h^{(0)}}{\tau_b} \tag{15}$$

substituting Eqs. (8) and (10) into Eq. (15) we have the equation which describes the cooling of short pulses:

$$\frac{\partial T}{\partial t} = -\frac{15}{4\pi^4}\left(\frac{c\varepsilon_c}{c_h T}\right)^3 T e^{-\varepsilon_c/T} \tau_b^{-1}, \tag{16}$$

where τ_b^{-1} is found from Eq. (5).[15] When $T \ll 1$ K and we have a narrow cone, $\Omega_p \ll 1$, the creation rate τ_b^{-1} at the saturated vapor pressure can be written in the form[15]

$$\tau_b^{-1} = 3.27 \times 10^9 \, \Omega_p^2 T^5 \, \text{s}^{-1}, \tag{17}$$

where here and below the temperature T is in Kelvin.

For our cone $\Omega_p = 0.115$ sterad we obtain

$$\tau_b^{-1} = 4.33 \times 10^7 T^5 \, \text{s}^{-1}. \tag{18}$$

Note that in Ref. 6 the value of τ_b^{-1} was for a pulse with a larger solid angle than 0.115 sterad. As we shall see this does not produce any qualitative change and only a small quantitative difference. Substituting Eq. (18) into Eq. (16) and integrating with the initial condition (13) we find an expression for the time dependence of the temperature of the l phonons:

$$T e^{-\varepsilon_c/T} = T_0 e^{-\varepsilon_c/T_0}(1 + t/t_s)^{-1}, \tag{19}$$

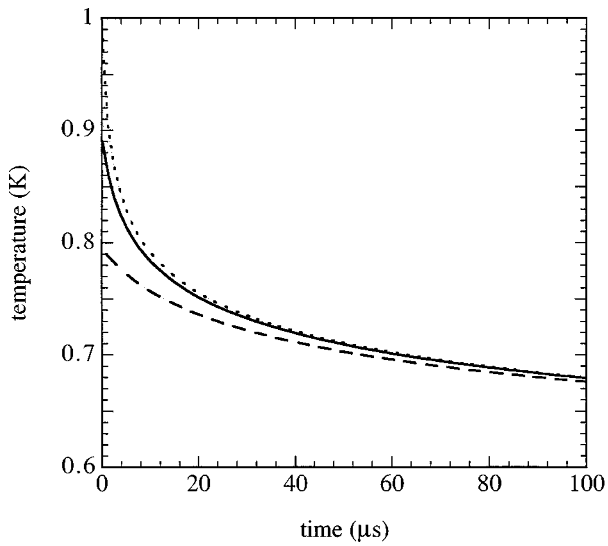

FIG. 3. The temperature of a short phonon pulse as a function of propagation time, for three different initial temperatures $T_0 = 0.8$ K (dashed), 0.9 K (solid), and 1.0 K (dotted), obtained from Eqs. (19) and (20). We see that the cooling rate is a strong function of temperature.

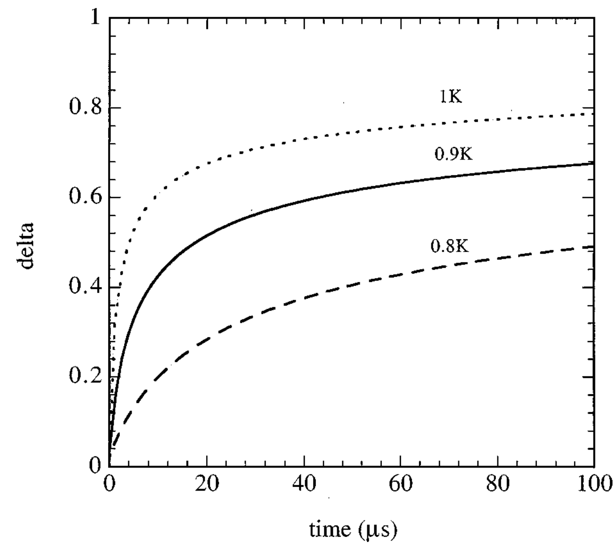

FIG. 4. The fraction of energy Δ in the h phonons relative to the total energy for a short pulse, is shown as a function of time for three initial temperatures T_0 that correspond to the cooling curves in Fig. 3. Note that a major part of the initial energy is converted to h phonons.

where the time t_s is given by

$$t_s = \frac{4E_l^{(0)}(T_0)T_0\tau_b(T_0)}{E_h^{(0)}(T_0)\varepsilon_c} \tag{20}$$

has a straight forward physical meaning. At the time $t = t_s$ the l phonon cloud has already transformed a major part of its energy into h phonons. We see $t_s \gg \tau_b$ because $E_l^{(0)}/E_h^{(0)} \gg 1$, e.g., if $T = 1$ K then $t_s/\tau_b \sim 30$ and $t_s = 0.66\ \mu$s which is much shorter than the propagation time of the pulse which is about 60 μs.

