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Effective actions and phase fluctuations in *d*-wave superconductors

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We study effective actions for order-parameter fluctuations at low temperature in layered *d*-wave superconductors such as the cuprates. The order parameter lives on the bonds of a square lattice and has two amplitude and two phase modes associated with it. The low-frequency spectral weights for amplitude and relative phase fluctuations is determined and found to be subdominant to quasiparticle contributions. The Goldstone phase mode and its coupling to density fluctuations in charged systems is treated in a gauge-invariant manner. The Gaussian phase action is used to study both the *c*-axis Josephson plasmon and the more conventional in-plane plasmon in the cuprates. We go beyond the Gaussian theory by deriving a coarse-grained quantum XY model, which incorporates important cutoff effects overlooked in previous studies. A variational analysis of this effective model shows that in the cuprates, quantum effects of phase fluctuations are important in reducing the zero-temperature superfluid stiffness, but thermal effects are small for $T \ll T_c$.

I. INTRODUCTION

The high-temperature cuprate superconductors (SC's) differ from conventional SC's in several respects: a *d*-wave gap with gapless quasiparticle excitations, a small superfluid phase stiffness, a short coherence length and strong electron interactions. It is, therefore, of interest to examine some of these unconventional aspects and their interplay in simple models of high T_c systems. With this motivation, we study in this paper, the low-temperature collective properties of charged, layered *d*-wave SC's with a short coherence length and small superfluid stiffness.

The superfluid phase stiffness, $D_s = n_s/m^*$, is a fundamental property characterizing SC's,[1] which is directly related to λ, the experimentally measured magnetic penetration depth[2] in the London limit. The low-temperature behavior of D_s contains information on the low-lying excitations in these systems. Experimentally, $\lambda(T)$ is found to increase linearly with T in the high-T_c SC's,[3] implying a linearly decreasing D_s. This linear drop in D_s has been attributed to quasiparticle excitations near the nodes of the *d*-wave gap. Alternatively, it has been suggested that this effect could arise entirely from classical thermal phase fluctuations[4,5] and quasiparticles can be ignored.[4,5] It is then clearly of interest to identify the important low-energy excitations in these systems, from the point of view of understanding the penetration depth data, as well as other thermodynamic properties and response functions.

From a theoretical perspective, the physics of a system with a small superfluid stiffness and short coherence length has been studied in detail in case of neutral *s*-wave SC's.[6,7] In this case, the fermionic excitations and fluctuations in the order-parameter amplitude are gapped, and phase fluctuations are the only important excitation at low temperature. It is of interest to compare this with the behavior in models which support an anisotropic order parameter with low lying fermionic excitations, such as a *d*-wave SC.

We approach the problem by deriving and analyzing effective actions for a *d*-wave SC within a functional integral framework, which allows us to focus on the collective (order parameter) degrees of freedom. We note that our effective actions are derived by looking at fluctuations around a BCS mean-field solution. We believe that such an approach is valid for the SC state of the high-T_c materials, at least for $T \ll T_c$. There is considerable experimental evidence for sharply defined quasiparticle excitations about the *d*-wave SC state, and thus the ground-state and low-lying excitations appear to be adiabatically connected to their BCS counterparts. We thus expect that while strong correlations will modify the *coefficients* of the phase action, they will not change its qualitative form.

Our main results can be summarized as follows:

(1) We find that the *d*-wave SC state is characterized in terms of an order parameter which lives on the bonds of a square lattice. The bond order parameter leads to two amplitude and two phase modes in contrast to *s*-wave SC's. One of the phase modes is identified as the usual (Goldstone) phase mode while the other, which we call the ''bond phase,'' is the relative phase between the x and y bonds at a site. The latter can be thought of as representing fluctuations from the *d*-wave state towards an extended *s*-wave state. The amplitude and bond-phase fields have spin zero and couple to the particle-particle channel.

(2) We study the spectral weight for fluctuations of the amplitude and bond-phase fields and find that they are *not gapped* but rather exhibit power laws down to zero energy. However, the low-energy spectral weight in these fluctuations is very small compared to the quasiparticle contribution.

(3) We derive an effective Gaussian action for the usual phase variable in charged systems, since this couples directly to the electromagnetic potentials. The large in-plane plasma frequency, which is relatively unaffected by superconductivity, and the low-energy *c*-axis Josephson plasmon at zero and finite temperatures are studied in a unified manner within the same formalism. We emphasize the relation between unusual

0163-1829/2000/62(10)/6786(14)/$15.00 PRB 62

aspects of the c-axis optical conductivity and the Josephson plasmon. We also discuss the plasmon dispersion in layered systems.

(4) We extend the above formalism to consider the effect of phase fluctuations beyond the Gaussian level on the superfluid stiffness in charged systems. The quantum XY model[8] is usually used for such an analysis, motivated by studies of Josephson-junction arrays and granular SC's. We emphasize that there are important differences when considering low-temperature bulk SC's, and derive a quantum XY phase action suitable for our problem, correctly taking into account appropriate momentum and frequency cutoffs, missed in earlier studies.

(5) The low-temperature renormalization of D_s by phase fluctuations is studied within a self-consistent harmonic approximation. For parameter values relevant to the cuprates near optimal doping, quantum phase fluctuations are shown to lead to a sizeable renormalization of the superfluid stiffness. However, thermal fluctuations are found to have no effect at low temperatures, unlike the results of earlier studies.[4,5] These studies focused on the effect of thermal phase fluctuations, but Coulomb effects were considered to be unimportant, in contrast to the present work.

(6) As part of our analysis, we also touch upon certain formal issues which may be of some general interest. Among these are: (a) how gauge invariance can be understood in a simple manner within the functional integral language; (b) the role of the linear time derivative term $i\rho\partial\tau\theta$ in the phase action; and (c) the problems involved in deriving local phase actions which respect 2π periodicity starting from a fermionic model.

The paper is organized as follows. In Sec. II, we present the Hamiltonian for our model, and discuss the effective action and mean-field theory in Sec. III. Sec. IV contains a discussion of fluctuations of the amplitude and bond-phase fields with some of the details discussed in Appendix A. In Sec. V, we turn to phase fluctuations and derive effective phase-only actions for neutral and charged systems. The linear time derivative term in the action which arises in this context is briefly discussed in Appendix B. We then derive gauge-invariant density and current correlations, leaving details of the algebra to Appendix C. In Sec. VI we discuss collective in-plane and c-axis plasmons. In Sec. VII we present the derivation of a quantum XY model appropriate for charged, layered SC's. We analyze this action and compute the renormalization of the phase stiffness by longitudinal phase fluctuations in Sec. VIII and discuss experimental implications. We conclude in Sec. IX with a discussion and summary of our results.

II. THE HAMILTONIAN

We consider a system of fermions with kinetic energy $K=\sum_{k,\sigma}\xi_{\mathbf{k}}c^{\dagger}_{\mathbf{k},\sigma}c_{\mathbf{k},\sigma}$ [where $\xi_{\mathbf{k}}=\epsilon_{\mathbf{k}}-\mu$ with $\epsilon_{\mathbf{k}}$ begin the two-dimensional (2D) dispersion and μ the chemical potential] interacting via a separable potential which is attractive in the d-wave channel. We will show that in coordinate space this interaction leads to the superexchange term of the t-J model.

Let us begin with

$$H'_{\mathrm{pair}}=-\frac{g}{N}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\varphi_d(\mathbf{k})\varphi_d(\mathbf{k}')\times c^{\dagger}_{\mathbf{k}+\mathbf{q}/2\uparrow}c^{\dagger}_{-\mathbf{k}+\mathbf{q}/2\downarrow}c_{-\mathbf{k}'+\mathbf{q}/2\downarrow}c_{\mathbf{k}'+\mathbf{q}/2\uparrow}, \tag{1}$$

where $\varphi_d(\mathbf{k})=(\cos k_x-\cos k_y)$. and we work on a 2D square lattice with lattice spacing a. Ω denotes the volume of the system with N sites. We set $a=1$ in most equations, but retain it in some for the sake of clarity. Fourier transforming to real space, we get

$$H_{\mathrm{pair}}=-\frac{g}{4}\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}B^{\dagger}_{\mathbf{r},\mathbf{r}'}B_{\mathbf{r},\mathbf{r}'}=\frac{g}{2}\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}\left(\mathbf{S}_{\mathbf{r}}\cdot\mathbf{S}_{\mathbf{r}'}-\frac{1}{4}n_{\mathbf{r}}n_{\mathbf{r}'}\right). \tag{2}$$

[The prime on H_{pair} is omitted in going from Eq. (1) to Eq. (2) for reasons explained below.] Here $\langle\mathbf{r},\mathbf{r}'\rangle$ are nearest-neighbor sites, and $B^{\dagger}_{\mathbf{r},\mathbf{r}'}\equiv c^{\dagger}_{\mathbf{r},\uparrow}c^{\dagger}_{\mathbf{r}',\downarrow}-c^{\dagger}_{\mathbf{r},\downarrow}c^{\dagger}_{\mathbf{r}',\uparrow}$ creates a singlet on the bond $(\mathbf{r},\mathbf{r}')$, while $\mathbf{S}_{\mathbf{r}}$ and $n_{\mathbf{r}}$ are the spin and number operators. Clearly this is the interaction term of the t-J model with $g=2J$.

There is a subtlety involved here; on transforming (2) back to $\mathbf{k}$ space we do *not* recover the original expression (1) we had started with. Instead we obtain, using from now on $g=2J$,

$$H_{\mathrm{pair}}=-\frac{J}{N}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}[\varphi_d(\mathbf{k})\varphi_d(\mathbf{k}')+\varphi_s(\mathbf{k})\varphi_s(\mathbf{k}')]\times c^{\dagger}_{\mathbf{k}+\mathbf{q}/2\uparrow}c^{\dagger}_{-\mathbf{k}+\mathbf{q}/2\downarrow}c_{-\mathbf{k}'+\mathbf{q}/2\downarrow}c_{\mathbf{k}'+\mathbf{q}/2\uparrow}, \tag{3}$$

where $\varphi_s(\mathbf{k})=(\cos k_x+\cos k_y)$. The reason for two different $\mathbf{k}$-space interactions leading to the same real-space expression is the following operator identity on a 2D square lattice:

$$\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}[\varphi_d(\mathbf{k})\varphi_d(\mathbf{k}')-\varphi_s(\mathbf{k})\varphi_s(\mathbf{k}')]\times c^{\dagger}_{\mathbf{k}+\mathbf{q}/2\uparrow}c^{\dagger}_{-\mathbf{k}+\mathbf{q}/2\downarrow}c_{-\mathbf{k}'+\mathbf{q}/2\downarrow}c_{\mathbf{k}'+\mathbf{q}/2\uparrow}\equiv 0. \tag{4}$$

It can be shown that Eqs. (2) and (3) both lead to the same self-consistent BCS gap equation. Thus we will use the interaction in Eq. (3), and not Eq. (1). From the form of Eq. (3), it is clear that H_{pair} has attraction in both the d-wave channel, with a $\varphi_d(\mathbf{k})$ order parameter, and in the extended s-wave (s^*) channel, with a $\varphi_s(\mathbf{k})$ order parameter.

We will now analyze the Hamiltonian $H=K+H_{\mathrm{pair}}$. Later, we will also add to it the Coulomb interaction appropriate to layered systems (see Sec. V B).

