Wave transport along surfaces with random impedance

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We study transport and diffusion of classical waves in two-dimensional disordered systems and in particular surface waves on a flat surface with randomly fluctuating impedance. We derive from first principles a radiative transport equation for the angularly resolved energy density of the surface waves. This equation accounts for multiple scattering of surface waves as well as for their decay because of leakage into volume waves. We analyze the dependence of the scattering mean free path and of the decay rate on the power spectrum of fluctuations. We also consider the diffusion approximation of the surface radiative transport equation and calculate the angular distribution of the energy transmitted by a strip of random surface impedance.

I. INTRODUCTION

The propagation of wave energy in random media can be analyzed with radiative transport theory in the regime of weak random fluctuations and propagation distances that are long compared to the wavelength. This transition of waves to transport has been explored for more than 30 years.[1–5] An equation for the correlation of wave functions, the Bethe-Salpeter equation,[3] can be obtained using diagram techniques. For propagation distances that are long compared to the wavelength and for weak fluctuations, the Bethe-Salpeter equation simplifies to a radiative transport equation[3] in the ladder approximation. A systematic and efficient way to obtain transport equations directly from random wave equations is presented in Refs. 6 and 7. The mathematics behind the asymptotic limit involved in this process is explored in Refs. 8 and 9. A recent collection of interesting papers on various aspects of waves in random media is given in Ref. 10.

Propagation of surface waves on a randomly inhomogeneous surface is a typical example of propagation in a two-dimensional random medium. Its study is of considerable interest both for the many applications in electronics, acoustics, and solid-state physics, and for the understanding of the effects of dimensionality on general optical and quantum-mechanical disordered systems. Surface excitations on rough surfaces have been analyzed extensively since the early 1980s.[11] Interest in this problem accelerated, and is still significant, after it was realized that the backscattering enhancement from slightly (compared to the wavelength) perturbed surfaces is due to the coherent interference of multiply scattered surface waves.[12–15] As a result, coherent surface effects and in particular the localization of surface waves have been studied in detail.[12,16,13] Backscattering enhancement from the coherence of the double scattering of surface polaritons on a weakly random boundary is considered in Ref. 17. Surprisingly, little is known about the transport regime in two dimensions and the diffusion of surface excitations on random interfaces. Part of the reason for this may be the well-known conclusion of the scaling theory of conductivity, that all electron states in two-dimensional (2D) disordered systems are localized.[18,19] Recently, however, the generality of this result has been questioned because a metal-insulator transition has been observed experimentally in two-dimensional electronic systems (see, for example, Refs. 20 and 21). In the case of classical (electromagnetic or elastic) waves, the localization length for 2D surface excitations can be much longer than the scattering mean free path and, in contrast to one-dimensional propagation, there exists a range of distances where the localization effects are not important and transport of energy takes place.

In this paper we derive from first principles a radiative transport equation for scalar surface waves on a flat surface with randomly varying impedance. We give explicit expressions for the scattering cross section of surface waves as well as the leakage rate to volume waves in terms of the statistics of the impedance fluctuations. We also analyze the surface transport equation in the small mean free path limit and obtain a formula for energy transmission by a strip of random surface impedance.

The radiative transport equation for the angularly resolved surface wave energy density $W(\mathbf{x},\mathbf{k})$ is given by Eq. (24) in Sec. III. In Sec. IV we discuss the relative strength of surface scattering versus leakage to volume waves for different power spectral densities of the impedance fluctuations. In Sec. V we analyze the small mean free path, diffusion limit of the transport equation. The main result is formula (40), which gives the angular distribution of surface wave energy transmitted by a strip of random surface impedance. In Appendix A we explain briefly the formal perturbation analysis that leads to the transport equation (24). In Appendix B we give a brief presentation of the diffusion approximation for the transport equation in a strip. Diffusion approximations are analyzed in various contexts in Refs. 22–27.

II. HIGH-FREQUENCY SURFACE-WAVE PROBLEM

We start from the three-dimensional wave equation

$$\Delta\Psi + k_0^2\Psi = 0, \quad z > 0, \tag{1}$$

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where Ψ is a component of the electric field of a monochromatic (with frequency ω) electromagnetic wave in a spatially homogeneous dielectric medium with c the speed of light in vacuum and $k_0=\omega/c$. Scalar surface waves propagating along the plane $z=0$ are solutions of Eq. (1) subject to the impedance boundary conditions

$$\frac{\partial\Psi}{\partial z}=H(x,y)\Psi,\quad z=0, \tag{2}$$

which decay exponentially in the upper half-space $z>0$. Here H is the surface impedance that we assume to have the form

$$H(\mathbf{x})=-\eta_0(\mathbf{x})-\eta(\mathbf{x}),\quad \mathbf{x}=(x,y), \tag{3}$$

where $\eta_0(\mathbf{x})>0$ is the nonrandom part and $\eta(\mathbf{x})$ models the impedance perturbations that may by caused by fluctuations of the dielectric constant of the background media or by small surface roughness. We allow consideration of a coordinate-dependent impedance $\eta_0(\mathbf{x})$, but this dependence should be slow compared to the wavelength so that the standard high-frequency approximation is valid for surface waves when the fluctuations are zero. We assume that the fluctuations $\eta(\mathbf{x})$ have mean zero, $\langle\eta(\mathbf{x})\rangle=0$, and are spatially homogeneous and isotropic random process with the correlation function $R(\mathbf{x})$,

$$\langle\eta(\mathbf{x})\,\eta(\mathbf{y})\rangle=R(|\mathbf{x}-\mathbf{y}|),$$

and power spectrum $\hat{R}(k)$,

$$\frac{1}{(2\pi)^2}\langle\hat{\eta}(\mathbf{p})\,\hat{\eta}(\mathbf{k})\rangle=\hat{R}(k)\,\delta(\mathbf{p}+\mathbf{k}),$$

$$\hat{R}(k)=\int d\mathbf{y}\,e^{-i\mathbf{k}\cdot\mathbf{y}}R(|\mathbf{y}|),\quad k=|\mathbf{k}|. \tag{4}$$

We use the standard form of the Fourier transform:

$$\hat{f}(\mathbf{k})=\int d\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{x}),\quad f(\mathbf{x})=\int\frac{d\mathbf{k}}{(2\pi)^2}e^{i\mathbf{k}\cdot\mathbf{x}}\hat{f}(k).$$

Since η has dimensions $[\text{length}]^{-1}$, we see that the power spectral density $\hat{R}$ is dimensionless. In the absence of fluctuations [$\eta=0$ in Eq. (3)] and when, in addition, $\eta_0=\text{const}>0$, Eq. (1) with the boundary conditions (2) admits surface-wave solutions of the form

$$\Psi(\mathbf{x},z)=e^{i\mathbf{k}\cdot\mathbf{x}-\eta_0 z},$$

with

$$|\mathbf{k}|^2=p_s^2=k_0^2+\eta_0^2. \tag{5}$$

We assume that the fluctuations $\eta(\mathbf{x})$ are weak,

$$\varepsilon=R(0)/\eta_0^2\ll 1, \tag{6}$$

and the wavelength and correlation length l_{cor} are of the same order, which means that scattering has an appreciable effect at distances of order ε^{-1}. Therefore, we rescale the spatial variable $\mathbf{x}\to\mathbf{x}/\varepsilon$, and problem (1),(2) becomes, in rescaled variables,

$$\varepsilon^2\Delta\Psi_\varepsilon+k_0^2\Psi_\varepsilon=0,\quad z>0, \tag{7}$$

$$\begin{aligned}\varepsilon\frac{\partial\Psi_\varepsilon}{\partial z}(\mathbf{x},0)&=\left[-\eta_0(\mathbf{x})-\sqrt{\varepsilon}\,\eta\left(\frac{\mathbf{x}}{\varepsilon}\right)\right]\Psi_\varepsilon(\mathbf{x},0)\\&:=-\eta_\varepsilon(\mathbf{x})\Psi_\varepsilon(\mathbf{x},0).\end{aligned} \tag{8}$$

Here we have also replaced $\eta\to\sqrt{\varepsilon}\,\eta$, consistent with Eq. (6), regarding the strength of the fluctuations.

We assume that solutions of Eq. (8) are outgoing or decaying waves at infinity so that the function $\Psi_\varepsilon(\mathbf{x},z)$ has the form

$$\Psi_\varepsilon(\mathbf{x},z)=\int_{\mathbb{R}^2}\frac{d\mathbf{k}}{(2\pi)^2}\,e^{[i\mathbf{k}\cdot\mathbf{x}+i\phi(\mathbf{k})z]/\varepsilon}B_\varepsilon(\mathbf{k}), \tag{9}$$

where $\phi(\mathbf{k})$ is given by

$$\phi(\mathbf{k})=\begin{cases}\sqrt{k_0^2-k^2}, & |\mathbf{k}|\leqslant k_0\\ i\sqrt{k^2-k_0^2}, & |\mathbf{k}|>k_0\end{cases}. \tag{10}$$

Then the boundary condition (8) implies that the amplitudes $B_\varepsilon(\mathbf{k})$ satisfy the equation

$$i\phi(\mathbf{k})B_\varepsilon(\mathbf{k})=-\int\frac{d\mathbf{q}}{(2\pi)^2}\,\hat{\eta}_\varepsilon(\mathbf{q})B_\varepsilon(\mathbf{k}-\varepsilon\mathbf{q}), \tag{11}$$

which we rewrite as

$$i\phi(\varepsilon\mathbf{k})\,\hat{\psi}_\varepsilon(\mathbf{k})=-\int\frac{d\mathbf{q}}{(2\pi)^2}\,\hat{\eta}_\varepsilon(\mathbf{q})\,\hat{\psi}_\varepsilon(\mathbf{k}-\mathbf{q}), \tag{12}$$

where $\hat{\psi}_\varepsilon(\mathbf{k})$ stands for the Fourier transform of the function $\psi_\varepsilon(\mathbf{x})\equiv\Psi_\varepsilon(\mathbf{x},0)$. If we define $\omega(\mathbf{x,k})$ by

$$\omega(\mathbf{x},\mathbf{k})=-i\phi(\mathbf{k})-\eta_0(\mathbf{x}), \tag{13}$$

then we can write Eq. (12) in the form

$$\int\frac{d\mathbf{k}}{(2\pi)^2}\,e^{i\mathbf{k}\cdot\mathbf{x}}\omega(\mathbf{x},\varepsilon\mathbf{k})\,\hat{\psi}_\varepsilon(\mathbf{k})=\sqrt{\varepsilon}\,\eta\left(\frac{\mathbf{x}}{\varepsilon}\right)\psi_\varepsilon(\mathbf{x}). \tag{14}$$

In the presence of random inhomogeneities $\eta\neq 0$, the surface waves undergo scattering both into surface and volume waves. The second process produces effective decay of surface waves even if the medium itself is lossless, since volume waves propagate in a homogeneous medium.