So the l phonon cloud cools as it propagates, as shown in Fig. 3, and creates the trailing cloud of h phonons. As energy is conserved between the h and l phonons, the fraction of energy in the h phonons is given by

$$\Delta(t) = \frac{E_l^{(0)}(T_0) - E_l^{(0)}[T(t)]}{E_l^{(0)}(T_0)}. \tag{21}$$

The fraction of energy in the h phonons for three typical starting temperatures is shown in Fig. 4. It can be seen that a significant part of the energy in the l phonons is rapidly transformed into h phonons. This very effective up-scattering mechanism is due to the h phonons being lost out of the trailing end of the l cloud where they accumulate without decay.

The physical causes of such a strong effect becomes clear if we rewrite the expression (21) in a form that follows from Eq. (15):

$$\Delta(t) = \frac{1}{E_l^{(0)}(T_0)} \int_0^t \frac{E_h^{(0)}[T(t)]}{\tau_b[T(t)]} dt. \tag{22}$$

The term under integration in Eq. (22) is the energy of the created h phonons per unit time. If the time t is much larger than τ_b the value of the integral in Eq. (22) is comparable to the initial energy of the l phonons $E_l^{(0)}(T_0)$.

So we see that as T drops, the creation falls and becomes negligible at $T \sim 0.7$ K. When the creation rate becomes small the h and l phonon clouds separate as indicated in Fig. 2(b) in Ref. 6 for some time between t_2 and t_3.

V. DYNAMICS OF *h* PHONONS IN LONG PULSES

The time dependence of the h phonons in long pulses can be obtained from Eq. (7) if we ignore the second term on the left-hand side. This then describes all points in the pulse except those near to the front $(z = 0)$ of the pulse where E_h is changing relatively rapidly with z. So we have

$$\frac{\partial E_h}{\partial t} = \frac{E_h^{(0)}}{\tau_b} - \frac{E_h}{\tau_d}. \tag{23}$$

Assuming the temperature is constant we can integrate Eq. (23) using the initial condition (14)

$$E_h = \frac{\tau_d}{\tau_b} E_h^{(0)}(1 - e^{-t/\tau_d}). \tag{24}$$

Equation (24) describes how the energy density E_h develops with time. This is shown in Fig. 5 for $T = 1$ K. We see that the energy density of h phonons reaches the value $E_h^{(0)}$, which corresponds to the energy in thermal equilibrium given by the Bose-Einstein equilibrium distribution (4), in a time $\tau_b = 2.3 \times 10^{-8}$ s [Eq. (18)]. After a time longer than $\tau_d = 8.7 \times 10^{-7}$ s, which is found from Eq. (6),[15] we get a dynamic equilibrium within the pulse and

$$E_h^{(e)} = \frac{\tau_d}{\tau_b} E_h^{(0)}, \tag{25}$$

where the energy in the h phonons, $E_h^{(e)}$, is more than an order of magnitude larger than $E_h^{(0)}$.

This remarkable effect where the density of h phonons increases beyond the usual equilibrium density, we call a

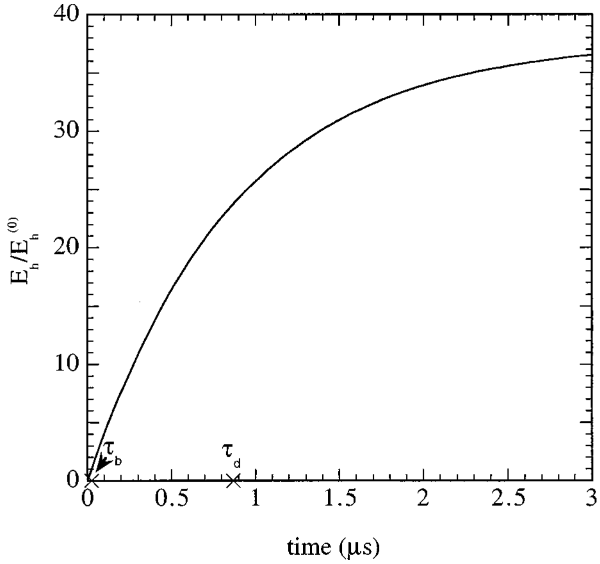

FIG. 5. The energy in the h phonons relative to that in a Bose-Einstein distribution $E_h/E_h^{(0)}$ as a function of time in a long pulse according to Eq. (24). It reaches a maximum of 37.7 for $\varepsilon=\varepsilon_c$. The time constants for creation (τ_b) and decay (τ_d) are indicated by crosses.

''suprathermal'' distribution. It is the result of the asymmetry between the creation and decay lifetimes of the h phonons, Eqs. (5) and (6), which arises because the h phonons are in a narrow cone, $\Omega_p \ll 1$.