III. MEAN-FIELD THEORY

The partition function at a temperature T is written as the standard coherent state path integral with the action $\int_0^{1/T}d\tau[\sum_{\mathbf{r},\sigma}c^{\dagger}_{\mathbf{r},\sigma}\partial_\tau c_{\mathbf{r},\sigma}+H(c,c^{\dagger})]$. We decouple H_{pair} with a complex field $\Delta_{\mathbf{r},\mathbf{r}'}(\tau)$ using the Hubbard-Stratonovich transformation:

$$\exp\left(\frac{J}{2}B^{\dagger}_{\mathbf{r},\mathbf{r}'}(\tau)B_{\mathbf{r},\mathbf{r}'}(\tau)\right)=\int\mathcal{D}(\Delta\Delta^*)\exp[-\mathcal{L}(\mathbf{r},\mathbf{r}';\tau)], \tag{5}$$

where

$$\mathcal{L}(\mathbf{r},\mathbf{r}';\tau)=\frac{1}{8J}|\Delta_{\mathbf{r},\mathbf{r}'}(\tau)|^2-\frac{1}{4}(\Delta_{\mathbf{r},\mathbf{r}'}(\tau)B^{\dagger}_{\mathbf{r},\mathbf{r}'}(\tau)+\text{H.c.}). \tag{6}$$

We thus obtain the action

$$S=\int_0^{1/T}d\tau\left[\sum_{\mathbf{k},\sigma}c^{\dagger}_{\mathbf{k},\sigma}(\tau)(\partial_\tau+\xi_{\mathbf{k}})c_{\mathbf{k},\sigma}(\tau)+\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}\mathcal{L}(\mathbf{r},\mathbf{r}';\tau)\right]. \tag{7}$$

The fermion fields can then be integrated out to obtain the effective action $\exp(-S_{\text{eff}}[\Delta,\Delta^*])=\int\mathcal{D}(c,c^{\dagger})e^{-S}$.

The d-wave saddle point is given by $\Delta_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}}(\tau)=-\Delta_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}}(\tau)=\Delta_d$, a $(\mathbf{r},\tau)$-independent real number, obtained from $\delta S_{\text{eff}}/\delta\Delta_d=0$, which leads to the BCS gap equation

$$\frac{1}{J}=\frac{1}{N}\sum_{\mathbf{k}}\frac{\varphi_d^2(\mathbf{k})}{2E_{\mathbf{k}}}\tanh(E_{\mathbf{k}}/2T), \tag{8}$$

where $E_{\mathbf{k}}=\sqrt{\xi^2_{\mathbf{k}}+\Delta^2_{\mathbf{k}}}$ and $\Delta_{\mathbf{k}}=\Delta_d\varphi_d(\mathbf{k})/2$. The same gap equation can be obtained by starting from the momentum space potential in Eq. (3) and decoupling in the d-wave channel.

Given H_{pair} of Eq. (3) one could equally well look for possible extended s-wave (s^*) saddle points with $\Delta_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}}(\tau)=\Delta_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}}(\tau)=\Delta_s$. However, for our choice of dispersion[9] (which includes nearest- and next-nearest-neighbor hopping with opposite signs) we have found by numerical solution of the gap equation that the d-wave saddle point is stable relative to the s^* solution. The reason for this can be seen as follows: $\varphi_s(\mathbf{k})$ is small over most of the Fermi surface for the fillings of interest, while $\varphi_d(\mathbf{k})$ vanishes only on the nodes. Thus the condensation energy gained by the s^* state is smaller than the d-wave state. Further, if we consider the large on-site repulsion between electrons (which we have not done here, but is certainly an essential part of the full t-J model) and demand $\langle c^{\dagger}_{\mathbf{r},\uparrow}c^{\dagger}_{\mathbf{r},\downarrow}\rangle=0$, then we have $\Sigma_{\mathbf{k}}\Delta_{\mathbf{k}}/2E_{\mathbf{k}}=0$. This is automatically satisfied for the d-wave state at any filling, but not for the s^* state. In this work, we rely on the former ''Fermi-surface effect'' to stabilize the d-wave state.

IV. FLUCTUATIONS

To treat fluctuations in the order parameter we write $\Delta_{\mathbf{r},\mathbf{r}'}(\tau)=|\Delta_{\mathbf{r},\mathbf{r}'}(\tau)|e^{i\Phi_{\mathbf{r},\mathbf{r}'}(\tau)}$. The phase $\Phi_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}}(\tau)=0$ and $\Phi_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}}(\tau)=\pi$ at the d-wave saddle point. We now divide the phase field into two parts; following Ref. 10, we set $\Phi_{\mathbf{r},\mathbf{r}+\hat{x}}(\tau)=\theta(\mathbf{r},\tau)$ and $\Phi_{\mathbf{r},\mathbf{r}+\hat{y}}(\tau)=\pi+\phi(\mathbf{r},\tau)+\theta(\mathbf{r},\tau)$.

We next assume that the spatial variation of $\theta(\mathbf{r},\tau)$ is small on the scale of the lattice spacing, which allows us to set $\Phi_{\mathbf{r},\mathbf{r}+\hat{x}}(\tau)\approx\frac{1}{2}[\theta(\mathbf{r},\tau)+\theta(\mathbf{r}+\hat{x},\tau)]$ and $\Phi_{\mathbf{r},\mathbf{r}+\hat{y}}(\tau)\approx\pi+\phi(\mathbf{r},\tau)+\frac{1}{2}[\theta(\mathbf{r},\tau)+\theta(\mathbf{r}+\hat{y},\tau)]$. While we lose the $\theta(\mathbf{r},\tau)\rightarrow\theta(\mathbf{r},\tau)+2\pi$ invariance of the action with this approximation, it is nevertheless useful in isolating that part of the phase field which couples to electromagnetic fields as we will see below. We can now transform to new fermion variables given by $c^{\dagger}_{\mathbf{r}}(\tau)\rightarrow c^{\dagger}_{\mathbf{r}}(\tau)e^{-i\theta(\mathbf{r},\tau)/2}$.[11,12] As a result of this ''gauge transformation'' the action of Eq. (7) gets modified to

$$S=\int_0^{1/T}d\tau[\mathcal{L}_0+\mathcal{L}_1] \tag{9}$$

with

$$\mathcal{L}_0=\sum_{\mathbf{r},\sigma}c^{\dagger}_{\mathbf{r},\sigma}(\tau)e^{-i\theta(\mathbf{r},\tau)/2}(\partial_\tau-\mu)c_{\mathbf{r},\sigma}(\tau)e^{i\theta(\mathbf{r},\tau)/2}$$
$$-\frac{1}{2}\sum_{\mathbf{r},\mathbf{r}',\sigma}t(\mathbf{r}-\mathbf{r}')[c^{\dagger}_{\mathbf{r},\sigma}(\tau)c_{\mathbf{r}',\sigma}(\tau)$$
$$\times e^{-i[\theta(\mathbf{r},\tau)-\theta(\mathbf{r}',\tau)]/2}+\text{H.c.}] \tag{10}$$

where $t(\mathbf{r}-\mathbf{r}')$ is the hopping matrix element between point $\mathbf{r}$ and $\mathbf{r}'$, so that $\epsilon_{\mathbf{k}}=\Sigma_{\mathbf{r}-\mathbf{r}'}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}t(\mathbf{r}-\mathbf{r}')$ and

$$\mathcal{L}_1=\frac{1}{8J}\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}|\Delta_{\mathbf{r},\mathbf{r}'}(\tau)|^2-\frac{1}{4}\sum_{\mathbf{r}}|\Delta_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}}(\tau)|(B^{\dagger}_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}}(\tau)$$
$$+\text{H.c.})+\frac{1}{4}\sum_{\mathbf{r}}|\Delta_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}}(\tau)|(B^{\dagger}_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}}(\tau)e^{i\phi(\mathbf{r},\tau)}+\text{H.c.}). \tag{11}$$

In the following sections we shall integrate out the fermions and examine the resulting effective actions for the amplitude, ϕ and θ fields.

A. Amplitude fluctuations

Amplitude fluctuations can be considered by setting $|\Delta_{\mathbf{r},\mathbf{r}+\hat{\alpha}}(\tau)|=\Delta_d[1+\eta_\alpha(\mathbf{r},\tau)]$ in Eq. (11), where $\alpha=x,y$. The transformation from $\Delta_\alpha,\Delta^*_\alpha$ to η_α,θ,ϕ has a Jacobian $4\Delta_d^4(1+\eta_x)(1+\eta_y)$ at every point $(\mathbf{r},\tau)$, leading to an additional term

$$\int_0^{1/T}d\tau\mathcal{L}_2=T\int_0^{1/T}d\tau\sum_{\mathbf{r},\alpha}\ln(1+\eta_\alpha(\mathbf{r},\tau))$$
$$\approx T\int_0^{1/T}d\tau\sum_{\mathbf{r},\alpha}\left[\eta_\alpha(\mathbf{r},\tau)-\frac{1}{2}\eta^2_\alpha(\mathbf{r},\tau)\right] \tag{12}$$

in the action in Eq. (9). For $(\mathbf{q},\omega)\neq(0,0)$, the linear term in Eq. (12) can be set to zero and only the quadratic term contributes. However, even this term is zero at $T=0$ and can be ignored at low T.

From Eq. (11) we see that the spin-zero amplitude fields η_α couple to singlet pairs. Their coupling to the ϕ field can be shown to be small at small momentum and frequency. In particular, for static uniform distortions of ϕ and η_α the energy has to be even under $\phi\rightarrow-\phi$ and terms like $\phi\eta_\alpha$ cannot appear in the action on integrating out the fermions. The mixing of η_α with the phase θ and electromagnetic potentials can also be shown to be negligible at small $\mathbf{q},\omega$ since η_α couples to the particle-particle channel while the θ and electromagnetic potentials couple to the particle-hole channel. The mixing then involves integrals over products of ordinary and anomalous Green's functions which vanish using particle-hole symmetry near the Fermi surface and the $\mathbf{k}$ dependence of $\Delta_{\mathbf{k}}$. This lack of mixing of amplitude and phase modes is similar to the well-known weak-coupling result for s-wave superconductors.[7]

Unlike the s-wave case, however, the amplitude excitations have low-frequency spectral weight in d-wave SC's which we now estimate. Setting the phase fields to saddle-point values and integrating out the fermions we obtain an effective action for amplitude fluctuations. Transforming to new variables $\eta_s=(\eta_x+\eta_y)/\sqrt{2}$ and $\eta_d=(\eta_x-\eta_y)/\sqrt{2}$, diagonalizes the action for η fields at $\mathbf{q}=0$ and is a good starting point to consider small $\mathbf{q}$ fluctuations. We obtain, to Gaussian order, $S[\eta]=(1/2T)\Sigma_{\mathbf{q},n,i=s,d}\eta_i^* M_i^{-1}(\mathbf{q},i\omega_n)\eta_i$, where $\omega_n=2\pi nT$. We have made the approximation of ignoring the coupling between η_s and η_d, which can be shown to be negligible at small momentum.

The low-energy density of states is given by $N_i(\omega)=(1/N)\Sigma_{\mathbf{q}}\mathrm{Im}\,M_i(\mathbf{q},\omega)/\pi$ with $|\mathbf{q}_x|,|\mathbf{q}_y|<\pi/\xi_0$. The cutoffs arise since the fluctuations must have an energy lower than the condensation energy as discussed in more detail later, in the context of phase fluctuations. From a numerical calculation of $N_{s,d}(\omega)$ we find that $N_s(\omega)/N_{qp}(\omega)\sim\omega^2/(\Delta_d v_F/a)$ and $N_d(\omega)/N_{qp}(\omega)\sim\omega^4/(\Delta_d^3 v_F/a)$ where $N_{qp}(\omega)\equiv k_F\omega/(\pi v_F\Delta_d)$ is the density of states per spin for quasiparticle excitations. These results can also be understood from an approximate analysis of the form of $M_{s,d}(\mathbf{q},\omega)$ discussed in Appendix A.

We see that both $N_s(\omega)$ and $N_d(\omega)$ are much smaller than $N_{qp}(\omega)$ for $\omega\ll\Delta_d$, and thus conclude that amplitude fluctuations are unimportant for low temperature properties which will be dominated by the quasiparticle contribution.

B. The ''bond-phase'' field ϕ

We next study the field ϕ. We note from Eq. (11) that a uniform $\phi=\pi$ would lead to extended s-wave (s^*) order. One can therefore think of ϕ as representing fluctuations of s^* character about the d-wave saddle point. From Eq. (11) we see ϕ is a spin zero field which couples to pairs.

