Creep and depinning in disordered media

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Elastic systems driven in a disordered medium exhibit a depinning transition at zero temperature and a creep regime at finite temperature and slow drive f. We derive functional renormalization-group equations which allow us to describe in detail the properties of the slowly moving states in both cases. Since they hold at finite velocity v, they allow us to remedy some shortcomings of the previous approaches to zero-temperature depinning. In particular, they enable us to derive the depinning law directly from the equation of motion, with no artificial prescription or additional physical assumptions, such as a scaling relation among the exponents. Our approach provides a controlled framework to establish under which conditions the depinning regime is universal. It explicitly demonstrates that the random potential seen by a moving extended system evolves at large scale to a random field and yields a self-contained picture for the size of the avalanches associated with the deterministic motion. At finite temperature $T>0$ we find that the effective barriers grow with length scale as the energy differences between neighboring metastable states, and demonstrate the resulting activated creep law $v \sim \exp(-Cf^{-\mu}/T)$ where the exponent μ is obtained in a $\epsilon = 4-D$ expansion (D is the internal dimension of the interface). Our approach also provides quantitatively an interesting scenario for creep motion as it allows us to identify several intermediate length scales. In particular, we unveil a ''depinninglike'' regime at scales larger than the activation scale, with avalanches spreading from the thermal nucleus scale up to the much larger correlation length R_V. We predict that $R_V \sim T^{-\sigma} f^{-\lambda}$ diverges at small drive and temperature with exponents σ, λ that we determine.

I. INTRODUCTION

Understanding the statics and dynamics of elastic systems in a random environment is a long-standing problem with important applications for a host of experimental systems. Such problems can be split into two broad categories: (i) propagating interfaces such as magnetic domain walls,[1] fluid invasion in porous media,[2] or epitaxial growth;[3] (ii) periodic systems such as vortex lattices,[4] charge-density waves,[5] or Wigner crystals of electrons.[6] In all these systems the basic physical ingredients are identical: the elastic forces tend to keep the structure ordered (flat for an interface and periodic for lattices), whereas the impurities locally promote the wandering. From the competition between disorder and elasticity emerges a complicated energy landscape with many metastable states. This results in glassy properties such as hysteresis and history dependence of the static configuration. In the dynamics, one expects of course this competition to have important consequences on the response of the system to an externally applied force.

To study the statics, the standard tools of statistical mechanics could be applied, leading to a good understanding of the physical properties. Scaling arguments and simplified models showed that even in the limit of weak disorder, the equilibrium large scale properties of disordered elastic systems are governed by the presence of impurities. In particular, below four (internal) dimensions, displacements grow unboundedly[7] with the distance, resulting in rough interfaces and loss of strict translational order in periodic structures.[4] To go beyond simple scaling arguments and obtain a more detailed description of the system is rather difficult and at present only main two methods, each with its own shortcomings, have been developed. The first one is to perform a perturbative renormalization-group calculation on the disorder, and is valid in $4-\epsilon$ dimensions to first order in ϵ. In this functional renormalization-group (FRG) approach,[8,9] the whole correlation function of the disorder is renormalized. The occurrence of glassiness is signaled by a nonanalyticity appearing at a finite length scale during the flow, specifically a cusp in the force correlator. This yields nontrivial predictions for the roughness exponents of interfaces.[8] Another approach relies on the replica method to study either the mean-field limit (i.e., large number of components) or to perform a Gaussian variational approximation of the physical model. Using this variational approach both for manifolds[10] and for periodic systems,[11,12] correlation functions and thermodynamic properties could be obtained. It confirms the existence of glassy properties, with energy fluctuations growing as L^{θ} where θ is a positive exponent. To obtain the glass phase in this method, one must break the replica symmetry. At a qualitative level, this is in good agreement with the physical intuition of such systems as being composed of many low-lying metastable states separated by high barriers. As was clearly shown in the case of periodic manifolds, the correlation functions can be obtained by both the FRG and variational approach and are found to be in very reasonable agreement, bridging the gap between the two-methods.[11,13] Taken together, these two approaches thus provide a very coherent picture for the statics.[14,13] In particular, they allow us to understand that although disorder leads to glassy features in both the manifold and the periodic systems, these two types of problems belong to quite different universality classes in other respects, such as the large distance behavior of the correlations.[14]

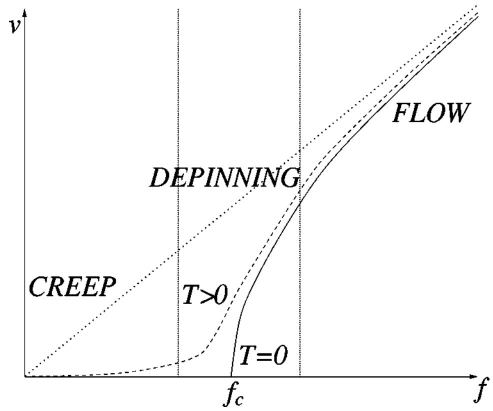

FIG. 1. Typical force-velocity characteristics, exhibiting pinning at $T=0$ with a threshold force f_c and creep at $T>0, f<f_c$. At large drive, the system flows as if there were no disorder.

These properties have drastic consequences for the dynamics of driven systems in the case, important in practice, where an elastic description holds (i.e., when plastic deformations can be neglected). Determining the response to an externally applied force is not only an interesting theoretical question, but also one of the most important experimental issues. Indeed, in most systems the velocity v versus force f characteristics is directly measurable and is simply linked to the transport properties [voltage-current for vortices, current-voltage for charge-density wave (CDW) and Wigner crystals, velocity-applied magnetic field for magnetic domain walls]. In the presence of disorder it is natural to expect that, at zero temperature, the system remains pinned and only polarizes under the action of a small applied force, i.e., moves until it locks on a local minimum of the tilted energy landscape. At larger drive, the system follows the force f and acquires a nonzero asymptotic velocity v. In the simplest cases, the effect of disorder at large velocity is washed out and one recovers the viscous flow, as in the pure case. In the thermodynamic limit, it is believed that there exists a threshold force f_c separating both states, and that a dynamical transition occurs at f_c called *depinning*, where the velocity is continuously switched on, like an order parameter of a second-order transition in an equilibrium system,[15] leading to a v-f characteristics such as the one shown in Fig. 1.

An estimate of f_c can be obtained via scaling arguments[16] or with a criterion for the breakdown of the large velocity expansion.[17,18] Beyond f_c, if one describes the depinning as a conventional dynamical critical phenomenon, the important quantities to determine are of course the depinning exponent β giving the velocity $v \sim (f-f_c)^\beta$ and the dynamical exponent z which relates space and time as $t \sim r^z$.

An even more challenging question, and experimentally at least as relevant, is the response at finite temperature $T>0$. In the most naive description, the system can now overcome barriers via thermal activation, leading to a thermally assisted flow[19] and a linear response at small force of the form $v \sim e^{-\Delta/T} f$, where Δ is some typical barrier. It was realized[20–23] that because of the glassy nature of the static system, the motion is actually dominated by barriers which *diverge* as the drive f goes to zero, and thus the flow formula with finite barriers is incorrect. Well below the threshold critical force, the barriers are very high and thus the motion, usually called ''creep'' is extremely slow. Scaling arguments, relying on strong assumptions such as the scaling of energy barriers and the use of statics properties to describe an out of equilibrium system, were used to infer the small f response. This led to a nonlinear response, characteristic of the creep regime, of the form $v \sim \exp(-Cf^{-\mu}/T)$ where $\mu = (D-2+2\zeta_{eq})/(2-\zeta_{eq})$ and ζ_{eq} is the roughening exponent for the static D-dimensional system.

Obtaining a detailed experimental confirmation of this behavior is a nontrivial feat, in reasons of the range in velocity required. Although in vortex systems these highly nonlinear flux creep behaviors have been measured ubiquitously, it is rather difficult to obtain clean determination of the exponents, given the many regimes of lengthscales which characterize type-II superconductors.[4] In some recent measurements, some agreement with the creep law in the Bragg glass regime was obtained.[24] Probably the most conclusive evidence for the above law was obtained, not in vortex systems, but for magnetic interfaces. Quite recently Lemerle *et al.*[1] successfully fitted the force-velocity characteristics of a magnetic domain wall driven on a random substrate by a stretched exponential form $v \sim \exp - f^{-0.25}$ over 11 decades in velocity. This provided evidence not only of the stretched exponential behavior, but of the validity of the exponent as well.

Given the phenomenological aspect of these predictions and the uncontrolled nature of the assumptions made, both for the creep and for the depinning, it is important to derive this behavior in a systematic way from the equation of motion. Less tools are available than for the statics, and averages over disorder should be made using dynamical methods. Fortunately, it is still possible to use a functional renormalization-group (FRG) approach for the dynamical problem. Such an approach has been used at $T=0$ to study depinning.[25,26] It allowed for a calculation of the depinning exponents, in $D=4-\epsilon$. However this approach is still rather unsatisfactory. The FRG flow used in Refs. 25 and 26 is essentially the static one, the finite velocity being only invoked to remove—by hand—some ambiguities and to cut off the flow, with no real controlled way to show that this is the correct procedure. Furthermore in these approaches it is also necessary to assume, instead of deriving them from the FRG, some scaling relations in order to obtain the exponents. Another rather problematic point is that, with no additional input, the method of Ref. 26 would yield three universality classes for the depinning: two universality classes depending on the nature of the disorder (random bond versus random field) for manifolds and one for periodic systems, while numerics and physical arguments[25] suggested that only two (random field and periodic) universality classes could exist. In addition, since this is also intrinsically a $T=0$ (and $v=0$) approach, it cannot be used to tackle the creep behavior.

We propose here a single theory for describing all the regimes of a moving elastic system, including depinning and the nonzero temperature regimes. Our FRG equations contain from the start the finite velocity and finite temperature. They thus allow to address questions which are beyond the reach of either approximate scaling theories, or $v=0$ FRG flow. For the depinning we are able to determine the conditions required for the existence of a universal depinning behavior, as well as computing the depinning exponents (and estimating f_c). We show in particular that only two universality classes exist (out of the three) for the depinning since we explicitly find that random bond systems flow to the random-field universality class. We can also extract from our equations the characteristic length scales of the depinning. The main advantage of our approach is of course to address the finite T small v regime as well. The method allows to *derive* the creep formula directly and thus allows to confirm the assumptions made on the scaling of the energy barriers. In addition we show that the creep is followed by a depinninglike regime and determine its characteristic length scales. A short account of some of these results was presented in Ref. 27.

The paper is organized as follows: in Sec. II we present the equation of motion and the types of disorder studied here. Section III is devoted to a brief review of scaling arguments and a summary of useful results from perturbation theory, presented in Appendix B. Section IV contains the field-theoretical formulation of the problem and the associated renormalization-group flow equations, derived in Appendix C. The static case is studied in Sec. IV C, focusing on the appearance of the cusp. The effect of the temperature is studied in detail in Appendixes D and E. In the next sections, we study the depinning (V) and creep (VI) regimes. Both sections contain the outline of the derivation and a physical discussion. Appendix G is devoted to the effect of a small velocity on the FRG. We conclude in Sec. VII, referring to an extension of our work proposed in Appendix F. In Appendix A we fix the notations used throughout the paper.

II. ELASTICITY AND DISORDER

Elastic systems are extended objects which ''prefer'' to be flat or well ordered. We are dealing with two different types of elastic systems which, however, can be treated in the same way. On the one hand, *interfaces*, i.e., surfaces with a stiffness that makes local distortions energetically expensive, on the other hand, *lattices* with elastic displacements allowed about a regularly ordered configuration.

The first type is the easiest to visualize. The interface is assumed to have no overhangs and is thus described by a height function u_r defined at each point r (see Fig. 2). Its energy is proportional to its area $\int_r \sqrt{1+|\nabla u|^2}$ and in the elastic limit $|\nabla u| \ll 1$, reduces to

$$H_{\rm el}[u] = \int_r \frac{c}{2} |\nabla u|^2 \tag{2.1}$$

relative to the flat $u_r=0$ configuration (notations are defined in Appendix A). We denote by c the stiffness, or elastic constant.

Periodic structures, such as flux-line lattices or charge-density waves (CDW), can be described by the same type of elastic Hamiltonian. For each point (or line) in the elastic periodic system one can introduce a (vector) displacement field u_R that gives the shift from the reference position R (see Fig. 2). The elastic energy for small displacements is given by a quadratic form in the differences $u_R - u_{R'}$ between neighboring points and thus can be written as Eq. (2.1) in a continuum description (r being a generic point in space).

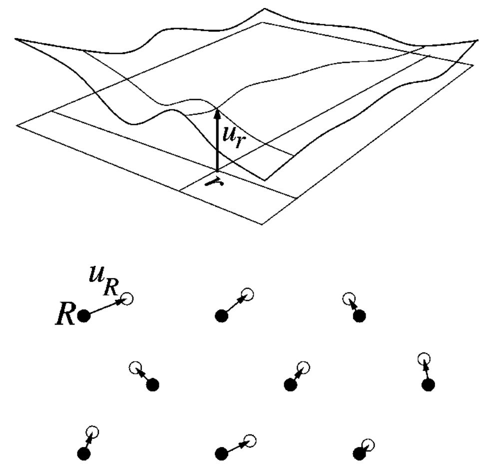

FIG. 2. Top: an interface with height field u_r above r. We denote by r the (internal) coordinates along the interface and by z the height coordinates. Bottom: a lattice with reference positions R and displacements u_R from R.

When u has more than one component, c should be understood as a tensor (see Appendix F).

To take the quenched disorder into account in such systems it is necessary to express the energy of the above elastic structure in the presence of impurities. The coupling to a substrate or to local fields is easily written for interface models and is more subtle for lattices. Quite generally the coupling to disorder leads to an energy

$$H_{\rm dis} = \int_r V(r,u_r) \tag{2.2}$$

which gives rise to a pinning force $F(r,u) = -\partial_u V(r,u)$ acting on the displacement u_r. Depending on the microscopic origin of the disorder term V, the coupling (2.2) leads to quite different physics.

In the case of interfaces Eq. (2.2) originates from

$$H_{\rm dis} = \int_{r,z} V(r,z)\rho(r,z), \tag{2.3}$$

$$\rho(r,z) = \int_{\kappa_z} e^{i\kappa_z\cdot(z-u_r)} = \delta(z-u_r) \tag{2.4}$$

in terms of the density $\rho(r,z)$. One then usually distinguishes two cases: either ''random bond'' (RB) when $V(r,z)$ is short range (random exchange for magnetic domain walls), or ''random field'' (RF) as discussed below, where $V(r,z)$ has long-range correlations.

In the case of periodic structures, the density $\rho(r)$ can be expressed using the set of vectors κ of the reciprocal lattice and Eq. (2.2) originates from

$$H_{\rm dis} = \int_r W(r)\rho(r), \tag{2.5}$$

$$\rho(r) \simeq \rho_0 \sum_\kappa e^{i\kappa\cdot(r-u_r)}, \tag{2.6}$$

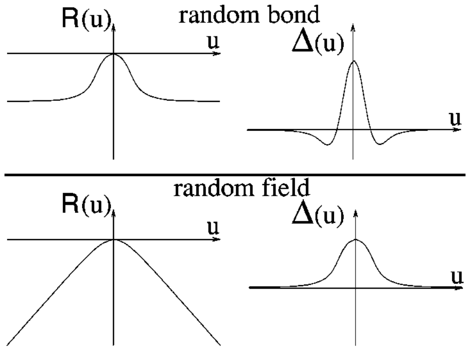

FIG. 3. Up: RB case, down: RF case. Right: correlator of the potential, left: correlator of the force.

where ρ_0 the average density.[11] The potential W is random, of short range r_f (e.g., point impurities for a vortex lattice or a CDW). We call this case ''random periodic'' (RP).

In both cases, using Eqs. (2.3)–(2.6) and (2.2) one obtains for the correlations of V in Eq. (2.2)

$$\overline{[V(r,u)-V(r',u')]^2}=-2\delta_{rr'}\mathsf{R}(u-u'), \qquad (2.7)$$

where $\mathsf{R}(u)$ is a periodic function with the periodicity a of the lattice in the periodic (RP) case.[28] The δ function is cut off at the microscopic scale r_f.

For an interface, $\mathsf{R}(u)$ has the shape shown in Fig. 3. In that case, the width r_f of $\mathsf{R}(u)$ is typically given by the width of the interface or the size of impurities. The force resulting from such a random bond disorder has correlations[29]

$$\overline{F(r,u)F(r',u')}=\delta_{rr'}\Delta(u-u') \qquad (2.8)$$

as shown in Fig. 3 where

$$\Delta(u)=-\mathsf{R}''(u). \qquad (2.9)$$

The signature of such a RB disorder for the interface is that $\int\Delta=0$ since $R'(u)$ decreases to zero at infinity.

Another type of disorder occurs in the case of interfaces separating two phases, like, e.g., a domain wall in a disordered magnet. A random field couples differently to the two phases on the right and left of the interface, thus the energy resulting from the coupling to disorder involves an integral in the bulk of the system and not just *at* the interface position. The correlation of the force can still be expressed by Eq. (2.8) and Δ still decreases to zero above a scale r_f as shown on Fig. 3. Contrarily to the RB case, $\int\Delta$ does not vanish. For a single-component displacement field u, the RF, of correlator (2.8), is still *formally* the derivative of a potential $V(r,u)=-\int^u du' F(r,u')$. The correlations of this fictitious potential are of the form (2.7) with $\mathsf{R}(u) =-\int_0^u du' \int_0^{u'} du'' \Delta(u'')$, and one has $\mathsf{R}(u)\simeq -\frac{1}{2}|u|\int\Delta$ for $|u|\gg r_f$ which can be visualized as a random walk (where u plays the role of ''time'' and the *random-field strength* $\int\Delta =-2\mathsf{R}'(\infty)$ is the ''diffusion constant''). Contrarily to the RB for which $\mathsf{R}(u)$ is short range, $\mathsf{R}(u)$ for the RF grows at large u as shown on Fig. 3.

In this paper we study the overdamped driven motion of such elastic systems which obey

$$\eta\partial_t u_{rt}=c\nabla^2 u_{rt}+F(r,u_{rt})+\zeta_{rt}+f, \qquad (2.10)$$

where η is a friction, f is the external driving force density and ζ_{rt} a Langevin noise. The correlation $\langle\zeta_{rt}\zeta_{r't'}\rangle =2\eta T\delta_{rr'}\delta_{tt'}$ defines as usual a temperature T for this out of equilibrium system. The long time behavior of Eq. (2.10) at zero drive $f=0$, reduces to the thermodynamics at temperature T. In Eq. (2.10) the bare[30] pinning force $F(r,u)$ is Gaussian with zero average and correlator given by Eq. (2.8). We will consider three universality classes for Δ corresponding to an interface in a random potential (RB), in a random field (RF), or a periodic system in a random potential (RP). Physical realizations of such disorders would be, respectively, a random anisotropy for a magnetic domain wall,[1] the random-field Ising systems,[31] and vortex lattices or CDW's.[11,25]

It is also useful to rewrite Eq. (2.10) in the comoving frame at average velocity $v=\overline{\langle\partial_t u_{rt}\rangle}$. In the remainder of this paper, we switch to $u_{rt}\to u_{rt}+vt$ and thus study the following equation of motion:

$$\begin{cases}\overline{\langle\partial_t u_{rt}\rangle}=0\\ (\eta\partial_t-c\nabla^2)u_{rt}=F(r,vt+u_{rt})+\zeta_{rt}+\tilde{f},\end{cases} \qquad (2.11)$$

where $\tilde{f}=f-\eta v$ is the average pinning force and r belongs to a D-dimensional internal space. From now on we specialize to an unidimensional displacement field u_r as would be the case for an interface model or a single Q CDW. This simpler case already captures the main physics at small velocity, investigated here. Extensions to many-component systems will be briefly discussed.

Before giving a quantitative treatment using renormalization group, let us review the qualitative arguments which have been given previously to describe the physics originating from Eq. (2.10).

III. PRELIMINARY ARGUMENTS

A. Statics

In the absence of drive, Eq. (2.10) is equivalent to the equilibrium problem at temperature T. The state of the system results from the competition between elasticity, pinning, and thermal fluctuations. The physics of such problems can be investigated by a host of methods[4,11,16,7,8] and here we only recall the salient points. Temperature does not play an important role as will become clear and we begin with the $T=0$ case.