The suprathermal distribution occurs in pulses which are long enough for the probability of escape of the h phonon to be very much smaller than the probability of decay. The time to escape t_{esc} is given by

$$t_{\mathrm{esc}}=\frac{L}{u_c}=t_p\frac{c}{u_c} \tag{26}$$

and must be much longer than τ_d so

$$\frac{L}{u_c}\gg\tau_d. \tag{27}$$

This asymmetry between decay and creation follows from the conservation of momentum,

$$\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4, \tag{28}$$

and energy,

$$\varepsilon_1+\varepsilon_2=\varepsilon_3+\varepsilon_4, \tag{29}$$

in the $4pp$. The relationship between energy and momentum can be expressed as

$$\varepsilon_i=c(p_i+f_i), \tag{30}$$

where $f_i=f(p_i)$ is a term describing the nonlinearity of the phonon dispersion. From Eqs. (28)–(30) we obtain to first order in f_i

$$\xi_{21}=\frac{p_3p_4}{p_1p_2}\xi_{34}+\frac{p_1+p_2}{p_1p_2}\phi, \tag{31}$$

where

$$\xi_{jk}=1-\frac{\mathbf{p}_j\cdot\mathbf{p}_k}{p_jp_k}=1-\cos\Theta_{jk} \tag{32}$$

and

$$\phi=f_3+f_4-f_2-f_1. \tag{33}$$

It is clear from Eq. (31) that there is an asymmetry between Θ_{21} and Θ_{34} when $p_{2,3,4}<p_c$, and $p_i\geqslant p_c$. From Eq. (31), as $\phi>0$ there is a minimum value of ξ_{12},

$$\xi_{21}^{\min}=\frac{p_1+p_2}{p_1p_2}\phi. \tag{34}$$

When an h phonon decays due to its interaction with l phonons in the pulse, we have the following inequality:

$$\xi_{21}^{\min}<\xi_p\ll 1, \tag{35}$$

where

$$\xi_p=1-\cos\Theta_p=\frac{\Omega_p}{2\pi}. \tag{36}$$

If $T<\sim 1$ K, then an average phonon $\langle p_2\rangle$ satisfies the inequality

$$\frac{p_1}{\langle p_2\rangle}\gg 1. \tag{37}$$

Note that the relation (37) is a consequence of the asymmetry of the problem: h phonons with $cp_1\geqslant cp_c=10$ K interact with phonons in a pulse with characteristic energy $c\langle p_2\rangle \sim 1$ K because $T\sim 1$ K. Therefore Eq. (34) is the product of a large parameter (37) and a small one,

$$\frac{\phi}{p_1}\ll 1, \tag{38}$$

as a result $\xi_{21}^{\min}$ is relatively large and the angular range,

$$\xi_{21}^{\min}\leqslant\xi_{21}\leqslant\xi_p\ll 1 \tag{39}$$

in which a h phonon can decay, becomes narrower.

The value of the integral (6) is greatly reduced by the inequality (39) and the decay time is then large. The decay rate in a narrow cone, $\Omega_p\ll 1$, for $T<1$ K and at the saturated vapor pressure, is from Eq. (6),[15]

$$t_d^{-1}(p_1=p_c)=8.70\times 10^7\,\Omega_p^2T^5\ \mathrm{s}^{-1}. \tag{40}$$

For our cone $\Omega_p=0.115$ steradians we have

$$\tau_d^{-1}(p_1=p_c)=1.15\times 10^6T^5\ \mathrm{s}^{-1}. \tag{41}$$

When p_1 increases, $\xi_2^{\min}$ increases so that at $p_1=p_0$ the inequalities (35) and (39) cannot be satisfied for any p_2. For our pulse with $\Omega_p=0.115$ steradians $\varepsilon(p_0)\sim 11$ K, so we have

$$t_d^{-1}(p_1\geqslant p_0)=0 \tag{42}$$

and therefore these h phonons do not interact with the l phonons in the pulse. In this case we must consider other decay processes for these phonons.[34] In this article we take

$$\tau_d^{-1}=t_d^{-1}(p_1=p_c)=1.15\times 10^6T^5\ \mathrm{s}^{-1} \tag{43}$$

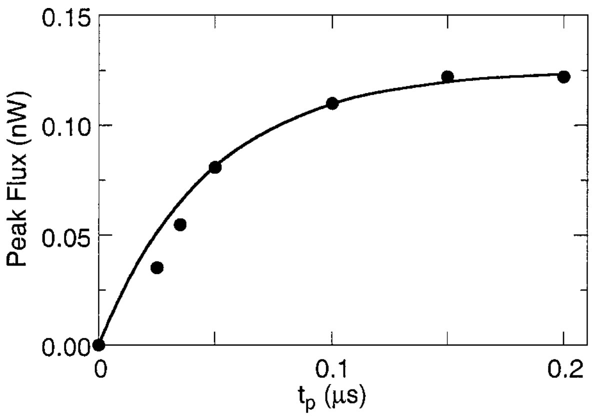

FIG. 6. The measured peak energy flux of the h phonon signal at various pulse lengths t_p for an electrical heating power of 16 mW/mm^2. The line has the form $1-\exp(-ct_p/u_c\tau_d)$ with $\tau_d=228$ ns.

for numerical evaluations. This value is of the same order as the value from the pulse length dependence, see Fig. 6.

