Critical equation of state of three-dimensional XY systems

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We address the problem of determining the critical equation of state of three-dimensional XY systems. For this purpose we first consider the small-field expansion of the effective potential (Helmholtz free energy) in the high-temperature phase. We compute the first few nontrivial zero-momentum n-point renormalized couplings, which parametrize such expansion, by analyzing the high-temperature expansion of an improved lattice Hamiltonian with suppressed leading scaling corrections. These results are then used to construct parametric representations of the critical equation of state which are valid in the whole critical regime, satisfy the correct analytic properties (Griffith's analyticity), and take into account the Goldstone singularities at the coexistence curve. A systematic approximation scheme is introduced, which is limited essentially by the number of known terms in the small-field expansion of the effective potential. From our approximate representations of the equation of state, we derive estimates of universal ratios of amplitudes. For the specific-heat amplitude ratio we obtain $A^+/A^- = 1.055(3)$, to be compared with the best experimental estimate $A^+/A^- = 1.054(1)$.

I. INTRODUCTION

In the theory of critical phenomena, continuous phase transitions can be classified into universality classes determined only by few basic properties characterizing the system, such as space dimensionality, range of interaction, number of components, and symmetry of the order parameter. Renormalization-group theory predicts that all systems belonging to a given universality class have the same critical exponents and the same scaling functions. Here we consider the XY universality class, which is characterized by a two-component order parameter and effective short-range interactions. The lattice spin model described by the Hamiltonian

$$\mathcal{H}_L = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j + \sum_i \vec{h}_i \cdot \vec{s}_i , \tag{1}$$

where $\vec{s}_i$ is a two-component unit vector, is a particular system belonging to the XY universality class. It may be viewed as a magnetic system with easy-plane anisotropy, in which the magnetization plays the role of order parameter and the spins are coupled to an external magnetic field h.

The superfluid transition of ^{4}He, occurring along the λ line $T_\lambda(P)$ (where P is the pressure), belongs to the three-dimensional XY universality class. The order parameter is here the complex quantum amplitude of helium atoms. Such a transition provides an exceptional opportunity for an experimental test of the renormalization-group predictions, thanks to the weakness of the singularity in the compressibility of the fluid and to the purity of the samples. Moreover, experiments may be performed in a microgravity environment, leading to a reduction of the gravity-induced broadening of the transition. Recently, a Space Shuttle experiment[1] performed a very precise measurement of the heat capacity of liquid helium to within 2 nK from the λ transition, obtaining an extremely accurate estimate of the exponent α and of the ratio A^+/A^- of the specific-heat amplitudes

$$\alpha = -0.01285(38), \quad A^+/A^- = 1.054(1). \tag{2}$$

These results represent a challenge for theorists because the accuracy of the test of the renormalization-group predictions is now limited by the precision of the theoretical calculations. We mention the best available theoretical estimates for α: $\alpha = -0.0150(17)$ obtained using high-temperature expansion techniques,[2] $\alpha = -0.0169(33)$ from Monte Carlo simulations using finite-size scaling techniques,[3] $\alpha = -0.011(4)$ from field theory.[4] The close agreement with the experimental data clearly supports the standard renormalization-group description of the λ transition.[5]

In this paper we address the problem of determining the critical equation of state for the XY universality class. The critical equation of state relates the thermodynamical quantities in the neighborhood of the critical temperature, in both phases. It is usually written in the form (see, e.g., Ref. 6)

$$\vec{H} = \vec{M} M^{\delta-1} f(x), \quad x \propto t M^{-1/\beta}, \tag{3}$$

where $f(x)$ is a universal scaling function [normalized in such a way that $f(-1)=0$ and $f(0)=1$]. The universal ratios of amplitudes involving quantities defined at zero momentum (i.e., integrated in the volume), such as specific heat, magnetic susceptibility, etc., can be obtained from the scaling function $f(x)$.

It should be noted that, for the λ transition in ^{4}He, Eq. (3) is not directly related to the conventional equation of state that relates temperature and pressure. Moreover, in this case the field $\vec{H}$ does not correspond to an experimentally accessible external field, so that the function appearing in Eq. (3) cannot be determined directly in experiments. The physically interesting quantities are universal amplitude ratios of quantities formally defined at zero external field.