One way to treat surface wave scattering with $\eta_0=\text{const}$ is to approximate Eq. (1) or (12) by a closed two-dimensional Schrödinger-type equation for the wave function of the surface waves in which the coupling of surface waves with outgoing volume waves is taken into account by means of an imaginary part of an effective potential.[13] Assuming that the correlation length of the fluctuations is much smaller than the wavelength $\lambda=2\pi/k_0$, this equation has the form

$$\frac{1}{2\eta_0}(\varepsilon^2\Delta+p_s^2)\psi-[\eta_{ss}(\mathbf{x})+\Sigma_v]\psi=0. \tag{15}$$

Here the random function $\eta_{ss}(\mathbf{x})$ has the same power spectrum $\hat{R}(p)$ as $\eta(\mathbf{x})$ in the vicinity of $p=2p_s$ and the complex nonrandom part of the potential, Σ_v, is given by

$$\Sigma_v = \int_{|\mathbf{p}|\le k_0} \frac{d\mathbf{p}}{(2\pi)^2} \hat{R}(|\mathbf{p}-\mathbf{k}|) G_{0v}(\mathbf{p}), \tag{16}$$

where $|\mathbf{k}| = p_s$, and G_{0v} is the Green's function of the three-dimensional wave equation (1) with $\eta_0 = \text{const}$ and $\eta = 0$, evaluated at the surface $z = 0$:

$$G_{0v}(p) = \frac{1}{i\sqrt{k_0^2 - p^2} + \eta_0}.$$

Once the approximate surface Schrödinger equation (15) has been obtained, one can get the Bethe-Salpeter equation for correlations of surface wave functions and then simplify this to a surface transport equation in the high frequency limit. The result is Eq. (24) below.

Here we take a different approach, going directly from Eq. (12) to the transport equation for surface waves in the high-frequency limit and bypassing the approximation (15), the Bethe-Salpeter equation for it, and its high-frequency limit. This allows us to identify the relations between the wavelength, the correlation length, and the relative strength of the impedance fluctuations that lead to the transport regime for surface waves and to avoid intermediate approximations. In particular, the assumption $kl_{\mathrm{cor}} \ll 1$, which is not necessary in the regime we consider here, was essential for the derivation of Eq. (15). Moreover, we have also allowed the background $\eta_0(\mathbf{x})$ to be nonuniform on a scale large compared to the wavelength, when an explicit expression for the Green's function used in the derivation of Eq. (15) is not available. In this regime the coherent interaction of multiply scattered waves is negligible and so are localization effects. They can be considered by analyzing Eq. (15) as a two-dimensional Schrödinger equation with random potential as is done in Ref. 13.

III. TRANSPORT EQUATION FOR SURFACE WAVES

We will derive the radiative transport equation for surface waves starting from the Wigner distribution[6] for the surface wave function $\psi_\varepsilon(\mathbf{x})$ satisfying Eq. (12). It is a function of position $\mathbf{x}$ and wave vector $\mathbf{k}$, and it is scaled with the parameter ε of Eq. (6):

$$W_\varepsilon(\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^2} \frac{d\mathbf{y}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{y}} \psi_\varepsilon\left(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}\right) \overline{\psi_\varepsilon}\left(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2}\right),$$
$$\mathbf{x},\mathbf{k} \in \mathbb{R}^2. \tag{17}$$

The Wigner distribution has many useful properties. It is real, its weak limit as $\varepsilon \to 0$ is a non-negative distribution, and energy and flux density may be expressed as

$$|\psi_\varepsilon(\mathbf{x})|^2 = \int d\mathbf{k}\, W_\varepsilon(\mathbf{x},\mathbf{k}),$$

$$\frac{i\varepsilon}{2}[\psi_\varepsilon \nabla \overline{\psi_\varepsilon} - \overline{\psi_\varepsilon} \nabla \psi_\varepsilon] = \int \mathbf{k} W_\varepsilon(\mathbf{x},\mathbf{k}) d\mathbf{k},$$

respectively. It is customary to interpret $W_\varepsilon(\mathbf{x},\mathbf{k})$ as an energy density in phase space in the high-frequency limit, even though only its limit as $\epsilon \to 0$ is positive. Detailed mathematical properties of the Wigner distribution may be found in Refs. 6 and 7 and references cited therein. We note that the weak limit $\varepsilon \to 0$ is equivalent to the ladder approximation in many situations when other diagrams are negligible because of phase cancelations.

We obtain an exact equation for $W_\varepsilon(\mathbf{x},\mathbf{k})$ using Eq. (12) and the definition (17):

$$\int\int \frac{d\mathbf{x}' d\mathbf{k}'}{(2\pi)^4} e^{i\mathbf{k}\cdot\mathbf{x}' + i\mathbf{x}\cdot\mathbf{k}'} \hat{\omega}(\mathbf{k}',\mathbf{x}') W_\varepsilon\left(\mathbf{x} + \frac{\varepsilon\mathbf{x}'}{2}, \mathbf{k} - \frac{\varepsilon\mathbf{k}'}{2}\right)$$
$$= \sqrt{\varepsilon} \int \frac{d\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon} \hat{\eta}(\mathbf{p}) W_\varepsilon\left(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2}\right), \tag{18}$$

which after using Eq. (13) becomes

$$\int \frac{dx'}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}'} \hat{\phi}(\mathbf{x}') W_\varepsilon\left(\mathbf{x} + \frac{\varepsilon\mathbf{x}'}{2}, \mathbf{k}\right)$$
$$- i \int \frac{d\mathbf{k}'}{(2\pi)^2} e^{i\mathbf{k}'\cdot\mathbf{x}} \hat{\eta}_0(\mathbf{k}') W_\varepsilon\left(\mathbf{x}, \mathbf{k} - \frac{\varepsilon\mathbf{k}'}{2}\right)$$
$$= i\sqrt{\varepsilon} \int \frac{d\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon} \hat{\eta}(\mathbf{p}) W_\varepsilon\left(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2}\right). \tag{19}$$

We want to find the asymptotic behavior of the average $\langle W_\varepsilon \rangle$ of the solution of this equation as $\varepsilon \to 0$, which is the high-frequency and weak-fluctuation regime. The small parameter ϵ appears in two different ways in this equation. In the terms on the left the $\mathbf{x}$ and $\mathbf{k}$ arguments of W_ε are shifted by an order ϵ term. The small-ϵ expansion for these terms is just a Taylor expansion. The term on the right is the random perturbation, and the shift in the argument of W_ε is not small, while the Fourier exponential $e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon}$ is rapidly oscillating and the whole term is of order $\sqrt{\varepsilon}$. To deal with this term we introduce a two-scale expansion of the form

$$W_\varepsilon(\mathbf{x},\mathbf{k}) = W(\mathbf{x},\mathbf{k}) + \sqrt{\varepsilon} W_1(\mathbf{x},\boldsymbol{\xi},\mathbf{k}) + \varepsilon W_2(\mathbf{x},\boldsymbol{\xi},\mathbf{k}) + \cdots,$$
$$\boldsymbol{\xi} = \frac{\mathbf{x}}{\varepsilon}, \tag{20}$$

with the leading term $W(\mathbf{x},\mathbf{k})$ independent of the fast variable $\boldsymbol{\xi}$. The elimination of the fast oscillatory dependence can be also done by integrating (averaging) the ladder approximation of Bethe-Salpeter equation over an area $|\Delta\mathbf{x}|$ such that $\lambda \ll |\Delta\mathbf{x}| \ll l_{\mathrm{sc}}$.[28] We substitute now expansion (20) into Eq. (19) and obtain in the leading order in ε

$$\omega(\mathbf{x},\mathbf{k}) W(\mathbf{x},\mathbf{k}) = 0 \tag{21}$$

or

$$\phi(\mathbf{k}) W(\mathbf{x},\mathbf{k}) = i\eta_0(\mathbf{x}) W(\mathbf{x},\mathbf{k}). \tag{22}$$

It follows from Eq. (22) that nontrivial solutions $W(\mathbf{x},\mathbf{k})$ exist only for wave vectors $\mathbf{k}$ defined by the dispersion equation $i\phi(\mathbf{k}) = -\eta_0(\mathbf{x})$. In other words, the Wigner distribution $W(\mathbf{x},\mathbf{k})$ is singular (a delta function) with support on the set

$$S = \{(\mathbf{x},\mathbf{k}) : i\phi(\mathbf{k}) = -\eta_0(\mathbf{x})\},$$

which is a circle in $\mathbf{k}$ space centered at the origin $\mathbf{k} = 0$ of radius $p_s(\mathbf{x}) = \sqrt{k_0^2 + \eta_0^2(\mathbf{x})}$ at every point $\mathbf{x}$ on the surface.

Physically, this means that only surface waves with wave number $k=p_s(\mathbf{x})$ are present in the leading order, and the local wave number $p_s(\mathbf{x})$ is given by the expression above. However, the higher-order terms in the expansion (20) contain volume waves with $k\leqslant k_0$. They are generated by scattering of the surface waves and result in an effective nondissipative attenuation.

