Intermittent synchronization of resistively coupled chaotic Josephson junctions

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Numerical simulations have been used to investigate the dynamics of a pair of resistively linked Josephson junctions with ac bias. For suitable choices of parameters, the chaotic states of the two junctions become intermittently synchronized. Intervals of synchronization are interleaved between bursts of desynchronized activity. The distributions of these laminar times and their dependence on the coupling strength are determined. The role of phase winding in the definition of synchronization intervals is considered.

I. INTRODUCTION

In a previous numerical study[1] we examined the case of a coupled pair of forced pendulums and found that intermittent synchronization of the chaotic motions occurred. We also reported[2] experimental results from a pair of chaotic pendulums coupled through their differential angular velocities. Again, intermittent synchronization was observed.

The possibility that intermittent synchronized chaos also might appear in configurations of linked Josephson junctions arises from the well-known isomorphism between the equation governing a torque-driven pendulum and that of a current-biased junction.[3] That is the question addressed in this paper. Thus, two phenomena which have been studied extensively but separately in connection with Josephson junctions—synchronized oscillations[4] and chaotic dynamics[5–10]—appear here in combination.

As a by-product of this investigation, we reflect on the manner in which the condition of synchronization is conventionally defined and suggest an alternative definition that is suitable for systems with a periodicity attribute.

II. THEORY

A. Coupled parallel-connected Josephson junctions

The arrangement shown in Fig. 1 consists of a pair of Josephson junctions wired in *parallel* with a linking resistor R_s. Each junction is characterized by an order parameter phase difference φ, a critical current i_c, capacitance C, and normal resistance R. The junctions are biased with identical ac current sources $i_0 \cos \omega t$, but no dc source is included. This ac-only driving scenario is commonly adopted to probe essential chaotic behavior in Josephson systems[5,8,11–14] and in driven pendulums.[15]

The dynamical equations for the two junctions are, in this case,

$$\frac{\hbar C_1}{2e}\frac{d^2\varphi_1}{dt^2}+\frac{\hbar}{2eR_1}\frac{d\varphi_1}{dt}+i_{c1}\sin\varphi_1=i_0\cos\omega t-i_s, \quad (1)$$

$$\frac{\hbar C_2}{2e}\frac{d^2\varphi_2}{dt^2}+\frac{\hbar}{2eR_2}\frac{d\varphi_2}{dt}+i_{c2}\sin\varphi_2=i_0\cos\omega t+i_s, \quad (2)$$

where the current flowing through the coupling resistor is given by

$$i_s=\frac{\hbar}{2eR_s}\left[\frac{d\varphi_1}{dt}-\frac{d\varphi_2}{dt}\right]. \quad (3)$$

The junction plasma frequencies are

$$\omega_{J1}=\sqrt{\frac{2ei_{c1}}{\hbar C_1}} \quad \text{and} \quad \omega_{J2}=\sqrt{\frac{2ei_{c2}}{\hbar C_2}}. \quad (4)$$

For this case, choose the normalized time scale

$$t^*=\omega_{J1}t. \quad (5)$$

The dimensionless damping parameter usually associated with the resistively shunted junction (RSJ) model is defined as

$$\beta_J=\frac{1}{R_1}\sqrt{\frac{\hbar}{2ei_{c1}C_1}}. \quad (6)$$

Therefore with $i_0^*=i_0/i_{c1}$, $\Omega=\omega/\omega_{J1}$ and

$$\alpha_s=\frac{R_1}{R_s}\beta_J, \quad (7)$$

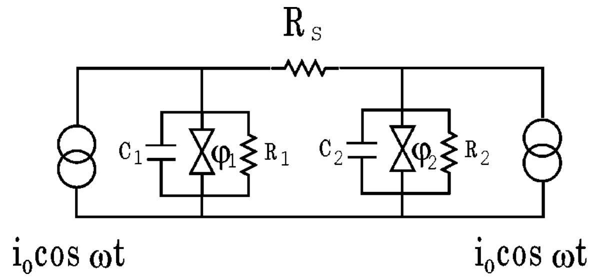

FIG. 1. Two Josephson junctions connected in parallel and linked by a resistor R_s.