As our starting point for the determination of the critical equation of state, we compute the first few nontrivial coefficients of the small-field expansion of the effective potential (Helmholtz free energy) in the high-temperature phase. For

this purpose, we analyze the high-temperature expansion of an improved lattice Hamiltonian with suppressed leading scaling corrections.[7,8] If the leading nonanalytic scaling corrections are no longer present, one expects a faster convergence, and therefore an improved high-temperature (IHT) expansion whose analysis leads to more precise and reliable estimates. We consider a simple cubic lattice and the ϕ^4 Hamiltonian

$$\mathcal{H} = -\beta \sum_{\langle x,y \rangle} \vec{\phi}_x \cdot \vec{\phi}_y + \sum_x [\vec{\phi}_x^2 + \lambda (\vec{\phi}_x^2 - 1)^2], \qquad (4)$$

where $\langle x,y \rangle$ labels a lattice link, and $\vec{\phi}_x$ is a real two-component vector defined on lattice sites. The value of λ at which the leading corrections vanish has been determined by Monte Carlo simulations using finite-size techniques,[3] obtaining $\lambda^* = 2.10(6)$. In Ref. 2 we already considered the high-temperature expansion (to 20th order) of the improved ϕ^4 Hamiltonian (4) for the determination of the critical exponents, achieving a substantial improvement with respect to previous theoretical estimates. IHT expansions have also been considered for Ising-like systems,[8] obtaining accurate determinations of the critical exponents, of the small-field expansion of the effective potential, and of the small-momentum behavior of the two-point function.

We use the small-field expansion of the effective potential in the high-temperature phase to determine approximate representations of the equation of state that are valid in the whole critical region. To reach the coexistence curve ($t < 0$) from the high-temperature phase ($t > 0$), an analytic continuation in the complex t plane[6,9] is required. For this purpose we use parametric representations,[10–12] which implement in a rather simple way the known analytic properties of the equation of state (Griffith's analyticity). This approach was successfully applied to the Ising model, for which one can construct a systematic approximation scheme based on polynomial parametric representations[9] and on a global stationarity condition.[8] This leads to an accurate determination of the critical equation of state and of the universal ratios of amplitudes that can be derived from it.[9,4,8] XY systems, in which the phase transition is related to the breaking of the continuous O(2) symmetry, present a new important feature with respect to Ising-like systems: the Goldstone singularities at the coexistence curve. General arguments predict that at the coexistence curve ($t < 0$ and $H \to 0$) the transverse and longitudinal magnetic susceptibilities behave, respectively, as

$$\chi_T = \frac{M}{H}, \quad \chi_L = \frac{\partial M}{\partial H} \sim H^{d/2-2}. \qquad (5)$$

In our analysis we will consider polynomial parametric representations that have the correct singular behavior at the coexistence curve.

By using parametric representations we obtain estimates of several universal amplitude ratios; in particular, of the experimentally important specific-heat amplitude ratio. Our final estimate

$$A^+/A^- = 1.055(3) \qquad (6)$$

is perfectly consistent with the experimental estimate (2), although not equally precise.

The paper is organized as follows. In Sec. II we study the small-field expansion of the effective potential (Helmholtz free energy). We estimate the first few nontrivial coefficients of such expansion by analyzing the corresponding IHT series. The results are then compared with other theoretical estimates. In Sec. III, using as input parameters the critical exponents and the known coefficients of the small-field expansion of the effective potential, we construct approximate representations of the critical equation of state. We obtain new estimates for many universal amplitude ratios. These results are then compared with experimental and other theoretical estimates.

II. THE EFFECTIVE POTENTIAL IN THE HIGH-TEMPERATURE PHASE

A. Small-field expansion of the effective potential in the high-temperature phase

The effective potential (Helmholtz free energy) is related to the (Gibbs) free energy of the model. If $\vec{M} \equiv \langle \vec{\phi} \rangle$ is the magnetization and $\vec{H}$ the magnetic field, one defines

$$\mathcal{F}(M) = \vec{M} \cdot \vec{H} - \frac{1}{V} \ln Z(H), \qquad (7)$$

where $Z(H)$ is the partition function and the dependence on the temperature is always understood in the notation.