For reasons similar to those explained above for amplitude fluctuations, the coupling of ϕ to other fields is weak and may be ignored at low momentum and frequency. We, therefore, derive a Gaussian action for ϕ by setting $\theta=\eta_\delta=0$, their saddle-point values, and integrating out the fermions. This leads to the action $S[\phi]=(1/T)\Sigma_{n,\mathbf{q}}M_\phi^{-1}(\mathbf{q},\omega_n)|\phi(\mathbf{q},\omega_n)|^2$. Since the ϕ field has low-frequency spectral weight, we compute its density of states $N_\phi(\omega)=(1/N\pi)\Sigma_{|\mathbf{q}|<\pi/\xi_0}\mathrm{Im}\,M_\phi(\mathbf{q},\omega+i0^+)$. From numerical calculations, as well as simpler approximations discussed in Appendix A, we find $N_\phi(\omega)/N_{qp}(\omega)\sim\omega^2/(\Delta_d v_F/a)$. We thus see that ϕ fluctuations are much less important than quasiparticles at low temperatures.

V. PHASE FLUCTUATIONS

From action of Eq. (10) we see that uniform shifts in θ do not cost any energy, and θ is the Goldstone mode of the superconducting state.

We now obtain the action for θ fluctuations coupled to fermions, setting $\eta_\alpha=\phi=0$ (their saddle-point values) in Eq. (10) and (11). For slow spatial fluctuations, the deviation from the mean field action, obtained from Eq. (10) with $\theta=0$, is given by

$$\begin{aligned}\delta S[c^\dagger,c,\theta]=&\frac{1}{2N}\int_0^{1/T}d\tau\sum_{\mathbf{k},\mathbf{q},\sigma}c^\dagger_{\mathbf{k},\sigma}(\tau)c_{\mathbf{k}-\mathbf{q},\sigma}(\tau)\\&\times[i\partial_\tau-i(\xi_{\mathbf{k}}-\xi_{\mathbf{k}-\mathbf{q}})]\theta_{\mathbf{q}}(\tau)-\frac{1}{8N}\int_0^{1/T}d\tau\\&\times\sum_{\mathbf{k},\mathbf{q},\mathbf{q}',\sigma}c^\dagger_{\mathbf{k},\sigma}(\tau)c_{\mathbf{k}-\mathbf{q}-\mathbf{q}',\sigma}(\tau)\theta_{\mathbf{q}}(\tau)\theta_{\mathbf{q}'}(\tau)\\&\times(\xi_{\mathbf{k}}+\xi_{\mathbf{k}-\mathbf{q}-\mathbf{q}'}-\xi_{\mathbf{k}-\mathbf{q}}-\xi_{\mathbf{k}-\mathbf{q}'}).\end{aligned}\tag{13}$$

Using $\theta_{\mathbf{q}}(\tau+1/T)=\theta_{\mathbf{q}}(\tau)$, an assumption discussed in detail in Appendix B, and making a small-$\mathbf{q}$ expansion, we arrive at

$$\begin{aligned}\delta S[c^\dagger,c,\theta]=&\frac{1}{2TN}\sum_{k,q,\sigma}c^\dagger_{k,\sigma}c_{k-q,\sigma}\theta_q[\omega_n-i\mathbf{v}_{\mathbf{k}\alpha}\mathbf{q}_\alpha]\\&-\frac{1}{8TN}\sum_{k,q,q',\sigma}c^\dagger_{k,\sigma}c_{k-q-q',\sigma}\\&\times m^{-1}_{\alpha\beta}(\mathbf{k})\mathbf{q}_\alpha\mathbf{q}'_\beta\theta_q\theta_{q'},\end{aligned}\tag{14}$$

where $k\equiv(\mathbf{k},i\nu_m)$ and $q\equiv(\mathbf{q},i\omega_n)$ with $\nu_m=(2m+1)\pi T$ and $\omega_n=2n\pi T$. We have used $\xi_{\mathbf{k}-\mathbf{q}}=\xi_{\mathbf{k}}-\mathbf{v}_{\mathbf{k}\alpha}\mathbf{q}_\alpha+\frac{1}{2}m^{-1}_{\alpha\beta}(\mathbf{k})\mathbf{q}_\alpha\mathbf{q}_\beta+\cdots$ where $\mathbf{v}_{\mathbf{k}\alpha}\equiv\partial\xi_{\mathbf{k}}/\partial\mathbf{k}_\alpha$ is the velocity and $m^{-1}_{\alpha\beta}(\mathbf{k})\equiv\partial^2\xi_{\mathbf{k}}/\partial\mathbf{k}_\alpha\partial\mathbf{k}_\beta$ is the inverse mass tensor.

A. Neutral systems

For neutral systems we integrate out the fermions in Eq. (14), using a cumulant expansion[13] controlled by small spatial and temporal gradients in θ, leading to

$$\begin{aligned}S_{neutral}[\theta]=&\frac{1}{T}\sum_{\mathbf{q},\omega_n}\frac{1}{8}[-\chi_0\omega_n^2+\Lambda_0^{\alpha\beta}\mathbf{q}^\alpha\mathbf{q}^\beta]\theta(\mathbf{q},i\omega_n)\\&\times\theta(-\mathbf{q},-i\omega_n).\end{aligned}\tag{15}$$

Here, $\chi_0\equiv-(1/T)\langle\rho(\mathbf{q},i\omega_n)\rho(-\mathbf{q},-i\omega_n)\rangle$ is the mean-field density-density correlation function given by

$$\begin{aligned}\chi_0(\mathbf{q},i\omega_n)=&\frac{2}{\Omega}\sum_{\mathbf{k}}(1-f-f')(uv'+vu')\left[\frac{uv'}{i\omega_n-E-E'}\right.\\&\left.-\frac{u'v}{i\omega_n+E+E'}\right]+\frac{2}{\Omega}\sum_{\mathbf{k}}(f-f')(vv'-uu')\\&\times\left[\frac{uu'}{i\omega_n-E+E'}+\frac{vv'}{i\omega_n+E-E'}\right]\end{aligned}\tag{16}$$

and $\Lambda_0^{\alpha\beta}\equiv(1/\Omega)\Sigma_{\mathbf{k}}m^{-1}_{\alpha\beta}(\mathbf{k})\langle\hat{n}_{\mathbf{k}}\rangle-(1/T)\langle j^\alpha(\mathbf{q},i\omega_n)j^\beta(-\mathbf{q},-i\omega_n)\rangle_0$ is the mean-field phase stiffness, with $\langle\hat{n}_{\mathbf{k}}\rangle=[1-\xi_{\mathbf{k}}/E_{\mathbf{k}}\tanh(E_{\mathbf{k}}/2T)]$ and the paramagnetic current correlator

$$\frac{1}{T}\langle j_\alpha j_\beta^*\rangle_0 = \frac{2}{\Omega}\sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}\alpha}\mathbf{v}_{\mathbf{k}\beta}(1-f-f')(vu'-uv') \times\left[\frac{uv'}{i\omega_n - E - E'} + \frac{u'v}{i\omega_n + E + E'}\right] + \frac{2}{\Omega}\sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}\alpha}\mathbf{v}_{\mathbf{k}\beta}(f'-f)(vv'+uu') \times\left[\frac{vv'}{i\omega_n + E - E'} - \frac{uu'}{i\omega_n - E + E'}\right]. \quad (17)$$

E, u, and v refer to standard BCS notation, $f=f(E)$ is the Fermi function, $E'\equiv E_{\mathbf{k}-\mathbf{q}}$ with $E\equiv E_{\mathbf{k}}$ and similarly for other primed variables. Analytically continuing $i\omega_n \to \omega + i\eta$, and working at $T=0$,[14] we obtain in the limit $\mathbf{q}\to 0, \omega\to 0$, the mean-field superfluid stiffness

$$\Lambda_0^{\alpha\beta}(T=0) = \frac{1}{\Omega}\sum_{\mathbf{k}} m_{\alpha\beta}^{-1}\left(1-\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \equiv D_s^0(T=0)\,\delta_{\alpha\beta} \quad (18)$$

and the mean-field compressibility

$$\chi_0(T=0) = -\frac{1}{\Omega}\sum_{\mathbf{k}} \frac{\Delta_{\mathbf{k}}^2}{E_{\mathbf{k}}^3} \equiv -\kappa. \quad (19)$$

We note that the effective action (15) is appropriate for phase distortions whose energy is smaller than the condensation energy $E_{\mathrm{cond}} = \frac{1}{8}D_s^0(\pi/\xi_0)^2$. If this energy (density) is exceeded the d-wave BCS saddle point would become unstable. This leads to the following restrictions in Eq. (15): $|\mathbf{q}|<q_c\equiv \pi/\xi_0$ and $\omega_n<\omega_c$ with $\kappa\omega_c^2/8 = E_{\mathrm{cond}}$. In the BCS limit,$E_{\mathrm{cond}}\simeq \Delta_d^2/v_F k_F$ which translates into $v_F q_c \sim \omega_c \sim \Delta_d$.

We emphasize that the coefficients in the phase action are *not* the physical correlation functions. The gauge-invariant correlation functions, obtained by including the effect of Gaussian phase fluctuations, are given by

$$\Lambda^{\alpha\beta} = \Lambda_0^{\alpha\beta} + \frac{\Lambda_0^{\alpha\mu}\Lambda_0^{\nu\beta}\mathbf{q}^\mu\mathbf{q}^\nu}{[\omega_n^2\chi_0 - \Lambda_0^{\mu\nu}\mathbf{q}^\mu\mathbf{q}^\nu]} \quad (20)$$

and

$$\chi(\mathbf{q},i\omega_n) = \frac{\mathbf{q}^\alpha\mathbf{q}^\beta\Lambda_0^{\alpha\beta}\chi_0}{\mathbf{q}^\alpha\mathbf{q}^\beta\Lambda_0^{\alpha\beta} - \omega_n^2\chi_0}, \quad (21)$$

as shown in Appendix C.

From Eqs. (20) and (21) we see that *Gaussian* phase fluctuations do not affect transverse correlation functions. In particular, the superfluid stiffness is unrenormalized. However, longitudinal correlations *are* affected in general. While Λ_0 does not satisfy the f-sum rule, Λ does, which implies restoration of gauge invariance. Further, from Eq. (21) we can see that χ has a pole for $\mathbf{q}\to 0$ unlike χ_0, which leads to a collective mode which we will discuss in the next section. However, the static compressibility given by $-\chi(\mathbf{q}\to 0, \omega_n=0)$ is unaffected at the Gaussian level.

We clearly see that the gauge-invariant $(\mathbf{q},\omega)$-dependent correlation functions are different from the mean-field correlations which appear as coefficients of the phase action. This is not surprising since the phase variable in fact contributes to the physical longitudinal correlation functions and modifies the mean-field result. This is true even for charged systems as will be shown below.

B. Charged systems

In charged systems we have to take into account the long-range Coulomb interaction with energy density

$$\frac{1}{\Omega}H_{\mathrm{coulomb}} = \frac{1}{2N}\sum_{\mathbf{q}} V_{\mathbf{q}}\rho_{\mathbf{q}}\rho_{-\mathbf{q}}, \quad (22)$$

where $\rho_{\mathbf{q}} \equiv (1/\Omega)\Sigma_{\mathbf{k},\sigma}c^\dagger_{\mathbf{k}+\mathbf{q},\sigma}c_{\mathbf{k},\sigma}$ is the electron density. In anisotropic layered systems $V_{\mathbf{q}}$ is given by[16,32]

$$V_{\mathbf{q}} = \frac{2\pi e^2 d_c}{q_\parallel \epsilon_b}\left[\frac{\sinh(q_\parallel d_c)}{\cosh(q_\parallel d_c) - \cos(q_\perp d_c)}\right], \quad (23)$$

where $q_\parallel, q_\perp$ denote in-plane and c-axis components of $\mathbf{q}$, d_c denotes the mean interlayer spacing, and ϵ_b is the background dielectric constant. We assume $q_\parallel a \ll 1$ always, where the in-plane lattice spacing $a=1$. For $q_\parallel d_c, q_\perp d_c \ll 1$ this reduces to the ordinary 3D result: $V_{\mathbf{q}} = 4\pi e^2/(\epsilon_b q^2)$.

We take our Hamiltonian to be $K + H_{\mathrm{pair}} + H_{\mathrm{coulomb}}$. Since the short-range attraction of H_{pair} is important for small center-of-mass momentum while the Coulomb effects are important for small momentum transfer, we believe the breakup of the actual interaction in this manner is physically sensible and does not lead to any ‘‘overcounting.’’