For h phonon creation, we see from Eq. (31) that Θ_{34} has values down to zero but that Θ_{12} is limited by Eq. (34). Because there are no restrictions on Θ_{34} we have

$$t_b^{-1} \gg t_d^{-1} \tag{44}$$

for all p_1. Furthermore the absence of restrictions on Θ_{34} leads to a relatively weak dependence of t_b^{-1} on p_1. In this article we take

$$\tau_b^{-1} = t_b^{-1}(p_1 = p_c) \tag{45}$$

for numerical evaluations.

VI. QUASISTATIONARY *h* PHONON POPULATIONS AND SATURATION OF THE *h* PHONON FLUX

When the h phonon density in the pulse has finished growing and E_h is changing slowly with time, we have a quasisteady situation for the h phonons so we may neglect the first term on the left-hand side of Eq. (7). In this case we have

$$u_c \frac{\partial E_h}{\partial z} = \frac{E_h^{(0)}}{\tau_b} - \frac{E_h}{\tau_d}. \tag{46}$$

The solution of Eq. (46) with the boundary conditions (12) gives $E_h(z)$ throughout the pulse, $0<z<L$,

$$E_h = \frac{\tau_d}{\tau_b} E_h^{(0)} (1 - e^{-z/u_c\tau_d}). \tag{47}$$

We see that we get a suprathermal distribution within the pulse for points $z \gg u_c\tau_d$ (see Sec. V). From Eq. (47) we can get an expression for the flux of h phonons generated by the l phonon cloud.

The amplitude of the h phonon signal at the detector is determined by the flux of energy Φ in the h phonons per unit time through unit area of the trailing edge of the pulse, hence

$$\Phi = u_c E_h (z=L). \tag{48}$$

From Eqs. (47) and (48) we see that the amplitude of the h phonon flux Φ increases with pulse length $t_p = L/c$ from a small value given by

$$\Phi = \frac{L}{\tau_b} E_h^{(0)}, \quad \text{where } L \ll u_c\tau_d \tag{49}$$

for the short pulse, up to the maximum value given by

$$\Phi_{\max} = u_c \frac{\tau_d}{\tau_b} E_h^{(0)}, \quad \text{where } L \gg u_c\tau_d \tag{50}$$

for the long pulse.
When

$$t_p > \tau_d \frac{u_c}{c} \tag{51}$$

the h phonon flux approaches its maximum value given by Eq. (50). For longer pulses the detected h phonon signal remains the same amplitude and only the length of the signal increases with t_p. This result corresponds to the experimental observation that the amplitude of the h phonon signal initially increases linearly with heater pulse length but then more slowly increases and eventually saturates, as is shown in Fig. 6.

VII. COOLING OF PULSES OF DIFFERENT LENGTHS

Equations (9) and (46) enable us to calculate how the temperature of pulses change with time as it creates h phonons in the quasistationary state. We write the left side of Eq. (9) equal to the negative of the left side of Eq. (46), and integrate over z,

$$u_c \int_0^L \frac{\partial E_h}{\partial z} dz = -\frac{\partial}{\partial t} \int_0^L E_l^{(0)} dz. \tag{52}$$

We can write

$$\int_0^L E_l^{(0)} dz = E_l^{(0)}(T) L, \tag{53}$$

where T is the average temperature of the pulse. Using Eqs. (52) and (53) with boundary conditions (12) we obtain

$$-L \frac{\partial E_l^{(0)}}{\partial t} = u_c E_h (z=L). \tag{54}$$

Equation (54) is applicable for all pulse lengths in this approximation using the average temperature of the cloud. It has a simple physical interpretation: the change in energy of the l-phonon cloud, in unit time, is equal to the energy of the h phonons which emerge in unit time from the trailing edge of the pulse (48).

Substituting Eq. (47) into Eq. (54) and taking into account expressions (8) and (10) we obtain equations which with the initial conditions (13) determine the cooling of pulses of different length,

$$\frac{\partial T}{\partial t} = -\frac{15}{4\pi^4} \left(\frac{c\varepsilon_c}{c_h T} \right)^3 T e^{-\varepsilon_c/T} \frac{u_c}{L} \frac{\tau_d}{\tau_b} (1 - e^{-L/u_c\tau_d}). \tag{55}$$

The second exponential term contains τ_d which is temperature dependent and this makes Eq. (55) difficult to integrate. However, for short and long pulses this problem does not exist and we can obtain solutions of Eq. (55). For short pulses, $L \ll u_c \tau_d$, the solution gives Eq. (19). For long pulses we have

$$T^4 e^{\varepsilon_c/T}\left(1+4\frac{T}{\varepsilon_c}\right)=T_0^4 e^{\varepsilon_c/T_0}\left(1+4\frac{T_0}{\varepsilon_c}\right)\left(1+\frac{t}{t_L}\right), \quad (56)$$

where

$$t_L=t_s\left(1+4\frac{T_0}{\varepsilon_c}\right)\frac{L}{u_c\tau_d(T_0)} \quad (57)$$

and $L \gg u_c \tau_d$. In Eqs. (56) and (57) we here retain the terms in T/ε_c because they are multiplied by the large coefficient 4.