An asymptotic analysis, presented in detail in Appendix A, leads to the following transport equation for the average Wigner distribution $\langle W\rangle$, which from now on we denote again by W:

$$\nabla_{\mathbf{k}}\omega\cdot\nabla_{\mathbf{x}}W(\mathbf{x},\mathbf{k})-\nabla_{\mathbf{x}}\omega\cdot\nabla_{\mathbf{k}}W(\mathbf{x},\mathbf{k})$$
$$=\int_{|\mathbf{p}|>k_0}\frac{d\mathbf{p}}{2\pi}\hat{R}(\mathbf{k}-\mathbf{p})[W(\mathbf{x},\mathbf{p})-W(\mathbf{x},\mathbf{k})]\delta(\omega(\mathbf{x},\mathbf{p})$$
$$-\omega(\mathbf{x},\mathbf{k}))-2\int_{|\mathbf{p}|\leqslant k_0}\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})$$
$$\times\mathrm{Im}\left[\frac{1}{\omega(\mathbf{x},\mathbf{p})-\omega(\mathbf{x},\mathbf{k})}\right]W(\mathbf{x},\mathbf{k}) \tag{23}$$

or, on using Eqs. (13) and (21),

$$\frac{\mathbf{k}}{\eta_0}\cdot\nabla_{\mathbf{x}}W(\mathbf{x},\mathbf{k})+\nabla_{\mathbf{x}}\eta_0\cdot\nabla_{\mathbf{k}}W(\mathbf{x},\mathbf{k})$$
$$=\int_{|\mathbf{p}|>k_0}\frac{d\mathbf{p}}{2\pi}\hat{R}(|\mathbf{p}-\mathbf{k}|)[W(\mathbf{x},\mathbf{p})-W(\mathbf{x},\mathbf{k})]\delta(\sqrt{k^2-k_0^2}$$
$$-\sqrt{p^2-k_0^2})-\int_{|\mathbf{p}|\leqslant k_0}\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(|\mathbf{p}-\mathbf{k}|)$$
$$\times\frac{2\sqrt{k_0^2-p^2}}{k_0^2-p^2+\eta_0(\mathbf{x})^2}W(\mathbf{x},\mathbf{k}). \tag{24}$$

The left side of Eq. (24) describes the streaming of energy in the phase space $(\mathbf{x},\mathbf{k})$ along rays that are in general curved for $\eta_0(\mathbf{x})\neq$ const. The rays satisfy

$$\frac{d\mathbf{X}}{ds}=\frac{\mathbf{K}}{\eta_0(\mathbf{X})},\quad \frac{d\mathbf{K}}{ds}=\nabla_{\mathbf{x}}\eta_0(\mathbf{X}),$$

so that the wave vector $\mathbf{K}$ also changes along the ray if $\eta_0(\mathbf{x})\neq$ const. However, the rays are tangent to the set $|\mathbf{K}|^2=k_0^2+\eta_0(\mathbf{X})^2$, on which $W(\mathbf{x},\mathbf{k})$ is supported.

The first term on the right side of the transport equation (24) accounts for the scattering of surface waves into surface waves with the same $|\mathbf{k}|=p_s$, but with different directions of propagation in the plane $z=0$, and has a form standard for any transport equation, as in Eq. (23). The second, absorption term appears because surface waves undergo scattering into outgoing volume waves with wave numbers $k\leqslant k_0$. The latter, however, leave the surface after scattering and do not contribute to the production of scattered surface waves. Therefore this term in Eq. (24) is a pure loss term and the loss rate is equal to $-2\,\mathrm{Im}\,\Sigma_v$ with Σ_v defined by Eq. (16).

Recall that $W(\mathbf{x},\mathbf{k})$ vanishes off the frequency shell $|\mathbf{k}|=p_s$, and hence we may look for solutions of Eq. (24) in the form

$$W(\mathbf{x},\mathbf{k})=I(\mathbf{x},\phi)\,\delta(k-p_s(\mathbf{x})), \tag{25}$$

with $\hat{\mathbf{k}}=(\cos\phi,\sin\phi)$. When we substitute this form of W into Eq. (24), the terms that involve $\delta'(k-p_s(\mathbf{x}))$ on the left side cancel and we obtain for $I(\mathbf{x},\phi)$ the equation

$$\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}I(\mathbf{x},\phi)+\frac{\eta_0}{p_s}(\hat{\mathbf{k}}^{\perp}\cdot\nabla_x\eta_0)\frac{\partial I}{\partial\phi}$$
$$=\frac{\eta_0^2}{p_s}\int_0^{2\pi}\frac{d\phi'}{2\pi}\hat{R}(p_s|\hat{\mathbf{k}}-\hat{\mathbf{k}}'|)[I(\mathbf{x},\phi')-I(\mathbf{x},\phi)]$$
$$+\frac{2\,\eta_0\,\mathrm{Im}\,\Sigma_v}{p_s}I(\mathbf{x},\phi), \tag{26}$$

where $\hat{\mathbf{k}}^{\perp}=(-\cos\phi,\sin\phi)$ and Σ_v is given by Eq. (16). This equation is the main result of this section.

Volume waves are generated only by scattering of surface waves, and hence their energy is of order $O(\varepsilon)$. This means that for $k\leqslant k_0$, $\langle W_0\rangle_{\mathrm{vol}}=\langle W_1\rangle_{\mathrm{vol}}=0$, and the first nonzero term in the expansion of the Wigner distribution for volume waves is $\langle W_2\rangle$. At the surface $(z=0)$ this is equal to

$$\langle W_2\rangle_{\mathrm{vol}}(\mathbf{x},z=0,\mathbf{k})=\frac{1}{\mathrm{Im}[\omega(\mathbf{x},\mathbf{k})]}\int\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{p}-\mathbf{k})$$
$$\times W(\mathbf{x},\mathbf{p})\mathrm{Im}\left[\frac{1}{\omega(\mathbf{x},\mathbf{p})-\omega(\mathbf{x},\mathbf{k})}\right], \tag{27}$$

or, on using Eqs. (13) and (21),

$$\langle W_2\rangle_{\mathrm{vol}}(\mathbf{x},z=0,\mathbf{k})=\int\frac{d\mathbf{p}}{(2\pi)^2}\frac{\hat{R}(|\mathbf{k}-\mathbf{p}|)}{p^2-k^2}W(\mathbf{x},\mathbf{p}),$$
$$k\leqslant k_0, \tag{28}$$

or, using Eq. (25),

$$\langle W_2\rangle_{\mathrm{vol}}(\mathbf{x},z=0,\mathbf{k})=\frac{p_s}{p_s^2-k^2}\int\frac{d\phi}{(2\pi)^2}$$
$$\times\hat{R}(|\mathbf{k}-p_s\hat{\mathbf{p}}|)I(\mathbf{x},\phi),$$
$$\hat{\mathbf{p}}=(\cos\phi,\sin\phi).$$

Here $I(x,\phi)$ is the solution of Eq. (26), that is, the intensity of surface waves at the point x, propagating in the direction ϕ. We see that in the absence of dissipation, energy is conserved. The loss of surface wave energy coming from the last term in Eq. (26) is due solely to its transformation into radiating volume waves in the upper half space.

IV. EFFECT OF THE POWER SPECTRUM

In order to compare the strength of scattering and leakage of surface waves in the transport equation (26), we introduce the scattering and radiation lengths defined by

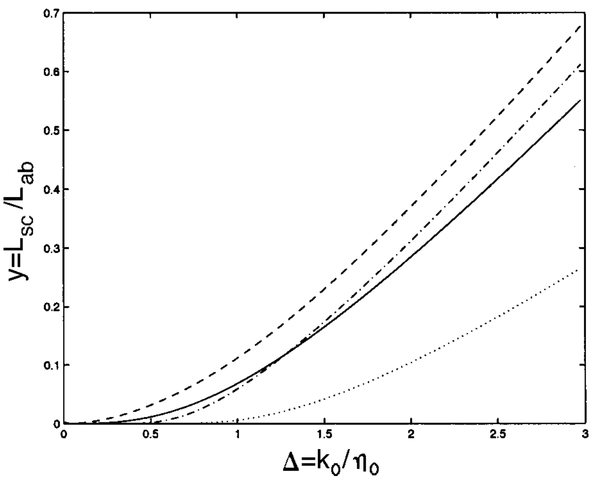

FIG. 1. The ratio $l_{\mathrm{sc}}/l_{\mathrm{rad}}$ defined by Eq. (29) for the power spectrum (32) as a function of $\Delta=k_0/\eta_0$ with $\sigma=1$ and various values of p_0: (a) Solid line for the δ-function correlation; (b) $p_0=0$, dash-dotted line; (c) $p_0=p_s$, dashed line; and (d) $p_0=3p_s$, dotted line.

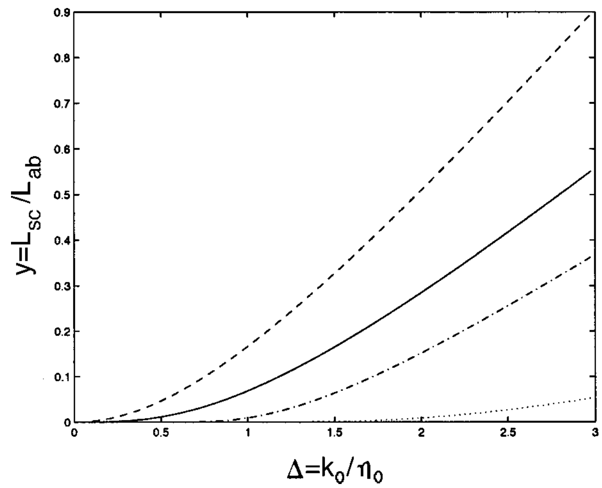

FIG. 2. The ratio $l_{\mathrm{sc}}/l_{\mathrm{rad}}$ defined by Eq. (29) for the power spectrum (32) as a function of $\Delta=k_0/\eta_0$ with $\sigma=5$ and various values of p_0, and δ-correlated: (a) Solid line for the δ-function correlation; (b) $p_0=0$, dash-dotted line; (c) $p_0=p_s$, dashed line; and (d) $p_0=3p_s$, dotted line.

$$l_{\mathrm{sc}}^{-1}=\frac{\eta_0^2}{p_s}\int_0^{2\pi}\frac{d\phi'}{2\pi}\hat{R}(p_s|\hat{\mathbf{k}}-\hat{\mathbf{k}}'|),$$

$$l_{\mathrm{rad}}^{-1}=\frac{\eta_0\gamma}{p_s}=\frac{2\eta_0}{p_s}\int_{|\mathbf{p}|\leq k_0}d\mathbf{p}\frac{\sqrt{k_0^2-p^2}}{\eta_0^2+k_0^2-p^2}\hat{R}(|\mathbf{p}-\mathbf{k}|), \tag{29}$$

where $|\mathbf{k}|=p_s$ and $\gamma=-2\,\mathrm{Im}\,\Sigma_v$. These expressions show an important difference that exists between the two-dimensional case we consider here and the one-dimensional case considered in Ref. 13. Both exhibit an exponential decay of energy (because of both localization and leakage in one dimension, and only due to leakage in two dimensions). However, while in the one-dimensional case scattering of surface waves into themselves may be easily suppressed by choosing a power spectrum supported away from $p=0$ and $p=2p_s$, which makes $l_{\mathrm{sc}}\to\infty$ in the one-dimensional case, while keeping l_{rad} finite, this mechanism does not apply in two dimensions. In the latter case total suppression of the surface wave scattering requires a power spectrum supported away from the whole interval $0\leq p\leq 2p_s$, which will also suppress the leakage into the volume by making both l_{sc} and l_{rad} tend to infinity. It is easier in the two-dimensional case to control the leakage of surface waves into volume by choosing a power spectrum that is supported outside the interval $p_s-k_0\leq p\leq p_s+k_0$. Then l_{sc} is finite, while $l_{\mathrm{rad}}\to\infty$, and energy does not decay exponentially as a function of the propagation distance r, but rather behaves as $1/r$ for large r.