The Coulomb interaction can now be decoupled using a field $\mathcal{U}_{\mathbf{q}}(\tau)$ as

$$\exp\left[-\sum_{\mathbf{q}>0} V_{\mathbf{q}}\rho_{\mathbf{q}}(\tau)\rho_{-\mathbf{q}}(\tau)\right] = \int \mathcal{D}(\mathcal{U}_{\mathbf{q}}(\tau),\mathcal{U}^*_{\mathbf{q}}(\tau))\exp\left[-\sum_{\mathbf{q}>0}\frac{1}{V_{\mathbf{q}}}\mathcal{U}^*_{\mathbf{q}}\mathcal{U}_{\mathbf{q}} + i\sum_{\mathbf{q}>0}(\mathcal{U}^*_{\mathbf{q}}\rho_{-\mathbf{q}} + \mathcal{U}_{\mathbf{q}}\rho_{\mathbf{q}})\right]. \quad (24)$$

To integrate over independent modes, since $\mathcal{U}_{-\mathbf{q}} = \mathcal{U}^*_{\mathbf{q}}$, we only sum over $\mathbf{q}>0$. The last term in Eq. (24) can be recast as $\Sigma_{\mathbf{r}}\mathcal{U}(\mathbf{r})\rho(\mathbf{r})$. Thus the density ρ couples to the scalar field $\mathcal{U}$ in the same way as it couples to $\partial_\tau\theta$ in Eq. (14).

On integrating out the fermions we arrive at an effective action for the phase θ_q and the scalar potential $\mathcal{U}_q$, given by

$$S[\theta,\mathcal{U}] = \frac{1}{8T}\sum_{\mathbf{q},i\omega_n}[\theta^*(\mathbf{q},i\omega_n)\mathcal{U}^*(\mathbf{q},i\omega_n)]\mathcal{M}^{-1}\begin{bmatrix}\theta(\mathbf{q},i\omega_n)\\ \mathcal{U}(\mathbf{q},i\omega_n)\end{bmatrix} \quad (25)$$

with

$$\mathcal{M}^{-1} = \begin{pmatrix} -\omega_n^2\chi_0 + \Lambda_0^{\alpha\beta}\mathbf{q}^\alpha\mathbf{q}^\beta & 2i\omega_n\chi_0 \\ -2i\omega_n\chi_0^* & 4(-\chi_0 + V_{\mathbf{q}}^{-1}) \end{pmatrix}, \quad (26)$$

where $\chi_0^* = \chi_0(-\mathbf{q},-i\omega_n)$. Integrating out the field $\mathcal{U}$ leads to the action

$$S_{charged}[\theta]=\frac{1}{8T}\sum_{\mathbf{q},\omega_n}(-\omega_n^2\chi_0^{RPA}+\Lambda_0^{\alpha\beta}\mathbf{q}^\alpha\mathbf{q}^\beta)\,\theta(\mathbf{q},i\omega_n)\times\theta(-\mathbf{q},-i\omega_n), \tag{27}$$

where the mean-field charged system density correlator $\chi_0^{RPA}=\chi_0/(1-V_{\mathbf{q}}\chi_0)$. This form of the phase action is independent of the order-parameter symmetry and has been obtained earlier for *s*-wave SC's.[12] We note that this form differs considerably from that assumed in Ref. 15, where the *physical* $(\mathbf{q},\omega)$-dependent longitudinal dielectric function appears as a coefficient in the action. As emphasized earlier, *physical* longitudinal correlations cannot appear as coefficients of the phase action.

The regime of validity for the above action is $\mathbf{q}\lesssim\pi/\xi_0$ as for neutral systems [see discussion below Eq. (19)]. The frequency cutoff is given by $|\chi_0^{RPA}(q)\omega_n^2|\lesssim E_{\mathrm{cond}}$ and thus depends on $\mathbf{q}$. In particular, for $\mathbf{q}\to 0$, the action remains valid even at frequencies larger than the gap Δ_d and can be used to obtain the $\mathbf{q}\to 0$ plasma mode.

The gauge-invariant correlations for the charged system are obtained from the neutral system results (20) and (21) by replacing $\chi_0\to\chi_0^{RPA}$, as discussed in Appendix C.

VI. PLASMONS

In the previous section we have found phase actions of the form $S[\theta]=(1/T)\Sigma_{\mathbf{q},\omega_n}M_\theta^{-1}|\theta(\mathbf{q},\omega_n)|^2$ for neutral systems [see Eq. (15)] and for charged SC's [see Eq. (27)]. The dispersion of the collective phase mode is defined by $\mathrm{Re}\,M_\theta^{-1}(\mathbf{q},\omega)=0$. For neutral systems, we note that this condition is identical to demanding a pole in physical density-density correlation χ given in Eq. (21). Phase and density fluctuations are thus coupled and share the pole of the collective mode. This is true even for charged systems, where the plasma frequency, ω_p, corresponds to the pole of the physical density correlator and is given by $\lim_{\mathbf{q}\to 0}\mathrm{Re}\,\chi^{-1}(\mathbf{q},\omega_p)=0$.

The phase action is valid for all frequencies such that $\omega^2\lesssim E_{\mathrm{cond}}V_{\mathbf{q}}$ and thus is valid even for large frequencies for $\mathbf{q}\to 0$. If the plasmon is at finite frequency for $\mathbf{q}\to 0$, Landau singularities do not occur at finite temperatures. One can thus use this action to obtain the $\mathbf{q}\to 0$ plasma mode at zero and finite temperatures. Further, to have a sharp plasmon the damping must be relatively small: $\mathrm{Im}\,M_\theta\ll\omega_p$.

In this section, we first briefly consider neutral systems followed by a discussion of charged systems. For charged systems, we study the in-plane plasma mode and then consider *c*-axis plasmons for systems with a finite *c*-axis superfluid stiffness. Our discussion of the *c*-axis plasmon is to a large extent independent of the details of any *c*-axis model. We also make an estimate of the *c*-axis plasma frequency for Bi2212 obtained within this phase action and compare it with experiment.

A. Neutral systems

For neutral systems we have $M_\theta^{-1}=\frac{1}{8}(\Lambda_0^{\alpha\beta}\mathbf{q}_\alpha\mathbf{q}_\beta-\chi_0\omega_n^2)$. At $T=0$, continuing to real frequency, the collective mode frequency obtained from above is given by $\omega(q)=cq$ where the sound velocity $c\equiv\sqrt{D_s^0/\kappa}$. This reduces to the standard result $c=v_F/\sqrt{d}$ in the weak-coupling limit in the continuum, where d is the spatial dimension and v_F is the Fermi velocity. At finite temperatures, there are Landau singularities in D_s^0 and χ_0 as seen from Eqs. (16) and (17), which prevents us from taking the $\mathbf{q}\to 0,\omega\to 0$ limit.

B. Charged systems

In the action (27), we can set $\chi_0^{RPA}\to -1/V_{\mathbf{q}}$ in the limit $\mathbf{q}\to 0$. Analytically continuing to the real frequency, and setting $\mathrm{Re}\,M_\theta^{-1}(\mathbf{q},\omega_p)=0$, we obtain

$$\lim_{\mathbf{q}\to 0}\left[q^\alpha q^\beta\mathrm{Re}\,\Lambda_0^{\alpha\beta}(\mathbf{q},\omega_p(\hat{q}))-\frac{1}{V_{\mathbf{q}}}\omega_p^2(\hat{q})\right]=0, \tag{28}$$

where ω_p depends on the direction of propagation (in-plane or *c*-axis) in anisotropic systems.

We first consider the in-plane plasmon. For $\mathbf{q}\to 0$ ($\hat{\mathbf{q}}$ in-plane) and finite ω we can write $\Lambda_0^{\alpha\beta}=\delta_{\alpha\beta}(-i\omega\sigma(\omega)/e^2)$ where $\sigma\equiv\sigma'+i\sigma''$ is the in-plane optical conductivity (including the effects of Gaussian phase fluctuations). The background dielectric constant enters in this definition of σ since the conduction electrons are affected by the electric field which has been screened by the background. We thus see that

$$\lim_{\mathbf{q}\to 0}V_{\mathbf{q}}\mathbf{q}^2\frac{\sigma''_{ab}(\omega_p)}{e^2}=\frac{4\pi\sigma''_{ab}(\omega_p)}{\epsilon_b}=\omega_p\,. \tag{29}$$

$\sigma(\omega)$ thus governs the location and damping of the plasma mode through a self-consistent equation. This equation is identical to demanding a zero in the real part of the longitudinal dielectric constant (ϵ), since $\epsilon=\epsilon_b+4\pi i\sigma/\omega$ in a *gauge-invariant theory*.

At this stage it is instructive to compare (a) the physical plasma frequency ω_p, (b) the conductivity sum rule plasma frequency ω_p^* defined by[17]

$$\int_0^\infty d\omega\,\sigma'(\omega)=\frac{e^2\pi}{2\Omega}\sum_{\mathbf{k}}m^{-1}(\mathbf{k})\langle n(\mathbf{k})\rangle\equiv\frac{\epsilon_b\omega_{p\alpha}^{*2}}{8}, \tag{30}$$

and (c) the superfluid plasma frequency ω_{ps} defined by $\omega_{ps}^2(T)\equiv 4\pi e^2D_s(T)/\epsilon_b$, where $D_s(T)$ is the T-dependent superfluid stiffness related to $\lambda(T)$, the penetration depth.[2] With this definition, the real part of σ can be represented as $\sigma'(\omega,T)=[\epsilon_b\omega_{ps}^2(T)/4]\delta(\omega)+\sigma_{reg}(\omega,T)$ where σ_{reg} is the regular part. Finally, we have the Kramers-Krönig relation for $\sigma(\omega)$:

$$\sigma''(\omega)=\frac{1}{\pi}\mathcal{P}\int_0^\infty d\omega'\,\sigma'(\omega')\frac{2\omega}{\omega^2-\omega'^2}. \tag{31}$$

We will now try to use these relations and the structure of $\sigma'(\omega)$ to obtain direct information about the behavior of the plasma mode, which is not directly seen in an optical conductivity measurement.

Conventional *clean* 3D *s*-wave SC's at $T=0$ have a very large superfluid stiffness which can be inferred from penetration depth measurements, and little spectral weight at higher energies. Ignoring interband transitions, we can then take

$\sigma'(\omega) \approx [\epsilon_b \omega_{ps}^2(0)/4]\delta(\omega)$ which leads to $\omega_p^* = \omega_{ps}(0)$ from Eq. (30). It also implies $\sigma''(\omega) = \epsilon_b \omega_{ps}^2(0)/4\pi\omega$ through the Kramers-Krönig relation (31). Using Eq. (29) we then get $\omega_p = \omega_{ps}(0)$, a large plasma frequency. In the presence of weak disorder and at finite T, ω_{ps} decreases and the spectral weight in $\sigma'(\omega)$ redistributes, leading to finite $\sigma(\omega)$ over energy scales $\tau_{\rm tr}^{-1}, \Delta$ which are the quasiparticle transport lifetime and SC gap, respectively. Since $\omega_p \gg \tau_{\rm tr}^{-1}$, Δ to begin with, it is unaffected by this low-energy redistribution. This is easy to see from Eq. (31) above for $\sigma''(\omega)$ where we can set $\omega' \approx 0$ in the denominator for the region of interest, and this along with Eqs. (30) and (29) leads to $\omega_p = \omega_p^*$, which is insensitive to the spectral weight redistribution. Further, the small $\sigma'(\omega_p)$ implies a sharp plasmon in this case. Thus for conventional s-wave SC's, we finally arrive at $\omega_p \approx \omega_p^* \geq \omega_{ps}(T)$. The last relation is satisfied as an equality only in a Galilean invariant system at $T=0$.

For the cuprate superconductors, the in-plane $\sigma'(\omega, T=0)$ has the following features: (i) a condensate contribution $[e^2 \pi D_s^0(T=0)]\delta(\omega)$ and (ii) absorption by quasiparticles which has low-frequency spectral weight (in a d-wave SC) with features around twice the maximum gap followed by other higher energy features. The condensate contribution *along* with the large low-energy spectral weight coming from quasiparticles is expected to lead to the large plasma frequency, as in the case of conventional SC's above. Ignoring interband transitions in calculating ω_p^*, we then arrive at $\omega_p \approx \omega_p^* > \omega_{ps}$.