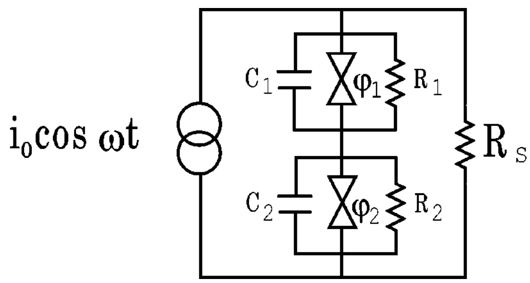

FIG. 2. Two Josephson junctions connected in series and shunted by a resistor.

Eqs. (1) and (2) become

$$\ddot{\varphi}_1+\beta_J\dot{\varphi}_1+\sin\varphi_1=i_0^*\cos\Omega t^*-\alpha_s[\dot{\varphi}_1-\dot{\varphi}_2], \tag{8}$$

$$\ddot{\varphi}_2+\left[\frac{R_1}{R_2}\frac{C_1}{C_2}\right]\beta_J\dot{\varphi}_2+\left[\frac{\omega_{J2}}{\omega_{J1}}\right]^2\sin\varphi_2 =\left[\frac{C_1}{C_2}\right]i_0^*\cos\Omega t^*-\left[\frac{C_1}{C_2}\right]\alpha_s[\dot{\varphi}_2-\dot{\varphi}_1]. \tag{9}$$

For the special case of junctions which are *identical*,

$$\ddot{\varphi}_1+\beta_J\dot{\varphi}_1+\sin\varphi_1=i_0^*\cos\Omega t^*-\alpha_s[\dot{\varphi}_1-\dot{\varphi}_2], \tag{10}$$

$$\ddot{\varphi}_2+\beta_J\dot{\varphi}_2+\sin\varphi_2=i_0^*\cos\Omega t^*-\alpha_s[\dot{\varphi}_2-\dot{\varphi}_1]. \tag{11}$$

Note that the coupling arises naturally as a direct consequence of the exchange of current through the resistor R_S and that it depends on the differential voltage $(\dot{\varphi}_1-\dot{\varphi}_2)$.[16] In the pendulum analog,[2] this translates to a differential angular velocity term which in that instance was generated by a magnet–eddy-current linkage module.

B. Shunted series-connected Josephson junctions

For completeness, we also present an obvious alternate configuration for linking a pair of Josephson junctions as illustrated in Fig. 2. In this case the devices are connected in *series*.[17–21] The appropriate equations for the phase variables are now

$$\frac{\hbar C_1}{2e}\frac{d^2\varphi_1}{dt^2}+\frac{\hbar}{2eR_1}\frac{d\varphi_1}{dt}+i_{c1}\sin\varphi_1=i_0\cos\omega t-i_s, \tag{12}$$

$$\frac{\hbar C_2}{2e}\frac{d^2\varphi_2}{dt^2}+\frac{\hbar}{2eR_2}\frac{d\varphi_2}{dt}+i_{c2}\sin\varphi_2=i_0\cos\omega t-i_s, \tag{13}$$

where i_s, the current flowing through the shunt resistor, is given by

$$i_s=\frac{\hbar}{2eR_s}\left[\frac{d\varphi_1}{dt}+\frac{d\varphi_2}{dt}\right]. \tag{14}$$

Notice that the cross-coupling of these equations now depends on the *sum* $(\dot{\varphi}_1+\dot{\varphi}_2)$, in contrast to the parallel-connected case which involved the *difference* $(\dot{\varphi}_1-\dot{\varphi}_2)$ as in Eqs. (10) and (11). Our interest in this paper lies with this latter form of interaction, and we will not pursue the series-connected case further.

It could be added that other coupling schemes lead to still different types of terms. For example, Doedel *et al.*[22] considered a system in which the mixing depended on the difference in the phases rather than in the phase derivatives.

III. NUMERICAL SIMULATIONS

Equations (10) and (11) were solved using a fourth-order Runge-Kutta routine in double-precision arithmetic. As noted in the Introduction, the principal goal is to explore possible modes of synchronized chaos. Clearly, this requires that parameters be chosen so that the individual junctions are operating chaotically. Confirmation of this state was provided by the appearance of the time series $\dot{\varphi}(t)$ and by the manifestation of strange attractors in the phase plane. For all the simulations reported here, we selected $\beta_J=0.25$, $i_0^*=1.20$, and $\Omega=0.60$. The computational time grid Δt^* was set at 0.005 of a drive cycle.