For pulses of arbitrary length we have numerically integrated Eq. (55). These give essentially the same results from those obtained in Sec. IV for $T(t)$ and $\Delta(t)$ for short pulses, as shown in Figs. 5 and 6, up to the pulse length where the pulse begins to saturate. We see that a significant fraction of the initial energy is transformed into *h* phonons within a few millimeters of the heater, and the energy in the *h* phonons increases with pulse length. All these results correspond with experimental observations.

The situation with long pulses is different. The results of the calculation are shown in Figs. 7(a) and (b) and Figs. 8(a) and (b) for 1- and 10-μs pulse lengths. We see that long pulses cool much more slowly than short ones, and from Eqs. (47) and (54) we find

$$\frac{\left(\dfrac{\partial E_l^{(0)}}{\partial t}\right)_L}{\left(\dfrac{\partial E_l^{(0)}}{\partial t}\right)_S}=\frac{u_c\tau_d}{L_L}\ll 1, \quad (58)$$

where L_L is the length of the long pulse. However, long pulses create larger fluxes of *h* phonons than short pulses; according to Eqs. (49) and (50) we find

$$\frac{\Phi_L}{\Phi_S}=\frac{u_c\tau_d}{L_S}\gg 1, \quad (59)$$

where L_S is the length of the short pulse.

A long heater pulse creates many *h* phonons over a long time, the sequence of events is shown schematically in Fig. 9. With a bolometric detector in the liquid the signal from *h* phonons will run into the large signal from *l* phonons because *h* phonons are still being created when the pulse reaches the detector, so we do not see well separated *h* and *l* phonon signals as we do with a short pulse. However, a superconducting tunnel junction detector is not sensitive to the *l* phonons so it would detect the long *h* phonon cloud more clearly.

VIII. DISCUSSION

At the outset of this work we knew that a single short heat pulse produces two phonon pulses after propagating ~10 mm through cold liquid helium. Experiments had shown that the faster pulse was due to low-energy phonons and the slower one was due to phonons with $\varepsilon \approx \varepsilon_C$, where $\varepsilon_C = 10$ K at zero pressure. From careful measurements on the time of flight it was clear that the *h* phonons were produced in front of the heater[20] and it had been realized that both $3pp$ and $4pp$ up-scattering were necessary to create *h* phonons from *l* phonons in liquid ^{4}He.

The theory presented in Ref. 6 and here circumvents considering the detailed $3pp$ up-scattering sequence because the low-energy (*l*) phonons are thermalized on a very short time scale, $\sim 10^{-10}$ s.[30,32,33] This ensures a thermal equilibrium spectrum of phonon energies ε up to ε_C. Only a single $4pp$ between two *l* phonons is required to create an *h* phonon with $\varepsilon > \varepsilon_C$, and the scattering rate for this process is known.[15] The *h* phonons escape from the trailing edge of the *l* cloud because they have a slower group velocity than the *l* cloud. The *h* phonon number density in the *l* cloud is therefore reduced from its equilibrium value and this deficiency is redressed $4pp$ up-scattering.

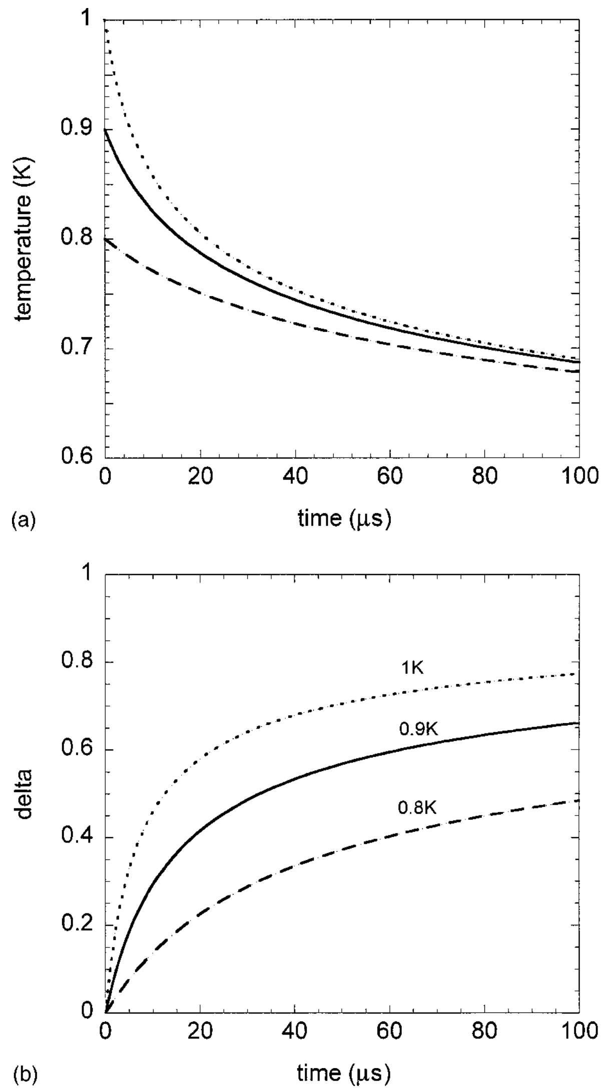

FIG. 7. (a) The temperature of a long pulse, $t_p = 1$ μs, as a function of its propagation time for three different initial temperatures; $T_0 = 0.8$ K (dashed), 0.9 K (solid), and 1.0 K (dotted), using Eq. (55). (b) the fraction of energy Δ in the *h* phonons relative to the total energy, is shown as a function of time for the three initial temperatures T_0 that correspond to the cooling curves in Fig. 9(a).