In the high-temperature phase the effective potential admits an expansion around $M = 0$:

$$\Delta\mathcal{F} \equiv \mathcal{F}(M) - \mathcal{F}(0) = \sum_{j=1}^{\infty} \frac{1}{(2j)!} a_{2j} M^{2j}. \qquad (8)$$

This expansion can be rewritten in terms of a renormalized magnetization φ

$$\Delta\mathcal{F} = \frac{1}{2} m^2 \varphi^2 + \sum_{j=2} m^{3-j} \frac{1}{(2j)!} g_{2j} \varphi^{2j}, \qquad (9)$$

where

$$\varphi^2 = \frac{\xi(t, H=0)^2 M(t,H)^2}{\chi(t, H=0)}, \qquad (10)$$

t is the reduced temperature, χ and ξ are, respectively, the magnetic susceptibility and the second-moment correlation length

$$\chi = \sum_x \langle \phi_\alpha(0) \phi_\alpha(x) \rangle, \qquad (11)$$

$$\xi = \frac{1}{6\chi} \sum_x x^2 \langle \phi_\alpha(0) \phi_\alpha(x) \rangle,$$

and $m \equiv 1/\xi$. In field theory φ is the expectation value of the zero-momentum renormalized field. The zero-momentum $2j$-point renormalized constants g_{2j} approach universal constants (which we indicate with the same symbol) for $t \to 0$. By performing a further rescaling

$$\varphi = \frac{m^{1/2}}{\sqrt{g_4}} z \tag{12}$$

in Eq. (9), the free energy can be written as

$$\Delta \mathcal{F} = \frac{m^3}{g_4} A(z), \tag{13}$$

where

$$A(z) = \frac{1}{2} z^2 + \frac{1}{4!} z^4 + \sum_{j=3} \frac{1}{(2j)!} r_{2j} z^{2j} \tag{14}$$

and

$$r_{2j} = \frac{g_{2j}}{g_4^{j-1}}, \quad j \geqslant 3. \tag{15}$$

One can show that $z \propto M t^{-\beta}$ and that the equation of state can be written in the form

$$H \propto t^{\beta\delta} F(z), \tag{16}$$

where

$$F(z) = \frac{\partial A(z)}{\partial z} = z + \tfrac{1}{6} z^3 + \sum_{m=3} F_{2m-1} z^{2m-1} \tag{17}$$

with

$$F_{2m-1} = \frac{1}{(2m-1)!} r_{2m}. \tag{18}$$

B. Zero-momentum renormalized couplings by IHT expansion

To compute the high-temperature series of the four-point coupling g_4 and of the effective-potential parameters r_{2j}, we rewrite them in terms of the zero-momentum connected $2j$-point Green's functions χ_{2j}

$$\chi_{2j} = \sum_{x_2, \ldots, x_{2j}} \langle \phi_{\alpha_1}(0) \phi_{\alpha_1}(x_2) \ldots \phi_{\alpha_j}(x_{2j-1}) \phi_{\alpha_j}(x_{2j}) \rangle_c \tag{19}$$

($\chi = \chi_2$). For generic values of N, the number of components of the spin vector, we have

$$g_4 = -\frac{3N}{N+2} \frac{\chi_4}{\chi_2^2 \xi^3} \tag{20}$$

and

$$r_6 = 10 - \frac{5(N+2)}{3(N+4)} \frac{\chi_6 \chi_2}{\chi_4^2}, \tag{21}$$

$$r_8 = 280 - \frac{280(N+2)}{3(N+4)} \frac{\chi_6 \chi_2}{\chi_4^2} + \frac{35(N+2)^2}{9(N+4)(N+6)} \frac{\chi_8 \chi_2^2}{\chi_4^3},$$

$$r_{10} = 15\,400 - \frac{7700(N+2)}{(N+4)} \frac{\chi_6 \chi_2}{\chi_4^2} + \frac{350(N+2)^2}{(N+4)^2} \frac{\chi_6^2 \chi_2^2}{\chi_4^4} + \frac{1400(N+2)^2}{3(N+4)(N+6)} \frac{\chi_8 \chi_2^2}{\chi_4^3} - \frac{35(N+2)^3}{3(N+4)(N+6)(N+8)} \frac{\chi_{10} \chi_2^3}{\chi_4^4}.$$

The formulas relevant for the XY universality class are obtained setting $N=2$.