A troublesome computational artifact can appear when and if the precise equality $\varphi_1=\varphi_2$ occurs. Then the mutual interaction terms exactly vanish and the pair of equations (10) and (11) become identical. It has been observed[1,23,24] that finite-precision calculations may indeed ''find'' such special solutions and when they do the junctions exactly mirror each other indefinitely. A protection against this spurious locking is afforded by the addition of very small levels of random noise to the ac drive terms.

IV. SYNCHRONIZATION

A representative time series for the voltage difference $(\dot{\varphi}_1-\dot{\varphi}_2)$ is shown in Fig. 3. As the figure clearly reveals, the junctions exhibit *intermittent synchronizaton* of their chaotic motions. When synchronized, $(\dot{\varphi}_1-\dot{\varphi}_2)$ is small. Desynchronizing bursts interrupt these laminar intervals in an obviously irregular fashion.

While it is intuitively sensible to associate the condition of being synchronized with something like smallness in the differential voltage, i.e., $|(\dot{\varphi}_1-\dot{\varphi}_2)|\leq\delta$, the selection of a specific threshold is problematic. In a situation where systems become synchronized and then remain synchronized, the particular value assigned to δ affects only the moment of the onset of locking. But when systems move in and out of synchronization, such as the case presented here, the choice has an impact on the perceived distribution of locking intervals.

This point is illustrated by the inset of Fig. 3 which shows a magnified view of the portion of the time series in the box and also two possible threshold levels: $\delta=\pm0.01$ and $\delta=\pm0.001$. If voltage differences which do not fall outside the range ±0.01 are regarded as meeting the test for synchronization, then clearly the junctions would be considered to have a laminar interval stretching from about 14.0 to about 16.7. However, thresholds set at $\delta=\pm0.001$ would imply much shorter laminar runs.

Further evidence of the difficulties posed by threshold-based testing is provided by the following results. Long

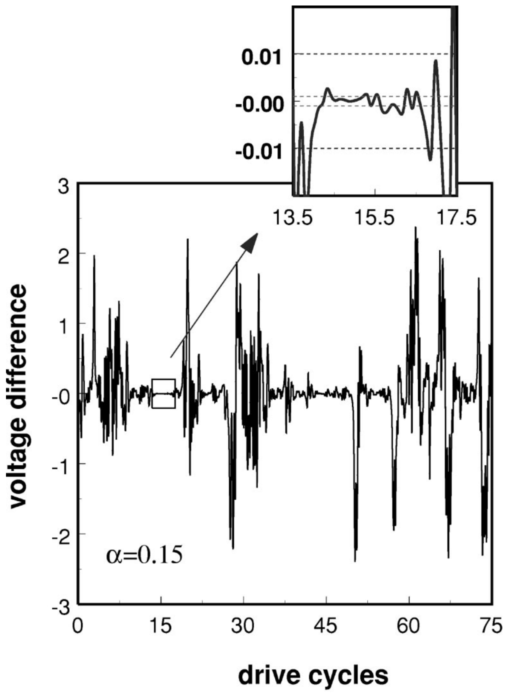

FIG. 3. Portion of the time series $\dot{\varphi}_1 - \dot{\varphi}_2$ showing intermittent synchronization between the junctions. The box highlights a region of strong locking. Inset: magnified portion of the time series. The dotted horizontal lines are at thresholds $\delta = \pm 0.01$ and $\delta = \pm 0.001$.

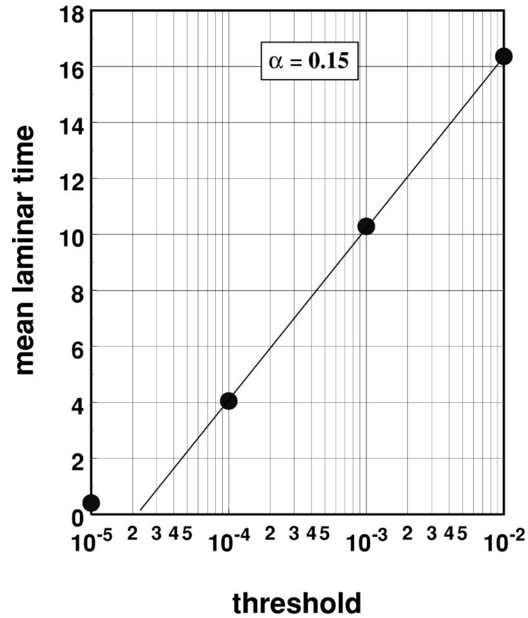

FIG. 5. Dependence of apparent mean laminar time on the choice of threshold.

simulation runs (10^6 drive cycles) were carried out. Using various thresholds δ, the time series was tested for the occurrence of different laminar intervals. Figure 4 shows how the choice of δ alters the apparent distribution of laminar times. In such a semilogarithmic plot, the inverse slope of the approximately linear portion of the data beyond small laminar values is proportional to the mean laminar time.[1] Figure 5 displays the manner in which the apparent mean laminar duration decreases as a function of the selected threshold value.