A subsystem of size R, with displacement $w(R) =\sqrt{\overline{(u_R-u_0)^2}}$, is submitted to a typical elastic force density $f_{\rm el}=cw(R)/R^2$ and to a typical pinning force density $f_{\rm pin} =\sqrt{\Delta(0)/R^D}$. Balancing these quantities, one obtains that elasticity wins at large scales for $D>4$, resulting in a flat interface with *a priori* bounded displacements. In $D<4$, systems of size R smaller than the Larkin length

$$R_c=\left(\frac{c^2 r_f^2}{\Delta(0)}\right)^{1/4-D} \qquad (3.1)$$

wander as predicted by the Larkin model:[7]

$$w(R) \sim r_f \left(\frac{R}{R_c} \right)^{4-D/2}. \tag{3.2}$$

At larger scales $R > R_c$, the system wanders further than the correlation length r_f of the disorder. This simple picture breaks down and the system can be viewed as made of Larkin domains of size R_c, which are independently pinned. First-order perturbation theory confirms this picture below the Larkin length. The static equilibrium (equal-time) correlation function at $T=0$ is (see Appendix B)

$$\overline{u_{-q,t} u_{q,t}} = \frac{\Delta(0)}{(cq^2)^2}. \tag{3.3}$$

The wandering computed from Eq. (3.3),

$$\frac{1}{2} \overline{(u_{r,t} - u_{0,t})^2} \sim \frac{\Delta(0)}{c^2} S_4 r^{\epsilon}/\epsilon, \tag{3.4}$$

for $D = 4 - \epsilon$ gives back Eq. (3.2), and we recover the scaling expression (3.1) by equating the wandering to r_f^2:

$$R_c = \left(\epsilon \frac{c^2 r_f^2}{S_4 \Delta(0)} \right)^{1/\epsilon}. \tag{3.5}$$

We used that $\int_q [(1 - \cos q \cdot r)/q^4] = A_D r^{4-D}$ for $2 < D < 4$ with $A_{D=4-\epsilon} = -\pi^{\epsilon/2-2} \Gamma[-\epsilon/2]/16 \sim S_4/\epsilon$ when $\epsilon \to 0^+$.

The remarkable feature is that at $T=0$, straight perturbation theory,[32] or the use of replicas or equilibrium dynamics,[33,34] gives that Eq. (3.3) is exact *to all orders* in Δ, and is identical to the correlation in the Larkin model.[7] Indeed, the naive perturbation series organizes as if the pinning energy were simply expanded in u [thus the pinning force is independent of u with $\overline{F(r) F(r')} = \Delta(0) \delta(r - r')$], resulting in a Gaussian model.

In fact, due to the occurrence of multiple minima beyond R_c, this perturbative result is incorrect[35] at large scale. It can be shown, for example, on discrete systems, that if a configuration $u_r^{\rm GS}$ which minimizes $H[u] = \int_r [(c/2)(\nabla u_r)^2 + V(r, u_r)]$ is defined on a volume larger than R_c, then the Hessian $(\delta^2 H/\delta u_r \delta u_{r'})[u^{\rm GS}]$ becomes singular.[36] Such instability appears clearly in a functional renormalization-group (FRG) treatment of the problem[8] which proves that Δ becomes nonanalytic beyond the length R_c, as will be discussed below. It can also be seen within variational or mean-field treatments using replicas[10] that replica symmetry breaking (RSB) is necessary to describe the physics beyond the Larkin length R_c. Using either replicas with RSB or the FRG it is possible to describe the physics at all scales and to obtain the correct roughness exponent $\zeta_{\rm eq}$ defined by

$$w(R) \sim r_f \left(\frac{R}{R_c} \right)^{\zeta_{\rm eq}}, \tag{3.6}$$

where the value of $\zeta_{\rm eq}$ depends on the statics universality class.[4] Since disorder induces unbounded displacements, the system is rough and the temperature is always formally irrelevant in $D > 2$. It is described by a $T=0$ fixed point, characteristic of a glass phase.

B. Depinning

An elastic system does not necessarily move under the action of a driving force. The disorder leads to the existence of a threshold force f_c at $T=0$ as shown in Fig. 1. A simple dimensional estimate of f_c can be obtained[16] by computing the sum of the independent pinning forces acting on the Larkin domains $(R/R_c)^D \sqrt{\Delta(0) R_c^D}$ and balancing it with the driving force acting on the same volume $R^D f$. This gives

$$f_c \sim \frac{c r_f}{R_c^2}. \tag{3.7}$$

Another estimate of f_c comes from the large velocity expansion[18,17] of the equation of motion (2.11) (from the criterion $\tilde{f} \simeq \eta v$). It coincides with Eq. (3.7).

For $f \gtrsim f_c$ the system moves with a small velocity, and it has been proposed[15] that depinning can be described in the framework of standard critical phenomena, with the velocity as an order parameter. This leads to the assumption of two independent critical exponents ζ and z, defined in Refs. 26, 25, and 37 through the correlation function in the comoving frame (in the stationary state for $f \to f_c^+$)

$$\overline{(u_{r,t} - u_{0,0})^2} = r^{2\zeta} \mathcal{C}(t/r^z), \tag{3.8}$$

$\mathcal{C}(x) \to \text{cst}$ for $x \to 0$ and $\mathcal{C}(x) \sim x^{2\zeta/z}$ for $x \to \infty$. The dynamical roughening exponent ζ close to the threshold *a priori* differs from its equilibrium value $\zeta_{\rm eq}$. Several related exponents can be also introduced such as: (i) the depinning exponent β; (ii) the correlation length exponent ν describing the divergence of the length ξ defined from the equal-time velocity-velocity correlation function. They satisfy

$$v \sim (f - f_c)^{\beta}, \tag{3.9}$$

$$\xi \sim (f - f_c)^{-\nu}. \tag{3.10}$$

Numerically[38] the motion of the system looks like a deterministic succession of avalanches of size ξ with characteristic time $\tau \sim (f - f_c)^{-z\nu}$. From the argument $u \sim v \tau$ and the statistical tilt symmetry[26,25] (see below), the exponents β and ν are usually determined from ζ, z by the scaling relations

$$\nu = \frac{1}{2 - \zeta} = \frac{\beta}{(z - \zeta)}. \tag{3.11}$$

To obtain these exponents analytically, one needs to perform an FRG analysis of the equation of motion. This will be discussed in more details in Sec. V.

C. Creep

At finite temperature $T > 0$, motion occurs at any drive. For low temperatures and very small drive $f \ll f_c$ one expects the motion to be very slow, and thus, although it is a dynamical problem, a qualitative understanding can be obtained by considering thermal activation over barriers determined from *statics* arguments. An original estimate[19] of such barriers led to linear, albeit activated, response. However, the effects linked to the glassy nature of the problem were understood at a qualitative level[20–23] using scaling arguments.

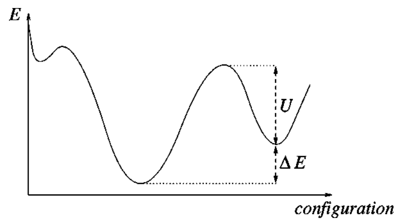

FIG. 4. Energy landscape, with many metastable states in the valleys, differing by ΔE, and barriers U between them.

The argument proceeds as follows: systems larger than R_c have a (static) roughness $w(R) \sim r_f (R/R_c)^{\zeta_{\rm eq}}$ and hence the energy has typical fluctuations of order

$$E(R) \sim U_c \left(\frac{R}{R_c}\right)^{D-2+2\zeta_{\rm eq}} \tag{3.12}$$

with $U_c = c R_c^{D-2} r_f^2$ the energy scale of a Larkin domain. Assuming that the energy landscape is characterized by a *unique* energy scale, and thus that the energy differences between neighboring metastable states is the same as the energy barrier separating them as schematically shown in Fig. 4, one obtains that the barriers height scale with an exponent $D-2+2\zeta_{\rm eq}$. Since the motion is very slow, it is usually argued that the effect of the drive is just to tilt the energy landscape, and the effective barrier becomes

$$U_c \left(\frac{R}{R_c}\right)^{D-2+2\zeta_{\rm eq}} - f R^D r_f \left(\frac{R}{R_c}\right)^{\zeta_{\rm eq}}. \tag{3.13}$$

The maximum of Eq. (3.13), obtained at $R_{\rm opt} \sim R_c (f/f_c)^{-1/(2-\zeta_{\rm eq})}$, gives via Arrhenius law the largest time spent in the valley by the thermally activated system and thus yields the velocity

$$v \sim \exp\left[-\frac{U_c}{T}\left(\frac{f}{f_c}\right)^{-\mu}\right], \quad \mu = \frac{D-2+2\zeta_{\rm eq}}{2-\zeta_{\rm eq}}, \tag{3.14}$$

known as the *creep* motion, characterized by the stretched exponential with exponent μ. Note that the effective barrier given by the above formula vanishes at a scale $R_0 \sim R_c (f/f_c)^{-1/(2-\zeta_{\rm eq})}$ which diverges as fast as $R_{\rm opt}$, the typical size of a thermally activated excitation (see Fig. 5).

This elegant scaling argument leading to the creep formula relies, however, on strong assumptions and does not yield any information on the detailed behavior, in particular on what happens after the thermal jumps. The fact that *static* barriers and valleys scale with the same exponent is already a nontrivial hypothesis about the structure of the infinite-dimensional energy landscape. Refined simulations[39–41] of a directed polymer $D=1$ in $N=1$ and $N=2$ are consistent with the ''equal scaling assumption'' for this particular case, but a general proof is still lacking. The second, and more delicate, hypothesis is the validity of the Arrhenius description: (i) the system being out of equilibrium, it is not clear that *dynamical* barriers can be determined purely from the statics; (ii) one assumes that the motion is dominated by a *typical* barrier. These assumptions can turn incorrect for some specific problems. For example, in the case of a point moving in a one-dimensional random potential, the v-f characteristics at low drive is *not*[42] of Arrhenius type. Although this $0+1$ case is peculiar since the particle has no freedom to pass aside impurities (it is dominated at $T=0$ by the highest slope of the potential and at finite T by the rare highest barriers), one should also address the question of the distribution of barriers in higher dimensions.

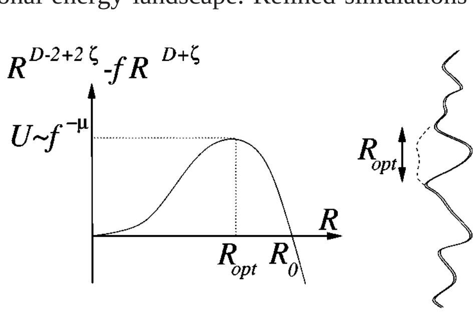

FIG. 5. Effective barrier and motion by nucleation.

IV. DYNAMICAL ACTION AND RENORMALIZATION

A. Formalism and exact relations

Let us now study the equation of motion (2.11) using a full FRG treatment. This will enable us to describe the physics at all length scales and in particular the depinning and creep regime.

A natural framework for computing perturbation theory in off-equilibrium systems is the dynamical formalism.[43,44] After exponentiating the equation of motion (2.10) using a *response field* $\hat{u}$, the average over thermal noise and disorder can safely be done and yields the simple (''unshifted'') action

$$S_{\rm uns}(u,\hat{u}) = \int_{rt} i\hat{u}_{rt}(\eta\partial_t - c\nabla^2)u_{rt} - \eta T \int_{rt} i\hat{u}_{rt} i\hat{u}_{rt} - f\int_{rt} i\hat{u}_{rt} - \frac{1}{2}\int_{rtt'} i\hat{u}_{rt} i\hat{u}_{rt'} \Delta(u_{rt}-u_{rt'}). \tag{4.1}$$

Disorder and thermal averages $\overline{\langle A[u]\rangle} = \langle A[u]\rangle_{S_{\rm uns}}$ of any observable $A[u]$ can be computed with the weight $e^{-S_{\rm uns}}$. Furthermore, response functions to an external perturbation h_{rt} added to the right-hand side of Eq. (2.10) are simply given by correlations with the response field: $\langle A[u] i\hat{u}_{rt}\rangle = (\delta/\delta h_{rt})\langle A[u]\rangle$. It can be checked that causality is satisfied: $\langle A[\{u_{t'}\}_{t'<t},\hat{u}] i\hat{u}_{rt}\rangle$ vanishes. In the time continuum, the response to a perturbation at time t of an observable depending on u_t is ill defined. We choose Ito convention for the equation of motion, which ensures that equal-time response functions, and hence any diagram occurring in perturbation theory containing a loop of response functions, vanish. The continuum field theory necessarily breaks down at small scales and it becomes necessary to cut off the integrals over the modes at large q, using a large wave vector Λ. A full summary of the notations can be found in Appendix A.

It proves more convenient to work in the *comoving frame* [i.e., with Eq. (2.11)]. The corresponding action is

$$S(u,\hat{u})=\int_{rt} i\hat{u}_{rt}(\eta\partial_t - c\nabla^2)u_{rt} - \eta T\int_{rt} i\hat{u}_{rt}i\hat{u}_{rt} - \tilde{f}\int_{rt} i\hat{u}_{rt}$$
$$-\frac{1}{2}\int_{rtt'} i\hat{u}_{rt}i\hat{u}_{rt'}\Delta[u_{rt}-u_{rt'}+v(t-t')], \qquad (4.2)$$

where the field u satisfies $\overline{\langle\partial_t u_{rt}\rangle}=\langle\partial_t u_{rt}\rangle_S=0$. This condition fixes $\tilde{f}\equiv f-\eta v$ in Eq. (4.2). This quantity is the (macroscopic) pinning force, since it shifts the viscous law $f=\eta v$ by the amount of $\tilde{f}$.

Several exact relations can be derived directly from Eq. (4.2). For any static field h_r (vanishing at infinity)

$$S\left(u+\frac{1}{c}\nabla^{-2}h,\hat{u}\right)=S(u,\hat{u})-\int_{rt} i\hat{u}_{rt}h_r .$$

Performing the change of variable $u\to u+(1/c)\nabla^{-2}h$ gives

$$\int Du\, D\hat{u}\, u_{rt}\, e^{-S(u,\hat{u})}=\int Du\, D\hat{u}\left(u_{rt}+\frac{1}{c}\nabla^{-2}h_r\right)$$
$$\times e^{-S(u,\hat{u})+\int_{rt} i\hat{u}_{rt}h_r}.$$

Applying $\delta/\delta h_r|_{h=0}$ yields the exact relation

$$\int_t \mathcal{R}_{qt}=\frac{1}{cq^2}, \qquad (4.3)$$

where we denote by $\mathcal{R}_{rt}$ the exact response function. This symmetry, known as statistical tilt symmetry, ensures that the elasticity is *not* corrected during the renormalization.

Another important relation can be derived from

$$\frac{d}{df}\langle\partial_t u_{rt}\rangle_{S_{\rm uns}}=\int_{r't'}\partial_t\langle u_{rt}i\hat{u}_{r't'}\rangle_{S_{\rm uns}}. \qquad (4.4)$$

This leads to the identity between the macroscopic mobility and the slope of the v-f characteristics at any drive and any temperature:

$$\frac{d}{df}v(f)=\lim_{\omega\to 0} -i\omega\mathcal{R}_{q=0,\omega}. \qquad (4.5)$$

This exact result can also be checked explicitly in the case of a particle moving in a one-dimensional environment.[45]

To extract the physical properties from the action (4.2) it is necessary to build a perturbative approach in the disorder. A particularly simple case[46] occurs when the velocity is very large. In that case the disorder operator in the action can be formally replaced by

$$-\frac{1}{2}\int_{rtt'} i\hat{u}_{rt}i\hat{u}_{rt'}\Delta[v(t-t')] \qquad (4.6)$$

since one may neglect the $u_{rt}-u_{rt'}$ compared to $v(t-t')$. This trick suppresses the nonlinearity and the remaining action is quadratic. Furthermore, at large velocity, $\Delta[v(t-t')]$ can be replaced by $\delta[v(t-t')]\int\Delta=(1/v)\delta(t-t')\int\Delta$ and the disorder operator transforms into a temperature operator (because it becomes local in time $t=t'$). The resulting action is the dynamical action associated to the Edwards-Wilkinson equation[47] describing the motion of an elastic system in a purely thermal noise

$$\eta\partial_t u_{rt}=c\nabla^2 u_{rt}+\nu_{rt} \qquad (4.7)$$

with $\langle\nu_{rt}\nu_{r't'}\rangle=2\eta(T+T_{\rm ew})\delta_{rr'}\delta_{tt'}$, Langevin noise[46] of additional temperature $T_{\rm ew}=\int\Delta/2\eta v$.

Note that at $T=0$ the results at large v coincide with the perturbative expansion in powers of the disorder. The equal-time correlation function in the driven system with force f crosses over from the static $1/q^4$ Larkin behavior at small scale to a thermal $1/q^2$ behavior at larger scale

$$\overline{u_{-q,t}u_{q,t}}\simeq\begin{cases}\dfrac{\Delta(0)}{(cq^2)^2} & \text{for} \quad q^2\gg\dfrac{\eta v}{cr_f}\\[2ex] \dfrac{T_{\rm ew}}{cq^2} & \text{for} \quad q^2\ll\dfrac{\eta v}{cr_f}\end{cases} \qquad (4.8)$$

with the same $T_{\rm ew}$, generated at length scales $r\gg\sqrt{\dfrac{cr_f}{\eta v}}$.

B. Renormalization

We renormalize the theory using Wilson's momentum-shell method. As the cutoff $\Lambda_l=\Lambda e^{-l}$ is reduced, corresponding to a growing microscopic scale $R_l=e^l/\Lambda$ in real space, the parameters of the effective action for slow fields (whose modes q are smaller than Λ_l) are computed by integration over the fast part of the fields (whose modes q lie between Λ_l and Λ). This iterative integration gives rise to *flow* equations, better expressed in terms of the *reduced* quantities

$$\tilde{\Delta}_l(u)=\frac{S_D\Lambda_l^D}{(c\Lambda_l^2e^{\zeta l})^2}\Delta_l(ue^{\zeta l}),$$

$$\tilde{T}_l=\frac{S_D\Lambda_l^D}{c\Lambda_l^2e^{2\zeta l}}T_l,$$

$$\lambda_l=\frac{\eta_l v}{c\Lambda_l^2e^{\zeta l}},$$

$$\tilde{f}_0=f-\eta_0 v, \qquad (4.9)$$

where S_D is the surface of the unit sphere in D dimensions divided by $(2\pi)^D$. The exponent ζ is for the moment arbitrary and will be fixed later so that the reduced parameters flow next to appropriate fixed points. In one case (RB) we will need an l-dependent ζ, and it is understood that everywhere the rescaling factors $e^{\zeta l}$ [appearing, e.g., in Eq. (4.9)] should then be replaced by $\exp\int_0^l dl'\,\zeta_{l'}$. The reduced quantities $\tilde{\Delta},\tilde{T}$ are homogeneous to u^2 and λ to u. The parameter λ_l, which plays a crucial role below, can simply be expressed as the following ratio:

$$\frac{v\,\tau(R)}{\delta u(R)}=\frac{\lambda(R)}{r_f} \qquad (4.10)$$

of the distance (along u) travelled by the center of mass of the interface during $\tau(R)$ and the roughness $\delta u(R) = r_f(R/R_c)^{\zeta}$. We have defined $\tau(R)=\eta(R)R^2/c$ as the characteristic relaxation time in the model renormalized up to scale R.

The details of the renormalization procedure can be found in Appendix C. The flow equations read

$$\begin{aligned}\partial\tilde{\Delta}(u) &= (\epsilon-2\zeta)\tilde{\Delta}(u)+\zeta u\tilde{\Delta}'(u)+\tilde{T}\tilde{\Delta}''(u)\\ &+\int_{s>0,s'>0} e^{-s-s'}(\tilde{\Delta}''(u)\{\tilde{\Delta}[(s'-s)\lambda]\\ &-\tilde{\Delta}[u+(s'-s)\lambda]\}-\tilde{\Delta}'(u-s'\lambda)\tilde{\Delta}'(u+s\lambda)\\ &+\tilde{\Delta}'[(s'+s)\lambda][\tilde{\Delta}'(u-s'\lambda)-\tilde{\Delta}'(u+s\lambda)]),\end{aligned}\tag{4.11}$$

$$\partial \ln\lambda = 2-\zeta-\int_{s>0} e^{-s}s\tilde{\Delta}''(s\lambda),$$

$$\partial \ln\tilde{T} = \epsilon-2-2\zeta+\int_{s>0} e^{-s}s\lambda\tilde{\Delta}'''(s\lambda),$$

$$\partial\tilde{f} = e^{-(2-\zeta)l}c\Lambda_0^2\int_{s>0} e^{-s}\tilde{\Delta}'(s\lambda),$$

where $\epsilon=4-D$ and ∂ denotes $\partial/\partial l$.