Using the ϕ^4 lattice Hamiltonian (4), we have calculated χ and $m_2 \equiv \Sigma_x x^2 \langle \phi(0) \phi(x) \rangle$ to 20th order, χ_4 to 18th order, χ_6 to 17th order, χ_8 to 16th order, and χ_{10} to 15th order, for generic values of λ. The IHT expansion, i.e., with suppressed leading scaling corrections, is achieved for $\lambda = 2.10(6)$.[3] In Table I we report the series of m_2, χ, χ_4, χ_6, χ_8, and χ_{10} for $\lambda = 2.10$. Using Eqs. (20) and (21) one can obtain the high-temperature (HT) series necessary for the determination of g_4 and r_{2j}. We analyzed them using the same procedure applied to the IHT expansions of Ising-like systems in Ref. 8. In order to estimate the fixed-point value of g_4 and of the coefficients r_{2j}, we considered Padé, Dlog-Padé and first-order integral approximants of the series in β for $\lambda = 2.10$, and evaluated them at β_c. We refer to Ref. 8 for the details of the analysis. Our estimates are

$$g_4 = 21.05(3+3), \tag{22}$$

$$\bar{g} \equiv \frac{5}{24\pi} g_4 = 1.396(2+2), \tag{23}$$

$$r_6 = 1.951(11+3), \tag{24}$$

$$r_8 = 1.36(6+3). \tag{25}$$

We quote two errors: the first one is related to the spread of the approximants, while the second one gives the variation of the estimate when λ varies between 2.04 and 2.16. In addition, we obtained a rough estimate of r_{10}, i.e., $r_{10} = -13(7)$. For comparison, we anticipate that the analysis of the critical equation of state using approximate parametric representations leads to the estimate $r_{10} = -10(3)$. From the estimates of g_4^* and r_{2j} one can obtain corresponding estimates for the zero-momentum renormalized couplings $g_{2j} = r_{2j} g_4^{j-1}$ with $j > 2$.

Table II compares our results (denoted by IHT) with the estimates obtained using other approaches, such as the HT expansion of the standard lattice spin model (1),[13–15] field-theoretic methods based on the fixed-dimension $d=3$ g expansion[4,16,17] and on the ϵ-expansion.[14,18,19] The fixed-dimension field-theoretic estimates of g_4 have been obtained from the zero of the Callan-Symanzik β-function, whose expansion is known to six loops.[20] In the same framework g_6 and g_8 have been estimated from the analysis of the corresponding four- and three-loop series, respectively.[16] The authors of Ref. 16 argue that the uncertainty on their estimate of g_6 is approximately 0.3%, while they consider their value for g_8 much less accurate. The ϵ-expansion estimates have been obtained from constrained analyses of the four-loop series of g_4 and of the three-loop series of r_{2j}.

TABLE I. Coefficients of the high-temperature expansion of m_2, χ, χ_4, χ_6, χ_8, χ_{10}. They have been obtained using the Hamiltonian (4) with $\lambda = 2.10$.

i	m_2	χ	χ_4
0	0	0.82305062235187783163838	−0.18361621025768068492172
1	1.01611849043072013077966	2.03223698086144026155932	−1.81350523351772810305606
2	5.01790173559352893587925	4.41339999108761956819390	−10.24033284918732045378
3	17.137288764521068686709 2	9.49465465075720590174531	−44.89280738007436266906 82
4	50.572960916471706433779 8	19.843208475785079507388 0	−168.75329431053571275853 2
5	136.97572915318563479793 9	41.310531607716926967502 4	−572.89932535107241202815 5
6	351.54795520401554387353 3	84.865766534658754351794 0	−1806.05498365521624417337
7	867.882609960590948513531	173.919877080583932668034	−5384.52486767607368575700
8	2082.72988964511892008013	353.805866554914925628451	−15362.4317708302607812112
9	4887.89348922134368695104	718.544573182206654200415	−42306.4841196583464482445
10	11271.6038365544561486584	1452.55696588171500306095	−113150.461045583807761395
11	25618.1126339930289598217	2932.78243299692312158659	−295305.926162214852952064
12	57532.0447298988841107389	5902.70244465199387875428	−754781.109517718395587745
13	127889.749773305781861722	11869.0080781568501056056	−1894806.32404641680977111
14	281828.971329949201651321	23811.0622160012068230227	−4682769.64431772632678941
15	616366.979493029224503143	47733.2907485305100325248	−11414677.5171522019349233
16	1339133.76964227829126980	95522.7109197203117005540	−27486804.4332023911210055
17	2892420.77412897864771425	191043.098791148001757550	−65472437.4333935066305991
18	6215040.60947250914305650	381560.204019249641798944	−154435238.761288809747194
19	13292241.3642748890173429	761691.955581673385411208	
20	28309542.5693146536002667	1518865.25572011244933658	