The *normal state* in-plane plasma frequency has been measured to be large (~ 1 eV) in the cuprates,[18] while the spectral weight rearrangement in σ' in going from the normal state to the SC state is over smaller energy scales.[19] The high-energy normal state plasmon is thus expected to smoothly go over into a high-energy SC state plasmon expected from our above discussion, similar to conventional SC's.

C. Josephson plasmons along c axis

To study the c-axis plasmon we assume a nonvanishing c-axis stiffness $D_\perp$. We set $\hat{\mathbf{q}}$ along the c axis in (29), leading to $\omega_{p,c} = 4\pi\sigma_c''(\omega_{p,c})/\epsilon_b$. We know $\omega_{ps,c}$ is small in the high-T_c systems as seen from the large c-axis penetration depths, and $\sigma_c'(\omega)$ is measured to be very small over a very large energy range.[20] This is partly due to the form of the c-axis dispersion which is proportional to $(\cos k_x - \cos k_y)^2$,[21] so that nodal quasiparticles which lead to low energy spectral weight in-plane have a much smaller contribution along the c axis. Thus at $T=0$, the only ''free carriers'' come from the condensate, and $\omega_{p,c}^* \simeq \omega_{ps,c}$. On using Eqs. (31) and (30), this leads to $\omega_{p,c}(T=0) = \omega_{ps,c}(T=0)$. With increasing temperature, the spectral weight in σ_c'' gets transferred to very high energies.[22] The plasma frequency then continues to be given by $\omega_{p,c}(T) = \omega_{ps,c}(T)$, which decreases with increasing T and vanishes above T_c as seen in experiment.[20,23,24] Thus the c-axis plasma oscillations are seen only below T_c, justifying the use of the term ''Josephson plasmon.''

A model which assumes disordered quasiparticle transport along the c axis[25] and a model which only permits pair tunneling[26] both appear to lead to this behavior for $\omega_{pc}(T)$. Presumably the main effect of disorder in the first model is to suppress single-particle tunneling leading to pair tunneling as the dominant process below T_c; it then appears that the two models are similar in spirit. The disorder scale required to reproduce the experimental results in the first model appears to be large, of the scale of the one-particle tunneling bandwidth. One reason for the large disorder scale could be that the c-axis dispersion in this calculation has been chosen to be independent of k_x, k_y, while the actual $(\cos k_x - \cos k_y)^2$ dependence would already suppress single-particle tunneling near the nodes in the clean case. This might then lead to a smaller disorder scale required to suppress this tunneling process completely.

We finally proceed to compare the c-axis plasma frequency with the stiffness obtained from penetration depth experiments in Bi2212. We consider only the in-phase c-axis plasmon and ignore the ''optical mode'' corresponding to out-of-phase fluctuations arising from the bilayer structure, which is expected to be at a higher energy.[27,28] This leads to

$$\omega_{p,c}^2 = \frac{4\pi e^2}{\epsilon_b} D_\perp = \frac{c^2}{\epsilon_b \lambda_c^2}, \tag{32}$$

where c is the velocity of light and λ_c is the low-temperature c-axis penetration depth. λ_c in Bi2212 has been measured[29] to be about 100 μm. Using this and setting $\epsilon_b \approx 10$, we get $\omega_{p,c} \approx 7$ K. This is in reasonable agreement with $\omega_{p,c} \approx 8-10$ K extracted from experiment[30–32] given experimental errors, and uncertainties in the estimate of ϵ_b.

D. Plasmon dispersion

In order to understand the plasmon dispersion and the variation of the plasma energy with direction of propagation, we consider a simplified model for the in-plane and out-of-plane conductivity. Since $\omega_{p,ab} \approx \omega_p^*$ and is large for the in-plane plasmon, we set the in-plane conductivity $\sigma'(\omega) = (\epsilon_b \omega_p^{2*}/4)\delta(\omega)$, consistent with the conductivity sum rule, which then reproduces the large in-plane plasma frequency. We emphasize that this simplified form for the in-plane conductivity is valid *only* at large frequency, for studying the high-energy in-plane plasmon. It is *not valid* for considering low-energy in-plane physical observables, such as the superfluid stiffness. Using this form for the real conductivity, $\sigma_{ab}''(\omega) = \epsilon_b \omega_p^{*2}/4\pi\omega$. The c-axis conductivity is given by $\sigma_c' = (\epsilon_b \omega_{ps,c}^2/4)\delta(\omega)$ as discussed in the last section, which leads to $\sigma_c''(\omega) = \epsilon_b \omega_{ps,c}^2/4\pi\omega$. This simplified model is tailored to capture the correct plasma frequency for propagation along the in-plane and c-axis directions. For an arbitrary direction of propagation, we expect to obtain a reasonable interpolation.

For general $\mathbf{q}$, in the absence of detailed information on the $\sigma(\mathbf{q},\omega)$, we assume that the conductivity is independent of $\mathbf{q}$. This appears to be a reasonable assumption at high energy, and leads to the plasma frequency being given by

$$\omega_p(\mathbf{q}) = \frac{V_{\mathbf{q}}}{e^2}[\sigma_{ab}''(\omega_p)q_\parallel^2 + \sigma_c''(\omega_p)q_\perp^2]. \tag{33}$$

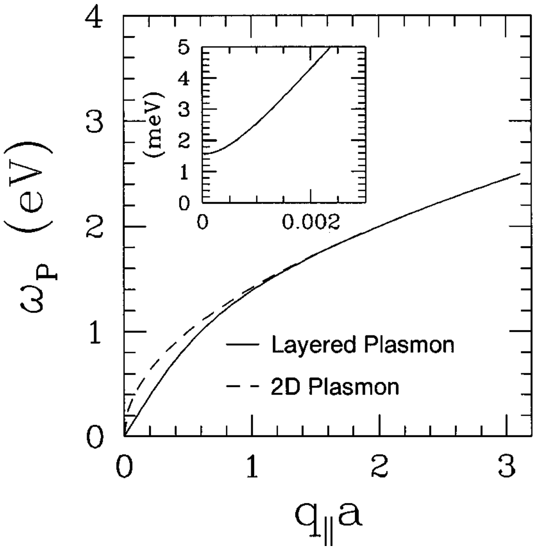

FIG. 1. The plasmon dispersion in a layered system (Bi2212), as a function of $\mathbf{q}_\parallel$ with $q_\perp = \pi/d_c$ (see text for details and parameters used). The $\sqrt{q_\parallel}$ dispersion in the 2D limit is plotted for comparison. The dispersion appears nearly acoustic for small $q_\parallel$ due to the *very small* Josephson plasma frequency ($\sim$10 K) but rapidly crosses over to high energies ($\sim$eV) with increasing $q_\parallel$. The inset shows the behavior of the dispersion (in meV) for $q_\parallel \to 0$.

The plasma frequency for $|\mathbf{q}| \to 0$ is a function of the angle of propagation, ψ, measured with respect to the ab plane and given by $\omega_p^2(\psi) = [\omega_p^{*2}\cos^2(\psi) + \omega_{ps,c}^2 \sin^2(\psi)]$ and varies smoothly from the low-energy Josephson plasmon for c-axis propagation to the high-energy plasmon in-plane. In order to examine the plasmon dispersion in a particular case, we consider the limit $q_\perp = \pi/d_c$ and study ω_p as a function of $(\mathbf{q}_\parallel a)$. Normal state data for LSCO (Ref. 18) shows $\omega_{p,\parallel}(q_\perp = 0, \mathbf{q}_\parallel \to 0) \sim 1$ eV; we assume a similar value for Bi2212 with $\omega_{p,\parallel}(q_\perp = 0, \mathbf{q}_\parallel \to 0) \approx \omega_p^*$. Using $d_c/a \approx 4$ and $\omega_{pc} = \omega_{ps,c} \sim 1.0$ meV,[30–32] we plot the plasma frequency in Fig. 1. This crosses over from a low-energy Josephson plasmon for $q_\parallel \to 0$, corresponding to c-axis propagation, to a nearly two dimensional plasmon dispersing as $\sqrt{q_\parallel}$ at larger $q_\parallel$. At small $q_\parallel$, the dispersion is known to be acoustic for $\omega_{ps,c} = 0$.[33] It appears to be acoustic in Bi2212 at small $q_\parallel$ (see Fig. 1) due the extremely small value of $\omega_{ps,c}$, but finally levels off leading to a finite Josephson plasmon gap for $q_\parallel \to 0$ as shown in the inset in Fig. 1. The 2D $\sqrt{q_\parallel}$ dispersion is obtained mathematically in the limit of large d_c/a and is given by $\omega_{p,2D}(\mathbf{q}_\parallel) = (\omega_p^*/\sqrt{2})\sqrt{q_\parallel d_c}$ and we plot this in Fig. 1 for comparison. It must be emphasized that plasmon damping would be important in the real system and would have nontrivial dependence on the angle of propagation. The sharp plasmon we obtain in the above cases is an artifact of our simplified model for the conductivity.

VII. QUANTUM *XY* MODEL

The superfluid stiffness obtained above $D_s^0(T) = (1/\Omega)[\Sigma_{\mathbf{k}} m_{xx}^{-1}(1 - \xi_{\mathbf{k}}/E_{\mathbf{k}}) - 2\Sigma_{\mathbf{k}} \mathbf{v}_x^2(-\partial f/\partial E_{\mathbf{k}})]$ is unaffected by Gaussian phase fluctuations. Corrections to this result are unimportant in conventional superconductors which have a large coherence length and a large $D_s^0(T=0)$. However, as we show below, effects beyond the Gaussian approximation could lead to large corrections in systems with a small coherence length and small D_s^0. To study such effects we derive in this section a quantum XY model and analyze quantum and thermal phase fluctuations in the following section. For clarity of presentation, we outline the derivation for a d-dimensional isotropic system; the generalization to the anisotropic case is straightforward.

The quantum XY model describes the dynamics of the phase variables $\theta_{\mathbf{R}}(\tau)$ defined on a lattice with lattice spacing ξ_0, the coherence length. The simplest action periodic under $\theta_{\mathbf{R}}(\tau) \to \theta_{\mathbf{R}}(\tau) + 2\pi$ is given by

$$S_{XY}[\theta] = \sum_{\mathbf{Q},\omega_n} A(\mathbf{Q})\omega_n^2 |\theta(\mathbf{Q}, i\omega_n)|^2 + B\int_0^{1/T} d\tau \sum_{\langle \mathbf{R},\mathbf{R}'\rangle} [1 - \cos(\theta_{\mathbf{R}}(\tau) - \theta_{\mathbf{R}'}(\tau))], \tag{34}$$

where $\langle \mathbf{R},\mathbf{R}'\rangle$ are neighboring sites. It is important to emphasize that, given the $\cos(\theta_{\mathbf{R}} - \theta_{\mathbf{R}'})$ form, there are no constraints on the spatial gradient of the phases defined on the coarse-grained scale of ξ_0. This is in contrast to the Gaussian action (27), derived on the scale of the microscopic lattice spacing a, which could only describe slow spatial fluctuations of the phase whose energy did not exceed the condensation energy.