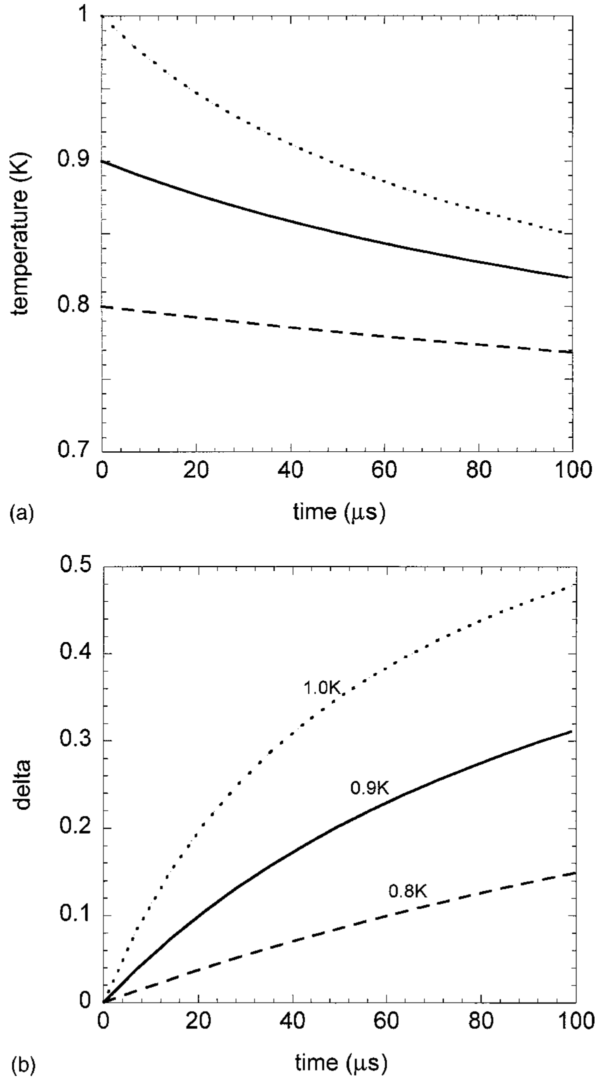

FIG. 8. (a) The temperature of a long pulse, $t_p = 10\,\mu$s, as a function of its propagation time for three different initial temperatures; $T_0 = 0.8$ K (dashed), 0.9 K (solid), and 1.0 K (dotted), using Eqs. (56) and (57). (b) the fraction of energy Δ in the *h* phonons relative to the total energy is shown as a function of time for the three initial temperatures T_0 that correspond to the cooling curves in (a).

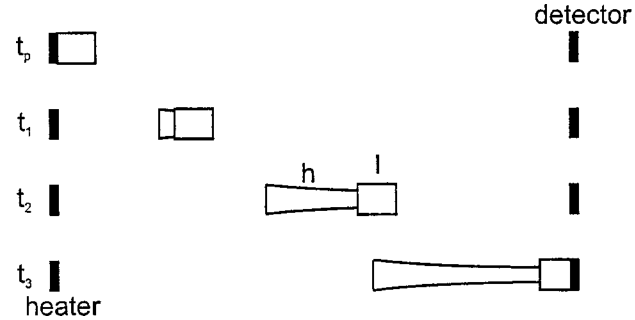

FIG. 9. A schematic diagram for a long pulse which should be compared with Fig. 1(b) in Ref. 6 for a short pulse. Note that *h* phonons are generated throughout the propagation path of the *l* phonons when the pulse is long.

The theory presented here addresses the hitherto unexplained experimental observations of this phenomenon, which are: (i) short heat pulses, $t_p \leqslant 10^{-7}$ s, are best for generating *h* phonon signals, longer pulses give signals that are not of greater amplitude, but are broader; the *h* phonon generation saturates as the heater pulse length is increased beyond 2×10^{-7} s; (ii) the conversion of *l* phonons to *h* phonons is very efficient especially for short pulses, a major part of the energy in the *l* phonons can be converted to *h* phonons; (iii) *h* phonons are created in the liquid, within a few millimeters of the heater for short pulses, but for long pulses the generation continues all the way from the heater to the detector but it decreases with distance from the heater.