Both the scattering and radiation lengths may be computed explicitly for δ-correlated fluctuations of $\eta(\mathbf{x})$, $R(\mathbf{x})=\sigma^2\delta(x/l_{\mathrm{cor}})$:

$$l_{\mathrm{sc}}^{-1}=\frac{\eta_0^2}{p_s}(\sigma l_{\mathrm{cor}})^2,$$

$$l_{\mathrm{rad}}^{-1}=\frac{\eta_0(\sigma l_{\mathrm{cor}})^2}{\pi p_s}\left[k_0-\eta_0\tan^{-1}\frac{k_0}{\eta_0}\right], \tag{30}$$

so that

$$\frac{l_{\mathrm{sc}}}{l_{\mathrm{rad}}}=\frac{1}{\pi}(\zeta-\tan^{-1}\zeta),\qquad \zeta=\frac{k_0}{\eta_0}. \tag{31}$$

Therefore, in the δ-correlated case scattering dominates leakage for wave numbers $k_0\ll\eta_0$, while for high wave numbers $k_0\gg\eta_0$ leakage dominates scattering.

In Figs. 1–4 we present numerical results for the ratio $l_{\mathrm{sc}}/l_{\mathrm{rad}}$ for power spectra of the form

$$\hat{R}(p)=C\exp[-\sigma(p-p_0)^2] \tag{32}$$

for various values of σ and p_0. Scattering dominates leakage when $k_0/\eta_0\ll 1$ for a wide range of parameters σ and p_0, while the converse is true for $k_0/\eta_0\gg 1$, as we noted above for the δ-correlated case. Leakage is strongest when, the power spectrum is centered at $p_0=p_s$ as is also clear from the definition of l_{rad}. Indeed, leakage is produced by scattering of a surface wave with wave vector $\mathbf{k}$ on the circle of

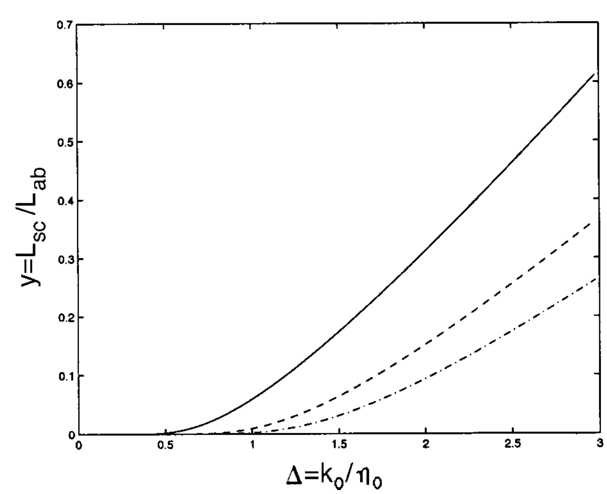

FIG. 3. The ratio $l_{\mathrm{sc}}/l_{\mathrm{rad}}$ defined by Eq. (29) for the power spectrum (32) as a function of $\Delta=k_0/\eta_0$ with $p_0=0$ for various values of σ: (a) $\sigma=0.1$, solid line; (b) $\sigma=5$, dashed line; and (c) $\sigma=10$, dash-dotted line.

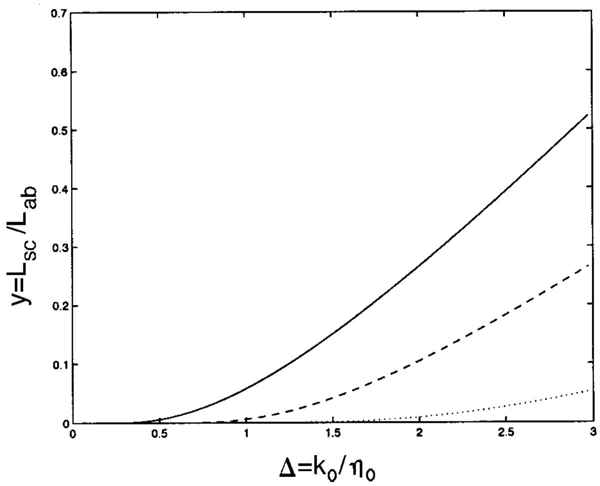

FIG. 4. The ratio l_{sc}/l_{rad} defined by Eq. (29) for the power spectrum (32) as a function of $\Delta = k_0/\eta_0$ with $p_0 = 3p_s$ for various values of σ: (a) $\sigma = 0.1$, solid line; (b) $\sigma = 1$, dashed line; and (c) $\sigma = 5$, dotted line.

radius p_s into a volume wave with wave vector $\mathbf{p}$ inside the disk of radius k_0 centered at $k=0$ with the probability of such an event being proportional to $\hat{R}(\mathbf{k}-\mathbf{p})$. This probability is enhanced if $\hat{R}(p)$ has a maximum at $p=p_s$. We also see that there are two ways to increase the radiation length relative to the scattering length for a given ratio k_0/η_0. First, with a power spectrum of the form (32) and with large enough σ, for some fixed p_0, as Figs. 3 and 4 show. This will both reduce leakage into volume waves and make surface wave scattering be mostly forward in the case $p_0=0$. Another way, which seems to be more efficient, is to fix the variance σ and let the centering parameter p_0 be sufficiently large, as can be seen by comparing these figures with Figs. 1 and 2. This would produce mostly backward scattering. However, while using the tails of Gaussian power spectra tends to reduce leakage, the constant C in Eq. (32) has to be very large to make scattering significant.

V. TRANSMITTED DISTRIBUTION AND DIFFUSION LIMIT

There exist several numerical methods to solve Eq. (26) and we refer to Ref. 22 for details. The main difficulty lies in the large number of degrees of freedom, 2 in space $(\mathbf{x})$ and 1 in the direction of propagation (k) for each frequency. We consider in this section an approximation to the transport equation which is valid in the diffusive regime. It is well known[23,24,6] that the energy density diffuses more than it transports when the following conditions are satisfied: (i) the scattering length, or mean free path, is small compared to the distance of propagation, and (ii) the absorption length is large compared to the distance of propagation. We have seen in Sec. IV that these requirements can be satisfied for specific power spectra and frequencies.

To simplify the presentation we consider here a flat power spectrum $\hat{R}=M$ and strip geometry. That is, we assume that the surface is unperturbed outside of a strip of width L with its boundaries perpendicular to the x axis. We also assume that $\eta_0=\text{const}$. The thickness of the slab L is large compared

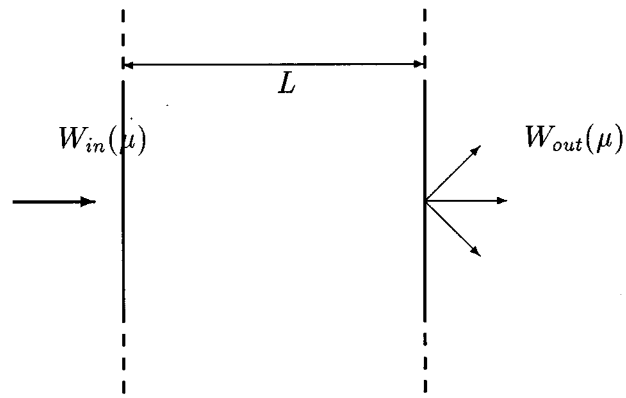

FIG. 5. Slab geometry.

to the scattering length, that is,

$$l_{sc}=\frac{2\pi p_s}{\eta_0^2 M}\ll L,$$

where l_{sc} is the transport mean free path. The absorption length is also much larger than both l_{sc} and L, and is given by

$$l_{rad}=\frac{1}{l_{sc}\gamma}=\frac{p_s}{2\eta_0|\mathrm{Im}\,\Sigma_v|}.$$

In particular, we have in the above scaling

$$l_{sc}l_{rad}\gtrsim L^2.$$

We consider propagation of a surface wave through a random surface layer occupying an interval $[0,L]$ (see Fig. 5) and are interested in the asymptotic distribution of the outgoing energy density $W_{out}(L,\mu)$ radiated out from this layer given the incident energy distribution $W_{in}(0,\mu)$, where $\mu=\cos\phi$ and ϕ is the angle between the direction of propagation and the x axis, $\mu\in[-1,1]$. In this notation the radiative transfer equation (26) takes the form

$$\mu\partial_x W(x,\mu)+l_{sc}\gamma W(x,\mu)+\frac{1}{l_{sc}}\left(W(x,\mu)-\frac{1}{\pi}\int_{-1}^{1}\frac{W(x,\mu')}{\sqrt{1-(\mu')^2}}\right)=0,$$

$$W(0,\mu)=W_{in}(\mu)\quad\text{for } 0<\mu<1, \tag{33}$$

$$W(L,\mu)=0\quad\text{for } -1<\mu<0, \tag{34}$$

where $W(x,\mu)$ is the energy density at position $x\in(0,L)$ propagating with direction cosine $\mu\in[-1,1]$. We assume that no energy enters the domain at $x=L$, which means that the outgoing distribution is given by $W_{out}(\mu)=W(L,\mu)$ for $0<\mu<1$.