Our task now is to determine the coefficients $A(\mathbf{Q})$ and B of this effective action. Unlike in some other cases[34] it is not possible to directly derive the quantum XY action from the underlying fermionic Hamiltonian, since the cumulant expansion we used to derive effective phase actions was controlled by the smallness of spatiotemporal gradients in θ. We therefore proceed as follows: we compare the action (34) in the limit of slow spatial variations on the scale of the coherence length and match coefficients with those of the Gaussian action:

$$S[\theta] = \frac{1}{8T}\sum_{\mathbf{q},\omega_n}{}'' \frac{\omega_n^2 a^d}{V(\mathbf{q})}|\theta(\mathbf{q}, i\omega_n)|^2 + \frac{1}{8}\int_0^{1/T} d\tau \sum_{\mathbf{r},\alpha} D_s^0 a^{d-2}[\theta_{\mathbf{r}}(\tau) - \theta_{\mathbf{r}+\alpha}(\tau)]^2, \tag{35}$$

where $V(\mathbf{q})$ is the generalized d-dimensional Coulomb interaction. For the $(\mathbf{q}, i\omega_n)$ of interest, we have set $\chi_0^{RPA} \approx -1/V(\mathbf{q})$ and $\Lambda_0 = D_s^0(T)$. The double prime on the summation denotes momentum and frequency cutoffs

$$|\mathbf{q}| < q_c \equiv \pi/\xi_0 \quad \text{and} \quad \omega_n^2 \leqslant (2\pi n_c T)^2 \equiv D_s^0\left(\frac{\pi}{\xi_0}\right)^2 V_{\mathbf{q}}, \tag{36}$$

which arise from demanding that the energy cost of the terms in Eq. (35) to be less than the condensation energy $E_{\rm cond} = \frac{1}{8}D_s^0(\pi/\xi_0)^2$. The $\mathbf{q}$ cutoff can also be viewed as a representation of the $\mathbf{q}$-dependent stiffness being roughly constant ($\approx D_s^0$) for $\mathbf{q} < q_c$ and decreasing to zero for $\mathbf{q} > q_c$.

In arriving at Eq. (35), we have Fourier transformed the gradient term from the $(\mathbf{q}, i\omega_n)$ to $(\mathbf{r}, \tau)$ variables. While the above τ-local form of this term is true when *all* Matsubara frequencies are present, we now determine its regime of validity given the frequency cutoff in Eq. (36). With the cutoff in Eq. (36), the gradient term on Fourier transforming is given by

$$\frac{T}{8}\sum_{\mathbf{q}} D_s^0 a^d \mathbf{q}^2 \int_0^{1/T} d\tau\, d\tau'\, \theta(\mathbf{q},\tau)\,\theta(\mathbf{q},\tau') K[T(\tau-\tau')]. \tag{37}$$

In terms of the dimensionless quantity $z=\tau T$ the kernel is given by $K(z)=\sin[(2n_c+1)\pi z]/\sin(\pi z)$ where $n_c\equiv n_c(\mathbf{q})$ given by Eq. (36). The kernel $K(z)$ is periodic in z, $K(z+1)=K(z)$, and it is sharply peaked around $z=0$ for large n_c. The width of the peak can be estimated from the first zero of $K(z)$ as $z_0=1/(2n_c+1)$. For $n_c\gtrsim 10$, $z_0\ll 1$ which is true in the low-temperature regime that we shall be interested in. We thus approximate $K(z)$ as a delta function in ''time'' and work with a local-τ action in Eq. (35).

To determine the coupling B in Eq. (34), we consider static θ configurations and apply a small external twist Φ which will be distributed uniformly over the system. For the Gaussian model (35), with lattice spacing $a=1$ and $N=L^d$ sites, the phase twist per link is (Φ/L), while for the the XY model (34), on the ''coarse-grained'' lattice with lattice spacing ξ_0 and $(L/\xi_0)^d$ sites, there is a larger phase gradient $\Phi/(L/\xi_0)$. Since the total energy cost for this phase twist is the same in the two cases, one obtains $D_s^0 a^{d-2}(\Phi/L)^2 L^d/8=B\,(\Phi\xi_0/La)^2\,(La/\xi_0)^d/2$, which leads to $B=D_s^0\xi_0^{d-2}/4$.

Similarly, for a τ-dependent phase fluctuation at a frequency ω_n, $(L/\xi_0)^d$ phases contribute in the ''coarse-grained'' XY model, as opposed to L^d in the Gaussian case. We thus get $A(\mathbf{Q})\omega_n^2(La/\xi_0)^d=(\omega_n^2 a^d/8V_{\mathbf{q}})L^d$ on equating the first term in the two actions. This leads to $A(\mathbf{Q})=\xi_0^d/8V_{\mathbf{q}}$. Finally, noting that the momentum $\mathbf{Q}=\mathbf{q}\xi_0/a$ since the distances in the ''coarse grained'' lattice are in units of ξ_0, this can also be written as $A(\mathbf{Q})=\xi_0^d/8\tilde{V}(\mathbf{Q})$, where $\tilde{V}(\mathbf{Q})=V(\mathbf{Q}a/\xi_0)=V(\mathbf{q})$.

Having obtained $A(\mathbf{Q})$ and B in the isotropic d-dimensional case, we generalize to anisotropic systems. We work with a layered 3D system with lattice spacing $a=1$ in-plane and d_c along the c axis. The in-plane coherence length $\xi_0\gg a$ and the c-axis coherence length $\xi_\perp=d_c$. In this case, since $\xi_\perp=d_c$, we proceed exactly as above but coarse grain *only* the in-plane variables. Denoting the Gaussian model stiffness by $D_\parallel^0$ in-plane and $D_\perp^0$ along the c axis, we arrive at the final action

$$\begin{aligned} S[\theta]=&\frac{1}{8T}\sum_{\mathbf{Q},\omega_n}{}' \frac{\omega_n^2\xi_0^2 d_c}{\tilde{V}_{\mathbf{Q}}}\theta(\mathbf{Q},\omega_n)\,\theta(-\mathbf{Q},-\omega_n)\\ &+\frac{D_\parallel^0 d_c}{4}\int_0^{1/T}d\tau \sum_{\mathbf{r},\alpha=x,y}\{1-\cos[\theta(\mathbf{r},\tau)\\ &-\theta(\mathbf{r}+\alpha,\tau)]\}+\frac{D_\perp^0 d_c}{4}\left(\frac{\xi_0}{d_c}\right)^2\int_0^{1/T}d\tau\\ &\times\sum_{\mathbf{r}}\{1-\cos[\theta(\mathbf{r},\tau)-\theta(\mathbf{r}+\hat{\mathbf{z}},\tau)]\}, \end{aligned} \tag{38}$$

where $\tilde{V}(\mathbf{Q})=V(\mathbf{Q}a/\xi_0)$ and $V(\mathbf{Q})$ is the Coulomb interaction for layered systems given in Eq. (23). While all $|\mathbf{Q}|\leqslant\pi$ contribute in Eq. (39) above, the prime on the summation denotes the Matsubara cutoff consistent with Eq. (36). The couplings in this action depend crucially on ξ_0 and we shall examine the consequences of this below.

VIII. RENORMALIZATION OF THE STIFFNESS

The quantum XY model action has both longitudinal fluctuations and transverse (vortex) excitations. Near a finite temperature phase transition, the dynamics is unimportant and we recover the classical XY model with the possibility of a phase transition in the 3D-XY universality class. In this section, we examine low-temperature properties and the effect of quantum dynamics. We ignore vortex-antivortex pair excitations since these have a core energy cost and would be exponentially suppressed at low temperatures. We deal with the longitudinal fluctuations within a self-consistent harmonic approximation (SCHA).

To examine the low-temperature in-plane properties, we assume $D_\perp^0=0$ in Eq. (39) since it is very small in highly anisotropic systems with a large $\lambda_\perp$. The c-axis stiffness would become important if $\lambda_\perp\lesssim\lambda_\parallel(\xi_0/d_c)$, which implies $D_\parallel\lesssim D_\perp(\xi_0/d_c)^2$, leading to the c-axis contribution being important in Eq. (39). For Bi2212, detailed calculations, which we omit here, show that it does not affect our in-plane results.

The SCHA (Refs. 35,4) is carried out by replacing the above action by a trial harmonic theory with the renormalized stiffness $D_\parallel$ chosen to minimize the free energy of the trial action. This leads to $D_\parallel=D_\parallel^0\exp(-\langle\delta\theta^2\rangle/2)$ where $\delta\theta\equiv(\theta_{\mathbf{r},\tau}-\theta_{\mathbf{r}+\alpha,\tau})$ and the expectation value is evaluated in the renormalized harmonic theory. Explicitly, we get

$$\langle\delta\theta^2\rangle=2T\int_{-\pi}^{\pi}\frac{d^3\mathbf{Q}}{(2\pi)^3}\sum_{n=-n_c}^{n_c}\frac{\epsilon_{\mathbf{Q}}}{\omega_n^2\xi_0^2 d_c/\tilde{V}_{\mathbf{Q}}+D_\parallel d_c\epsilon_{\mathbf{Q}}}, \tag{39}$$

where $\epsilon(\mathbf{Q})=4-2\cos Q_x-2\cos Q_y$.

We have analyzed the above equations to extract information about the importance of quantum and thermal phase fluctuations. Our numerical results can be simply summarized as follows: $\langle\delta\theta^2\rangle(T=0)\sim\sqrt{(e^2/\epsilon_b\xi_0)/[D_\parallel(0)d_c]}$ is a measure of quantum fluctuations, while thermal fluctuations become important above a crossover scale $T_\times\sim\sqrt{[D_\parallel(0)d_c](e^2/\epsilon_b\xi_0)}$. These quantities are simply understood as the zero-point spread and the energy-level spacing of an oscillator respectively, in the renormalized harmonic theory. The scale $T_\times$ is also the temperature at which $n_c(\mathbf{Q}=\pi)\sim 1$ [giving a broad kernel $K(z)$], with nonlocal effects in τ becoming important.

It is easy to see that phase fluctuation effects are negligible in the BCS limit of large ξ_0. With $e^2/\epsilon_b a\sim D_\parallel d_c\sim E_F$ and $\xi_0\sim v_F/\Delta$, one obtains the standard result $E_{\mathrm{cond}}\sim\Delta^2/E_F$ per unit cell, and $\langle\delta\theta^2\rangle(T=0)\sim\sqrt{\Delta/E_F}\ll 1$ and $T_\times\sim\sqrt{E_F\Delta}\gg T_c$.

For the cuprates the short coherence length and small $D_\parallel$ act together to increase $\langle\delta\theta^2\rangle$, but they push $T_\times$ in opposite directions. For optimal Bi2212 we use $e^2/\epsilon_b a\approx 0.3$ eV with

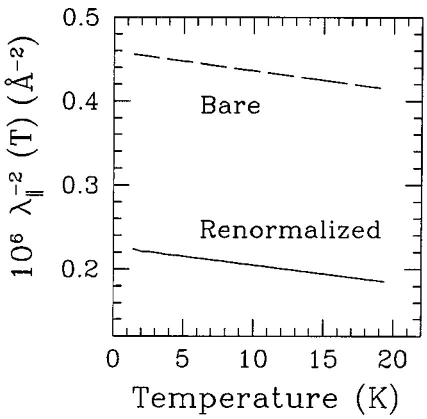

FIG. 2. The bare and renormalized $1/\lambda_\parallel^2$ plotted as a function of temperature near optimal doping for Bi2212. We have chosen bare values such that the renormalized results, $\lambda_\parallel(0)\approx 2000$ Å and $d\lambda_\parallel/dT\approx 10$ Å/K, are in agreement with experiment.

$\epsilon_b\approx 10$ and $\xi_0/a\approx 10$. Bi2212 has a bilayer stacking structure with the planes within a bilayer being much closer than the distance between bilayers. Assuming the phase within the bilayer to be fully correlated, we set d_c to be the mean inter-*bilayer* spacing and thus $d_c/a\approx 4$. Using $\lambda_\parallel(0)\approx 2000$ Å, we then get the bilayer stiffness $D_\parallel(0)d_c\approx 80$ meV.[2] These values lead to $E_{\rm cond}\approx 6$ K/unit cell which is somewhat larger than estimates for optimally doped Y-Ba-Cu-O from specific-heat measurements;[36] we are unaware of similar data for Bi2212. We find that the crossover scale $T_\times\approx 350$ K $\gg T_c$ and a better estimate is then $T_x\sim T_c$. Thus thermal fluctuations are unimportant at low temperatures. Quantum fluctuations are important since we find $\langle\delta\theta^2\rangle(T=0)\sim 1$ at optimal doping.