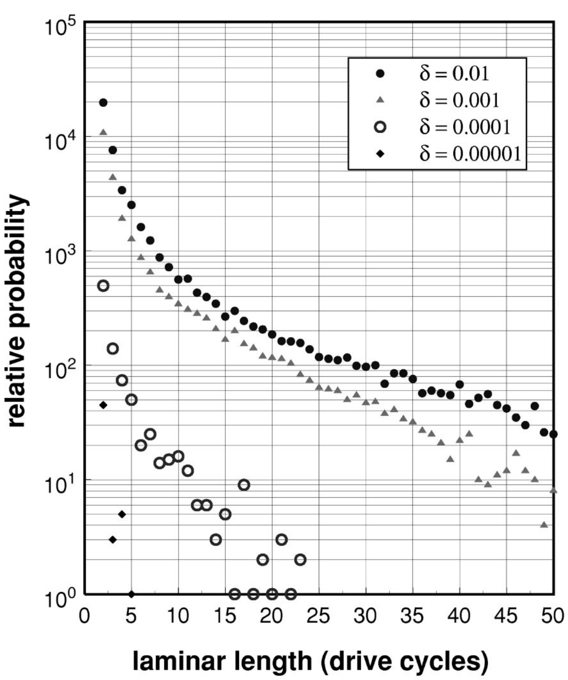

FIG. 4. Distribution of laminar times for coupled junctions. Laminar lengths are measured in forcing drive cycles. Results for $\alpha = 0.15$ and four different threshold values (δ) are shown.

In connection with these issues, we note that to achieve synchronization, real physical systems require feedback signals (however small) which are produced from differences in corresponding dynamical coordinates. One might think of such differences as providing an ongoing exchange of information about the respective dynamical states of each subsystem. Without this exchange, inherent noise will decouple even perfectly matched trajectories, resulting in eventual desynchronization. In essence, then, physical systems are never absolutely synchronized.

An alternative definition of laminar intervals can be constructed as follows. First note that as with voltage differences, the phase differences ($\varphi_1 - \varphi_2$) also wander, even in intervals of obviously high-quality synchronization (see the upper portion of Fig. 6). Let the domain of ($\varphi_1 - \varphi_2$) be divided into zones $\cdots(-3\pi,-\pi)(-\pi,\pi)(\pi,3\pi)\cdots$. Then synchronization can be viewed as continued residence within any zone. A shift from one zone to an adjacent one defines a *phase winding* event and every such event marks both the end of one synchronizing interval *and* the beginning of the next. Thus laminar times are the intervals between phase winding events, and so are defined uniquely and without recourse to an arbitrarily chosen threshold value.

These ideas are illustrated in lower portion of Fig. 6. The winding number W increments and decrements according to the direction of the 2π slips in angle between the oscillators.

The definition of synchronization just proposed is perhaps somewhat counterintuitive in that it does not associate the near equality of coordinates with the *condition* of being synchronized—but only with the *quality* of the synchronization. That is to say, synchronization is considered to hold even in situations where motions do not track very closely; it