This complicated set of equations require a few comments: (i) as for the statics[8] it is necessary to renormalize the whole function Δ, instead of just keeping few couplings as in standard field theory, (ii) the elasticity c is not renormalized $\partial c=0$ due to the statistical tilt symmetry; (iii) our equations correctly show that no temperature can be generated at $v=0$ since the fluctuation dissipation theorem holds at equilibrium.

Setting both $T=0$ and $v=0$ in Eq. (4.11) gives back the simplified set of equations used in Refs. 26 and 25 (setting only $v=0$ also yields equations found in Ref. 48). But compared to the previous FRG approaches of the depinning transition, our equations correctly take into account the effect of the velocity on the flow itself (instead of being treated simply as a cutoff as in Ref. 26). Other attempts[49] to incorporate velocity and temperature in the FRG equations did not obtain the first equation giving the renormalization of the disorder at $T>0$ and $v>0$. To be able to tackle the full dynamical problem and study the depinning and the creep regime, one cannot avoid keeping track of the velocity and of the temperature in the flow, as will become clear later, since they yield nontrivial effects which are unreachable by simple scaling arguments.

Our flow equations allow in principle to compute the whole v-f characteristics at low temperature. In the following we analyze them in the three regimes corresponding to the statics ($v=0$), to the depinning at zero temperature ($T=0, f\sim f_c$) and the creep regime ($T>0, f\sim 0$).

C. Statics: The cusp

At zero velocity, our approach is a dynamical formulation of the equilibrium problem. It thus allows to recover the known results about the statics, avoiding the use of replicas. The standard derivation of the statics using the FRG consists in writing a replicated Hamiltonian for the elastic system pinned in a random potential with correlator $\overline{[V(r,u)-V(r',u')]^2}=-2\delta^D(r-r')\mathsf{R}(u-u')$. After averaging over V the replicated action reads[8]

$$S[\vec{u}]=\frac{1}{2T}\sum_a\int_r|\nabla u_r^a|^2-\frac{1}{2T^2}\sum_{ab}\int_r\mathsf{R}(u_r^a-u_r^b),\tag{4.12}$$

where a,b are the n replica indices. Performing an FRG analysis of Eq. (4.12) yields for the flow of R and T (remarkably independent of n):

$$\begin{aligned}\partial\tilde{\mathsf{R}}(u) &= (\epsilon-4\zeta)\tilde{\mathsf{R}}(u)+\zeta u\tilde{\mathsf{R}}'(u)+T\tilde{\mathsf{R}}''(u)+\frac{1}{2}\tilde{\mathsf{R}}''(u)^2\\ &-\tilde{\mathsf{R}}''(0)\tilde{\mathsf{R}}''(u),\end{aligned}\tag{4.13}$$

$$\partial\ln\tilde{T}=\epsilon-2-2\zeta$$

with $\tilde{\mathsf{R}}_l(u)=e^{-4\zeta l}[S_D\Lambda_l^D/(c\Lambda_l^2)^2]\mathsf{R}_l(ue^{\zeta l})$ and $\tilde{T}_l = e^{-2\zeta l}(S_D\Lambda_l^D/c\Lambda_l^2)T_l$, which are the same redefinitions as Eq. (4.9) where the correlator Δ of the force is related to R by Eq. (2.9). It is easy to see that Eq. (4.13) coincides with our Eq. (4.11) when $v=0$ which reads

$$\begin{aligned}\partial\tilde{\Delta}(u) &= (\epsilon-2\zeta)\tilde{\Delta}(u)+\zeta u\tilde{\Delta}'(u)+\tilde{T}\tilde{\Delta}''(u)\\ &+\tilde{\Delta}''(u)[\tilde{\Delta}(0)-\tilde{\Delta}(u)]-\tilde{\Delta}'(u)^2,\end{aligned}\tag{4.14}$$

$$\partial\ln\tilde{T}=\epsilon-2-2\zeta.$$

Thus the two methods give the same results for the static and equal-time physical quantities. The additional information conveyed by the flow of the friction η in the dynamical formalism is discussed later in Secs. V A and V B.

The temperature in the static system is an irrelevant operator, since it decreases exponentially fast with l. One thus commonly restricts to the $T=0$ version of the above equations. In that case, as is obvious from the closed equation

$$\partial\tilde{\Delta}''(0)=\epsilon\tilde{\Delta}''(0)-3\tilde{\Delta}''(0)^2\tag{4.15}$$

the curvature $\Delta''(0)<0$ (see Fig. 3) of the correlator, for any initial condition, *blows up* at a finite length scale for $D<4$

$$l_c=\frac{1}{\epsilon}\ln\left(1+\frac{\epsilon}{3|\tilde{\Delta}_0''(0)|}\right)\tag{4.16}$$

which corresponds to

$$R_c=e^{l_c}/\Lambda\simeq\left(\epsilon\frac{c^2}{3S_D|\Delta_0''(0)|}\right)^{1/\epsilon}\sim\left(\epsilon\frac{c^2r_f^2}{S_D\Delta_0(0)}\right)^{1/\epsilon}\tag{4.17}$$

when approximating $|\Delta_0''(0)|$ by $\Delta_0(0)/r_f^2$. One thus recovers the Larkin length (3.5). The blowup of the curvature of Δ corresponds to the generation of a *cusp* singularity: Δ becomes nonanalytic at the origin and acquires for $l>l_c$ a nonzero $\Delta'(0^+)<0$. However, the flow equation for the running nonanalytic correlator *still makes sense*. The nonanalyticity just signals the occurrence of metastable states. A well-defined fixed point function $\mathsf{R}^*(u)$ exists for each of the RB, RF, RP cases when a suitable ζ is chosen.

In the RP case, $\zeta=\zeta_{\rm eq}=0$ so as to conserve the period a, and the fixed point is given by[11]

$$\Delta^*(ax)=\frac{\epsilon a^2}{6}\left(\frac{1}{6}-x(1-x)\right) \qquad (4.18)$$

for $x\in[0,1)$.

In the RF case, $\zeta=\zeta_{\rm eq}=\epsilon/3$ so as to conserve the RF strength $\int\Delta$ and the fixed point is given by[8]

$$\frac{x^2}{2}=y-1-\ln y, \qquad (4.19)$$

where $y\equiv\Delta^*(u)/\Delta^*(0)$, $x\equiv u\sqrt{\epsilon/[3\Delta^*(0)]}$ and $\Delta^*(0)\simeq 0.5\epsilon^{1/3}(\int\Delta_0)^{2/3}$ [see Eq. (E1)].

In the RB case, it has been shown[8] by numerical integration of the fixed-point equation that $\zeta=\zeta_{\rm eq}\simeq 0.2083\,\epsilon$ yields a physical fixed point, for which no analytical expression is available.

Despite the irrelevance of the temperature, this operator has important transient effects during the flow, even if we are left asymptotically with the $T=0$ cuspy fixed point. It can be shown (see Appendix D) that the temperature hinders the flow from becoming singular at a finite scale. The running correlator evolves smoothly towards its cuspy fixed point and remains analytic, as was also noticed in Ref. 48. As shown in Appendix D, the rounding due to temperature occurs in a boundary layer of width proportional to $\tilde{T}$ around the origin. This is confirmed by the existence of a well-defined expansion in T [see Eq. (D3)]. This effect is missed by simple perturbation theory that would naively suggest that the rounding occurs on a width proportional to $\sqrt{\tilde{T}}$. Indeed the correlation function is proportional to T and smoothes Δ by $\Delta_\kappa\to\Delta_\kappa e^{-\tilde{T}\kappa^2}$. Although not crucial for the statics this rounding has drastic consequences for the creep as analyzed in Sec. VI.

Let us return to the differences between the static and dynamical formalisms. Within the static approach (4.13) in the $T\to 0$ limit, despite the occurrence of the cusp at l_c, the RG equation for $\mathsf{R}_l(u)$ still makes sense after l_c and flows to a fixed point controlled by $\epsilon=4-D$. However, the physical meaning of the cusp is delicate.[13] On the other hand, the use of the dynamical formalism allows to put $T=0$ from the beginning but adds to the problem a time dimension and the corresponding parameter, the friction η. In this dynamical version of the problem, the cusp has strong physical consequences which are more immediate: after l_c, the cusp generates infinite corrections to the friction. This feature marks the onset of a nonzero threshold force at scales larger than the Larkin length and signals that an infinite time is needed to go from one metastable state to another. Metastability thus appears very clearly in the dynamical formulation of the statics problem.

A simple physical picture of the cusp in the statics at $T=0$ was given in Ref. 13. The renormalized potential $V_{\rm ren}(r,u)$ at scales $R>R_c$ develops "shocks" [i.e., discontinuities of the force $-\partial_u V_{\rm ren}(r,u)$ of typical magnitude $f_{\rm disc}(R)$ at random positions]. Let us now extend this description to draw the link with the critical force and to include thermal effects.

The force correlator for small $u-u'$ is dominated by the configurations with a shock present between u and u':

$$\overline{[F_{\rm ren}(r,u)-F_{\rm ren}(r,u')]^2}\sim f_{\rm disc}(R)^2\frac{dp}{du}|u-u'|, \qquad (4.20)$$

where dp/du denotes the probability to find a shock between u and $u+du$. Identifying the right-hand side with $R^{-D}|\Delta'_{\rm ren}(0^+)(u-u')|$ one finds, using the rescalings (4.9), that the discontinuity in the force has the following scale dependence

$$f_{\rm disc}(R)\sim f_c\left(\frac{R}{R_c}\right)^{-(2-\zeta)}\equiv f_c^{\rm eff}(R) \qquad (4.21)$$

and can thus be identified with an "effective critical force" $f_c^{\rm eff}(R)$ at scale R, which will play a role in the following (see Sec. VI C). At $R=R_c$, $f_c^{\rm eff}(R)$ reduces to the true critical force f_c.

The renormalized problem at scale R being the one of an interface in a potential $V_{\rm ren}(r,u)$ with the above characteristics, one can now easily understand the result that the cusp of $\tilde{\Delta}_l(u)$ is rounded on a width $\tilde{T}_l/\chi$ at $T>0$. Extending the previous argument, one expects a rounding of a shock if the barrier between u and u' is of order T. Since near a shock the potential is linear of slope $f_{\rm disc}(R)$, the barrier is $f_{\rm disc}(R)|u-u'|$, and the thermal rounding should thus occur in a boundary layer of width u given by

$$f_{\rm disc}(R)\,u\,R^D\sim T. \qquad (4.22)$$

Using the rescalings (4.9), this is indeed equivalent to the expression $\tilde{T}_l/\chi$ for the width of the boundary layer in rescaled variables found in Appendix D.

V. DEPINNING

At $T=0$ and $v\to 0$, our flow equations give a self-contained picture of the depinning transition. Thanks to our formalism, the problem is reduced to the mathematical study of Eq. (4.11), which although complicated, requires no additional physical assumptions. To focus on the depinning transition, we must analyze the solutions of these equations in the regime of small velocity where, using Eq. (4.9), $\lambda_{l=0}$ is small. We will examine the various regimes in the RG flow keeping in mind that λ_l increases monotonically with l.

Equations (4.11) involve averages over a range $u\sim\lambda_l$ and thus one naturally expects that, at least at the beginning of the flow, $\Delta_l(u)$ remains close to the $v=0$ solution. The two functions will differ in a boundary layer around $u=0$ of width denoted by ρ_l. Although the precise form of the solution for $|u|<\rho_l$ (e.g., whether the cusp persists at $v>0$) is very hard to obtain analytically, fortunately most of our results will not depend on such details. As we discuss below, the main issue will be to decide whether $\rho_l\ll\lambda_l$ or not, which is a well-posed mathematical question.

Let us start by analyzing the flow up to the Larkin scale l_c of the statics, at which the cusp occurs and the corrections to the friction become singular in the $v=0$ flow. Here at $v \gtrsim 0$ one enters at l_c a regime where $\tilde{\Delta}_l$ is close to its fixed point (see Appendix H). Within the boundary layer, the effect of the velocity is to decrease the singularities of the statics. As shown in Appendix G, the blow up of the curvature $\tilde{\Delta}''(0)$ is slowed down by the velocity as

$$\partial\tilde{\Delta}''(0)=\epsilon\tilde{\Delta}''(0)-3\tilde{\Delta}''(0)^2-9\lambda^2\tilde{\Delta}''(0)\tilde{\Delta}^{\rm iv}(0)+\mathcal{O}(\lambda^4)$$

and the same is true for the friction

$$\partial \ln \eta=-\tilde{\Delta}''(0)-3\lambda^2\tilde{\Delta}^{\rm iv}(0)+\mathcal{O}(\lambda^4).$$

If the blurring of the singularity results in a suppression of the cusp, i.e., if $\tilde{\Delta}_l$ remains analytic, one should wonder whether the $v\simeq 0$ flow can really *remain* close to Δ^* since the convergence to the fixed point is crucially dependent on the existence of the nonanalyticity and in particular on the term $-\tilde{\Delta}'(0^+)^2$ in the flow of $\tilde{\Delta}(0)$ in Eq. (4.14). A hint that $\tilde{\Delta}_l$ can stabilize for a while at $v>0$ is obtained by noting that one has [see Eq. (G2)]

$$\partial\tilde{\Delta}(0)=(\epsilon-2\zeta)\tilde{\Delta}(0)$$
$$-\int_{s>0}e^{-s}\left[\int_{s'>0}e^{-s'}\left(\frac{\tilde{\Delta}(\lambda s)-\tilde{\Delta}(\lambda s')}{\lambda}\right)\right]^2$$

which has indeed the correct sign to give the same effect.

Hence it is natural to expect for $l>l_c$ that $\tilde{\Delta}_l(u)$ has reached everywhere a fixed-point form except in the boundary layer. The correction to the friction, crucial to determine the v-f characteristics, reads

$$\partial_l \ln \eta_l=-\int_{s>0}e^{-s}s\tilde{\Delta}_l''(s\lambda_l) \tag{5.1}$$

and thus depends on the values of $\tilde{\Delta}_l(u)$ for $u\sim\lambda_l$. To estimate this expression, one must know whether the width ρ_l of the boundary layer is smaller than λ_l or not.

To summarize these preliminary remarks, the flow in the *Larkin regime* $l<l_c$ is similar to the $v=0$ flow and $\tilde{\Delta}_{l\gtrsim l_c}$ is close to Δ^* except for $|u|<\rho_l$. We will now analyze in details the flow for $l>l_c$ under the assumption that

$$\rho_l \ll \lambda_l. \tag{5.2}$$

As mentionned above, the validity of Eq. (5.2) can in principle be established by a mathematical or a numerical analysis of our equations. It turns out that Eq. (5.2) leads to the most physically reasonable results. The alternative case will be discussed below.

A. Derivation of the depinning law

For $l>l_c$, called the *depinning regime*, and relying on Eq. (5.2), the flow of η becomes

$$\partial_l \ln \eta_l \simeq -\Delta^{*\prime\prime}(0^+). \tag{5.3}$$

The friction is renormalized downwards with a nontrivial exponent $-\Delta^{*\prime\prime}(0^+)=-(\epsilon-\zeta)/3$ with $\zeta=\epsilon/3$ for the RF case [see Eq. (E1)] and $\zeta=0$ for the RP case [see Eq. (E2)]. For the random bond one would naively take the static $\zeta_{\rm eq}$. However, our flow equations show that during the Larkin regime, the form of the disorder correlator evolves to a RF, and thus $\zeta=\epsilon/3$ also in this case. This nontrivial effect of the transformation for the dynamical properties of a RB into a RF is discussed in detail in Sec. V B.

Since λ_l keeps on growing in the depinning regime, the assumption that $\tilde{\Delta}_l(u)$ can be replaced by Δ^* will cease to be valid. This occurs when λ_l reaches the range $r_f(l)$ of $\tilde{\Delta}_l(u)$, correlation length of the running disorder. This defines a scale $l_V=\ln \Lambda R_V$ given by $\lambda_{l_V}=r_f(l_V)$. Above this scale, one enters a regime where the corrections due to disorder are simply washed out by the velocity, since the integrals over s,s' in Eq. (4.11) average completely over the details of $\tilde{\Delta}_l(u)$. One thus enters the *Edwards Wilkinson regime*. Perturbation theory (4.8) shows that the interface is flat for these large scales for $D>2$, the disorder leading only[46] to the effective temperature $T_{\rm ew}$.

The family of systems indexed by $0\leq l<\infty$ have all the same velocity v and the same slope $df/dv(v)$. However, they have lesser and lesser singular behavior $f(v)$. We can thus iterate the FRG flow up to a point where the theory can solved perturbatively (e.g., above l_V). For the depinning, one can simply use the fact that the renormalized action at $l=\infty$ is Gaussian and its friction η_∞ is, from Eq. (4.5) equal to the slope df/dv of the depinning characteristics. Using the flow of λ_l in Eq. (4.11), the expressions for $\lambda_{l_V},\lambda_{l_c}$ with Eqs. (4.9) and (H2) lead to

$$\frac{\lambda_{l_V}}{\lambda_{l_c}}\approx\begin{cases}\exp[(2-\zeta)\beta(l_V-l_c)]\\ \dfrac{F_c}{\eta_{l_c}v}\end{cases} \tag{5.4}$$

with

$$\beta=1-\Delta^{*\prime\prime}(0^+)/(2-\zeta) \tag{5.5}$$

which will turn to be the depinning exponent and we have defined a characteristic force $F_c=cr_f/R_c^2$. Note that F_c is not exactly the critical force f_c. Solving Eq. (5.4) gives

$$\frac{R_V}{R_c}\approx\left(\frac{F_c}{\eta_{l_c}v}\right)^{1/(2-\zeta)\beta}, \tag{5.6}$$

$$\frac{\eta_{l_V}}{\eta_{l_c}}\approx\left(\frac{\eta_{l_c}v}{F_c}\right)^{1/\beta-1}. \tag{5.7}$$

Since the system at l_V is nearly pure, one has $\eta_{l_V}\approx\eta_\infty$ and, integrating over v the derivative $df/dv=\eta_\infty\approx\eta_{l_V}$, one gets

$$\frac{\eta_{l_c}v}{F_c}=\left(\frac{f-f(v=0^+)}{F_c}\right)^\beta \tag{5.8}$$

which shows that the depinning is characterized by an exponent β and a pinning force $f_c=f(v=0^+)$ (yet to be determined).

The flow of $\tilde{f}_l$ allows to fix the value of f_c. Instead of just computing f_c we also show that the integration of the flow of $\tilde{f}_l$ provides a second way to derive the depinning law (5.8).

Indeed, as discussed below, in our formalism the term proportional to v which was problematic in the previous approaches[25] cancels naturally.