To study the temperature dependence of $\lambda_\parallel(T)$ and the bilayer stiffness $D_\parallel(T)d_c$,[2] we set the bare stiffness $d_cD_\parallel^0(T)=d_cD_\parallel^0(0)-2\alpha^0T$, where the linear decrease arises purely from nodal quasiparticle excitations within a single layer. This implies $1/\lambda_{\parallel,0}^2(T)=1/\lambda_{\parallel,0}^2(0)-(4\pi e^2/\hbar^2c^2d_c)2\alpha^0T$. We plot the results of a numerical calculation of $1/\lambda_\parallel^2(T)$ in Fig. 2. Phase fluctuations are seen to lead to a large quantum renormalization of $1/\lambda_\parallel^2(0)$ and to a small decrease in the slope of $1/\lambda_\parallel^2(T)$. The small renormalization of the slope of $1/\lambda_\parallel^2(T)$ which we find,[38] is true for a range of parameter values around our specific choice which has been constrained by experiment. It is, however, not the case more generally, and the slope could be renormalized by quantum fluctuations for a different choice of parameter values. This is to be contrasted with the effect of *classical* thermal phase fluctuations which would lead to no change in $\lambda_\parallel(0),D_\parallel(0)$ and to an *increase* in the slope of $1/\lambda_\parallel^2$ relative to the bare value.

We note that in the absence of quasiparticles, the superfluid stiffness in this model would have an exponentially small temperature dependence, arising from phase fluctuations which are gapped. While it might appear that there could be low-temperature crossovers resulting from the c-axis plasmon being at low energy (~ 10 K for Bi2212), the phase space for these low-lying fluctuations is extremely small. Even in a purely 2D system, which supports (gapless) low-energy plasmons dispersing as $\sqrt{q_\parallel}$, the phase stiffness would decrease slowly, with a large power law ($\sim T^5$). Quasiparticles are thus crucial in obtaining the observed linear temperature dependence.

We find that we have to choose $d_cD_\parallel^0(0)\approx 150$ meV corresponding to a bare $\lambda_{\parallel,0}(0)\approx 1500$ Å and a slope $\alpha_0\approx 0.38$ meV/K to obtain the renormalized values $d_cD_\parallel\approx 80$ meV (implying $\lambda_\parallel\approx 2000$ Å) and $\alpha\approx 0.35$ meV/K, in agreement with experiment. Thus, quantum effects lead to a 50% decrease of $1/\lambda_\parallel^2(0)$ and to little change in its linear T slope. The bare slope α^0 within a theory of noninteracting Bogolubov quasiparticles is given by $(k_B\ln 2/\pi)(\hbar v_Fk_F/\Delta_d)$. Using the measured angle-resolved photoemission spectroscopy (ARPES) dispersion,[9] this leads to $\alpha^0\approx 0.8$ meV/K which is much larger than the ''bare'' value we have used above to obtain agreement with penetration depth experiments. This points to the inadequacy of the noninteracting quasiparticle picture. The above discrepancy could be accounted for by considering quasiparticle interaction effects at the mean-field level before considering the effect of phase fluctuations. These interaction effects become more important as one underdopes to approach the Mott insulator.[37,38]

IX. CONCLUSIONS

In this paper, we have focused on the excitations of a short coherence length d-wave superconductor. These are nodal fermions and the fluctuations of the amplitude and phase of the order parameter. Using an effective phase-only action we have discussed collective plasma modes and renormalization of superfluid stiffness by anharmonic longitudinal phase fluctuations. We summarize below some of our main conclusions.

We have found that the important excitations are the low-lying fermionic states near the nodes and quantum phase fluctuations of the order parameter. Although the d-wave state supports in addition, two amplitude fields and a bond-phase field, these have been shown to have negligible spectral weight at low energy and are unimportant for the low-temperature thermodynamics. They could possibly be probed in experiments such as Raman spectroscopy measurements designed to detect these fluctuations.

Our discussion and derivation of the plasma modes emphasizes a unified way of looking at the in-plane and c-axis plasmons, in a manner which is relatively independent of detailed models of c-axis propagation. The very different nature of the two plasma modes, with a small c-axis plasma frequency governed by the c-axis stiffness and a large in-plane plasma frequency not directly related to the in-plane stiffness, can both be understood within our phase action. A microscopic derivation of c-axis conductivity sum-rules and T-dependent spectral weight transfers would depend on specific models,[25,26] and we have not discussed these.

Our derivation and treatment of the effective phase-only action emphasizes the crucial role played by the coherence length in imposing momentum and frequency cutoffs in the phase fluctuations. This allows us to interpolate from the BCS limit where phase fluctuations are unimportant to a regime of strong quantum fluctuations in the short coherence length limit. We find that quantum and thermal fluctuations cannot both be present at low temperatures; The short coherence length increases quantum fluctuations while pushing up the temperature scale at which one crosses over to thermal phase fluctuations. The strong longitudinal quantum fluctuations of the phase predicted by our calculation would also imply dynamical charge density fluctuations at low temperatures in the SC phase, which could possibly be probed in experiments.

It has been pointed out that there is a discrepancy in the magnitude of the linear T slope of the measured penetration depth and the value calculated using ARPES data assuming free quasiparticles.[37] However, phase-fluctuation effects had not been taken into account before comparing data from the two measurements. We find that even including the relevant quantum phase fluctuations, a discrepancy is present which points to strong quasiparticle interactions even at optimal doping. A theory to account for these quasiparticle interactions is, however, lacking. One possibility is to invoke a phenomenological superfluid Fermi-liquid theory description for the quasiparticles.[39,38] It turns out however, that such a theory has a large number of free parameters and lacks predictive power although the experimental results may be easily rationalized.

We have restricted our study in this paper to the low-temperature properties of the SC without addressing the issue of what happens at higher temperatures within the SC state and in the normal state. This leads naturally to the problem of fermions interacting with a strongly fluctuating order parameter, which is at present an important open problem.

Note added in proof. We have recently studied, in some detail, the effect of ohmic dissipation on the phase fluctuations.[40] Such dissipation, arising from a finite low-frequency optical conductivity, is seen to reduce the magnitude of quantum fluctuations and reduce our estimate for the thermal crossover scale. Nevertheless, we find that the crossover scale is still large, so that our conclusions about quasiparticles dominating the low-temperature behavior of response functions remains unchanged.

ACKNOWLEDGMENTS

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APPENDIX A

In this Appendix, we present an approximate analysis of the density of states for the amplitude fields $\eta_{s,d}$ and the bond-phase field ϕ. This analysis gives us insight into the nature of fermionic excitations which contribute to the low-energy spectral weight for these fields and recovers the power law for the low energy DOS obtained in the numerics.

1. Amplitude fluctuations

The low-energy density of states, $N_i(\omega)=(1/N)\Sigma_{\mathbf{q}} -\mathrm{Im}\,M_i(\mathbf{q},\omega)/\pi \quad i=s,d$ with the restriction $|\mathbf{q}_x|,|\mathbf{q}_y| <\pi/\xi_0$. The spectral weight $\mathrm{Im}\,M_i(\mathbf{q},\omega)$ at $T=0$ arises from summing over low-lying pair excitations at *all* momenta $(\mathbf{k},\mathbf{k}-\mathbf{q})$. This contributes to absorption at frequencies $\omega=E_{\mathbf{k}}+E_{\mathbf{k}-\mathbf{q}}$. For $\mathbf{q}=0$, the spectral weight extends to $\omega=0$ coming from low lying pair excitations from momenta $\mathbf{k}$ arbitrarily close to the node. At finite $\mathbf{q}$ the absorption sets in beyond a minimum threshold which corresponds to $\mathbf{k}$ at the node and $\mathbf{k}-\mathbf{q}$ near the node (since $\mathbf{q}$ is small due to the momentum cutoff). The threshold is given by $\omega_{min}(\mathbf{q}) =\sqrt{q_1^2\Delta_d^2+q_2^2v_F^2}$ where q_1,q_2 refer to components of $\mathbf{q}$ parallel and perpendicular to the Fermi surface, respectively, at the node.

We find numerically that the spectral weight for $\mathbf{q}\neq 0$ is nearly the same as for $\mathbf{q}=0$ beyond the absorption threshold. We, therefore, approximate

$$N_i(\omega)\approx\left(\frac{-1}{\pi}\right)\int_{-\pi/\xi_0}^{\pi/\xi_0}\frac{d^2\mathbf{q}}{(2\pi)^2}\mathrm{Im}\,M_i(\mathbf{q}=0,\omega)\times\Theta(\omega-\omega_{min}(\mathbf{q})), \tag{A1}$$

where $\Theta(x)$ is the unit step function which is 1 for $x>0$ and 0 otherwise. For $\mathbf{q}=0$, the inverse propagator

$$M_{s,d}^{-1}=\frac{\Delta_d^2}{4J}-\frac{\Delta_d^2}{2N}\sum_{\mathbf{k}}\varphi_{s,d}^2(\mathbf{k})\frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}(\omega_n^2+4E_{\mathbf{k}}^2)}, \tag{A2}$$

where $\varphi_s(\mathbf{k})=\cos k_x+\cos k_y$ and $\varphi_d(\mathbf{k})=\cos k_x-\cos k_y$. Analytically continuing $i\omega_n\to\omega+i0^+$ and working at low frequency $(\omega\ll\Delta_d)$ leads to

$$-\frac{1}{\pi}\mathrm{Im}\,M_{s,d}=c_{s,d}^2\frac{1}{N}\sum_{\mathbf{k}}\delta(E_{\mathbf{k}}-\omega/2)\varphi_{s,d}^2(\mathbf{k})\xi_{\mathbf{k}}^2/E_{\mathbf{k}}^2, \tag{A3}$$

where $2/c_{s,d}\equiv\Delta_d(1/N)\Sigma_{\mathbf{k}}(\varphi_d^2/E_{\mathbf{k}}-\varphi_{s,d}^2\xi_{\mathbf{k}}^2/E_{\mathbf{k}}^3)$ We evaluate the ω-dependent momentum sum in Eq. (A3) analytically by converting it to a Fermi-surface integral and compute the constant $c_{s,d}$ numerically. Finally, doing the $\mathbf{q}$ sum to obtain the density of states leads to

$$\frac{N_s(\omega)}{N_{qp}(\omega)}=c_s^2\,\frac{\varphi_s^2(\mathbf{k}_{F,n})}{16\pi}\,\frac{\omega^2}{\Delta_d v_F}, \tag{A4}$$

$$\frac{N_d(\omega)}{N_{qp}(\omega)}=c_d^2\,\frac{1}{256\pi}\,\frac{\omega^4}{\Delta_d^3 v_F},$$

where $\varphi_s(\mathbf{k}_{F,n})$ refers to φ_s evaluated at the gap node point on the Fermi surface and $N_{qp}(\omega)\equiv k_F\omega/(\pi v_F\Delta_d)$ is the quasiparticle DOS per spin which is linear in ω. We numerically estimate $c_{s,d}\sim 10$; the prefactors in the Eq. (A4) are then of order unity.

2. The ''bond-phase'' field ϕ

For the ϕ field, pair excitations similar to that for amplitude excitations lead to low-energy spectral weight. This is easy to understand since both fields couple to the particle-particle channel with only different vertex factors. The behavior of $\mathrm{Im}\,M_\phi(\mathbf{q},\omega)$ is similar to that for amplitude fields with a vanishing threshold for $\mathbf{q}=0$ and a finite threshold for $\mathbf{q}\neq 0$. Following similar approximations, we set

$$N_\phi(\omega)=-\frac{1}{\pi}\int_{-\pi/\xi_0}^{\pi/\xi_0}\frac{d^2\mathbf{q}}{(2\pi)^2}\mathrm{Im}\,M_\phi(\mathbf{q}=0,\omega)\Theta(\omega-\omega_{min}(\mathbf{q})). \tag{A5}$$

For $T=0$, using particle-hole symmetry near the Fermi surface we find the inverse propagator

$$M_{\phi}^{-1}(\mathbf{q}=0,\omega_n)=\frac{\Delta_d^2}{16}\frac{1}{N}\sum_{\mathbf{k}}\left[\frac{\varphi_d^2(\mathbf{k})}{E_{\mathbf{k}}}-8\cos^2(k_y)\frac{E_{\mathbf{k}}}{\omega_n^2+4E_{\mathbf{k}}^2}\right]. \tag{A6}$$

Doing the integrals as before, we finally get

$$N_{\phi}(\omega)/N_{qp}(\omega)=\frac{1}{8\pi}c_{\phi}^2\varphi_s^2(\mathbf{k}_{F,n})\frac{\omega^2}{\Delta_d v_F}, \tag{A7}$$

where $1/c_{\phi}\equiv(\Delta_d/N)\Sigma_{\mathbf{k}}\varphi_s^2(\mathbf{k})/2E_{\mathbf{k}}$. The prefactor here is again of order unity.