The theory for short pulses shows that *h* phonons are lost from the *l* cloud as they are formed. This gives the maximum generation rate for *h* phonons as there are no decay processes for the *h* phonons within the cloud. Also, once an *h* phonon is left trailing behind the *l* cloud, it is completely stable because it cannot decay by $4pp$ as there are no suitable low-energy phonons to interact with. Numerical evaluation of the equations shows that a major fraction of the initial *l* cloud energy can be converted to *h* phonons. For a short pulse the *h* phonon generation process is so efficient that most of the energy is converted before the *l* cloud has moved many millimeters from the heater. Once the *l* cloud has lost most of its energy it essentially stops creating *h* phonons because the creation rate is a strong function of temperature. The initial *l* cloud temperature is ~1 K and after the cloud has cooled to ~0.7 K the generation rate is negligible. This is due to a large decrease with temperature of phonons with $\varepsilon \leqslant \varepsilon_C$ as well as the lower $4pp$ scattering rate. So this rapid cooling within a few millimeters of the heater makes a short, high amplitude, *h* pulse as observed in experiments.

For long pulses, there is a competing process to *h* phonons being lost from the *l* cloud. A created *h* phonon may decay by $4pp$ within the cloud before it is lost. For this to occur, there must be a low-energy phonon with its momentum at a large angle to the momentum of the *h* phonon. But there are few such phonons in the *l* cloud as the *l* phonon momenta are in a narrow cone from the injection process at the heater-liquid interface. However, for creation there is no such restriction on the angle between the two *l* phonons that combine to form the *h* phonon. This asymmetry between the decay and creation processes leads to a suprathermal distribution of *h* phonons within the pulse; the energy density per unit solid angle is more than an order of magnitude higher in the pulse than in a spherically symmetric Bose-Einstein distribution.

The competition between loss and decay means that the generation efficiency is less than for a short pulse. In a long pulse the *h* phonon density is determined by dynamic equilibrium between creation and decay, and the loss rate from the *l* cloud is simply proportional to the equilibrium density. As this is independent of pulse length for long pulses, the generation rate of the *h* cloud trailing the *l* cloud is also independent of pulse length. This explains why the *h* phonon signal amplitude saturates as a function of pulse length.

We can also see that the cooling rate for a long pulse is much slower than for a short pulse. The fraction of energy lost from the *l* cloud to the *h* cloud is smaller and decreases as the inverse of the pulse length. So the *l* cloud energy and temperature decrease more slowly with time. This means that the *h* phonon production rate decreases little over the whole propagation distance. So a very long *h* cloud is created by a long input pulse.

The model has developed some interesting ideas on its way to explaining the observed characteristics. It is now clear that the *l* cloud is a dynamic entity which evolves as it propagates. The model uses the notion of thermal equilibrium in a narrow momentum cone. This thermalization requires two conditions, first that the cloud exists for a long enough time and second that the necessary scattering processes can occur. The first condition is satisfied because the $3pp$ rate at normal isotropic phonon densities is ~5 orders of magnitude faster than the inverse of the propagation time, so the *l* cloud is thermalized soon after its creation by the heater. The second condition involves the scattering angles. In a $3pp$ there is a small angle between the two ingoing or outgoing phonons, and the scattering necessary for thermalization takes place only if this angle is smaller than the angle of the momentum cone. In fact if the cone is initially too narrow for this condition, the scattering will broaden it until the condition is satisfied. The phonon distribution in the cone is identical to a slice of an isotropic equilibrium distribution, and because it is a slice it has net momentum so it propagates at essentially the sound velocity.

Thermalization in the cone strictly applies only for phonon energies up to $\varepsilon_C^{(2)} \sim 8.3\,\mathrm{K}$, but is probably valid up to $\varepsilon \approx \varepsilon_C\,(=10\,\mathrm{K})$. However, it does not apply to phonons with $\varepsilon > \varepsilon_C$ because the necessary scattering processes cannot occur. As we have seen above, for $\varepsilon > \varepsilon_C$ only $4pp$ are possible and the phonons necessary for *h* phonon decay are not available in the narrow momentum cone. Because the *h* phonons cannot readily decay, their population increases dramatically and becomes about two orders of magnitude greater than the number density in an equivalent slice of an isotropic thermal equilibrium distribution at the temperature of the *l* cloud. Hence the generation rate of the *h* cloud increases by the same factor in long pulses. In fact a short pulse has to be very short for the population of *h* phonons in the *l* cloud to be smaller than the thermal equilibrium value. This unusual population increase depends on both the *l* phonons only occupying a cone and the details of $4pp$. Furthermore, the *l* phonons have a Bose-Einstein distribution of energies but the *h* phonons can only be in dynamic equilibrium with the *l* phonons in a long pulse. The model successfully accounts for the important characteristics of this phenomenon and describes the interesting physics of propagating phonon pulses in liquid ^{4}He.

IX. CONCLUSION

We have presented a theoretical model of energy transfer from low- to high-energy phonons in propagating phonon pulses in liquid ^{4}He when the low-energy phonons occupy a narrow cone in momentum space. The theory gives an excellent quantitative description of this unusual phenomenon.

ACKNOWLEDGMENT

It is a pleasure to acknowledge support for this collaborative work from EPSRC under Grant Nos. GR/M/22543 and GR/L/29149.