To calculate the distribution W_{out} in the diffusion regime, we write an asymptotic expansion of the transport equation (33) and its solution $W(x,\mu)$ in the form

$$\left(\frac{1}{l_{\rm sc}}\mathcal{L}_0+\mathcal{L}_1+l_{\rm sc}\mathcal{L}_2\right)\left[W_0+l_{\rm sc}W_1+l_{\rm sc}^2W_2+l_{\rm sc}^3W_3+b_{l_{\rm sc}}^0\left(\frac{x}{l_{\rm sc}},\mu\right)+b_{l_{\rm sc}}^L\left(\frac{L-x}{l_{\rm sc}},\mu\right)+O(l_{\rm sc}^4)\right]=0 \quad (35)$$

$$(W_0+l_{\rm sc}W_1+l_{\rm sc}^2W_2+l_{\rm sc}^3W_3)(0,\mu)+b_{l_{\rm sc}}^0(0,\mu)+O(l_{\rm sc}^4)=W_{\rm in}(\mu),\quad 0<\mu<1,$$

$$(W_0+l_{\rm sc}W_1+l_{\rm sc}^2W_2+l_{\rm sc}^3W_3)(L,\mu)+b_{l_{\rm sc}}^L(0,\mu)+O(l_{\rm sc}^4)=0,\quad -1<\mu<0,$$

where $b_{l_{\rm sc}}^0(y,\mu)$ and $b_{l_{\rm sc}}^L(y,\mu)$ are boundary layer terms defined for $y\geqslant 0$ and $\mu\in[-1,1]$ that decay exponentially fast when $y\to\infty$. Each of these terms also satisfies an asymptotic expansion of the form

$$b_{l_{\rm sc}}=b_0+l_{\rm sc}b_1+l_{\rm sc}^2b_2+l_{\rm sc}^3b_3+O(l_{\rm sc}^4).$$

The operators $\mathcal{L}_i$ for $i=0,1,2$ are given by

$$\mathcal{L}_0W(x,\mu)=W(x,\mu)-\int_{-1}^{1}\sigma(\mu')W(x,\mu')d\mu',$$

$$\mathcal{L}_1W(x,\mu)=\mu\partial_xW(x,\mu),$$

$$\mathcal{L}_2W(x,\mu)=\gamma W(x,\mu),$$

where we have introduced

$$\sigma(\mu)=\frac{1}{\pi}\frac{1}{\sqrt{1-\mu^2}}. \quad (36)$$

The asymptotic analysis, presented in detail in Appendix B 1, shows that the leading-order term W_0 is independent of the direction μ and has the form

$$W_0(x)=W_0(0)\frac{\exp(-\sqrt{2\gamma}x)-\exp[-\sqrt{2\gamma}(2L-x)]}{1-\exp(-2\sqrt{2\gamma}L)}\sim W_0(0)\frac{L-x}{L}\quad\text{as}\quad\gamma\to 0,$$

with

$$W_0(0)=\overline{W}_{\rm in}=\frac{\sqrt{2}}{\pi}\int_0^1\frac{\mu}{\sqrt{1-\mu^2}}H(\mu)W_{\rm in}(0,\mu)d\mu. \quad (37)$$

For the outgoing distribution $W_{\rm out}(L,\mu)=W(L,\mu)$ in the first-order (in $l_{\rm sc}$) approximation, we obtain

$$W_{\rm out}(L,\mu)=\frac{l_{\rm sc}}{\sqrt{2}}H(\mu)[-\partial_xW_0(L)]+O(l_{\rm sc}^2). \quad (38)$$

Here $H(\mu)$ is Chandrasekhar's function,[25] which solves the nonlinear equation

$$1=H(\mu)\int_0^1\frac{\mu'\sigma(\mu')H(\mu')}{\mu'+\mu}d\mu'. \quad (39)$$

We can rewrite Eq. (38) in the form

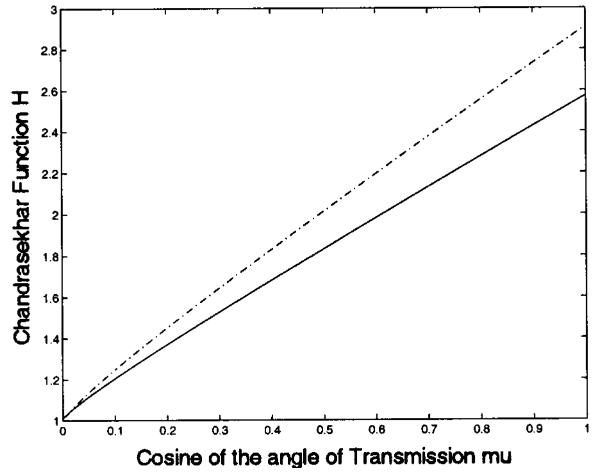

FIG. 6. Chandrasekhar's H functions in two (solid line) and three (dash-dotted line) dimensions.

$$W_{\rm out}(L,\mu)=\overline{\mu}_{\rm in}H(\mu)\frac{l_{\rm sc}\sqrt{\gamma}}{\sinh\sqrt{2\gamma L^2}}=\overline{u}_{\rm in}H(\mu)\sqrt{\frac{l_{\rm sc}}{l_{\rm rad}}}\left[\sinh\left(\sqrt{\frac{2L^2}{l_{\rm sc}l_{\rm rad}}}\right)\right]^{-1}. \quad (40)$$

Expression (40) provides a physical interpretation of Chandrasekhar's H function, for it is, up to some constant, the outgoing distribution of radiation for a source term located at infinity. This is also known as the law of darkening[25] or the Milne problem.[26]

We present now numerical calculations of the various quantities involved in the above derivation. It is interesting to compare them with their analog in three dimensions, obtained by replacing $\sigma(\mu)$ in Eq. (36) by 1. The computation of the H function is obtained by solving Eq. (39), which was already given for three dimensional problems in Ref. 25. We have plotted in Fig. 6 the H functions in two and three dimensions for isotropic scattering. The constant $\alpha=1/\sqrt{2}$ in two dimensions that appears in the definition (38) of $W_{\rm out}$ is replaced by $1/\sqrt{3}$ is three dimensions. Therefore, the transmitted flux is larger in two dimensions than in three dimensions, even though the H function is slightly smaller in two dimensions than in three. This is compatible with the extrapolation length $L_{\rm ex}=\Lambda(\mu)$ that appears in Eqs. (B7) and (B8). The extrapolation length gives the energy density in the diffusion approximation at $x=L$. Approximate values of Eq. (B12) in 2D and 3D are

$$L_{\rm ex}^{\rm 2D}\sim 0.8164,\quad L_{\rm ex}^{\rm 3D}\sim 0.7104, \quad (41)$$

respectively.

These values for the extrapolation lengths were obtained from the asymptotic analysis of the boundary layer in transport theory. It is interesting to compare them with the classical approximation of the extrapolation lengths obtained by assuming that the diffusion regime is valid up to the boundary. Consider W linear in μ (diffusion approximation), $W(x,\mu)=W_0(x)-\mu\partial_xW_0(x)$. We set here $l_{\rm sc}=1$ for simplicity. The boundary condition $W(L,\mu)=0$ for $\mu<0$ cannot be satisfied exactly since $W_0(x)$ does not depend on μ.

This requires a boundary layer and is done carefully in Appendix B. However, this boundary condition can be satisfied on average. Multiplying the transport equation (33) by $\sigma(\mu)\theta(x)$, where θ is a test function, and integrating over $(0,L)\times(-1,1)$ yields

$$\int_0^L\int_{-1}^1(\mu\partial_x W\theta+\gamma W\theta)\sigma\,dx\,d\mu=0.$$

Assume that $\theta(0)=0$. We have after integrating by parts

$$\int_0^L\int_{-1}^1(-\mu W\partial_x\theta+\gamma W\theta)\sigma\,dx\,d\mu+\int_{-1}^1\mu\sigma W(L)\theta(L)d\mu=0.$$

The condition that the mean incoming flux be zero is

$$\int_{-1}^0\mu\sigma[W_0(L)-\mu\partial_x W_0(L)]d\mu=0.$$

With σ given by Eq. (36) this is equivalent to

$$W_0+\frac{\pi}{4}\frac{\partial W_0}{\partial x}=0.$$

In 3D, where $\sigma=1$, the constant $\pi/4$ is replaced by 2/3. Therefore, we have in the diffusion regime the following approximations for the extrapolation lengths:

$$L_{\rm diff}^{\rm 2D}=\frac{\pi}{4}\sim0.7854,\quad L_{\rm diff}^{\rm 3D}=\frac{2}{3}\sim0.6667, \tag{42}$$

which are rather close to the asymptotically exact extrapolation lengths given in Eq. (41).

VI. CONCLUSIONS

We have studied analytically and numerically transport and diffusion of surface waves on random interfaces. Starting from the general three-dimensional wave equation with impedance boundary conditions, we derived the radiative transport equation (23) for surface waves on a flat surface with randomly fluctuating impedance. The transport equation accounts for both scattering of surface waves and leakage into volume waves that results in an effective loss of surface waves. The scattering cross section and the ''absorption'' length are expressed in terms of the power spectrum of the random fluctuations. We have studied the effect of the power spectrum on the relative strengths of scattering and leakage, and we have examined ways to decrease leakage by choosing an appropriate power spectrum, in particular by shifting its peak. We have also considered the diffusion approximation, which is valid when the scattering mean free path is much smaller than the absorption length, which must in turn be much larger than the propagation distance. We have obtained an analytical expression for the angular distribution of the energy of surface waves transmitted through a strip of random impedance in this regime. We have also computed the extrapolation length (41) for the diffusion approximation, which provides asymptotically correct boundary conditions for the diffusion equation.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE TRANSPORT EQUATION

We present now the perturbation analysis of Eq. (18). It is convenient to take the imaginary part of Eq. (18) using the fact that W_ϵ and η are real valued:

$$\begin{aligned}\int\frac{d\mathbf{x}'d\mathbf{k}^b}{(2\pi)^4}&[e^{i\mathbf{k}\cdot\mathbf{x}'+i\mathbf{x}\cdot\mathbf{k}'}\hat\omega(\mathbf{k}',\mathbf{x}')\\&+e^{-i\mathbf{k}\cdot\mathbf{x}'-i\mathbf{x}\cdot\mathbf{k}'}\overline{\hat\omega}(\mathbf{k}',\mathbf{x}')]W_\epsilon\left(\mathbf{x}+\frac{\epsilon\mathbf{x}'}{2},\mathbf{k}-\frac{\epsilon\mathbf{k}'}{2}\right)\\&=\sqrt{\epsilon}\int\frac{d\mathbf{p}}{(2\pi)^2}e^{ip\cdot\mathbf{x}/\epsilon}\hat\eta(\mathbf{p})\left[W_\epsilon\left(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}\right)\right.\\&\left.\quad-W_\epsilon\left(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}\right)\right].\end{aligned}\tag{A1}$$

The leading-order term $[O(1)]$ in Eq. (A1) is simply the imaginary part of Eq. (21). The order $O(\sqrt{\epsilon})$ term in Eq. (A1) is

$$\begin{aligned}\int\frac{d\mathbf{x}'d\mathbf{k}'}{(2\pi)^4}&[e^{i\mathbf{k}\cdot\mathbf{x}'+i\mathbf{x}\cdot\mathbf{k}'}\hat\omega(\mathbf{k}',\mathbf{x}')\\&-e^{-i\mathbf{k}\cdot\mathbf{x}'-i\mathbf{x}\cdot\mathbf{k}'}\overline{\hat\omega}(\mathbf{k}',\mathbf{x}')]W_1\left(\mathbf{x},\boldsymbol{\xi}+\frac{\mathbf{x}'}{2},\mathbf{k}\right)\\&=\int\frac{d\mathbf{p}}{(2\pi)^2}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\hat\eta(\mathbf{p})\left[W\left(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}\right)-W\left(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}\right)\right].\end{aligned}$$

Therefore, the Fourier transform of W_1 in the fast variable $\boldsymbol{\xi}$ is given by

$$\hat W_1(\mathbf{x},\mathbf{q},\mathbf{k})=\frac{\hat\eta(\mathbf{q})[W(\mathbf{k}-\mathbf{q}/2)-W(\mathbf{k}+\mathbf{q}/2)]}{\omega(\mathbf{x},\mathbf{k}+\mathbf{q}/2)-\overline{\omega}(\mathbf{x},\mathbf{k}-\mathbf{q}/2)-i\theta}.\tag{A2}$$

Here $\theta>0$ is a regularization parameter. We shall later take the limit $\theta\to0$.