In the theory renormalized up to l_V, the short scale cutoff is R_V and one can use first-order perturbation theory. One has [see Eqs. (B7) and (B11)]

$$\tilde{f}_{l_V} = -\int_t \Delta'_{l_V}(vt) R_{0t}^{l_V}. \tag{5.9}$$

Since in the renormalized theory the disorder is close to Δ^* [with the rescalings (4.9)], and the friction hidden in the response function is such that λ_{l_V} matches the range of $\tilde{\Delta}_{l_V}$, the velocity disappears from Eq. (5.9) which gives

$$\tilde{f}_{l_V} \simeq e^{-(2-\zeta)(l_V - l_c)} A \epsilon, \tag{5.10}$$

where A is some constant and the only v-dependent quantity is l_V. To connect $\tilde{f}_{l_V}$ to the initial parameters, one has to integrate the flow $\tilde{f}_{l_V} - \tilde{f}_0 = \int_0^{l_V} dl\, \partial_l \tilde{f}_l$. Expanding $\partial_l \tilde{f}_l$ in Eq. (4.11) at small velocity and using $\tilde{\Delta}'(s\lambda) = \tilde{\Delta}'(0^+) + s\lambda \tilde{\Delta}''(s\lambda) + \mathcal{O}(\lambda^2)$ one recognizes in the second term the correction to η. Thus for $l_c < l < l_V$ one has

$$\partial_l \tilde{f}_l \simeq e^{-(2-\zeta)l} c \Lambda_0^2 \Delta^{*\prime}(0^+) - v\, \partial_l \eta_l, \tag{5.11}$$

where we dropped the subdominant terms in velocity. The integration of the flow gives

$$\tilde{f}_{l_V} - \tilde{f}_0 \simeq -f_c(1 - e^{-(2-\zeta)(l_V - l_c)}) - v(\eta_{l_V} - \eta_0), \tag{5.12}$$

where we defined $f_c = c\Lambda_0^2 e^{-(2-\zeta)l_c} |\Delta *'(0^+)|/(2-\zeta)$. Injecting $\tilde{f}_0 = f - \eta_0 v$ and Eq. (5.10), we note that quite remarkably the $\eta_0 v$ *cancel each other*. We are left with

$$f - f_c \simeq e^{-(2-\zeta)(l_V - l_c)}(A\epsilon - f_c) + \eta_{l_V} v. \tag{5.13}$$

We already know from Eq. (5.7) that $\eta_{l_V} \sim v^{1/\beta - 1}$ and from Eq. (5.6) $e^{l_V} \sim v^{1/(2-\zeta)\beta}$ thus both terms on the right-hand side of Eq. (5.13) scale like $v^{1/\beta}$. This leads to the following result to lowest order in ϵ:

$$f_c \simeq \begin{cases} \dfrac{\epsilon}{2} \dfrac{c r_f}{R_c^2} & \text{RF} \\[2ex] \dfrac{\epsilon}{12} \dfrac{c a}{R_c^2} & \text{RP,} \end{cases} \tag{5.14}$$

$$v \sim (f - f_c)^{\beta}, \tag{5.15}$$

$$\beta = \begin{cases} 1 - \dfrac{\epsilon}{9} & \text{RF} \\[2ex] 1 - \dfrac{\epsilon}{6} & \text{RP,} \end{cases} \tag{5.16}$$

where we used the fact that $\zeta = \epsilon/3$ (RF or RB) or $\zeta = 0$ (RP) and the link between $\chi = |\Delta^{*\prime}(0^+)|$ and r_f (RF) or a (RP) stated in Appendix H. In addition, we assumed that η_{l_c} has a regular behavior when $v \to 0$, a nontrivial point which we discuss in Sec. V C.

B. Discussion

The approach of the previous Sec. V A allows us to obtain the characteristics of the $T=0$ depinning. We extract the depinning exponent β, the pinning force f_c and the characteristic length scales from the equation of motion without any additional physical hypothesis or scaling relation. Although the depinning problem, the exponent β, and the critical force were determined in previous studies,[25,26] our method is an improvement in several ways.

To get the depinning exponent and critical force, two main derivations exist in the literature. One of them extends the static FRG formalism to the out of equilibrium depinning problem at zero temperature,[25] using an ''expansion'' around an unknown mean-field solution.[15] Instead of directly looking at the renormalized correlator of the disorder, the method obliges one to deal with the time correlation of the force, $C[v(t-t')]$ in Ref. 25. This procedure does not allow for a precise enough calculation of the v-f characteristics to demonstrate the cancellation of the $\eta_0 v$ term [in our Eqs. (5.12), and (5.13)]. In order to obtain a depinning exponent β different from its ''mean-field'' value $\beta = 1$, it is necessary in Ref. 25 to neglect *by hand* in the small v limit a term proportional to v against a term proportional to $v^{1/\beta}$ with $\beta < 1$. Our method, that directly uses averaging over the disorder and properly takes into account the velocity in the flow of the renormalized action, allows one to show explicitly the needed cancellation.

The other analytical study[26,50] of depinning does not consider the renormalization before the Larkin length and assumes that the singularity is fully developed beyond this length scale. This amounts to taking as a starting point the equation of motion *at zero velocity* with a cuspy correlator for the force, and the Larkin length as the microscopic cutoff. Since the anomalous exponent of the friction is $-\Delta^{*\prime\prime}(0)$ which is ill defined for a cuspy correlator, one is forced in this method to argue that it should be replaced by $-\Delta^{*\prime\prime}(0^+)$ which is finite. This prescription and scaling relations linking the roughness, the depinning, and the time exponent $2 - \Delta^{*\prime\prime}(0^+)$, allows us to extract the depinning exponent. In our method, the ambiguities that existed in Ref. 26 to write the flow of η beyond l_c when using the zero velocity equations, and the trick $0 \to 0^+$ becomes a well-defined mathematical property of our finite velocity RG equations: if Eq. (5.2) is confirmed, our approach directly shows that the $-\Delta^{*\prime\prime}(0^+)$ prescription is the correct one and allows us to *prove* directly the scaling relations, instead of *assuming* them, to obtain the exponent.

Furthermore, the occurrence of the asymptotic Edwards Wilkinson regime in Ref. 26 has to be put by hand as a cut in the $v=0$ RG flow. The important correlation length R_V (denoted L_V in Ref. 26) at which this regime takes place is thus not well under control and has to be estimated from dimensional analysis. In our case the depinning regime is naturally cut when our λ, which tells how fast the system runs on the disorder, reaches the range of the flowing correlator. The scale R_V at which it occurs, and above which the nonlineari-

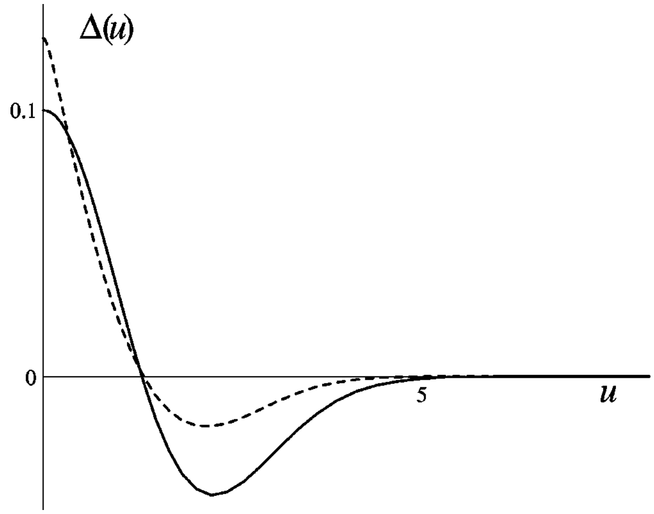

FIG. 6. The evolution of a random bond to a random-field correlator obtained by numerical integration of the flow. The initial condition $l=0$ of the flow is a RB $[\Delta(u)$ shown as a full line on $u \geqslant 0]$. Following Eq. (4.11), the running correlator transforms into a RF as shown on the snapshot of $\Delta_l(u)$ near l_c (dashed line), as can be seen by comparing with the characteristic shapes of RB/RF shown in Fig. 3.

ties are washed out, can clearly be identified with the correlation length of the moving interface (or more precisely, of the velocity-velocity correlation). The physical interpretation of Eq. (5.6), i.e.,

$$R_V \sim R_c \left(\frac{f_c}{\eta v} \right)^{1/(z-\zeta)}, \tag{5.17}$$

with $z=2-(\epsilon-\zeta)/3$ is the following: R_V is the scale at which ''avalanches'' occur in the driven deterministic system. The motion proceeds in a succession of such processes, where pieces of interface of typical size R_V depin over a distance $r_f(R_V/R_c)^{\zeta}$ during a time $r_f(R_V/R_c)^{\zeta}/v$.

In addition to providing a clean derivation of the depinning exponents and of the critical force, our equations contain new physics that was unreachable by the previous methods.

Although in principle one would expect three universality classes (RF, RB, RP) for the depinning exponent, it was conjectured by Narayan and Fisher[37] that the roughness exponent of the system at the depinning transition for RB or RF is equal to the roughness exponent of the static RF case, $\zeta = \epsilon/3$. This result *cannot* be obtained by the approach of Narayan and Fisher or that of Nattermann *et al.* since these authors did not include the velocity in their RG analysis, and simply treated the small v limit as $v=0$. On the contrary, our flow equations for the correlator shows directly that a RB disorder does indeed evolve during the flow towards a RF disorder, leaving only two different universality classes (RF, RP) for the dynamics against three for the statics (RB, RF, RP). Such evolution is shown on Fig. 6, where an initial RB becomes *dynamically* a RF. In Appendix G we show that the correction to $\int \Delta$, which measures the RF strength of the disorder, grows as

$$\partial \int \tilde{\Delta} = 2\lambda^2 \int \tilde{\Delta}''^2 + \mathcal{O}(\lambda^4)$$

where we have used $\zeta=\epsilon/3$. This ensures that a moving system, even at *arbitrary small* velocity, sees an effective *random field* at large scale.

C. Open questions

Our FRG equations prompt for several remarks and questions. In the previous sections, we have examined the consequences of the property (5.2) and established in that case that the values of the exponents were the ones proposed in Refs. 25, 26, and 37. Although we consider it as unlikely, we have not been able to rule out the possibility that either $\rho_l \sim \lambda_l$ (or even worse, $\rho_l > \lambda_l$) and thus we should examine the consequences of a violation of property (5.2). If $\rho_l \sim \lambda_l$, it is not excluded *a priori* that there exists another ''fixed-point'' behavior (e.g., with a scaling function of u/λ_l). However, in that case, the exponents should differ from the standard ones [unless some hidden and rather mysterious sum rule would fix the value of the integral in Eq. (5.1)]. In the absence of an identified fixed point, it is not clear whether universality would hold. Again this crucial point (5.2) can be definitely answered by an appropriate integration of Eq. (4.11). Thus the present approach, which clearly takes v into account, identifies as Eq. (5.2) the condition under which the trick used in Refs. 25 and 26 gives the correct exponents.

Another intriguing point concerns the continuity between the $v=0$ and the $v \to 0$ problems. Indeed, to derive the depinning law (5.14) we have assumed that η_{l_c} remains finite as $v \to 0$. However, we should recall that in the nondriven case ($v=0$ and $f=0$), η_l diverges at l_c and thus $\eta_{l_c}=\infty$.[51] If there is any continuity in the RG flow as $v \to 0$ then $\eta_{l_c} \to \infty$ in this limit. In that case the consequence would be [see Eq. (5.8)] a modification of the exponent $\beta \to \beta/(1-\alpha)$ if $\eta_{l_c} \sim v^{-\alpha}$ (or weaker logarithmic multiplicative corrections). We would then find for the depinning a different result from the conventional one. Since we are unable to solve analytically accurately enough the equation for η around l_c, one should resort to a numerical solution of our flow Eqs. (4.11) to resolve this question. Using Eq. (4.11) it is necessary to check that η_{l_c} does not diverge as $v \to 0$ like a power of v so as to recover the standard depinning exponent (5.5). The question is of particular importance since, if really a finite-scale behavior, occurring near R_c, would control the macroscopic asymptotic behavior, then again one could wonder whether universality would hold.

Therefore the description of depinning in terms of a standard critical phenomenon may be risky. Indeed as clearly appears in our FRG approach, since the fixed point at $v=0$ is characterized by a *whole function* Δ^* (i.e., an infinite number of marginal directions in $D=4$) rather than a single coupling constant (as in usual critical phenomena) the effect of an additional relevant perturbation, here the velocity, can be more complex due the feedback of v itself on the shape of the function during the flow. This is particularly clear in the RB case which dynamically tranforms into RF.

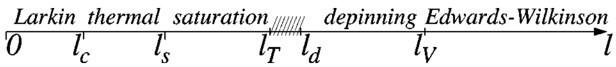

FIG. 7. Characteristic scales and regimes for creep motion.

VI. CREEP

We now deal with the nonzero-temperature case. The system can jump over any energy barrier and overcome the pinning forces, thus it *moves* with $v>0$ for any drive $f>0$ and never gets pinned. Let us now show how our Eqs. (4.11) allow us to investigate the *creep* regime that occurs when the system moves very slowly with $f \ll f_c$, at low temperature.

A. Derivation from FRG

As for the depinning, we are interested in infinitesimal velocities. The bare λ_0 is thus very small. The main difference compared to Sec. V A is that the temperature is now finite as well. The main effect of T is to round the cusp in the flow. Since we are interested in extremely small velocities, we will consider λ_0 as the smallest quantity to start with. A nonzero temperature thus gives rise to a new regime in the RG flow, where the rounding of the cusp is due to temperature and not to velocity. This leads to the following regimes in the FRG flow shown in Fig. 7. We will examine the various regimes in the RG flow keeping in mind that again, λ_l increases monotonically with l.

Just as in the previous case, we expect a *Larkin regime* for $0<l<l_c$ with small corrections. Above l_c the disorder reaches a regime where scaling is imposed by the temperature. Indeed since $\lambda_{l_c} \ll \tilde{T}_{l_c}/\chi$ one can forget about the velocity and the FRG equations are very similar to the $v=0$ and $T>0$ case. In Appendix D we show that the temperature rounds the cusp on a boundary layer $u \sim \tilde{T}_l/\chi$ and we obtain the explicit scaling form (D1),

$$\tilde{\Delta}_l(u) \simeq \tilde{\Delta}_l(0) - \tilde{T}_l f(u\chi/\tilde{T}_l)$$

$$f(x) = \sqrt{1+x^2} - 1,$$

$$\chi = |\Delta^{*\prime}(0^+)|, \qquad (6.1)$$

which in the statics holds at all scales larger than a scale of order l_c. Here, because we focus on $v \to 0$, the scanning scale λ_{l_c} is smaller than the width of the boundary layer, and the flow of the friction reads in this regime,

$$\partial \ln \eta_l \simeq -\tilde{\Delta}_l''(0) \simeq \frac{\chi^2}{\tilde{T}_l}. \qquad (6.2)$$

The temperature being irrelevant by power counting, the initial flow of $\tilde{T}$ is

$$\partial \ln \tilde{T} = -\theta \qquad (6.3)$$

since the anomalous correction to $\tilde{T}$ vanishes as $\lambda \to 0$. Here and in the following, $\theta = D - 2 + 2\zeta_{\rm eq}$ denotes the energy fluctuation exponent of the *static* problem. Together with Eq. (6.2) it shows that the friction grows extremely fast, like $\exp e^{\theta l}$. This is the *thermal regime* where motion only occurs via thermal activation over barriers. The velocity is so small that the center-of-mass motion is unimportant and the temperature essentially flows as in the $v=0$ problem. We have determined the flow in its initial stages, and we now determine the scale at which this behavior ceases to hold.

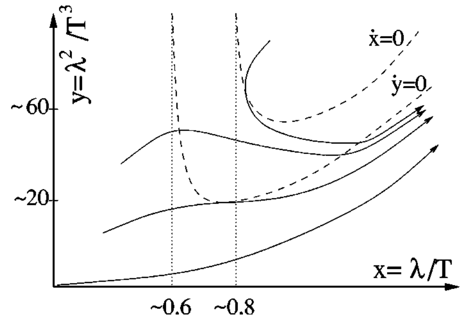

FIG. 8. The integration of the flow using Eq. (6.1) after l_c and reduced variables $x_l \equiv \chi\lambda_l/\tilde{T}_l$ and $y_l \equiv \chi^4\lambda_l^2/\tilde{T}_l^3$. The dotted lines indicate the set of points where $\partial_l x_l = 0$ or $\partial_l y_l = 0$. Some trajectories are displayed, with an arrow showing the direction of growing l. The initial conditions for creep are close to the origin, and closer to the x axis.

The flow Eq. (C6) for η_l together with the scaling function (6.1) for $\tilde{\Delta}$ for $u \sim \tilde{T}_l/\chi$ shows that Eq. (6.2) holds only until the new scale $l_T = \ln \Lambda R_T$ defined as

$$\lambda_{l_T} \sim T_{l_T}/\chi. \qquad (6.4)$$

For $l<l_T$ the temperature remains the main source of rounding of the cusp. Above that scale one must take the velocity into account.

In fact, this simple picture is not complete since, before reaching l_T another phenomenon occurs, leading to another length scale. In the thermal regime the correction to $\tilde{T}$ due to disorder competes with the simple exponential decay and Eq. (6.3) breaks down. This physically expresses that motion in a disordered landscape generates a thermal noise (provided some thermal noise is already present). Using Eq. (6.1), one has $\partial \ln \tilde{T} \approx -\theta + 6\chi^4\lambda^2/\tilde{T}^3$ at small λ. Thus the correction to $\tilde{T}$ reverts at a scale $l_s = \ln \Lambda R_s$ such that $\lambda_s \sim \tilde{T}_{l_s}^{3/2}/\chi^2$. Note that $l_s < l_T$. Above l_s the temperature does not decrease any more due to heating by motion. One can show using Eq. (4.11) that $\tilde{T}$ saturates and does not vary much until the scale l_T. We call this intermediate regime $l_s<l<l_T$ the *saturation regime*. We checked it using a numerical integration of the flow in this regime with the scaling form of the disorder (6.1). Analytically, if we suppose that after l_s, the correction of $\tilde{T}$ due to disorder dominates $-\theta$, then one would have in this regime an invariant of the flow $\partial_l(\tilde{T}_l^2 - 6\chi^2\lambda_l^2) \approx 0$. If this were true, it is clear that the flow could *never* realize the condition $\lambda_l \sim T_l/\chi$, possibility that is excluded on physical basis and by the numerics shown in Fig. 8.

Despite the saturation of the temperature, Eq. (6.2) remains true after l_s. Thus the friction and λ keep on growing and one finally reaches the scale l_T at which the scanning length λ_l crosses the boundary layer width $\tilde{T}_l/\chi$.

Above l_T, a rigorous analytical analysis of Eq. (4.11) becomes difficult. We, however, expect, since the velocity controls now the boundary layer, a regime similar to the depinning regime at $T=0$ to occur. Using the same arguments than for the depinning, one obtains in that regime

$$\partial \ln \eta = -\Delta^{*\prime\prime}(0^+), \tag{6.5}$$

$$\partial \ln \tilde{T} = 2 - D - 2\zeta, \tag{6.6}$$

leading again to a decrease of the temperature, even slightly accelerated by a negative $\mathcal{O}(\epsilon^{4/3})$ exponent. Let us call l_d the *depinning scale* at which one enters such a depinning regime. From the above discussion it is very reasonable to expect that one goes directly from the saturation to the depinning regime, i.e., $l_d - l_T \sim \text{cst}$. However, we cannot strictly rule out the possibility of an intermediate regime (divergent $l_d - l_T$ when $v \to 0$) during which the correction to the friction goes smoothly from positive (as in the thermal and saturation regimes) to negative values (depinning regime). Again, it would be useful to settle this point through a numerical solution of our flow equations. Note that in the RF and RP cases, the exponent ζ and the fixed point Δ^* in Eq. (6.5) are the same as in the statics. However, in the RB case, we have used a l-dependent ζ which crosses over between ζ_{eq} for $l<l_T$ and $\zeta=\epsilon/3$ for $l>l_d$ corresponding to the change from RB to RF fixed points Δ^*.

In the depinning regime, motion now proceeds in a similar way than for the one studied in Sec. V. Here again at large enough scale, velocity will wash out the disorder for $l>l_V$ with l_V determined by $\lambda_l \sim r_f(l)$. One then enters the Edwards-Wilkinson regime.

Let us now compute from the flow (4.11) the length scales defined above (see Fig. 7). In the thermal regime $l_c<l<l_s$ one can compute $\lambda_{l_s}/\lambda_{l_c}$ either by integrating its flow or by equating the boundary values to their expression. This gives

$$\frac{\lambda_{l_s}}{\lambda_{l_c}} \approx \begin{cases} \exp\left[(2-\zeta_{\text{eq}})(l_s-l_c) + \dfrac{U_c}{T}\left(e^{\theta(l_s-l_c)}-1\right)\right] \\ \left(\dfrac{T}{U_c}\right)^{3/2} \dfrac{f_c}{\eta_{l_c} v} e^{-(3/2)\theta(l_s-l_c)} \end{cases}$$

where we defined $f_c \equiv \epsilon(c r_f/R_c^2)$ and $U_c \equiv \epsilon^2 R_c^D (c r_f^2/R_c^2)$. Expressing the scales as a function of the velocity leads to

$$\left(\frac{R_s}{R_c}\right)^{\theta} \approx \frac{T}{U_c} \ln\left[\left(\frac{T}{U_c}\right)^{3/2} \frac{f_c}{\eta_{l_c} v}\right], \tag{6.7}$$

$$\frac{\eta_{l_s}}{\eta_{l_c}} \approx \frac{f_c}{\eta_{l_c} v}\left(\frac{U_c}{T}\right)^{1/\mu} \ln\left[\left(\frac{T}{U_c}\right)^{3/2} \frac{f_c}{\eta_{l_c} v}\right]^{-3/2-1/\mu} \tag{6.8}$$

with $\mu \equiv \theta/(2-\zeta_{\text{eq}})$.