[1] Y. Korczynskyj and A. F. G. Wyatt, J. Phys. (Paris), Colloq. **39**, C6-230, Suppl. 8 (1978).
[2] A. F. G. Wyatt, J. Phys.: Condens. Matter **1**, 8629 (1989).
[3] M. A. H. Tucker and A. F. G. Wyatt, J. Phys.: Condens. Matter **6**, 2813 (1994).
[4] M. A. H. Tucker and A. F. G. Wyatt, J. Phys.: Condens. Matter **6**, 2825 (1994).
[5] See reviews by L. J. Challis, J. Phys. C **7**, 481 (1974); A. F. G. Wyatt, in *Proceedings of Nato Summer School on Non-Equilibrium Superconductivity, Phonons and Kapitza Boundaries*, edited by K. E. Gray (Plenum, New York, 1981), Chap. 2.
[6] I. N. Adamenko, K. E. Nemchenko, A. V. Zhukov, M. A. H. Tucker, and A. F. G. Wyatt, Phys. Rev. Lett. **82**, 1482 (1999).
[7] H. J. Maris and W. E. Massey, Phys. Rev. Lett. **25**, 220 (1970).
[8] W. G. Stirling, *Proceedings 75th Jubilee Conference on ^{4}He*, edited by J. G. M. Armitage (World Scientific, Singapore, 1983), p. 109.
[9] N. G. Mills, R. A. Sherlock, and A. F. G. Wyatt, Phys. Rev. Lett. **32**, 978 (1974).
[10] T. J. Sluckin and R. M. Bowley, J. Phys. C **7**, 1779 (1974).
[11] See review by H. J. Maris, Rev. Mod. Phys. **49**, 341 (1977).
[12] R. C. Dynes and V. Narayanamurti, Phys. Rev. B **12**, 1720 (1975).
[13] F. Mezei, C. Lartigue, and B. Farago, in *Excitations in Two-Dimensional and Three-Dimensional Quantum Fluids*, edited by A. F. G. Wyatt and H. J. Lauter (Plenum Press, New York, 1991), p. 19.
[14] M. A. H. Tucker and A. F. G. Wyatt, J. Phys.: Condens. Matter **4**, 7745 (1992).
[15] M. A. H. Tucker, A. F. G. Wyatt, I. N. Adamenko, K. E. Nemchenko, and A. V. Zhukov (unpublished).
[16] M. J. Baird, F. R. Hope, and A. F. G. Wyatt, Nature (London) **304**, 325 (1983).
[17] M. Brown and A. F. G. Wyatt, J. Phys.: Condens. Matter **2**, 5025 (1990).
[18] M. A. H. Tucker and A. F. G. Wyatt, Physica B **165&166**, 493 (1990).
[19] A. C. Forbes and A. F. G. Wyatt (unpublished); see A. F. G. Wyatt, J. Phys.: Condens. Matter **8**, 9249 (1996), Fig. 13.
[20] M. A. H. Tucker and A. F. G. Wyatt, J. Low Temp. Phys. **113**, 62 (1998).
[21] A. F. G. Wyatt and M. Brown, Physica B **165&166**, 495 (1990).
[22] A. F. G. Wyatt, N. A. Lockerbie, and R. A. Sherlock, J. Phys.: Condens. Matter **1**, 3507 (1989).
[23] I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).
[24] R. A. Sherlock, A. F. G. Wyatt, N. G. Mills, and N. A. Lockerbie, Phys. Rev. Lett. **29**, 1299 (1972).
[25] I. N. Adamenko and I. M. Fuks, Zh. Eksp. Teor. Fiz. **59**, 2071 (1970) [Sov. Phys. JETP **32**, 1123 (1971)].
[26] I. M. Khalatnikov and I. N. Adamenko, Zh. Eksp. Teor. Fiz. **63**, 745 (1972) [Sov. Phys. JETP **36**, 391 (1973)].
[27] R. A. Sherlock, N. G. Mills, and A. F. G. Wyatt, J. Phys. C **8**, 300 (1975).

[28] A. F. G. Wyatt and G. J. Page, J. Phys. C **11**, 4927 (1978).
[29] A. F. G. Wyatt and G. N. Crisp, J. Phys. (Paris), Colloq. **39**, C6-244, Suppl. 8 (1978).
[30] I. N. Adamenko, K. E. Nemchenko, A. V. Zhukov, M. A. H. Tucker, and A. F. G. Wyatt, Physica B **284-288**, 33 (2000).
[31] H. J. Maris, Phys. Rev. A **9**, 1412 (1974).
[32] M. A. H. Tucker, A. F. G. Wyatt, I. N. Adamenko, K. E. Nemchenko, and A. V. Zhukov, Low Temp. Phys. **25**, 488 (1999).
[33] I. N. Adamenko, K. E. Nemchenko, A. V. Zhukov, M. A. H. Tucker, and A. F. G. Wyatt, Physica B **284-288**, 35 (2000).
[34] A. F. G. Wyatt, M. A. H. Tucker, I. N. Adamenko, K. E. Nemchenko, and A. V. Zhukov, Physica B **280**, 36 (2000).