We insert expression (A2) into the order $O(\varepsilon)$ term in Eq. (A1) and average. The left side of Eq. (A1) becomes then

$$\left(\frac{1}{2i}\nabla_{\mathbf{k}}\omega+\frac{1}{2i}\nabla_{\mathbf{k}}\overline{\omega}\right)\cdot\nabla_{\mathbf{x}}W-\left(\frac{1}{2i}\nabla_{\mathbf{x}}\omega+\frac{1}{2i}\nabla_{\mathbf{x}}\overline{\omega}\right)\cdot\nabla_{\mathbf{k}}W.\tag{A3}$$

We denoted here $\langle W\rangle$ by W again for simplicity. We average the right side of Eq. (A1) under the assumption that $\langle\eta\eta W\rangle\approx\langle\eta\eta\rangle\langle W\rangle$, which is equivalent to the ladder approximation in the diagram expansion. We will also assume that statistical averages are equal to the the spatial averages with respect to the fast variable $\boldsymbol{\xi}$. These formal assumptions may be justified rigorously in the high frequency limit.[9,8] The second assumption implies that the $O(\varepsilon)$ term involving W_2 becomes

$$\left\langle \int \frac{d\mathbf{x}'d\mathbf{k}'}{(2\pi)^4}[e^{i\mathbf{k}\cdot\mathbf{x}'+\mathbf{x}\cdot\mathbf{k}'}\hat{\omega}(\mathbf{k}',\mathbf{x}')+e^{-i\mathbf{k}\cdot\mathbf{x}'-i\mathbf{x}\cdot\mathbf{k}'}\overline{\hat{\omega}}(\mathbf{k}',\mathbf{x}')]W_2\left(\mathbf{x},\boldsymbol{\xi}+\frac{\mathbf{x}'}{2},\mathbf{k}\right)\right\rangle=[\omega(\mathbf{x},\mathbf{k})-\overline{\omega}(\mathbf{x},\mathbf{k})]\langle W_2(\mathbf{x},\boldsymbol{\xi},\mathbf{k})\rangle=0 \tag{A4}$$

for $k=p_s$ since $\omega(\mathbf{x},\mathbf{k})$ is real for such $\mathbf{k}$. Therefore, we get in the order $O(\varepsilon)$

$$\left(\frac{1}{2i}\nabla_{\mathbf{k}}\omega+\frac{1}{2i}\nabla_{\mathbf{k}}\overline{\omega}\right)\cdot\nabla_{\mathbf{x}}W-\left(\frac{1}{2i}\nabla_{\mathbf{x}}\omega+\frac{1}{2i}\nabla_{\mathbf{x}}\overline{\omega}\right)\cdot\nabla_{\mathbf{k}}W=\left\langle\int\frac{d\mathbf{p}}{(2\pi)^2}e^{i\mathbf{p}\cdot\mathbf{x}}\hat{\eta}(\mathbf{p})\left[W_1\left(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}\right)-W_1\left(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}\right)\right]\right\rangle. \tag{A5}$$

We insert expression (A2) into the right side of Eq. (A5) and use Eq. (4) to obtain that it is equal to

$$\int\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})[W(\mathbf{p})-W(\mathbf{k})]\times\left[\frac{1}{\overline{\omega}(\mathbf{x},\mathbf{k})-\omega(\mathbf{x},\mathbf{p})+i\theta}+\frac{1}{\overline{\omega}(\mathbf{x},\mathbf{p})-\omega(\mathbf{x},\mathbf{k})+i\theta}\right]. \tag{A6}$$

We split the above integral in two parts:

$$\int=\int_{|p|<k_0}+\int_{|p|\geq k_0}. \tag{A7}$$

In the first region $\omega(\mathbf{x},\mathbf{p})$ has a nonzero imaginary part and $W(\mathbf{x},\mathbf{p})=0$, while in the second $\omega(\mathbf{x},\mathbf{p})$ is real and $W(\mathbf{x},\mathbf{p})\neq 0$. Furthermore, $\omega(\mathbf{x},\mathbf{k})$ is real since $W(\mathbf{x},\mathbf{k})$ vanishes outside the frequency shell $k=p_s$, and thus $k>k_0$. Then the first term in Eq. (A7) is

$$\begin{aligned}-\int_{|\mathbf{p}|\leq k_0}&\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})W(\mathbf{k})\left[\frac{1}{\omega(\mathbf{x},\mathbf{k})-\omega(\mathbf{x},\mathbf{p})}+\frac{1}{\overline{\omega}(\mathbf{x},\mathbf{p})-\omega(\mathbf{x},\mathbf{k})}\right]\\&=2i\int_{|\mathbf{p}|\leq k_0}\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})\times\mathrm{Im}\left[\frac{1}{\omega(\mathbf{x},\mathbf{p})-\omega(\mathbf{x},\mathbf{p})}\right]W(\mathbf{x},\mathbf{k}).\end{aligned} \tag{A8}$$

The integral over the second region in Eq. (A7) is

$$\begin{aligned}\int_{|\mathbf{p}|\geq k_0}&\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})[W(\mathbf{p})-W(\mathbf{k})]\times\left[\frac{1}{\omega(\mathbf{x},\mathbf{k})-\omega(\mathbf{x},\mathbf{p})+i\theta}+\frac{1}{\omega(\mathbf{x},\mathbf{p})-\omega(\mathbf{x},\mathbf{k})+i\theta}\right]\\&=\int_{|\mathbf{p}|\geq k_0}\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})[W(\mathbf{p})-W(\mathbf{k})]\times\frac{(-2i\theta)}{[\omega(\mathbf{x},\mathbf{k})-\omega(\mathbf{x},\mathbf{p})]^2+\theta^2}\\&\to(-i)\int_{|\mathbf{p}|\geq k_0}\frac{d\mathbf{p}}{(2\pi)}\times\hat{R}(\mathbf{k}-\mathbf{p})[W(\mathbf{p})-W(\mathbf{k})]\delta[\omega(\mathbf{x},\mathbf{p})-\omega(\mathbf{x},\mathbf{k})]\end{aligned} \tag{A9}$$

as $\theta\to 0$. Combining expressions (A8) and (A9) for the right side of Eq. (A5) yields the radiative transport equation for the average Wigner distribution:

$$\begin{aligned}\nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})&\cdot\nabla_{\mathbf{x}}W(\mathbf{x},\mathbf{k})\nabla_{\mathbf{x}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{k}}W(\mathbf{x},\mathbf{k})\\&=\int_{|\mathbf{p}|\geq k_0}\frac{d\mathbf{p}}{2\pi}\hat{R}(|\mathbf{p}-\mathbf{k}|)[W(\mathbf{x},\mathbf{p})-W(\mathbf{x},\mathbf{k})]\\&\quad\times\delta[\omega(\mathbf{x},\mathbf{p})-\omega(\mathbf{x},\mathbf{k})]-2\int_{p\leq k_0}\frac{d\mathbf{p}}{(2\pi)^2}\\&\quad\times\hat{R}(\mathbf{p}-\mathbf{k})\mathrm{Im}\left[\frac{1}{\omega(\mathbf{x},\mathbf{p})-\omega(\mathbf{x},\mathbf{k})}\right]W(\mathbf{x},\mathbf{k}),\end{aligned}$$

which is Eq. (23).

We derive now expression (28) for $\langle W_2\rangle_{\mathrm{vol}}$ for $k\leq k_0$, that is, for the angular distribution of the energy for volume waves at the surface $z=0$. Once again we average Eq. (A1), but now for $k\leq k_0$, and not for $k=p_s$. Recall that $W(\mathbf{x},\mathbf{k})$ vanishes for $k\leq k_0$ and therefore the leading order term in Eq. (A1) is $O(\epsilon)$. It is given by

$$[\omega(\mathbf{x},\mathbf{k})-\overline{\omega}(\mathbf{x},\mathbf{k})]\langle W_2\rangle_{\mathrm{vol}}(\mathbf{x},\mathbf{k})=\left\langle\int\frac{d\mathbf{p}}{(2\pi)^2}e^{i\mathbf{p}\cdot\boldsymbol{\xi}}\hat{\eta}(\mathbf{p})\left[W_1\left(\mathbf{x},\boldsymbol{\xi},\mathbf{k}-\frac{\mathbf{p}}{2}\right)-W_1\left(\mathbf{x},\boldsymbol{\xi},\mathbf{k}+\frac{\mathbf{p}}{2}\right)\right]\right\rangle. \tag{A10}$$

The left side is the same as in Eq. (A4). It does not vanish now since $\omega(\mathbf{x,k})$ has a nonzero imaginary part for $k<k_0$. The right side of Eq. (A10) is given by Eq. (A6) with $W(\mathbf{x,k})=0$:

$$\int \frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})W(\mathbf{x,p})\left[\frac{1}{\overline{\omega}(\mathbf{x,k})-\omega(\mathbf{x,p})} - \frac{1}{\omega(\mathbf{x,k})-\overline{\omega}(\mathbf{x,p})}\right]. \tag{A11}$$

Then Eq. (28) follows from Eqs. (A10) and (A11) since $\omega(\mathbf{x,p})$ is real.