In the saturation regime $l_s<l<l_T$ we proceed in the same manner and obtain

$$\frac{\lambda_{l_T}}{\lambda_{l_s}} \approx \begin{cases} \exp\left[\left(2-\zeta_{\text{eq}} - \dfrac{\chi^2}{\tilde{T}_{l_s}}\right)(l_T - l_s)\right] \\ \left(\dfrac{U_c}{T}\right)^{1/2} e^{(\theta/2)(l_s-l_c)} \end{cases}.$$

Thus

$$\frac{R_T}{R_s} \approx 1, \tag{6.9}$$

$$\frac{\eta_{l_T}}{\eta_{l_s}} \approx \ln\left[\left(\frac{T}{U_c}\right)^{3/2} \frac{f_c}{\eta_{l_c} v}\right]^{1/2}. \tag{6.10}$$

Assuming $l_d \sim l_T$, the depinning regime $l_d<l<l_V$ follows directly and

$$\frac{\lambda_{l_V}}{\lambda_{l_d}} \approx \begin{cases} \exp[(2-\zeta)\beta(l_V - l_d)] \\ \dfrac{1}{\epsilon}\dfrac{U_c}{T} e^{\theta(l_s-l_c)} \end{cases}$$

leads to

$$\frac{R_V}{R_d} \approx \left(\frac{1}{\epsilon}\ln\left[\left(\frac{T}{U_c}\right)^{3/2} \frac{f_c}{\eta_{l_c} v}\right]\right)^{1/(2-\zeta)\beta}, \tag{6.11}$$

$$\frac{\eta_{l_V}}{\eta_{l_d}} \approx \left(\frac{1}{\epsilon}\ln\left[\left(\frac{T}{U_c}\right)^{3/2} \frac{f_c}{\eta_{l_c} v}\right]\right)^{1-1/\beta} \tag{6.12}$$

with $\beta \equiv [2-\zeta-\Delta^{*\prime\prime}(0^+)]/(2-\zeta)$ the depinning exponent (and ζ the dynamical roughness exponent).

We are now in a position to compute the characteristics $f(v)$. We fix a small velocity v and solve the flow equations for λ_l, $\tilde{\Delta}_l$, and $\tilde{T}_l$ up to l_V. This allows us to relate $\tilde{f}_{l_V}$ to the unknown $\tilde{f}_0$. We can now use the fact that at the scale l_V, the disorder is essentially washed out and a perturbative calculation of $\tilde{f}_{l_V} \approx \tilde{f}_\infty = 0$ is possible. Solving backwards we determine $\tilde{f}_0$, which is simply $f-\eta v$ where f is the real force applied on the system and $\eta=\eta_0$ the bare friction.

The correction to $\tilde{f}$ cannot be neglected during the depinning regime, thus, using $\tilde{f}_0 = f - \eta_0 v$, $\tilde{f}_\infty = 0$ and expressing $\int_0^\infty dl\, \partial_l \tilde{f}_l$ one has

$$f - \eta_0 v \approx -\int_0^\infty dl\, \partial_l \tilde{f}_l \approx \frac{c\Lambda_0^2 \chi}{2-\zeta} e^{-(2-\zeta_{\text{eq}})l_d}. \tag{6.13}$$

In the thermal regime there is essentially no correction to the flow of $\tilde{f}$. Thus Eq. (6.13) is controlled by the depinning regime and one should integrate essentially between l_d and l_V. In fact, due to the exponentially decreasing behavior of the integrand in Eq. (6.13) the whole integral depends in fact only on the behavior *at* the scale l_d. Assuming that $l_d \sim l_T$, using Eqs. (6.7) and (6.9), one sees that $e^{-(2-\zeta_{\text{eq}})l_d} \ll v$ for $v \to 0$ and thus one obtains

$$\frac{\eta v}{f_c} \approx \exp\left[-\frac{U_c}{T}\left(\frac{f}{f_c}\right)^{-\mu}\right], \tag{6.14}$$

$$\mu = \frac{D-2+2\zeta_{\text{eq}}}{2-\zeta_{\text{eq}}}. \tag{6.15}$$

The prefactor in front of the exponential cannot be obtained reliably at this order. Note that for the creep, contrarily to the depinning, the possible divergence of η_{l_c} when $v \to 0$ (and $T \to 0$) does not affect the argument of the exponential but only the prefactor.

B. Alternative method and open questions

For the depinning it was possible to recover the depinning law using both the integration of the flow of $\tilde{f}$ and of the friction η and the relation (4.5). Although one can also use, in principle, this method for the creep it gives poor results in this case. Indeed, contrarily to the derivation involving $\tilde{f}$ one needs here the flow of η in *all* regimes including the depinning regime $l>l_d$, where η is still renormalized. Since the renormalization of η goes from large positive growth (first like $\exp e^{\theta l}$, then exponentially) in the thermal/saturation regime to negative in the depinning regime (where the system accelerates with subdiffusive $z<2$) a precise knowledge of the behavior around l_T would be needed. Unfortunately, the lack of precise analytical methods available above R_T prevents us from computing precisely such a crossover. A crude estimate of the flow can thus only give a bound of the exact result. If we use (e.g., in the RF or RP cases) the estimates of each regime, and the perturbative estimate of η_{l_V} in the theory at l_V: $\eta_\infty \simeq \eta_{l_V} + \int_t \Delta''_{l_V}(vt)tR_{0t}^{l_V} \sim e^{-(2-\zeta)(l_V-l_c)}\epsilon$ (it will appear that η_{l_V} diverges faster than $e^{-(2-\zeta)(l_V-l_c)}$ when $v\to 0$). The product of Eqs. (6.8), (6.10), and (6.12) is equal to $(1/\eta_{l_c})(df/dv)$. Integrated from 0 to v, it yields

$$\frac{\eta_{l_c}v}{f_c} \approx \exp\left\{ -\left[\frac{U_c}{T}\left(\frac{f}{\epsilon^{1/\beta-1}f_c} \right)^{-\mu} \right]^{1/(1+\mu(1/\beta-1))} \right\}. \tag{6.16}$$

One would thus find, using the η method, a non-Arrhenius law for the creep regime. Even if one cannot, strictly speaking, exclude this result, as discussed above it is most likely an artifact of the approximate integration of the flow, and only a lower bound of the barrier height. Indeed compared to the integration of the flow of $\tilde{f}$, this procedure is much more sensitive to the neglect of the crossover $l_T<l<l_d$. A more precise integration of the flow would very likely show a compensation between the latent growth of the friction during the decrease of $\partial \ln \eta$ (for $l_T<l<l_d$) and the reduction of the friction occurring in the depinning regime $l_d<l<l_V$. Note that if df/dv were equal to η_{l_T} then, one would recover Eq. (6.14). It would be useful to check explicitly on a numerical integration of the flow that such a cancellation does occur and verify that the η method confirms also the result (6.14).

We also note that the precise determination of the length scales for $R>R_T$ depend on obtaining an accurate solution of the RG flow equations. In the previous section, we have obtained the formulas (6.9) and (6.11) under some assumptions about the mathematical form of the solutions of the flow in the region where λ_l and $\widetilde{T_l}$ cross. These assumptions, discussed in the previous section, should be checked further, e.g., via numerical integration. Although this should not affect the creep exponent derived above, the precise determination of these length scales is important to ascertain the exact value of the scale R_V (i.e., the avalanche scale discussed below).

C. Discussion

Since our flow Eqs. (4.11) include finite temperature and velocity, they allow us to treat the regime of slow motion at finite temperature, directly from Eq. (2.10). As for the depinning, we derive directly from the equation of motion the force-velocity law and we obtain interesting physics.

The first important result is, of course, the creep formula itself [Eq. (6.14)]. Our method allows one to prove the main physical assumptions, reviewed in Sec. III 3, needed for the phenomenological estimate, namely: (i) the equal scaling of the barriers and the valleys; (ii) the fact that velocity is dominated by activation over the barriers correctly described by an Arrhenius law. In our derivation such law comes directly from the integration of the flow equations in the thermal regime; (iii) the fact that one can use the static exponents in the calculation of the barriers. This appears directly in the formula (6.14) but can also be seen from the fact that in the thermal regime the velocity can essentially be ignored in the flow equations. We also recover the characteristic length scale predicted by the phenomenological estimate. Indeed, one can identify the scale (6.7) and (6.9) $R_T \sim R_c(f/f_c)^{-1/(2-\zeta_{\rm eq})}$ as the $R_{\rm opt}$ of Sec. III 3.

Our equations allow us to obtain additional physics in the very slow velocity regime. In particular, we see that the slow motion consists of two separate regimes. At small length scales $R<R_T$ the motion is controlled by thermal activation over barriers as would occur at $v=0$. This is the regime described by the phenomenological theory of the creep. Qualitatively, the main interesting result obtained here is that the thermally activated regime is followed by a depinning regime, as shown by our equations. This leads to the following physical picture: at the length R_T, bundles can depin through thermal activation. When they depin they start an avalanchelike process, reminiscent of the $T=0$ depinning, up to a scale R_V. The propagation of the avalanche proceeds on larger scales in a deterministic way. Thus one is left with a depinninglike motion, and the size of the avalanches is determined by the natural cut of the RG ($\lambda=r_f$), i.e., the scale at which the propagating avalanche motion is overcome by the regular motion of the center of mass. One recovers qualitatively and quantitatively some features of the $T=0$ case at intermediate scale. The typical nucleus jumps over an energy barrier $U_b \sim U_c(R_T/R_c)^\theta$ resulting in $v\sim\exp -(U_c/T)(f/f_c)^{-\mu}$. This jump of a region of size R_T initiates an avalanche spreading over a much larger size R_V which we find to be [see Eqs. (6.7), (6.9), and (6.11)]

$$\frac{R_V}{R_c} \sim \left(\frac{U_c}{T}\right)^{\nu/\beta}\left(\frac{R_T}{R_c}\right)^{1+\theta\nu/\beta} \tag{6.17}$$

with $\nu=\beta/(z-\zeta)=1/(2-\zeta)$ and $z=2-(\epsilon-\zeta)/3$ the critical exponents of the depinning, and θ the energy exponent of the statics. Note that the correlation length R_V diverges at small drive and temperature as $R_V \sim T^{-\sigma}f^{-\lambda}$ with $\sigma=\nu/\beta=1/(z-\zeta)$ and $\lambda=1/(2-\zeta_{\rm eq})+\mu/(z-\zeta)$.

To push the analogy further one can consider that the avalanches at length scales $R>R_T$ are similar to the ones occurring in a regular $T=0$ depinning phenomenon due to an excess driving force $(f-f_c)_{\rm eff}$. Considering a minimal block size R_T instead of R_c for this ''creepy'' depinning, $R_V/R_T \sim (f-f_c)_{\rm eff}^{-\nu}$, one obtains for this effective excess force

$$\frac{\eta_{\rm eff} v}{f_c^{\rm eff}} \sim \left(\frac{f-f_c}{f_c}\right)_{\rm eff}^{\beta} \sim \frac{T}{U_{\rm b}} \tag{6.18}$$

linking the creepy motion at $T>0$ and the threshold depinning at $T=0$. As explained before, there might be an uncertainty in the value of the avalanche exponent, which could be changed by a quantity of $\mathcal{O}(\epsilon)$. To confirm Eq. (6.17), one would need to further check the precise behavior of the solution of the RG equations for $R>R_T$.

One can understand qualitatively that the problem at scale $R>R_T$ looks like depinning according to Eq. (6.18). The tilted barrier (see Sec. III 3) $E(R,f)=U_c(R/R_c)^{\theta} - fR^D r_f(R/R_c)^{\zeta_{\rm eq}}$ to be overcome in order to move a region of size R (all barriers corresponding to smaller scales having been eliminated), vanishes at[52] $R_0(f) \gtrsim R_T$. For the $T=0$ depinning problem, one can define a scale dependent effective threshold force $f_c^{\rm eff}(R) \sim f_c(R/R_c)^{-(2-\zeta_{\rm eq})}$ such that $E(R,f_c^{\rm eff})=0$ (also defined in Sec. IV C), which corresponds to the force needed to depin scales larger than R [the true threshold $f_c=f_c^{\rm eff}(R_c)$ being controlled in that case by the Larkin length]. A possible scaling derivation of Eq. (6.18) is obtained by noting that at $T>0$, *nonactivated* motion at scale R occurs when the tilted barrier $E(R,f)$ is of the order of T. This yields a T-dependent effective threshold force such that

$$\frac{f_c^{\rm eff}(R)-f_c^{\rm eff}(R,T)}{f_c^{\rm eff}(R)} \sim \frac{T}{U_c(R/R_c)^{\theta}}. \tag{6.19}$$

At $R=R_0(f)$, one has $f=f_c^{\rm eff}(R)$ and Eq. (6.19) is identical to Eq. (6.18) to zeroth order in ϵ (i.e., $\beta=1$). In fact, to apply the above static barrier argument, it might be better to work in the comoving frame where the velocity of the interface vanishes. This amounts to replacing f by $f-\eta v$ in the previous argument, and $E[R_0(f),f]=0, E[R_0(f),f-\eta v] = T$ gives back Eq. (6.18).

The crossover between thermally activated processes and depinninglike motion can also be recovered by noting that the condition $\lambda_l \sim \tilde{T}_l/\chi$ which appears in the FRG flow can be rewritten as [using Eqs. (4.9) and (4.10)]

$$f_c^{\rm eff}(R)\, v\, \tau(R)\, R^D \sim T, \tag{6.20}$$

where the left-hand side is a natural energy scale involved in the depinning due to driving effect of the center of mass. If it is much larger than T, depinning effects dominate, while if it is smaller, the dynamics is activated.

Finally many open questions still remain. Technically it would be interesting to reconciliate the two methods based on η and $\tilde{f}$ which proved to be equivalent for the study of depinning. In fact, although the two methods should formally agree, the comparison at a given order in the RG is more subtle. Indeed $(d/dv)\delta\tilde{f} = -\delta\eta$, by integration over l between 0 and ∞ and derivation with respect to v, gives back $\eta_\infty = df/dv$ provided that $\tilde{f}_\infty = 0$. However, one should notice that in $(d/dv)\delta\tilde{f} = -\delta\eta$, the derivative is understood at fixed parameters *at the given scale*. The occurrence of this hidden dependence in the velocity in the running parameters makes the equivalence between both approaches delicate. However, the additional term is of higher order in disorder. Thus, as pointed out above, it is very likely that a careful integration of the flow of η should resolve this discrepancy, but this remains to be explicitly checked.

As for the $T=0$ depinning, the existence of the *depinning regime* at l_d depends on the precise form of the boundary layer in the presence of a velocity. Note that the alternative scenario discussed in Sec. V C, e.g., whether or not the depinning regime is universal, would not affect the creep exponents, but only the subleading corrections.

VII. CONCLUSION

We examine in this paper the dynamics of disordered elastic systems such as interfaces or periodic structures, driven by an external force. We take into account both the effect of a finite temperature and of a finite velocity to derive the general renormalization-group equations describing such systems. We extract the main features of the analytical solution to these equations both in the case of the $T=0$ depinning (shown on Fig. 1) and in the ''creep'' regime (small applied force f and finite temperature).

Our RG equations, when properly analyzed, allow us to recover the depinning law $v \sim (f-f_c)^{\beta}$ and the depinning exponent β also obtained by other methods. However, contrarily to previous approaches that needed additional physical assumptions, such as scaling relations among exponents or by hand regularization, our approach is self-contained, all quantities being derived directly from the equation of motion. It thus provides a coherent framework to solve the difficulties and ambiguities encountered in the previous analytical studies.[25,26] In addition, our method allows us to establish the universality classes for driven systems. It shows *explicitly* that a random-bond-type disorder gives rise close to a random-field critical behavior at the depinning. Thus the dynamics is characterized by only two universality classes [random field (RF) for interfaces and random periodic (RP) for periodic systems] instead of three. Since this phenomenon is an intrinsically dynamical one, it was out of the reach of the previous analytical approaches that used $v=0$ flow equations together with additional physical prescriptions using, e.g., the velocity as a cutoff on the $v=0$ RG flow.

Of course one of the great advantages of the present set of RG equations is to allow for the precise study of the small applied force regime at finite T, for which, up to now, only phenomenological scaling arguments could be given. Our FRG study confirms the existence of a creep law at small applied force

$$\frac{\eta v}{f_c} \approx \exp\left[-\frac{U_c}{T}\left(\frac{f}{f_c}\right)^{-\mu}\right] \tag{7.1}$$

with a creep exponent related to the static ones $\mu=(D-2+2\zeta_{\rm eq})/(2-\zeta_{\rm eq})$, where $\zeta_{\rm eq}$ is the statics roughening exponent. It provides a framework to demonstrate, directly from the equation of motion, the main assumptions used in the phenomenological scaling derivation of the creep, namely: (i) the existence of a single scaling for both the barriers and the minima of the energy landscape of the disordered system; (ii) the fact that the motion is characterized by an activation (Arrhenius) law over a typical barrier.

In addition, our study unveils a ''depinninglike regime'' within the creep phenomena, not addressed previously, even at the qualitative level since the phenomenological creep ar-

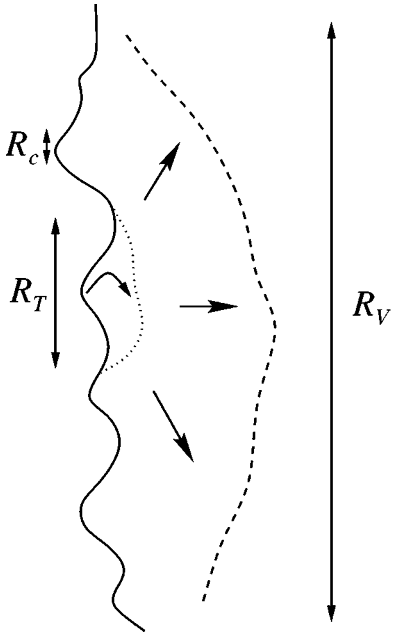

FIG. 9. Schematic picture of the creep process emerging from the present study: while thermally activated motion occurs between scales R_c (Larkin length) and R_T (thermal nucleus size), depinning-like motion occurs up to the avalanche size R_V.

guments did not address what happens *after* the thermally activated jump of the optimal nucleus. Although the velocity is dominated by the time spent to thermally jump over the barriers, our equations show that the small f behavior consists in fact of *two different regimes*. Up to a size R_T motion can only occur through thermal activation over barriers. This is the regime described by the phenomenological approach to the creep. The optimal nucleus of the scaling estimate is given directly by the RG derivation as $R_T \sim (1/f)^{1/(2-\zeta_{\rm eq})}$. Remarkably, another interesting regime exists above this length scale (see Fig. 9). It emerges directly from our RG equations and can be given the following simple physical interpretation. In some regions of the system, bundles of size R_T depin due to thermal activation. These small events then trigger much larger ones, and the motion above R_T proceeds in a *deterministic way*, much as the $T=0$ depinning. In particular, once the initial bundle depins it triggers an avalanche up to a size R_V which is given by $R_V/R_T \sim (U_c/T)^{\nu/\beta}(R_T/R_c)^{\theta\nu/\beta}$ where θ, β, and ν are the energy, depinning, and correlation length exponents, respectively.