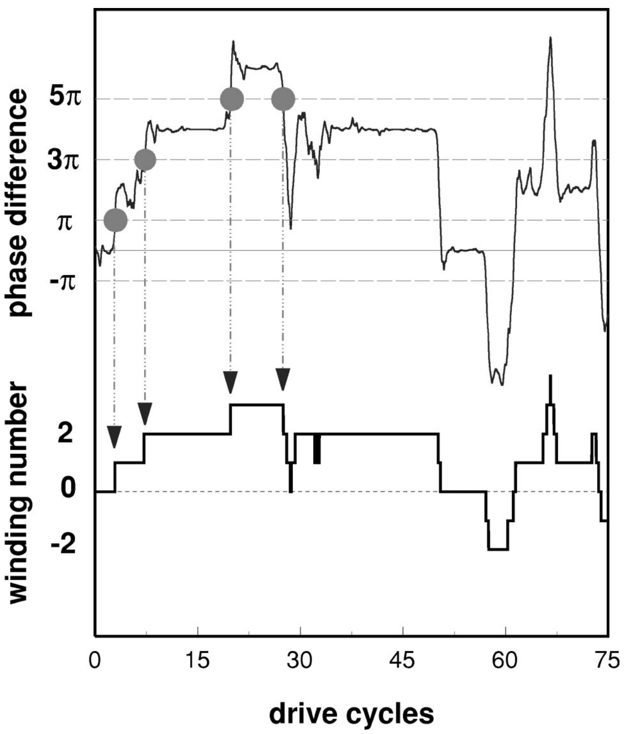

FIG. 6. Upper: time series for $(\varphi_1-\varphi_2)$ computed with $\alpha=0.15$. Lower: winding number corresponding to the upper trace. The locations of the first few transitions are indicated.

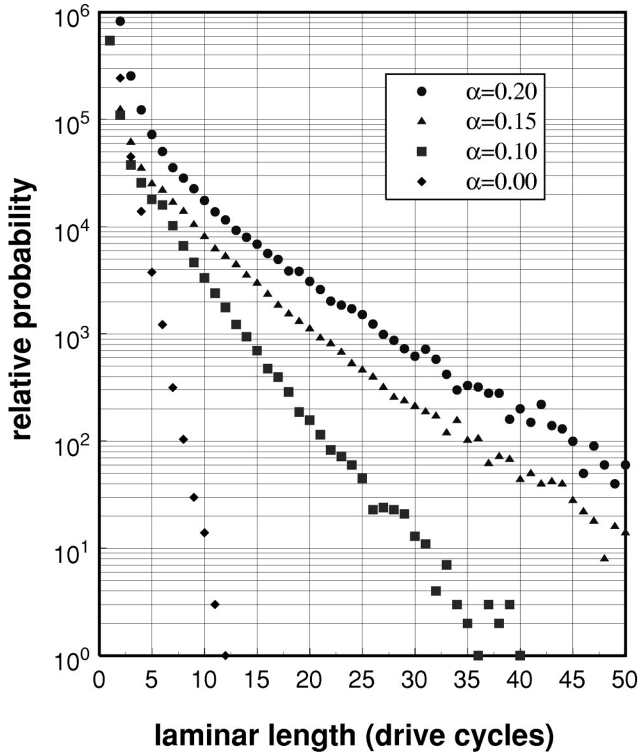

FIG. 7. Relative probability of occurrence of laminar times, where laminar intervals are defined as the time between transitions of the winding number. Data for four values of coupling strength are plotted.

is broken only when a phase slip event occurs.

A seeming paradox in this approach is posed by the hypothetical case of two lossless uncoupled pendulums which are both performing classical small oscillations. Neither pendulum achieves phase winding and there can be no phase slippage. Hence our test would say that they are synchronized even though there is no linkage connecting them. On the other hand, according to the more conventional threshold criterion, the pendulums might or might not be considered to be synchronized depending on the arbitrary choice of threshold magnitude. No matter how small a chosen threshold, still smaller uncoupled oscillations could be contrived which would still be judged to be synchronized according to the conventional test. In other words, to the question of whether these two uncoupled pendulums are synchronized, the phase winding test would say yes while the threshold test could say either yes or no.

Now consider a second pendulum pair to which some amount of interpendulum coupling has been added. The phase winding test applied to small oscillations in this new situation would yield the same answer as before—synchronized. The intuitive notion that synchronization is a behavior brought about by interaction may suggest that the verdict for the just-mentioned coupled case is acceptable while the verdict for the uncoupled case is unphysical. This point of view presupposes special knowledge of the relationship between system components. What if it was not known which pendulum pair was being observed?

In general, the available information concerning two oscillators is only that which is contained in simulation or experimental data sets $\varphi_1(t)$ and $\varphi_2(t)$ and thus the verdict as to their seeming synchronization (or absence of it) must be reached purely on the basis of tests executed on the observational data. Therefore, synchronization must be strictly a property of finite observational records. The quality and robustness of the synchronization, and its physical sources, are separate issues. This underscores the subtlety of the notion of synchronization.

Long simulations (10^6 drive cycles) were carried out for various values of the coupling coefficient α. In each case, data were accumulated on the relative frequency of occurrence of laminar lengths as now defined by the intervals between winding number transitions. The results are presented in Fig. 7. While the general form of these laminar time distributions is similar to that found when using the conventional threshold test, the winding number criterion is unambiguous and generates only one possible distribution for each selected value of α.