APPENDIX B: THE OUTGOING DISTRIBUTION

1. Perturbation analysis

We now carry out the asymptotic analysis of Eq. (35) that leads to Eqs. (37) and (38). Upon equating like powers of l_{sc} in Eq. (35), and knowing the boundary layer terms decay exponentially fast, we obtain the following equations inside the medium:

$$\begin{aligned} &\text{(i)} \quad \mathcal{L}_0 W_0=0,\\ &\text{(ii)} \quad \mathcal{L}_0 W_1+\mathcal{L}_1 W_0=0,\\ &\text{(iii)} \quad \mathcal{L}_0 W_2+\mathcal{L}_1 W_1+\mathcal{L}_2 W_0=0,\\ &\text{(iv)} \quad \mathcal{L}_0 W_3+\mathcal{L}_1 W_2+\mathcal{L}_2 W_1=0. \end{aligned} \tag{B1}$$

The boundary conditions are given by

$$\begin{aligned} &\text{(i)} \quad W_0(0,\mu)+b_0^0(0,\mu)=W_{\mathrm{in}}(\mu), \quad 0<\mu<1,\\ &\text{(ii)} \quad W_i(0,\mu)+b_i^0(0,\mu)=0, \quad 0<\mu<1, \quad 1\leqslant i\leqslant 3,\\ &\text{(iii)} \, W_i(L,\mu)+b_i^L(0,-\mu)=0, \quad -1<\mu<0, \quad 0\leqslant i\leqslant 3. \end{aligned} \tag{B2}$$

The boundary layer terms $b_j^{0,L}$ are solutions of the half-space problem

$$\mu b_x+b-\int_{-1}^{1}\sigma(\mu')b(x,\mu')d\mu'=0, \tag{B3}$$

and are decaying exponentially as $x\to\infty$. We will use below two important facts associated with Eq. (B3). First, it is known[27] that for a bounded incoming flux $b(0,\mu)=g(\mu)$ for Eq. (B3), $0<\mu<1$, the solution $b(x,\mu)$ converges exponentially rapidly to a constant b_∞ as $x\to\infty$. We let

$$b_\infty=\Lambda(g), \tag{B4}$$

where Λ is the linear functional that maps the incoming flux onto the solution at infinity. We require that $b_\infty=0$ for the boundary layers $b^{0,L}$. That is, the boundary layer terms decay to 0. Second, we define the response operator $\mathcal{R}$ by

$$\mathcal{R}[g](\mu)=b(0,-\mu), \quad 0<\mu<1, \tag{B5}$$

which maps the incoming flux onto the outgoing one.

Solving the above equations gives the asymptotic behavior of the energy density $W_{l_{\mathrm{sc}}}$ and that of its outgoing distribution W_{out}. Provided that the initial distribution $W_{\mathrm{in}}(\mu)$ is regular, all asymptotic expansions can be justified rigorously.[23,30] We deduce from (i) in Eqs. (B1) that

$$W_0(x,\mu)=W_0(x).$$

This means that the leading term in the expansion is actually independent of the direction of propagation away from the boundaries. This does not hold in the vicinity of the interface $x=0$ since the boundary condition (i) of Eqs. (B2) must be satisfied, which is impossible without the introduction of the boundary layer term b_0^0. The only way to have an exponentially decaying b_0^0 is to impose the condition that

$$W_0(0)=\Lambda(W_{\mathrm{in}}), \tag{B6}$$

with the linear functional Λ defined in (B4) above. Expression (B6) and the explicit expression (B12) for the linear functional Λ given below lead to the boundary condition (37). Since there is no incoming energy at $x=L$, we have $b_0^L\equiv 0$ and the second boundary condition

$$W_0(L)=0.$$

We now consider (ii) in Eqs. (B1). Since $\int_{-1}^{1}\mu\sigma(\mu)d\mu=0$, we deduce that W_1 is given by

$$W_1(x,\mu)=-\mu\partial_x W_0(x)+W_{10}(x),$$

where W_{10} is still undetermined, but depends only on x. The boundary condition (ii) in Eqs. (B2) at $x=0$ gives

$$-\mu\partial_x W_0(0)+W_{10}(0)+b_1^0(0,\mu)=0.$$

Exponential decay for b_1 is possible provided that

$$W_{10}(0)=\Lambda(\mu)\partial_x W_0(0), \tag{B7}$$

with the linear functional Λ defined in Eq. (B4). A similar relation at $x=L$ yields

$$W_{10}(L)=-\Lambda(\mu)\partial_x W_0(L). \tag{B8}$$

Since $\int_{-1}^{1}(\mathcal{L}_0 W_2)(x,\mu)\sigma(\mu)d\mu\equiv 0$ independently of W_2, we deduce from (iii) in Eqs. (B1) that $\int_{-1}^{1}(-\mu\partial_x\mu\partial_x W_0+\gamma W_0)\sigma(\mu)d\mu=0$, which gives the diffusion equation

$$-\frac{1}{2}W_0''+\gamma W_0=0,$$

$$W_0(0)=\Lambda(W_{\mathrm{in}}), \quad W_0(L)=0, \tag{B9}$$

since $\int_{-1}^{1}\mu^2\sigma(\mu)d\mu=1/2$. The solution to this equation is given by Eq. (37):

$$W_0(0)\frac{\exp(-\sqrt{2\gamma}x)-\exp[-\sqrt{2\gamma}(2L-x)]}{1-\exp(-2\sqrt{2\gamma}L)}$$

$$\sim W_0(0)\frac{L-x}{L} \quad \text{when} \quad \gamma\to 0.$$

The derivative at $x=L$ that enters Eq. (38) is

$$\partial_x W_0(L) = -\frac{W_0(0)}{L}\exp(-\sqrt{2\gamma}L)\frac{2\sqrt{2\gamma}L}{1-\exp(-2\sqrt{2\gamma}L)}$$

$$\sim -\frac{W_0(0)}{L} \quad \text{when} \quad \gamma\to 0. \tag{B10}$$

We do not analyze the boundary conditions in Eqs. (B2) for b_2^0 and b_2^L since they are not involved in the leading term of the expansion for $W_{\rm out}$.

It remains to find an equation for W_{10} so as to complete the description of the terms of order $l_{\rm sc}$. This is done by averaging (iv) of Eqs. (B1) multiplied by $\sigma(\mu)$ in μ over $[-1, 1]$, which gives

$$\frac{1}{2}\int_{-1}^{1}\mathcal{L}_1 W_2\sigma(\mu)d\mu + \gamma W_{10} = 0.$$

However, part (iii) of Eqs. (B1) implies that

$$\frac{1}{2}\int_{-1}^{1}\mu\frac{\partial W_2}{\partial x}d\mu = -\frac{1}{2}W_{10}'',$$

so we get diffusion equation for W_{10}:

$$-\frac{1}{2}W_{10}'' + \gamma W_{10} = 0,$$

$$W_{10}(0) = \Lambda(\mu)\partial_x W_0(0), \quad W_{10}(L) = -\Lambda(\mu)\partial_x W_0(L).$$

This equation can be solved since W_0 is known.

We are now ready to calculate the leading term in the asymptotic expansion of $W_{\rm out}$. Since $W_0(L)=0$, $b^L(0)=0$, and $b^0(L/l_{\rm sc})$ is exponentially small, $W_{\rm out}$ is at most of order $l_{\rm sc}$. The term of order $l_{\rm sc}$ does not vanish and is given by

$$W_1(L,\mu) + b^L(0,\mu)$$
$$= -\mu\partial_x W_0(L) + W_{10}(L) + b^L(0,-\mu) \quad \text{for} \quad -1<\mu<1.$$

The boundary condition for $b^L(y,\mu)$ at $y=0$ is

$$b^L(0,-\mu) = \mu\partial_x W_0(L) - W_{10}(x) \quad \text{for} \quad -1<\mu<0,$$

that is,

$$b^L(0,\mu) = -\mu\partial_x W_0(L) - W_{10}(x) \quad \text{for} \quad 0<\mu<1,$$

since the total incoming flux at $x=L$ is zero. Therefore, we have that

$$b^L(0,\mu) = -\mathcal{R}[\mu](-\mu)\partial_x W_0(L) - W_{10}(x)$$
$$\text{for} \quad -1<\mu<0,$$

since $\mathcal{R}[1](\mu)\equiv 1$. Here $\mathcal{R}[g]$ is the response operator defined in Eq. (B5). In other words, we obtain that, for $0<\mu<1$,

$$W_{\rm out}(\mu) = l_{\rm sc}[-\mu\partial_x W_0(L) + W_{10}(L) - \mathcal{R}[\mu](\mu)\partial_x W_0(L)$$
$$- W_{10}(L)] + O(l_{\rm sc}^2)$$
$$= l_{\rm sc}[\mathcal{R}[\mu](\mu)+\mu][-\partial_x W_0(L)] + O(l_{\rm sc}^2). \tag{B11}$$

To compute $W_{\rm out}$ numerically we need the linear functional Λ to determine $W_0(0)$ and the response operator acting on μ: $\mathcal{R}[\mu]$. Because of the relative simplicity of the linear transport equation in homogeneous half space, no numerical solution of the transport equation is actually necessary. We show in Appendix B 2 that the linear functional Λ is given by

$$\Lambda(f) = \frac{1}{\alpha}\int_0^1 \mu\sigma(\mu)H(\mu)f(\mu)d\mu, \tag{B12}$$

where $H(\mu)$ is Chandrasekhar's function, the solution of Eq. (39), and α is defined by

$$\alpha = \sqrt{2\int_0^1 \mu^2\sigma(\mu)d\mu} = \frac{1}{\sqrt{2}}. \tag{B13}$$

In Appendix B 2 we show also that the function $\mathcal{R}[\mu](\mu)$ is given by

$$\mathcal{R}[\mu](\mu) = \alpha H(\mu) - \mu \tag{B14}$$

and from this and Eq. (B11) we obtain expression (38) for the outgoing distribution $W_{\rm out}(\mu)$.