The present study also raises several interesting questions which deserve further investigation; some of them rely on being able to obtain a more accurate solution of our flow equations. We have shown explicitly how to recover from our equations the conventional depinning law (and the scaling creep exponents). It rested on a mathematical property, likely to hold, but not yet rigorously established, of the solution for the flow of the correlator of the disorder. Such behavior should be checked in detail. The equations being quite complicated, a numerical solution, albeit delicate, seems to be appropriate. If the constraint (5.2) on the flow defined in Sec. V B were found to be violated, then the conventional picture of the depinning would very likely fail, as we have analyzed in detail. A similar question arises concerning the flow of the friction η as discussed in Sec. VI C. If the solution of the flow is found to depend on the precise behavior at the Larkin length R_c, it is likely that even universality could be questioned. These issues are *a priori* less important for the first, thermally activated, part of the creep regime, but because of the existence of a second, depinning-like regime, they would also have consequences for creep. Again, these questions depend on the precise form of the flow and can be answered unambiguously by a detailed enough analysis of our equations. It would also be of great interest to develop a more detailed physical picture of the crossover between thermally activated and depinninglike motion since we found that both occur within the creep phenomenon.

Several applications and extensions of our work can be envisioned. First, extensions to many-dimensional displacement field (of dimension $N>1$), given in Appendix F, would be interesting to study within the methods used here. One could check whether the approximation used in Ref. 53 yields the correct result for the $N>1$ depinning. Second, the effect of additional KPZ nonlinearities could be investigated. In particular one could check the usual argument which yields that KPZ terms are unimportant for the depinning[31] since their coupling constant is proportionnal to the (small) velocity. Also, extensions to other types of disorder, such as correlated disorder,[54] are possible. Finally, it should allow one to describe in a systematic way the thermal rounding of the depinning, i.e., the study of the v-f characteristics for f close to the threshold and small T. If one assumes that one can simply carry naive perturbation theory in T around the $T=0$ solution of the RG flow near f_c (i.e., only keeping the contribution beyond l_V), one is led in Eq. (5.13) to an additional term proportional to T/v^2, which readily yields the value for the thermal rounding exponent $\rho = 1+2\beta$ proposed in Ref. 49 (i.e., a scaling form near $f=f_c$ and small T for the velocity $v \sim T^{\beta/\rho}\Phi[(f-f_c)/T^{1/\rho}]$). Although this exponent seems to be consistent with starting values $\lambda \ll T$, its validity could be further checked by solving our RG flow equations at small T.

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APPENDIX A: NOTATIONS

Here are some notations and conventions and diagrammatics we use in the text. The surface of the unit sphere in D dimensions divided by $(2\pi)^D$ is denoted by $S_D = 2(4\pi)^{-D/2}/\Gamma(D/2)$. The thermal average of any observable A is $\langle A \rangle$, the disorder average is $\overline{A}$, and the average with the dynamical action $S[u,\hat{u}]$ is denoted by $\langle A \rangle_S = \overline{\langle A \rangle}$. The Fourier transform of a function h_{rt} of (r,t) is $h_{q\omega} = \int_{rt} e^{-iq\cdot r + i\omega t} h_{rt}$ where $\int_{rt} \equiv \int dr\, dt$, and the inversion reads $h_{rt} = \int_{q\omega} e^{iq\cdot r - i\omega t} h_{q,\omega}$, where $\int_q \equiv \int [d^D q/(2\pi)^D]$, $\int_\omega \equiv \int (d\omega/2\pi)$. The Fourier transform of the correlator $\Delta(u)$ is $\Delta_\kappa = \int du\, e^{-i\kappa\cdot u}\Delta(u)$ in general or $\Delta_\kappa = \int_0^a du\, e^{-i\kappa u}\Delta(u)$ in the periodic case. One has thus $\Delta(u) = \int_\kappa e^{i\kappa\cdot u}\Delta_\kappa$, where $\int_\kappa \equiv \int (d\kappa/2\pi)$ or $(1/a)\Sigma_\kappa$ in the periodic case. Note that Δ_κ is a real and even function of κ.

The graphs are made of the following units (see Fig. 10):

FIG. 10. Conventions for the diagrammatics. The correlation function uu, which vanishes at $T=0$, is a full line with no arrow. The reponse $ui\hat{u}$ is a full oriented line. The vertex $-\frac{1}{2}i\hat{u}_{rt}i\hat{u}_{rt'}\Delta[u_{rt}-u_{rt'}+v(t-t')]$ is naturally split in two half vertices corresponding to the points (r,t) and (r,t'), and the dashed line means that both points have the same position. At $T=0$ the correlation vanishes.

a full line between points (r,t) and (r',t') is a correlation $\langle u_{rt}u_{r't'}\rangle_S$, an *oriented line* with an arrow from point (r',t') to point (r,t) is a response $\langle u_{rt}i\hat{u}_{r't'}\rangle_S$ (the arrow means that $t>t'$, for the function does not vanish by causality). The vertex is represented as a dashed line linking points (r,t) and (r,t'). The dashed line means that both points have the same position r. From each point emerges a $\hat{u}$ field. No arrow is needed for the full line or for the dashed line, since they are symmetric with respect to the exchange of their end points. The correlation being proportional to T vanishes at $T=0$. The graphs renormalizing the disorder (see Fig. 13) are made of vertices and responses, and they possess two external $i\hat{u}$ lines. It can be easily seen that arrows are no more necessary since the two external $\hat{u}$ lines provide an orientation to all the responses of the graph. Indeed, due to causality, each of the external $\hat{u}$ is root of a tree, whose branches are response functions, which are oriented in the direction of the root.

APPENDIX B: PERTURBATION THEORY

We derive here the direct perturbation theory at $T>0$ without the use of the MSR formalism. To organize the perturbation series, let us multiply the nonlinear part of the equation of motion $F(r,vt+u_{rt})$ by a fictitious small parameter α, which will be fixed to one at the end of the calculation. Directly on

$$\begin{cases} \overline{\langle u_{rt}\rangle}=0 \\ (\eta\partial_t-c\nabla^2)u_{rt}=\alpha F(r,vt+u_{rt})+\tilde{f}+\zeta_{rt}+h_{rt} \end{cases} \tag{B1}$$

we can formally expand $u=\Sigma_{n\geqslant 0}\alpha^n u^{(n)}, f-\eta v\equiv\tilde{f}=\Sigma_{n\geqslant 0}\alpha^n\tilde{f}^{(n)}$, solve recursively the system (B1), even at nonzero temperature, and compute the α expansion of every observable. Note that we added a source h_{rt} (with no constant uniform part) so as to compute the response function. As the force is Gaussian, the expansion of disorder averaged quantities is in powers of α^2, and is in fact an expansion in powers of Δ. We denote by $\mathcal{C}_{r-r',t-t'}=\overline{\langle u_{rt}u_{r't'}\rangle}$ the exact correlation and by $\mathcal{R}_{r-r',t-t'}=\overline{\langle\delta u_{r,t}/\delta h_{r',t'}\rangle}$ the exact response functions.

The first iterative steps are $\tilde{f}^{(0)}=\tilde{f}^{(1)}=0$ and

$$(\eta\partial_t-c\nabla^2)u^{(0)}_{rt}=\zeta_{rt}+h_{rt},$$

$$(\eta\partial_t-c\nabla^2)u^{(1)}_{rt}=F(r,vt+u^{(0)}_{rt}),$$

$$(\eta\partial_t-c\nabla^2)u^{(2)}_{rt}=\partial_u F(r,vt+u^{(0)}_{rt})u^{(1)}_{rt}+\tilde{f}^{(2)}.$$

These are sufficient to compute to first order in Δ the force, the correlation, and response.

In the absence of disorder the system moves with a linear characteristics $f=\eta v$ and one has the following correlation and response:

$$C_{q\omega}=\frac{2\eta T}{(cq^2)^2+(\eta\omega)^2},\quad C_{qt}=T\frac{e^{-cq^2|t|/\eta}}{cq^2}, \tag{B2}$$

$$R_{q\omega}=\frac{1}{cq^2-i\eta\omega},\quad R_{qt}=\frac{\theta(t)}{\eta}e^{-cq^2t/\eta} \tag{B3}$$

related by the fluctuation-dissipation theorem (FDT) $TR_{rt}=-\theta(t)\partial_t C_{rt}$. Note that $\mathcal{R}$ and $\mathcal{C}$ do *not* verify FDT at $v>0$.

To first order in Δ one obtains at $T=0$

$$f-\eta v=-\int_{\kappa q}\frac{i\kappa\Delta_\kappa}{cq^2-i\kappa\eta v}, \tag{B4}$$

$$\mathcal{C}_{q,\omega}=\frac{\frac{1}{v}\Delta_{\kappa=-\omega/v}}{(cq^2)^2+(\eta\omega)^2}, \tag{B5}$$

$$\mathcal{R}_{q,\omega}=R_{q,\omega}+\frac{1}{(cq^2)^2+(\eta\omega)^2}\int_t\Delta''(vt)R_{0t}(1-e^{i\omega t}). \tag{B6}$$

These results can be extended to any temperature T:

$$f-\eta v=-\mathcal{D}_1(\omega=0), \tag{B7}$$

$$\mathcal{C}_{q,\omega}=C_{q,\omega}+R_{q,\omega}\mathcal{D}_0(\omega)R_{-q,-\omega} +R_{q,\omega}[\mathcal{D}_2(\omega=0)-\mathcal{D}_2(\omega)]C_{q,\omega}+\text{H.c.}, \tag{B8}$$

$$\mathcal{R}_{q,\omega}=R_{q,\omega}+R_{q,\omega}[\mathcal{D}_2(\omega=0)-\mathcal{D}_2(\omega)]R_{q,\omega}, \tag{B9}$$

where we have introduced the effective vertices, nonlocal in time, smoothed by the temperature (see Fig. 11),

$$\mathcal{D}_0(t)=\int_\kappa\Delta_\kappa e^{i\kappa vt+(i\kappa)^2(C_{00}-C_{0t})}, \tag{B10}$$

$$\mathcal{D}_1(t)=\int_\kappa i\kappa\Delta_\kappa e^{i\kappa vt+(i\kappa)^2(C_{00}-C_{0t})}R_{0t}, \tag{B11}$$

$$\mathcal{D}_2(t)=\int_\kappa(i\kappa)^2\Delta_\kappa e^{i\kappa vt+(i\kappa)^2(C_{00}-C_{0t})}R_{0t} \tag{B12}$$

We now want to compute the corrections to the parameters c, η, $\tilde{f}$, T, $\Delta(u)$ so that v, $\mathcal{C}_{q\omega}$, $\mathcal{R}_{q\omega}$ remain unchanged while the physical (ultraviolet) cutoff Λ on the q integrations is reduced. To first order in Δ and T, one obtains

$$\partial c=0, \tag{B13}$$

$$\partial\eta=-\int_t tR^{>}_{0t}\Delta''(vt), \tag{B14}$$

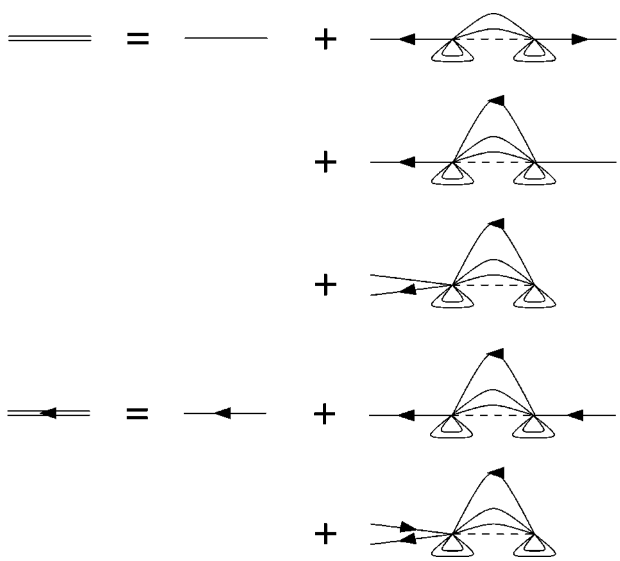

FIG. 11. Computation of the correlation (top) and response (bottom) functions to first order in perturbation theory. At $T>0$, the tadpoles and self-contractions of the vertices contain an arbitrary number of correlations.

$$\partial \tilde{f} = \int_t R^{>}_{0t}\Delta'(vt), \tag{B15}$$

$$\partial T = \frac{1}{\eta}\int_{t>0} t C^{>}_{0t}\partial_t \Delta''(vt), \tag{B16}$$

$$\partial \Delta(u) = C^{>}_{00}\Delta''(u) \tag{B17}$$

with $\partial \equiv -\Lambda(d/d\Lambda)$ and $R^{>}_{rt}, C^{>}_{rt}$ are the on-shell Gaussian response and correlation functions, i.e., with modes q lying only between $\Lambda - d\Lambda$ and Λ.

A completely different way for obtaining the perturbation expansion is presented in Refs. 55 and 4, as a first attempt to include thermal fluctuations in the large-velocity expansion of Ref. 18. It consists in splitting the displacement field into a $T=0$ part and a thermal part. This procedure is probably only true to first order in T and not controlled at higher T. Instead, the method presented here is really an expansion in disorder at any T.

Although the calculation can in principle be pushed to second order, the method is too cumbersome to do it in practice (see, however, Ref. 56 at $T=0$). It is easier to use the formalism of dynamical field theory as shown in Appendix C.

APPENDIX C: DERIVATION OF THE FLOW AT FINITE VELOCITY AND FINITE TEMPERATURE

Here we give the details of the renormalization procedure used for the moving system. We use the MSR formalism with action $S[u,\hat{u}]$ given by Eq. (4.2). Having shifted the field u_{rt} so that its average vanishes $\overline{\langle u_{rt}\rangle}=0$, we can do perturbation theory with the Gaussian part

$$S_0[u,\hat{u}] = \int_{rt}[i\hat{u}_{rt}(\eta\partial_t - c\nabla^2)u_{rt} - \eta T i\hat{u}_{rt}i\hat{u}_{rt}] \tag{C1}$$

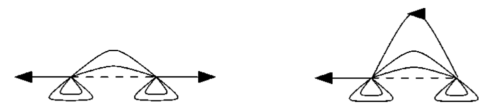

FIG. 12. First-order RG corrections. The internal lines carry fast fields.

of the action. The Gaussian correlation C_{rt} and response R_{rt} functions were defined in Appendix B.

The interaction part of the action contains the disorder correlator and also the pinning force $\tilde{f} = \mathcal{O}(\Delta)$:

$$S_i[u,\hat{u}] = -\tilde{f}\int_{rt} i\hat{u}_{rt} - \frac{1}{2}\int_{rtt'} i\hat{u}_{rt}i\hat{u}_{rt}\Delta[u_{rt}-u_{rt'} + v(t-t')]. \tag{C2}$$

The effective action for slow fields $u,\hat{u}$ is given by the following cumulant expansion where the averages are computed within the Gaussian part S_0 over the fast fields $u^{>},\hat{u}^{>}$

$$S_{<}[u,\hat{u}] = S_0[u,\hat{u}] + \langle S_i[u+u^{>},\hat{u}+\hat{u}^{>}]\rangle - \frac{1}{2}\langle S_i[u+u^{>},\hat{u}+\hat{u}^{>}]^2\rangle_c + \mathcal{O}(S_i^3). \tag{C3}$$

We now turn to the computation of the first- and second-order terms.

1. First order

To first order, the corrections arise from the graph shown in Fig. 12. They read

$$\langle S_i[u+u^{>},\hat{u}+\hat{u}^{>}]\rangle = -\tilde{f}\int_{rt} i\hat{u}_{rt} - \int_{rtt'\kappa}(i\kappa)\Delta_\kappa[u](r,t,t')R^{>}_{0t-t'}i\hat{u}_{rt} - \frac{1}{2}\int_{rtt'\kappa}\Delta_\kappa[u](r,t,t')i\hat{u}_{rt}i\hat{u}_{rt'} \tag{C4}$$

with the shorthand notation

$$\Delta_\kappa[u](r,t,t') \equiv \Delta_\kappa e^{i\kappa[u_{rt}-u_{rt'}+v(t-t')]}e^{(i\kappa)^2(C^{>}_{00}-C^{>}_{0t-t'})}.$$

The term (C4) appears to be the sum of a $i\hat{u}\mathcal{F}[u]$ term and a $i\hat{u}i\hat{u}\mathcal{G}[u]$ term. Let us begin to deal with the first type. A short time expansion of $e^{i\kappa(u_{rt}-u_{rt'})}$ yields the following operators:

$$-\left(\int_{rt} i\hat{u}_{rt}\right)\int_{\kappa t} i\kappa\Delta_\kappa e^{i\kappa v t}e^{(i\kappa)^2(C^{>}_{00}-C^{>}_{0t})}R^{>}_{0t} \tag{C5}$$

which is a correction to $\tilde{f}$ and

$$-\left(\int_{rt} i\hat{u}_{rt}\partial_t u_{rt}\right)\int_{\kappa t}(i\kappa)^2\Delta_\kappa e^{(i\kappa)^2(C^{>}_{00}-C^{>}_{0t})}tR^{>}_{0t} \tag{C6}$$

which is a correction to η. The elasticity operator $i\hat{u}\nabla^2 u$ is *not* corrected and *no* higher gradients like $i\hat{u}\nabla^n u$ are generated in the equation of motion. Note also that to this order, no KPZ term $i\hat{u}(\nabla u)^2$ is generated.[46]

The $i\hat{u}i\hat{u}\mathcal{G}[u]$ term can be rewritten as the sum of

$$-\frac{1}{2}\int_{rtt'} i\hat{u}_{rt}i\hat{u}_{rt'}\int_{\kappa}\Delta_{\kappa}e^{(i\kappa)^2C^{>}_{00}}e^{i\kappa[u_{rt}-u_{rt'}+v(t-t')]}$$

which has the form of a disorder correlator and yields a correction to $\Delta(u)$, and an operator quasilocal in time

$$\int_{rtt'} i\hat{u}_{rt}i\hat{u}_{rt'} \times\int_{\kappa}\Delta_{\kappa}e^{(i\kappa)^2C^{>}_{00}+i\kappa[u_{rt}-u_{rt'}+v(t-t')]}\left(\frac{1-e^{\kappa^2C^{>}_{0t-t'}}}{2}\right) \tag{C7}$$

which yields a correction to the $\int_{rt}i\hat{u}_{rt}i\hat{u}_{rt}$ term. The projection of Eq. (C7) on this thermal noise operator is

$$\left(\int_{rt}i\hat{u}_{rt}i\hat{u}_{rt}\right)\int_{\kappa t}\Delta_{\kappa}e^{(i\kappa)^2C^{>}_{00}}e^{i\kappa vt}\left(\frac{1-e^{\kappa^2C^{>}_{0t}}}{2}\right). \tag{C8}$$

To obtain the correction to the temperature T, one uses $\delta T/T=\delta\eta T/\eta T-\delta\eta/\eta$. An integration by parts of Eq. (C6), thanks to FDT for the ''pure'' R and C, yields $\delta T/T$.

To summarize,

$$\delta c=0,$$

$$\delta\tilde{f}=\int_{\kappa t}i\kappa\Delta_{\kappa}e^{(i\kappa)^2(C^{>}_{00}-C^{>}_{0t})}e^{i\kappa vt}R^{>}_{0t},$$

$$\delta\eta=-\int_{\kappa t>0}(i\kappa)^2\Delta_{\kappa}e^{(i\kappa)^2(C^{>}_{00}-C^{>}_{0t})}e^{i\kappa vt}tR^{>}_{0t},$$

$$\delta\Delta(u)=\int_{\kappa}\Delta_{\kappa}e^{(i\kappa)^2C^{>}_{00}}e^{i\kappa u},$$

$$\eta\delta T=\int_{\kappa t>0}i\kappa vt\Delta_{\kappa}e^{(i\kappa)^2C^{>}_{00}}e^{i\kappa vt}(1-e^{\kappa^2C^{>}_{0t}}).$$

The correction $\delta\tilde{f}$ has the same form as the perturbative expression for $\tilde{f}$, with opposite sign and shell-restricted functions C,R. Note that $\delta\eta=-(d/dv)\delta\tilde{f}$.

In the infinitesimal shell limit, the shell-restricted functions $C^{>},R^{>}$ which are evaluated at $r=0$, are of order dl. The differential flow is thus given by Eq. (4.11).