It is apparent that the mean laminar times (reciprocal slopes) decrease as the coupling decreases. In the limit of negligible coupling, the oscillators are essentially free-running independent chaotic systems whose *apparent* residual synchronization is merely an artifact of differencing two chaotic data sets which exhibit occasional accidental proximity of respective points in phase space.

V. SYNCHRONIZATION QUALITY

For Josephson devices, phase derivatives are of central importance because they are proportional to the junction voltages. If the two junctions could be perfectly synchronized, then $\dot{\varphi}_1(t)=\dot{\varphi}_2(t)$. However, as explained earlier, $\dot{\varphi}_1(t)\neq\dot{\varphi}_2(t)$ will always be the case for a physical system

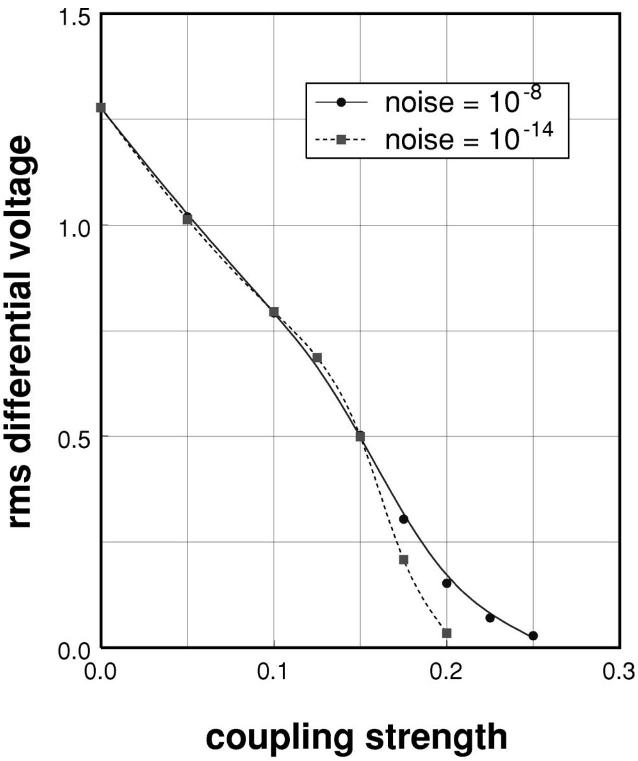

FIG. 8. Root-mean-squared differential voltage across the coupling resistor vs coupling strength. Numerical data for two noise amplitudes are plotted.

and therefore a differential voltage will exist across the coupling resistor. The magnitude of this differential voltage is a manifestation of the *quality of synchronization*. For Josephson junctions, measurements are complicated by the very high speed of the phase dynamics. Thus, feasible readings of junction voltages are in reality smoothed averages. The observed root-mean-squared differential voltage across the coupling resistor is represented by the quantity

$$\sqrt{\langle(\dot{\varphi}_1-\dot{\varphi}_2)^2\rangle}.$$

Numerical simulations extending to 500 000 drive cycles were used to find values of this rms differential voltage. These calculations were repeated for different values of the coupling coefficient α. The results shown in Fig. 8 indicate that the quality of synchronization improves as the coupling is increased. (We note that the quality factor plays a role that is somewhat analogous to that of the percentage of locking time as defined when using a traditional threshold criterion.) As noted earlier, small amounts of noise prevent false locking of the two chaotic oscillators. Figure 8 illustrates what happens when two different noise levels are injected. Not surprisingly, the smaller noise amplitude leads to improved synchronization quality, but the effect is only pronounced once the coupling strength exceeds about 0.15.

VI. CONCLUDING REMARKS

A pair of resistively coupled, parallel-connected Josephson junctions has been shown to exhibit intermittent synchronization of their chaotic states. The important dynamics considered here are intimately related to the attendant questions of laminar event definitions. In the present work we have sought a definition of synchronization which avoids the pitfall of arbitrariness associated with the notion of thresholds. Intervals of synchronization can be defined as periods which begin and end with phase-winding events. While developed here for coupled Josephson junctions, these ideas should be applicable to synchronization in any system possessing a coordinate with a wrapping property analogous to phase periodicity. Indeed, we have successfully used just this criterion to analyze synchronization in a coupled Rössler system.

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