2. Half-space problem

We present here the part of the perturbation analysis that leads to the half-space transport problem and analyze it to get Eqs. (B12) and (B14), and derive also Eq. (39) for Chandrasekhar's function. Half-space problems have been studied extensively in the physical[25,29] and mathematical literature.[30,31,32,33,34] Close to the boundary, the volume equations (B1) have to be modified to account for the boundary layers. The term of order $l_{\rm sc}^{-1}$ in Eq. (35) near the boundary is given by

$$\mathcal{L}_0 W_0 + \mathcal{L}_0(b_0^0 + b_0^L) + \mu\partial_y[b_0^0(y,\mu) + b_0^L((L-y),-\mu)] = 0$$

instead of (i) in Eqs. (B1). Since $\mathcal{L}_0 W_0 = 0$ and the boundary layers are exponentially decaying, we have

$$\mu\partial_y b_0^0(y,\mu) + \mathcal{L}_0 b_0^0(y,\mu) = 0 \quad \text{for} \quad y>0$$
$$\text{and} \quad \mu\in[-1,1],$$
$$b_0^0(0,\mu) = W_{\rm in} - W_0(0).$$

and a similar equation for b_0^L.

We want to analyze the half-space problem

$$\mu\partial_y b + b - \langle\sigma b\rangle = 0 \quad \text{in} \quad \mathbb{R}^+\times[-1,1],$$
$$b(0,\mu) = g(\mu) \quad \text{for} \quad 0<\mu<1. \tag{B15}$$

Here $\langle\cdot\rangle$ means averaging in μ over $[-1, 1]$:

$$\langle f\rangle = \int_{-1}^{1} f(x,\mu)d\mu,$$

and $\sigma(\mu)$ is a positive function defined on $[-1, 1]$ satisfying $\langle\sigma\rangle=1$ and $\sigma(-\mu)=\sigma(\mu)$, like, for instance, the function (36). We know from Refs. 23, 27 and 30–32 that Eq. (B15) admits a unique bounded solution, which converges exponentially fast as $y\to\infty$ to a constant

$$b_\infty = \Lambda(g).$$

This defines the linear functional Λ: $L^\infty(0,1)\to\mathbb{R}$. The reflection operator $\mathcal{R}\in\mathcal{L}(L^\infty(0,1))$ is denoted by

$$\mathcal{R}[g](\mu)=b(0,-\mu)\quad\text{for }0<\mu<1.$$

We now want to derive more explicit formulas for the reflection operator $\mathcal{R}$ and the linear functional Λ. Let $\lambda(\mu)$ be the solution of Eq. (B15) with $g(\mu)=\mu$. It is easily seen that $\lambda+(y-\mu)$ satisfies Eq. (B15) with $g(\mu)=0$. Let us note, however, that $\lambda+(y-\mu)$ is not uniformly bounded. We verify that $w(\mu)=\sigma(\mu)[\lambda(-\mu)+y+\mu]$ solves the equation

$$-\mu\partial_y+w-\sigma(\mu)\langle w\rangle=0\quad\text{for }y>0\quad\text{and}\quad\mu\in[-1,1],$$

$$w(0,\mu)=0\quad\text{for }-1<\mu<0.\tag{B16}$$

The function w actually solves an adjoint equation to Eq. (B15). Let us now multiply Eq. (B15) by w and Eq. (B16) by b, and subtract the latter from the former. Integrating $dy\,d\mu$ over $(0,x)\times[-1,1]$ yields

$$\langle\mu b(x,\mu)w(x,\mu)\rangle=\langle\mu b(0,\mu)w(0,\mu)\rangle.$$

This relation is valid for all $x\geqslant0$. Letting $x\to\infty$ yields

$$b_\infty\langle\mu w\rangle=\int_0^1\mu g(\mu)w(0,\mu).$$

We easily check that $\langle\mu\sigma\lambda\rangle=\langle\mu\sigma\rangle=0$. Therefore $\langle\mu w\rangle=\langle\sigma\mu^2\rangle$. Moreover, $w(0,\mu)$ is given for $0<\mu<1$ by $\sigma(\mu)[\mathcal{R}[\mu](\mu)+\mu]$ by definition of $\lambda(\mu)$. Defining α as in Eq. (B13) yields Eq. (B12):

$$\Lambda(g)=\frac{1}{\alpha}\int_0^1\mu\sigma(\mu)H(\mu)g(\mu)d\mu,\tag{B17}$$

where Chandrasekhar's function $H(\mu)$ is given by

$$H(\mu)=\frac{1}{\alpha}\{\mu+\mathcal{R}[\mu](\mu)\}.\tag{B18}$$

It remains to derive Eq. (39) for Chandrasekhar's function, defined by Eq. (B18). In order to do so we derive a new relation between the response operator $\mathcal{R}$ and the function H that also allows us to solve them easily numerically.

Let b be the solution of Eq. (B15), which converges to b_∞ when $y\to\infty$. Let v be the solution of Eq. (B15) with $g(\mu)$ replaced by $\mu g(\mu)$. We denote by u the function

$$u(y,\mu)=\mu b(y,\mu)-\int_0^y\langle\sigma b\rangle(s)ds.\tag{B19}$$

We verify that $\mu\partial_x u+u-\langle\sigma u\rangle=0$. Let

$$z(y,\mu)=v(y,\mu)-u(y,\mu)+b_\infty(\mu-y).\tag{B20}$$

We verify that z solves Eq. (B15) with boundary condition $g(\mu)=b_\infty\mu$. Since b converges to b_∞ exponentially, we see that $z(y,\mu)$ is bounded. Therefore, we can consider $\mathcal{R}[z(0,\mu>0)]$. Using Eq. (B20), the response operator can be expressed in two different ways:

$$\begin{aligned}z(0,\mu<0)&=b_\infty\mathcal{R}[\mu](-\mu)\\&=\mathcal{R}[\mu g](-\mu)-\mu\mathcal{R}[g](-\mu)+b_\infty\mu.\end{aligned}$$

In other words, we have according to Eq. (B18) that

$$\mathcal{R}[\mu g]+\mu\mathcal{R}[g]=b_\infty\alpha H.$$

Since $b_\infty=\Lambda(g)$, we deduce from Eq. (B17) that

$$\mathcal{R}[\mu g](\mu)+\mu\mathcal{R}[g](\mu)=H(\mu)\int_0^1\mu\sigma(\mu)H(\mu)g(\mu)d\mu.\tag{B21}$$

This relation holds for every function g. Consider now $g(\mu)=(\mu+\mu_0)^{-1}$ for some fixed $\mu_0\in[0,1]$ and apply Eq. (B21) at point $\mu=\mu_0$. Since $\mathcal{R}[1]=1$, we obtain

$$1=H(\mu)\int_0^1\frac{\mu'\sigma(\mu')H(\mu')}{\mu'+\mu}d\mu'.$$

This is relation (39).

[1] Yu. Barabanenkov, A. Vinogradov, Kravtsov. Yu, and V. Tatarskii, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **15**, 1852 (1972).

[2] I. M. Besieris and F. D. Tappert, J. Math. Phys. **14**, 704 (1973).

[3] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978), Vol. II.

[4] I. M. Besieris, W. Kohler, and H. Freese, J. Acoust. Soc. Am. **72**, 937 (1982).

[5] Yu. Barabanenkov, Yu. Kravtsov, V. Ozrin, and A. Saichev, Prog. Opt. **29**, 67 (1991).

[6] L. Ryzhik, G. Papanicolaou, and J. B. Keller, Wave Motion **24**, 327 (1996).

[7] G. Papanicolaou and L. Ryzhik, in *Hyperbolic Equations and Frequency Interactions*, edited by L. Caffarelli and W. E (AMS, Providence, 1998).

[8] H. Spohn, J. Stat. Phys. **17**, 385 (1977).

[9] L. Erdös and H. T. Yau, Pure Appl. Math **53**, 667 (2000).

[10] *Diffuse Waves in Complex Media*, edited by J. P. Fouque, Mathematical and Physical Sciences Series, Vol. 531 (Kluwer, Dordrecht, 1999).

[11] A. A. Maradudin, in *Electromagnetic Surface Excitations*, edited by R. Wallis and G. Stegeman (Springer-Verlag, New York, 1986), pp. 57–31.

[12] A. R. McGurn, A. A. Maradudin, and A. Celli, Phys. Rev. B **31**, 4866 (1985).

[13] V. Freilikher and I. Yurkevich, Phys. Lett. A **183**, 247 (1993).

[14] A. A. Maradudin and E. R. Méndez, Appl. Opt. **32**, 3335 (1993).

[15] H. Ogura and N. Takahashi, Waves Random Media **5**, 223 (1995).

[16] A. R. McGurn and A. A. Maradudin, J. Opt. Soc. Am. B **4**, 910 (1987).

[17] A. Shchegrov, Phys. Rev. B **57**, 4132 (1997).

[18] A. Abrikosov, *Fundamentals of the Theory of Metals* (North-Holland, Amsterdam, 1988).

[19] Y. Imry, *Introduction to Mesoscopic Physics* (Oxford University Press, New York, 1997).

[20] D. Simonian, S. Kravchenko, and M. Sarachik, Phys. Rev. B **55**, R13421 (1997); Phys. Rev. Lett. **79**, 2304 (1997).

[21] Y. Hainein, U. Meirav, D. Shahar, C. C. Li, D. C. Tsui, and H.

Strikman, Phys. Rev. Lett. **80**, 1288 (1998).
[22] J. J. Duderstadt and W. R. Martin, *Transport Theory* (Wiley-Interscience, New York, 1979).
[23] A. Bensoussan, J. L. Lions, and G. Papanicolaou, RIMS, Kyoto Univ. **15**, 53 (1979).
[24] E. W. Larsen and J. B. Keller, J. Math. Phys. **15**, 75 (1974).
[25] S. Chandrasekhar, *Radiative Transfer* (Oxford University Press, Cambridge, England, 1960).
[26] K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Plenum, New York, 1967).
[27] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology* (Springer-Verlag, Berlin, 1993), Vol. 6.
[28] F. Bass, V. Freilikher, and I. Fuks, IEEE Trans. Antennas Propag. **AP-22**, 288 (1974).
[29] M. C. W. van Rossum and Th. M. Nieuwenhuizen, Rev. Mod. Phys. **71**, 313 (1999).
[30] R. Sentis, Ph.D. thesis, Paris-Dauphine University, 1981.
[31] C. Bardos, R. Santos, and R. Sentis, Trans. Am. Math. Soc. **284**, 617 (1984).
[32] F. Golse (unpublished).
[33] G. Bal, SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **30**, 1208 (1999).
[34] G. Bal, Asymptot. Anal. **20**, 213 (1999).