2. Second order

The fast-modes average $\langle S_i^2\rangle_c$ can be decomposed into one term with $\tilde{f}$ in factor plus the rest which does not contain $\tilde{f}$. The former vanishes for the following reason: the contraction of the $\tilde{f}\int_{rt}i\hat{u}_{rt}$ with the $u_{r't_1}$ or $u_{r't_2}$ contained in the vertex operator involves a fast response $R^{>}_{r_i-r,t}$. But $\int_r R^{>}_{rt}=0$, since its modes live in the shell. The latter is the connected average of two disorder vertices. We now extract from it a correction to the disorder, i.e., a term which has the form $-\frac{1}{2}\int_{rtt'}i\hat{u}_{rt}i\hat{u}_{rt'}\delta\Delta[u_{rt}-u_{rt'}+v(t-t')]$. The corresponding diagrams are represented in Fig. 13.

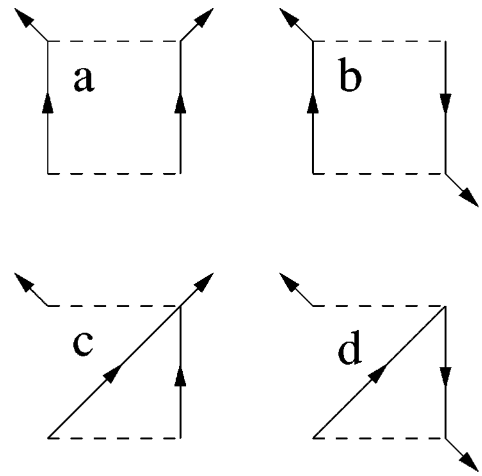

FIG. 13. Diagrams needed to compute the second-order corrections at $T=0$, and at any T in Wilson's scheme. Each of the two external lines (corresponding to a $\hat{u}$ field) is connected to a *tree* of response functions (the lines) due to causality, and provides an orientation to these lines: we drew the arrows just for clarity.

Each diagram has two external $i\hat{u}_{rt}$ $i\hat{u}_{r't'}$ legs, to which corresponds a functional half vertex of u_{rt} and $u_{r't'}$, respectively. Calling τ,τ' the (positive) time arguments of both response functions, denoting $U=u_{rt}-u_{r't'}+v(t-t')$, the diagrams have the following analytical expressions, integrated over $r,r',t,t',\rho,\tau,\tau'$:

$$a=-i\hat{u}_{rt}i\hat{u}_{r't'}\delta_{r'-r'}\Delta''(U)\Delta[U+v(\tau'-\tau)]R^{>}_{\rho\tau}R^{>}_{\rho\tau'},$$

$$b=-i\hat{u}_{rt}i\hat{u}_{r't'}\delta_{r'-r-\rho}\Delta'(U+v\tau')\Delta'(U-v\tau)R^{>}_{\rho\tau}R^{>}_{-\rho\tau'},$$

$$c=i\hat{u}_{rt}i\hat{u}_{r't'}\delta_{r'-r}\Delta''(U)\Delta[v(\tau'-\tau)]R^{>}_{\rho\tau}R^{>}_{\rho\tau'},$$

$$d=i\hat{u}_{rt}i\hat{u}_{r't'}\delta_{r'-r-\rho}\Delta'(U+v\tau')\Delta' \times[-v(\tau'+\tau)]R^{>}_{\rho\tau}R^{>}_{-\rho\tau'}.$$

After another short distance expansion of b and d, noting that $\int_{\rho}R^{>}_{\pm\rho\tau}R^{>}_{\rho\tau'}=\int_q R^{>}_{\mp q\tau}R^{>}_{q\tau'}=S_D\Lambda^D e^{-c\Lambda^2(\tau'+\tau)/\eta}dl/\eta^2$, a proper symmetry counting yields the term of order Δ^2 of Eq. (4.11). The results obtained here are consistent with the analysis of Ref. 57.

APPENDIX D: INTERESTING RESULTS IN THE NONDRIVEN CASE AT FINITE TEMPERATURE

We give here a detailed analysis of the functional renormalization-group flow at $T>0$ and zero velocity. The temperature is an irrelevant operator and flows exponentially fast to zero. We show, however, that the temperature rounds the cusp in a region of size proportional to T around the origin and that in this boundary layer, the disorder correlator takes a *superuniversal* (to lowest order in ϵ) scaling form. In addition, we show how to carry a systematic expansion at low T. As temperature decreases, the correlator of the disorder becomes more and more pinched, and eventually reaches its zero-temperature cuspy fixed point at infinity.

We show that during the renormalization at $v=0$ with a flowing temperature $T_l \to 0$, the cusp forms only asymptotically ($l \to \infty$), and $\Delta(u)$ has the following scaling form in the boundary layer $|u| \sim T_l/\chi$:

$$\Delta_l(u) \simeq \Delta_l(0) - T_l f(u\chi/T_l) \tag{D1}$$

with $f(x)=\sqrt{1+x^2}-1$ and where $\chi = |\Delta^{*\prime}(0^+)|$ measures the cusp.

Furthermore, we show that the following expansion in temperature for the solution of the FRG flow holds:

$$[\Delta(u)-\Delta(0)-T]^2 = \sum_{n \geq 2} T^n f_n(u/T) \tag{D2}$$

thus we obtained a fairly complete picture of the solution.

1. Curvature

The flow equation of the value at zero of the disorder correlator is

$$\partial_l \Delta_l(0) = (\epsilon - 2\zeta)\Delta_l(0) + T_l \Delta_l''(0). \tag{D3}$$

Since $\Delta_l \to \Delta^*$, the convergence of $\Delta_l(0)$ towards $\Delta^*(0)$ implies that $T_l \Delta_l''(0)$ also converges. From the fixed-point equation

$$(\epsilon - 3\zeta)\Delta^*(u) + \zeta[u\Delta^*(u)]' = \frac{1}{2}[\Delta^*(u)-\Delta^*(0)]^{2''}$$

one has simply $(\epsilon-2\zeta)\Delta^*(0) = \Delta^{*\prime}(0^+)^2$, and thus

$$-T_l \Delta_l''(0) \to \Delta^{*\prime}(0^+)^2. \tag{D4}$$

2. Scaling function in the boundary layer

We show here that the assumption that the curvature at zero of Δ_l diverges like a power of the inverse temperature implies that *all* the derivatives at zero also diverge and that there exists a well defined and particularly simple scaling function in the boundary layer around zero. Precisely, for any function T_l decreasing to zero and a function $\Delta_l(u)$ such that

$$\partial_l \Delta_l(u) = (\epsilon-2\zeta)\Delta_l(u) + \zeta u \Delta_l'(u) + T_l \Delta_l''(u) + \Delta_l''(u)[\Delta_l(0)-\Delta_l(u)] - \Delta_l'(u)^2 \tag{D5}$$

if $\Delta_l''(0) \sim -(\chi^2/T_l)^\alpha$ for some $\alpha>0$ and χ, then, defining the functions $f_l(x) = (1/T_l)[\Delta_l(0) - \Delta_l(xT_l^{(\alpha+1)/2}\chi^{-\alpha})]$, we obtain that all derivatives of f_l at $x=0$ converge to the corresponding derivatives of $f(x)=\sqrt{1+x^2}-1$, and that f is the only fixed possible fixed point for f_l.

A simple way to see the convergence to the scaling function f is to write the flow of f_l

$$T_l \Delta_l''(0) + \frac{1}{2}\chi^{2\alpha} T_l^{1-\alpha}(1+f_l)^{2''} = T_l\left(\partial_l f_l - 2 f_l + \frac{\alpha+1}{2} f_l'\right)$$

and eliminate at large l the right-hand side term which is subdominant (higher order in T_l) for $\alpha>0$, since T has been absorbed in the variable x of f_l. We have used that $\theta = 2 - \epsilon + 2\zeta$. Hence the fixed point equation for f_l is

$$\frac{1}{2}(1+f)^{2''} = 1$$

which has the solution $f(x)$ above since we know that $f(0) = 0$, $f''(0)=1$ and $f^{(4)}(0) = -3$ is easily checked.

This is confirmed by the study of the flow equations for the successive derivatives $a_n = \Delta^{(2n)}(0)$:

$$\partial a_n = [\epsilon + 2(n-1)\zeta]a_n + T a_{n+1} - \frac{1}{2}\sum_{k=1}^{n}\binom{2(n+1)}{2k} a_k a_{n+1-k}. \tag{D6}$$

From a trivial recurrence, the hypothesis $\Delta_l''(0) \sim -(\chi^2/T_l)^\alpha$ implies that $T^{n(\alpha+1)-1}a_n$ converges for any n. Moreover the limit $c_n = \lim_{l\to\infty} T^{n(\alpha+1)-1}\chi^{-2n\alpha}a_n$ can be obtained from Eq. (D6) and is $c_n = [1.3\cdots(2n-1)]^2/(2n-1) = f^{(2n)}(0)$.

To fix the value of α [$\alpha=1$ as strongly suggested by Eq. (D4)], we checked that the only values of $\beta>0, \gamma>0$ such that $g_l(x) = (1/T_l^\gamma)[\Delta^*(T_l^\beta x) - \Delta_l(T_l^\beta x)]$ has a meaningful fixed point are $(\beta,\gamma)=(1,1)$. For these values, the fixed point is $g(x) = \Delta^{*\prime}(0^+)x + \sqrt{1+[\Delta^{*\prime}(0^+)x]^2}$.

3. Next order in *T*

The procedure which gives us the leading behavior in the boundary layer controlled by temperature can be extended analytically with arbitrary accuracy in an expansion to any order in T. We study

$$\partial_l \Delta_l(u) = (\epsilon-2\zeta)\Delta_l(u) + \zeta u \Delta_l'(u) + T_l\Delta_l''(u) - \Delta_l''(u)[\Delta_l(u)\Delta_l(0)] - \Delta_l'(u)^2,$$

$$\partial_l \ln T_l = -\theta$$

with $\theta = 2-\epsilon+2\zeta$. For numerical purposes or for the following analytical computation, it is useful to switch to the function $y(u) = [\Delta(u)-\Delta(0)-T]^2$ which remains quadratic at the origin when $T\to 0$, since $y(u) = T^2 + |T\Delta''(0)|u^2 + \mathcal{O}(u^4)$ for $T>0$ and $y(u) = \Delta'(0^+)^2 u^2 + \mathcal{O}(u^4)$ for $T=0$. This function flows as

$$\partial_l y = 2(\epsilon-2\zeta)y + \zeta u y' + \sqrt{y}[y'' - y''(0) - 4T].$$

We can replace the scale l dependence of $y_l(u)$ by a T dependence since T and l are linked by $T_l = T_0 e^{-\theta l}$. The function $y_T(u)$ can be expanded in

$$y_T(u) = \sum_{n\geq 2} T^n f_n\left(\frac{u}{T}\right).$$

The expansion begins at $n=2$ since $y_T(0) \equiv T^2$ and we have

$$f_2(0)=1, \quad f_{n>2}(0)=0.$$

The equation for the f_n's reads

$$\sum_{n\geq 2} T^n\{[2(\epsilon-2\zeta)+n\theta]f_n + (\zeta-\theta)x f_n'\} = \sqrt{\sum_{n\geq 2} T^n f_n}\left(4T - \sum_{n\geq 2} T^{n-2}[f_n'' - f_n''(0)]\right).$$

One can solve this equation order by order in T. It is useful to divide Δ and T by χ^2 and u by χ. With these rescaled quantities, we have simply

$$y''_{T_l}(0) = -2T_l\Delta''_l(0) \to 2$$

and thus $f''_2(0)=2$. If we knew the full behavior of $y''_T(0)$, i.e., the $f''_n(0)$'s, we could completely solve the system. Here, we get

$$f_2(x) = 1+x^2,$$

$$f_3(x) = 4\left(1-\frac{\epsilon-\zeta}{3}\right)\left(\sqrt{1+x^2}-1-\frac{x^2}{2}\right)-[4-(\epsilon-\zeta)]$$

$$\times x(\mathrm{asinh}x - x) - \frac{\epsilon-\zeta}{3}x^2\sqrt{1+x^2}+f''_3(0)\frac{x^2}{2},$$

where we wrote $f_3(x)$ such that the three first lines are functions which vanish and have zero curvature at zero. Note that while $f_2(x)$ is universal, the last term $f_3(x)$ contains un unknown integration constant $f''_3(0)$ which presumably depends on the initial condition of the flow and is thus not universal. Indeed we observed a nonuniversal $f''_3(0)$ in a numerical integration of the flow of $y_T(u)$.

The procedure can be carried to any order in T and all the f_n's are accessible. The unknown coefficients of the expansion

$$-2T_l\Delta''_l(0) = 2+\sum_{n>0} T^n f''_{n+2}(0)$$

are similarly nonuniversal.

Both Secs. D 1 and D 2 thus provide a rather convincing and consistent picture for the solution of the $T>0$, $v=0$ FRG equations (awaiting a mathematical proof).

APPENDIX E: ANALYTICAL SOLUTIONS AT FIXED TEMPERATURE

We present here the analytical solutions of the fixed-point equations for RF and RP at fixed T. Thanks to the exact expression of these fixed points, we are able to check the scaling form derived in Appendix D within an ''adiabatic'' hypothesis where the running correlator at l is identified with the fixed point at $T=T_l$. Our families of fixed temperature fixed points (FTFP) give back the known fixed points at $T=0$ in both the RF (Ref. 8) and the RP (Ref. 11) cases. However, even if we obtain the same *form* (E6) for the RF $T=0$ fixed point as in Ref. 8, we disagree with the scaling in ϵ.

1. Random field

We look for a fixed point of

$$\partial_l\Delta_l(u) = (\epsilon-3\zeta)\Delta_l(u)+\zeta[u\Delta_l(u)]' \quad \text{(E1)}$$

$$-\frac{1}{2}[\Delta_l(u)-\Delta_l(0)-T]^{2\prime\prime} \quad \text{(E2)}$$

with *fixed* T and initial random-field condition $\int\Delta_0>0$. Since $\partial_l \ln\int du\,\Delta_l(u)=\epsilon-3\zeta$, a meaningful fixed point can be obtained only for $\zeta=\epsilon/3$. Fixing the RF strength $\int\Delta_0$ to one, we are led to the following problem: for any $T\geqslant 0$, find the fixed temperature fixed-point function (FTFP) $\Delta(T,u)$ such that

$$\frac{\epsilon}{3}[u\Delta(u)]' = \frac{1}{2}[\Delta(u)-\Delta(0)-T]^{2\prime\prime}, \quad \text{(E3)}$$

$$\int du\,\Delta(u) = 1. \quad \text{(E4)}$$

Integrating Eq. (E3) from 0 to ∞ yields $T\Delta'(0^+)=0$, hence the FTFP has a cusp for $T=0$ and no cusp for $T\neq 0$.

At $T=0$, integrating Eq. (E3) from 0 to u and dividing by $\Delta(u)$ yields $(\epsilon/3)u=\Delta'(u)-\Delta(0)\Delta'(u)/\Delta(u)$. Then, integrating again from 0 to u yields the $T=0$ FTFP, by imposing Eq. (E4)

$$\Delta(T=0,u) = \left(\frac{\epsilon}{3\left(\int y\right)^2}\right)^{1/3} y\left[\left(\frac{\epsilon}{3}\int y\right)^{1/3}u\right], \quad \text{(E5)}$$

where the function $y(x)$ is implicitly defined by[8]

$$\frac{x^2}{2} = y(x)-1-\ln y(x). \quad \text{(E6)}$$

Since $y(0)=1$ one has

$$\Delta(0,0) = \left(\frac{\epsilon}{3\left(2\sqrt{2}\int_0^1\sqrt{y-1-\ln y}\right)^2}\right)^{1/3}. \quad \text{(E7)}$$

It is easy to compute the number $\int y = \int_{-\infty}^{\infty}dx\,y(x) = 2\int_0^1 dy\,x(y) = 2\sqrt{2}\int_0^1 dy\,\sqrt{y-1-\ln y} \simeq 1.55$. Note the behavior near 0 given by $y(x)=1-|x|+x^2/3-|x^3|/36+\mathcal{O}(x^4)$ thus $\partial_u^2\Delta(0,0^+)=2\epsilon/9$. Note also the Gaussian decrease of correlations at infinity $y(x)\sim e^{-1-x^2/2}$.

An intriguing fact is the scaling of the $T=0$ fixed point with ϵ: its nth derivative at 0^+ scales like

$$\partial_u^n\Delta(T=0,u=0^+) \sim \epsilon^{(1+n)/3}. \quad \text{(E8)}$$

At $T>0$, there is no cusp $[\partial_u\Delta(T>0,0^+)=0]$ and the same double integration of Eq. (E3) yields

$$\Delta(T,u) = \Delta(T,0)\,y(T,u\sqrt{\epsilon/[3\Delta(0)]}) \quad \text{(E9)}$$

with $y(T,x)$ implicitly defined by

$$\frac{x^2}{2} = y-1-\left(1+\frac{T}{\Delta(0)}\right)\ln y. \quad \text{(E10)}$$

The value of $\Delta(T,0)$ is determined by condition (E4). Using $\int dx\,y(x)=\int dy\,x(y)$, this condition reads

$$\sqrt{\frac{24\Delta(T,0)^3}{\epsilon}}\int_0^1 dy\,\sqrt{y-1-\left(1+\frac{T}{\Delta(T,0)}\right)\ln y} = 1. \quad \text{(E11)}$$

This equation admits a unique solution $\Delta(T,0)>0$ for any $T>0$. Then there exists a unique FTFP $\Delta(T,u)$ for each $T>0$. Some of them are displayed in Fig. 14. Note that T

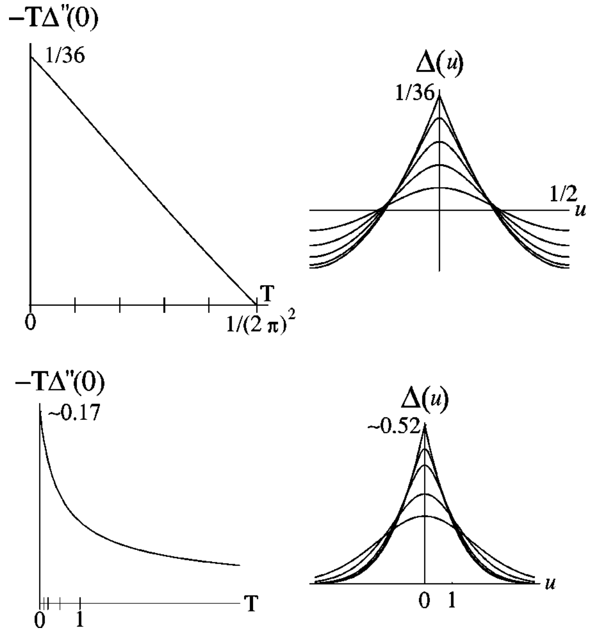

FIG. 14. Fixed points at fixed temperature $\Delta(T,u)$. Bottom: RF case, the solution $\Delta(T,u)$ to the reduced Eqs. (E14) and (E15) exists for any T. Right bottom: plot of the RF FTFP's for $T \in \{0,0.1,0.2,0.5,1\}$, these temperatures are located on the abscissas of the left bottom plot. On the left, plot of $-T\partial_u^2\Delta(T,u=0)$ versus T. Top: RP case, the one-periodic nontrivial solution $\Delta(T,u)$ to Eq. (E16) exists for $0 \leqslant T < (2\pi)^{-2}$. Right bottom: plot of the RP FTFP's for $T \in \{0,0.005,0.01,0.015,0.02\}$; these temperatures are located on the abscissas of the left top plot. The FTFP's are analytic for $T>0$ but tend to the cuspy $\Delta(T=0,u)$ as $T\to 0$. The curve on the left shows $-T\partial_u^2\Delta(T,u=0)$ as a function of T. It is *not* a straight line.

$=0$ in Eqs. (E10) and (E11) gives back the $T=0$ nonanalytic fixed point $\Delta(T=0,u)$ Eq. (E5). Hence the set of FTFP has a nice $T\to 0$ limit, even if there is a qualitative difference between the cuspy $T=0$ FTFP and the analytic $T>0$ FTFP's.

As is obvious from their analytical expression, or from Fig. 14, $\lim_{T\to 0}\Delta(T,u)=\Delta(0,u)$: the $T=0$ nonanalytic fixed point is approached smoothly by the set of analytic fixed T fixed points. When T approaches zero, the curvature of the FTFP's at the origin goes to $-\infty$ like

$$\lim_{T\to 0} -T\Delta''(T,0)=\frac{\epsilon}{3}\Delta(0,0) \tag{E12}$$

with $\Delta(0,0)$ given by Eq. (E7).