APPENDIX B

In this appendix, we briefly consider the linear time derivative term in the phase action, which we have dropped in the paper. For notational simplicity, we consider a neutral s-wave SC in 2D with lattice spacing $a=1$. In carrying out the Hubbard-Stratonovitch transformation as for the d-wave case, we introduce complex order parameter fields $\Delta_{\mathbf{r}}(\tau),\Delta_{\mathbf{r}}^*(\tau)$ which are bosonic variables and satisfy the constraint $\Delta_{\mathbf{r}}(0)=\Delta_{\mathbf{r}}(1/T)$. Writing $\Delta_{\mathbf{r}}=|\Delta_{\mathbf{r}}|e^{i\theta_{\mathbf{r}}}$, this translates into $|\Delta_{\mathbf{r}}|(1/T)=|\Delta_{\mathbf{r}}|(0)$, and $\theta_{\mathbf{r}}(1/T)=\theta_{\mathbf{r}}(0)+2\pi m_{\mathbf{r}}$, and the partition function involves an additional sum over $m_{\mathbf{r}}$. Working in small θ gradients and doing a cumulant expansion, one arrives at the following form of phase action for low momenta and frequencies:

$$S_{\theta}=\int_0^{1/T}d\tau\sum_{\mathbf{r}}\left[i\rho\dot{\theta}+\frac{1}{8}\kappa\dot{\theta}^2+\frac{1}{8}D_s(\nabla\theta)^2\right]. \tag{B1}$$

We now make the substitution $\theta(\mathbf{r},\tau)=\Theta(\mathbf{r},\tau)+2\pi T m_{\mathbf{r}}\tau$ which implies $\Theta(\mathbf{r},1/T)=\Theta(\mathbf{r},0)$. Substituting this in the action, we get

$$S=2\pi i\rho\sum_{\mathbf{r}}m_{\mathbf{r}}+\frac{1}{2}\pi^2\kappa T\sum_{\mathbf{r}}m_{\mathbf{r}}^2+\frac{D_s\pi}{6T}\sum_{\mathbf{r}}(\nabla m_{\mathbf{r}})^2$$

$$+\frac{1}{2}\pi D_s T\sum_{\mathbf{r}}\int_0^{1/T}d\tau\quad\tau(\nabla\Theta)\cdot(\nabla m_{\mathbf{r}}) \tag{B2}$$

$$+\frac{1}{8}\int_0^{1/T}\sum_{\mathbf{r}}[\kappa\dot{\Theta}^2+D_s(\nabla\Theta)^2], \tag{B3}$$

where the derivatives denote discrete derivatives on the lattice.

Now, at very low temperatures, $D_s/T\gg1$, we must set $\nabla m_{\mathbf{r}}=0$ while at high temperatures, $\kappa T\gg1$, we must set $m_{\mathbf{r}}=0$. In either case, the field Θ decouples from the field $m_{\mathbf{r}}$ and we get a Gaussian theory of phase fluctuations. The former condition ($D_s/T\gg1$) is equivalent to the condition that the spatial phase variation due to thermal effects is small; in particular, vortex configurations are unimportant. The latter condition ($\kappa T\gg1$) is just that the system starts behaving classically; since the extension along the imaginary time axis is $1/T$, there is essentially no dynamics if $1/T\to0$ and κ is finite.

In the presence of vortices, the core would described by a region where the magnitude of the order parameter $|\Delta_{\mathbf{r}}|$ decreases to zero. Since $|\Delta_{\mathbf{r}}|$ is a bosonic variable, the core would trace a closed loop in ''time'' $1/T$. In this case, all the electrons which lie inside the loop undergo a phase change of 2π each time the loop is traced while electrons outside the loop return to their original phase angle. This leads to a Berry phase factor $i\pi\rho S(\Gamma)$ for the loop Γ with area $S(\Gamma)$. The effect of this Berry phase factor on vortex dynamics was pointed out by Ao and Thouless,[42] except they obtained a coefficient of ρ_s (the superfluid density) instead of ρ (the total electron density). Their result is special to a Galilean invariant systems at $T=0$, where $\rho_s=\rho$. Our result has also been derived earlier by Gaitonde and Ramakrishnan.[41]

In the paper, we consider only slow spatial variations of the phase and work with just the periodic variable Θ which we refer to as θ. We note that the linear time derivative term will not be important in the critical regime around the finite temperature superconductor to normal-metal phase transition where dynamics is unimportant. It would, however, be important near quantum critical points at $T=0$.[43]

APPENDIX C:

Physically relevant correlation functions should be gauge invariant. The functional integral method leads to a very simple and elegant way of demonstrating the role of phase fluctuations in restoring gauge invariance. (This is, of course, well known from early work of Anderson and others.)[44,45]

To this end we introduce external gauge potentials $(\mathbf{A},A_0)$ in the Hamiltonian. This leads to the following modifications in the action (9). In the expression (10) for $\mathcal{L}_0$ we replace μ by $\mu+A_0(\mathbf{r},\tau)$ and $\delta\theta/2$ by $\delta\theta/2-A_{\mathbf{r},\mathbf{r}'}(\tau)$ where $\delta\theta\equiv[\theta(\mathbf{r},\tau)-\theta(\mathbf{r}',\tau)]$. Consider the gauge transformation $A_0(\mathbf{r},\tau)\to A_0(\mathbf{r},\tau)+i\partial_{\tau}\alpha(\mathbf{r},\tau)$ and $A_{\mathbf{rr}'}(\tau)\to A_{\mathbf{rr}'}(\tau)-[\alpha(\mathbf{r}',\tau)-\alpha(\mathbf{r},\tau)]$. Invariance under this gauge transformation implies that the phase field θ must transform as $\theta(\mathbf{r},\tau)\to\theta(\mathbf{r},\tau)+2\alpha(\mathbf{r},\tau)$ with $|\Delta|$, ϕ and the fermion fields unchanged. The correlation functions χ_0 and $\Lambda_0^{\alpha\beta}$ calculated at the mean-field level, setting $\theta\equiv0$ and $\phi\equiv0$ are *not* gauge invariant. From the above discussion, it is clear that this problem can be solved by allowing for θ fluctuations and integrating over these (rather than freezing $\theta\equiv0$). We now proceed to do this at the Gaussian level, which is entirely equivalent to the old RPA calculation.

Physical correlation functions are obtained by integrating out the fermions and functional differentiation of the resulting Gaussian effective action with respect to the external sources A_0 and $\mathbf{A}$. We emphasize that these sources couple minimally to the *original* fermion operators, before the transformation $c^{\dagger}\to c^{\dagger}e^{-i\theta/2}$ in Sec. IV. We then find the density-density correlation allowing for Gaussian phase fluctuations to be

$$\chi(\mathbf{q},i\omega_n)=\frac{\mathbf{q}^{\alpha}\mathbf{q}^{\beta}\Lambda_0^{\alpha\beta}\chi_0}{\mathbf{q}^{\alpha}\mathbf{q}^{\beta}\Lambda_0^{\alpha\beta}-\omega_n^2\chi_0}. \tag{C1}$$

This result obtains diagrammatically as follows. The fermions couple to the external source A_0 and to the phase field θ and we have to integrate out both θ and the fermions. However, θ itself does not have a propagator unless the fermions are integrated out. To simplify the diagrammatic calculation we first introduce a fake term $\lambda\theta(q)\theta(-q)$ in the action

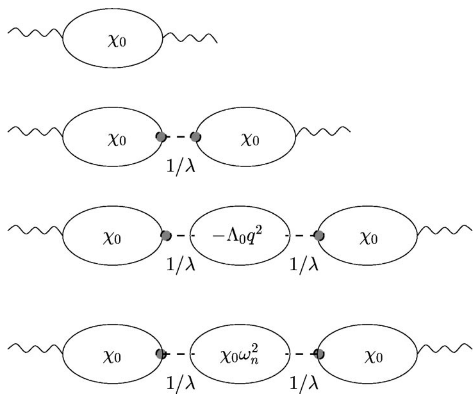

FIG. 3. Low-order diagrams for χ: The wavy lines indicate A_0, the external scalar potential. The dashed lines contribute $1/\lambda$, the false θ propagator. The heavy dots refer to vertex factors of ω_n arising from the vertex $i\rho\partial_\tau\theta$. Finally, the bubble contributions arising out of integrating out fermions are explicitly indicated.

which leads to the bare θ propagator $1/\lambda$; we will take the limit $\lambda\to 0$ at the end. The diagrams in Fig. 3 result for χ. Summing the geometric series leads to

$$\chi=\chi_0+\frac{\chi_0^2\omega_n^2}{\lambda}\left[1-\left(\frac{\chi_0\omega_n^2-\Lambda_0\mathbf{q}^2}{\lambda}\right)\right]^{-1}. \tag{C2}$$

Taking the limit $\lambda\to 0$ then leads to the result in Eq. (C1).

We note that the physical static compressibility is given by $\chi(\mathbf{q}\to 0,\omega_n=0)=\chi_0(\mathbf{q}\to 0,\omega_n=0)$, i.e., the mean-field result is unaffected by phase fluctuations at the RPA level.

Similarly, denoting the physical $(\mathbf{q},\omega)$ dependent stiffness by Λ, we get

$$\Lambda^{\alpha\beta}=\Lambda_0^{\alpha\beta}+\frac{\Lambda_0^{\alpha\mu}\Lambda_0^{\nu\beta}\mathbf{q}^\mu\mathbf{q}^\nu}{[\omega_n^2\chi_0-\Lambda_0^{\mu\nu}\mathbf{q}^\mu\mathbf{q}^\nu]}. \tag{C3}$$

We see that the transverse phase stiffness (for instance along the x direction) given by $\Lambda^{xx}(\omega_n=0,\mathbf{q}_\perp\to 0,q_x=0)$ is unaffected by the Gaussian phase fluctuations. However, the longitudinal part of the current correlation function is affected, and

$$\begin{aligned}\frac{1}{T}\langle j_x j_x\rangle(\omega_n=0,\mathbf{q}_\perp=0,q_x\to 0)\\ =\frac{1}{T}\langle j_x j_x\rangle_0(\omega_n=0,\mathbf{q}_\perp=0,q_x\to 0)\\ +\Lambda_0^{xx}(\omega_n=0,\mathbf{q}_\perp=0,q_x\to 0)\end{aligned} \tag{C4}$$

$$=\frac{1}{\Omega}\sum_{\mathbf{k}} m_{xx}^{-1}(\mathbf{k})\langle n_{\mathbf{k}}\rangle, \tag{C5}$$

now satisfies the f-sum rule (which was violated at mean-field level). Thus gauge invariance is restored.

The derivation remains unchanged in charged systems, with the only difference being that we have to make the replacement $\chi_0\to\chi_0^{RPA}\equiv\chi_0/(1-V_{\mathbf{q}}\chi_0)$ in all the equations (in this Appendix). This can be easily understood by comparing Eqs. (15) and (27) in the text. It is not hard to show that the longitudinal conductivity (σ_L) defined through the longitudinal dielectric function by

$$\epsilon\equiv\frac{1}{1+V_{\mathbf{q}}\chi}=1+\frac{4\pi i\sigma_L}{\omega} \tag{C6}$$

in a gauge-invariant theory, and the transverse conductivity defined by $\sigma_T(\omega)=i\Lambda e^2/(\epsilon_b\omega)$ (with Λ being the transverse part of $\Lambda^{\alpha\beta}$) are equal in the limit $\mathbf{q}\to 0$. In the text, we omit subscripts and refer to both conductivities by σ since we work at $\mathbf{q}\to 0$. It is easy to see that $\sigma_T(\omega)$ and hence $\sigma(\omega)$ is unaffected by phase fluctuations within RPA. However, it is only in a gauge-invariant theory, such as the RPA, that Eq. (C6) holds since it obtains from using the current conservation equation.

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