We also checked that the $\Delta(T,u)$ converge when $T\to 0$ to the zero-temperature fixed point with the predicted scaling form (D1)

$$\frac{\Delta(T,0)-\Delta\left(T,T\dfrac{x}{\chi}\right)}{T} \xrightarrow{T\to 0} \sqrt{1+x^2}-1, \tag{E13}$$

where $\chi=|\partial_u\Delta(0,0^+)|$ is given by the $T=0$ FTFP equation $\chi^2=(\epsilon/3)\Delta(0,0)$.

Some of the RF fixed T fixed points are shown on the right-bottom quarter of Fig. 14, including the cuspy (highest) $T=0$ fixed point. Absorbing ϵ in T and Δ, we chose to plot the nontrivial solution to the most reduced problem,

$$\frac{1}{3}[u\Delta(u)]'=\frac{1}{2}[\Delta(u)-\Delta(0)-T]^{2''}, \tag{E14}$$

$$\int du\,\Delta(u)=1. \tag{E15}$$

To restore ϵ and $\int\Delta$, one simply has to note that the ''dimensions'' are $T\equiv\Delta\equiv\epsilon^{1/3}(\int\Delta)^{2/3}$ and $u\equiv\epsilon^{-1/3}(\int\Delta)^{1/3}$. The left-bottom of Fig. 14 shows $-T\Delta''(T,0)$ as a function of T. This combination has a finite limit ($\simeq 0.17$) when $T\to 0$.

2. Random periodic

In the random periodic case, the conservation of the period a of Δ requires $\zeta=0$. After a suitable rescaling, $u\to u/a$, $\Delta\to\Delta/(\epsilon a^2)$ and $T\to T/(\epsilon a^2)$, the fixed-point equation reads for the one-periodic function $\Delta(u)$

$$\Delta(u)=\frac{1}{2}[\Delta(u)-\Delta(0)-T]^{2''} \tag{E16}$$

and is easily solved by quadrature, by analogy with a particle's position $X(u)=[\Delta(u)-\Delta(0)-T]^2$ at time u in a potential $V(X)=4X^{3/2}/3-2[\Delta(0)+T]X$ verifying $X''(u)=-V'[X(u)]$. The quadrature leads to the reciprocal function $u(X)$, parametrized by $\Delta(0)$ and T, as a sum of two elliptic functions. Then, imposing the solution $\Delta(u)$ be one-periodic fixes $\Delta(0)$ as a function of T.

The result is

(i) for $T\geqslant(2\pi)^{-2}$, the only solution is $\Delta(u)\equiv 0$.

(ii) For $0<T<(2\pi)^{-2}$, another solution arises, which resembles a cosinus function of linearly vanishing amplitude when $T\to(2\pi)^{-2}$. This nontrivial solution has no cusp but becomes pinched as T decreases [growing curvature $|\Delta''(0)|$ and higher harmonics]. As can be seen on the analytical expression (not given here), $-T\Delta''(0)\xrightarrow{T\to 0}1/36$. In particular, it remains finite when the temperature vanishes.

(iii) Eventually for $T\to 0$, the nontrivial solution uniformly tends to the zero-temperature fixed point

$$\Delta(u)=\frac{1}{6}\left(\frac{1}{6}-u(1-u)\right). \tag{E17}$$

The temperature $(2\pi)^{-2}$ in our units is *exactly* the critical temperature T_g of the random-field XY model[58] and the fixed points near T_g^- reproduce the line of fixed points of this problem (since we worked to second order, it is only an approximation). Indeed in $D=2$, the naive dimension of the temperature is zero and our FTFP has a direct physical meaning. Note that another random gradient term becomes relevant in $D=2$ but does not feed back on the flow of $\Delta(u)$.[58,59]

We can now use the exact FTFP's to check that an adiabatic hypothesis is consistent with the scaling form (D1). Indeed, one can numerically check that the correlator with a flowing temperature has the FTFP's have the scaling (D1) as $T\to 0$. To conclude about the problem with a flowing temperature $T_l \to 0$, it appears from these observations that no cusp occurs at finite scale for $T_0 > 0$. The cusp forms only asymptotically ($l \to \infty$), with

$$\lim_{l\to\infty} - T_l \Delta_l''(0) = \Delta(0,0) \tag{E18}$$

given by Eq. (E7) and it obeys a scaling form in the boundary layer $|u| < T_l/\chi$,

$$\Delta_l(u) \simeq \Delta_l(0) - T_l f(u\chi/T_l), \quad f(x) = \sqrt{1+x^2} - 1. \tag{E19}$$

We note that the precise form of the flow of the temperature (i.e., the value of θ) only affects subdominant behavior [i.e., the function $f_3(x)$ in Sec. (D 3)].

APPENDIX F: MULTIDIMENSIONAL CASE

We give here a possible extension of the FRG to a multidimensional displacement field. This study generalizes the approach of Ref. 53 by including the effect of $v > 0$ and $T > 0$ in the flow. For periodic structures, a similar study of the multidimensional displacement field was shown in Ref. 57 to yield interesting effects.

In a $D+N$-dimensional space, we distinguish between the *internal* or *longitudinal* space of dimension D, to which r belongs, and the *transverse* space of dimension N, to which u belongs. The elastic energy of an interface without overhangs defined by a height function u_r is quadratic in ∇u of the form $\frac{1}{2}\int_r c_{ij}^{\mu\nu}(\partial_\mu u_r^i)(\partial_\nu u_r^j)$.

The disorder: the random bond (RB) case corresponds to a random potential $V(r,u)$, with correlations $\overline{[V(r,u)-V(r',u')]^2} = -2\delta^D(r-r')\mathsf{R}(u-u')$. Function $\mathsf{R}(u)$ is even, vanishes at $u=0$ and goes to a negative constant for $|u| \gg r_f$. The random-field (RF) case corresponds to a force $F^i(r,u)$ with correlations $\overline{F^i(r,u)F^j(r',u')} = \delta^D(r-r')\Delta^{ij}(u-u')$, where the $\Delta^{ij}(u)$ vanish for $|u| \gg r_f$. A RB gives rise to a RF via $F^i = -\partial^i V$ and the correlators are related by $\Delta^{ij}(u) = -\partial^{ij}\mathsf{R}(u)$. Note th at this type of correlator deriving from a RB has $\int d^N u\, \Delta^{ij}(u) = 0$. Finally, the random periodic case (RP) occurs when u is defined up to a discrete set of translations forming a lattice of points P, e.g., when u is a phase, defined up to 2π shifts. In this case, the disorder is periodic and one has $\Delta^{ij}(u) = \Delta^{ij}(u+P)$ for any P of the lattice [or $\mathsf{R}(u) = \mathsf{R}(u+P)$].

The overdamped dynamics is given by

$$\eta_j^i \partial_t u_{rt}^j = c_j^{i\mu\nu}\partial_{\mu\nu} u_{rt}^j + F^i(r,u_{rt}) + \zeta_{rt}^i + f^i + h_{rt}^i,$$

where η is the friction tensor and ζ a Langevin noise, with correlations $\langle \zeta_{rt}^i \zeta_{r't'}^j \rangle = 2(\eta T)^{ij}\delta(r-r')\delta(t-t')$. The tensor T stands for the temperature(s) of this out of equilibrium system. We added a driving force f^i perpendicular to the interface and a source field h_{rt}^i, as an external excitation.

Without assuming any symmetry, let C_{rt}^{ij} and R_{rt}^{ij} be the Gaussian correlation and response functions. We obtain by the same procedure as for the $N=1$ case the following first-order corrections due to disorder

$$\delta c_{\mu\nu}^{ij} = 0,$$

$$\delta \tilde{f}^i = \int_{\kappa t} e^{i\kappa\cdot(C_{00}-C_{0t})\cdot i\kappa + i\kappa\cdot vt}\Delta_\kappa^{ik} i\kappa^l R_t^{lk},$$

$$\delta\eta^{ij} = -\int_{\kappa t} e^{i\kappa\cdot(C_{00}-C_{0t})\cdot i\kappa + i\kappa\cdot vt}\Delta_\kappa^{ik} i\kappa^l t R_t^{lk} i\kappa^j,$$

$$\delta(\eta T)^{ij} = \frac{1}{2}\int_{\kappa t} e^{i\kappa\cdot vt}(e^{i\kappa\cdot(C_{00}-C_{0t})\cdot i\kappa} - e^{i\kappa\cdot C_{00}\cdot i\kappa})\Delta_\kappa^{ij},$$

$$\delta\Delta^{ij}(u) = \int_\kappa \Delta_\kappa^{ij} e^{i\kappa\cdot C_{00}\cdot i\kappa + i\kappa\cdot u}.$$

Using $(\Delta_\kappa^{\alpha\beta})^* = \Delta_{-\kappa}^{\alpha\beta}$, $\Delta_{-\kappa}^{\alpha\beta} = \Delta_\kappa^{\beta\alpha}$, we write the on-shell corrections as (with the matrix product $A\cdot B = A_{\alpha\gamma}B_{\gamma\beta}$)

$$\delta c_{\mu\nu}^{ij} = 0,$$

$$\delta\tilde{f} = -\int_\kappa i\kappa \cdot \int_t R_{0t}^{>}\cdot \Delta_{-\kappa} e^{i\kappa\cdot vt},$$

$$\delta\eta = -\int_\kappa i\kappa\cdot \int_t t R_{0t}^{>}\cdot\Delta_{-\kappa} e^{i\kappa\cdot vt} i\kappa,$$

$$\delta(2\eta\cdot T) = -\int_\kappa i\kappa\cdot\int_t C_{0t}^{>}\cdot i\kappa \Delta_\kappa e^{i\kappa\cdot vt},$$

$$\delta\Delta_\kappa = i\kappa\cdot C_{00}^{>}\cdot i\kappa\Delta_\kappa.$$

The second-order correction to Δ reads

$$\begin{aligned}\delta\Delta^{ij}(u) = \int_{q\tau\tau'} & R_{q\tau}^{>mk} R_{q\tau'}^{>m'l}(\{\Delta^{kl}[v(\tau'-\tau)] - \Delta^{kl}[u+v(\tau' \\ & -\tau)]\}\partial^m\partial^{m'}\Delta^{ij}(u) - \partial^m\Delta^{il}(u+v\tau)\partial^{m'}\Delta^{kj}(u \\ & -v\tau') - \partial^m\Delta^{il}(u+v\tau)\partial^{m'}\Delta^{jk}[v(\tau'+\tau)] \\ & + \partial^m\Delta^{il}[v(\tau'+\tau)]\partial^{m'}\Delta^{kj}(u-v\tau')).\end{aligned}$$

Note that each of the first three terms are symmetric under $i\leftrightarrow j, u\leftrightarrow -u$ and that the fourth is exchanged with the fifth under this symmetry. Then Δ remains a correlator.

Of course this second-order correction to Δ gives back the expression already computed for a $D+1$ interface if $N=1$. At zero velocity, one gets the second derivative of the flow equation of Balents and Fisher.[9] If we assume that $\Delta(u)$ depends only on the component of u parallel to the velocity and send v to zero, then our expression reduces to the equations of Ertas and Kardar.[53]

To simplify the analysis, let us rely on the assumed symmetries of the system. If we suppose that the initial problem is rotationally invariant, i.e., has $O(N)$ symmetry, then the elasticity tensor c, the friction tensor η, and the temperature tensor T are only scalars and the force-force correlator Δ is covariant, i.e., for any $\mathcal{R}$ such that $\mathcal{R}^\dagger\cdot\mathcal{R} = 1$, $\mathcal{R}^\dagger\cdot\Delta(u)\cdot\mathcal{R} = \Delta(\mathcal{R}\cdot u)$.

During the flow, we expect from physical grounds that the running terms of the action will conserve their symmetries but the velocity v which is fixed once for all selects a particular direction in transverse space. The interesting symmetries are given by the little group of the velocity, i.e., the transformations $\mathcal{R}$ such that $\mathcal{R}^{\dagger}\cdot\mathcal{R}=1$ and $\mathcal{R}\cdot v=v$. Then one may decompose the tensors on a basis involving v [one has only two frictions, temperatures, response, and correlation functions and five[60] Δ_i's, functions of $(u^2,v^2,u.v)$].

Unfortunately, the full problem cannot be easily decoupled, even with the simplifications pointed out above. No closed equation, e.g., for the correlator restricted to displacements aligned with the velocity $\Delta(u\|v)$, has been found, and the problem even at zero temperature seems involved. The simplification used in Ref. 53 consists in assuming that Δ does not depend on the transverse coordinates. This assumption reduces the problem to the $N=1$ case, and it would be interesting to solve at finite T the behavior of transverse coordinates along the lines of our analysis.

APPENDIX G: THE FLOW OF THE DISORDER CORRELATOR AT SMALL VELOCITY

The effect of a small velocity on the FRG flow is mainly restricted to the boundary layer of width ρ_l about the origin. Analytically, it is rather difficult to give an estimate of ρ_l or to decide how $\tilde{\Delta}_l(u)$ precisely behaves in the boundary layer $|u|\sim\rho_l$. It is, however, possible to simplify the formidable second-order correction to the disorder correlator, displayed in Eq. (4.13), and to obtain analytically several results, giving some hints about this behavior.

The $\mathcal{O}(\tilde{\Delta}^2)$ term in Eq. (4.11) is written under a form involving two integrations over s,s', reflecting the presence of two response functions integrated over time. After some integrations by part, the $\mathcal{O}(\tilde{\Delta}^2)$ term becomes[61]

$$\tilde{\Delta}''(u)\int_{s>0}e^{-s}\left(\tilde{\Delta}(\lambda s)-\frac{\tilde{\Delta}(u+\lambda s)+\tilde{\Delta}(u-\lambda s)}{2}\right)+\int_{s>0}e^{-s}\frac{\tilde{\Delta}(u+\lambda s)-\tilde{\Delta}(u)}{\lambda}\int_{s>0}e^{-s}\frac{\tilde{\Delta}(u-\lambda s)-\tilde{\Delta}(u)}{\lambda}-\int_{s>0}e^{-s}\tilde{\Delta}'(\lambda s)\frac{\tilde{\Delta}(u+\lambda s)+\tilde{\Delta}(u-\lambda s)-2\tilde{\Delta}(u)}{\lambda}. \quad \text{(G1)}$$

Integrated over u, this correction becomes

$$\int_0^{\infty}du\,\tilde{\Delta}'(u)\int_{s>0}e^{-s}(2-s)\frac{\tilde{\Delta}(u+\lambda s)-\tilde{\Delta}(u-\lambda s)}{\lambda}.$$

For any noncrazy function Δ, this expression is positive. Assuming that Δ has no cusp, it can be safely expanded and we can check that it is of order λ^2:

$$\partial\int\tilde{\Delta}=(\epsilon-3\zeta)\int\tilde{\Delta}+2\lambda^2\int\tilde{\Delta}''^2+\mathcal{O}(\lambda^4).$$

Thus at $v\neq0$, the integral of $\tilde{\Delta}$ *grows* during the flow, whereas it was *conserved* in the statics.

Using Eq. (G1), one can also compare the flow of $\tilde{\Delta}_l$ at small velocity to the cuspy $v=0$ flow. In particular, one observes that the effect of the velocity is to reduce the blowup of the curvature $\tilde{\Delta}''(0)$:

$$\partial\tilde{\Delta}''(0)=\epsilon\tilde{\Delta}''(0)-\tilde{\Delta}''(0)\int_{s>0}e^{-s}\frac{\tilde{\Delta}(\lambda s)-\tilde{\Delta}(0)}{\lambda^2}-\int_{s>0}e^{-s}\left(\frac{\tilde{\Delta}'(\lambda s)}{\lambda}\right)^2$$
$$=\epsilon\tilde{\Delta}''(0)-3\tilde{\Delta}''(0)^2-9\lambda^2\tilde{\Delta}''(0)\tilde{\Delta}^{\text{iv}}(0)+\mathcal{O}(\lambda^4)$$

[note that $\tilde{\Delta}''(0)<0$ whereas $\tilde{\Delta}^{\text{iv}}(0)>0$]. The flow of the friction is similarly slowed down:

$$\partial\ln\eta=\int_{s>0}e^{-s}(2-s)\frac{\tilde{\Delta}(\lambda s)-\tilde{\Delta}(0)}{\lambda^2}$$
$$=-\tilde{\Delta}''(0)-3\lambda^2\tilde{\Delta}^{\text{iv}}(0)+\mathcal{O}(\lambda^4).$$

The term $-\tilde{\Delta}'(0^+)^2$ in the $v=0$ flow of $\tilde{\Delta}(0)$ in Eq. (4.14) at $v=0$ is replaced at $v>0$ by

$$\partial\tilde{\Delta}(0)=(\epsilon-2\zeta)\tilde{\Delta}(0)+\left(\int_{s>0}e^{-s}\frac{\tilde{\Delta}(\lambda s)}{\lambda}\right)^2-\int_{s>0}e^{-s}\left(\frac{\tilde{\Delta}(\lambda s)}{\lambda}\right)^2 \quad \text{(G2)}$$

which has the right sign (using Cauchy inequality) to slow down the exponential growth of $\tilde{\Delta}_l(0)$.

Obtaining a numerical integration of the flow is a highly nontrivial quest,[61] since all the interesting properties occur close to the origin yielding unaccurate results in real space. In Fourier space, the number of harmonics to be retained is huge if one wants to focus on the quasicuspy behavior ($\Delta^*_\kappa\sim\kappa^{-2}$ at the cuspy fixed point). However, we obtained, at least at the beginning of the flow (up to l_c) with small initial velocity, the curve shown in Fig. 6. The initial condition was a RB disorder (full line). It is obvious on the snapshot (dotted line) close to l_c that the flow transformed the RB into a RF.

APPENDIX H: BEFORE THE LARKIN LENGTH

We show here that at the scale l_c, $\tilde{\Delta}_l(u)$ is very close to the static zero-temperature fixed point $\Delta^*(u)$. This can be checked numerically, even in the presence of a small temperature and small velocity. Analytically, one cannot obtain an exact integration of the flow, but we can compare $\tilde{\Delta}_l(u)$ to the known $\Delta^*(u)$ by the following arguments.

Let us take, e.g., the RF case for which $\zeta=\epsilon/3$. For weak disorder one obtains from the integration of Eq. (4.17)

$$e^{\zeta l_c}=\left(1+\frac{\epsilon}{3|\tilde{\Delta}_0''(0)|}\right)^{1/3}\simeq r_f\left(\frac{\epsilon}{\int\tilde{\Delta}}\right)^{1/3}, \quad \text{(H1)}$$

where we have used $|\tilde{\Delta}_0''(0)|\simeq\tilde{\Delta}_0(0)/r_f^2\simeq\int\tilde{\Delta}/r_f^3$. We prove in Appendix E that $\chi=|\Delta^{*\prime}(0^+)|$ verifies $\chi\simeq\epsilon^{2/3}(\int\tilde{\Delta})^{1/3}$ and that $\Delta^*(0)\approx\epsilon^{1/3}(\int\tilde{\Delta})^{2/3}$. Thus the range $r_f^*\approx\Delta^*(0)/|\Delta^{*\prime}(0^+)|$ of Δ^* verifies

$$r_f^* \approx r_f e^{-\zeta l_c} \approx \chi/\epsilon. \tag{H2}$$

To determine the range of $\tilde{\Delta}_{l_c}(u)$ we use the fact that at the beginning of the flow one can neglect the nonlinear term in the flow Eq. (4.11). We are left with $\tilde{\Delta}_l(u) = e^{(\epsilon-2\zeta)l}\tilde{\Delta}_0(ue^{\zeta l})$ and thus the range of $\tilde{\Delta}_{l_c}(u)$ is simply

$$r_f(l_c) \simeq r_f e^{-\zeta l_c}. \tag{H3}$$

A comparison of Eqs. (H2) and (H3) shows that the two ranges are similar. Furthermore, in the RF case $\int\Delta$ is conserved by the flow at $v=0$, and thus the similarity of ranges shows that the shape of $\tilde{\Delta}_{l_c}$ is close to the shape of Δ^* (same integral, same range). Similarly in the RP case it is also true that $\tilde{\Delta}_{l_c}$ resembles Δ^*, but now $\chi = \epsilon a/6$ as can be seen on the fixed point $\Delta^*(u) = (\epsilon/6)[a^2/6 - u(a-u)]$.

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