i	χ_6	χ_8	χ_{10}
0	0.41651792496748423277383	−2.07276872395008612032057	17.8224779282297038884825
1	8.19357083273717982769144	−66.6409389277329689706620	845.24799121509307790503 5
2	81.695444131419045282007 0	−1011.50102952099242860043	17983.4651630934745766842
3	577.734561420010600288632	−10327.4974814820902607880	248053.171125799025029980
4	3288.503951894290088148 10	−81557.4477403308614541401	2572480.03564061669203194
5	16096.6021109696857145564	−536525.345841149579660234	21718902.5365937343351130
6	70442.2092664445625747715	−3075097.70592321924332973	156737903.631432308532083
7	282678.282078376456985396	−15817337.6236306593687791	998806344.593851737549288
8	1058442.95252211500336034	−74543191.4965166803297285	5750852421.81974869997198
9	3744677.84665924694446919	−326778866.937610399640322	30428100948.8992362353297
10	12635750.8707582942375013	−1347823157.99877343602884	149865516252.140441504474
11	40959287.0765118834279901	−5277067971.27466652475816	694034037396.324618345020
12	128268696.512235415964465	−19750695382.3283907899788	3046464352917.44305602911
13	389824946.322878356697650	−71066114622.3163753581174	12757807364908.6407194560
14	1153973293.83499223649654	−246972985495.952557381976	51244942192011.6695801488
15	3337461787.55665279189948	−832179913942.254869433972	198319718730601.629272462
16	9454326701.58705373223814	−2727546459564.59095483523	
17	26288539415.6469228911099		

TABLE II. Estimates of $\overline{g} \equiv 5g_4/(24\pi)$, r_6, and r_8.

	IHT	HT	$d=3$ g exp.	ϵ exp.
$\overline{g}$	1.396(4)	1.411(8), 1.406(8) (Ref. 13) 1.415(11) (Ref. 14)	1.403(3) (Ref. 4) 1.40 (Ref. 17)	1.425(24) (Refs. 19,14)
r_6	1.951(14)	2.2(6) (Ref. 15)	1.967 (Ref. 16)	1.969(12) (Refs. 19,18)
r_8	1.36(9)		1.641 (Refs. 16)	2.1(9) (Refs. 19,18)

III. THE CRITICAL EQUATION OF STATE

A. Analytic properties of the scaling equation of state

From the analysis of the IHT series we have obtained the first few nontrivial terms of the small-field expansion of the effective potential in the high-temperature phase. This provides the first few terms in the small-z expansion of the equation of state (16). The function $H(M,t)$, the external field in Eq. (16), satisfies Griffith's analyticity: it is analytic in M around $M=0$ for $t>0$ fixed and in t around $t=0$ for $M>0$ fixed. The first condition implies that $F(z)$ has an expansion in powers of z, see Eq. (17). The second condition implies that for $z\to\infty$, $F(z)$ has an expansion of the form

$$F(z)=z^{\delta}\sum_{n=0} F_n^{\infty} z^{-n/\beta}. \tag{26}$$

To reach the coexistence curve, i.e., $t<0$ and $H=0$, one should perform an analytic continuation in the complex t plane.[6,9] The spontaneous magnetization is related to the complex zero z_0 of $F(z)$. Therefore, the description of the coexistence curve is related to the behavior of $F(z)$ in the neighborhood of z_0.

B. Goldstone singularities at the coexistence curve

The physics of the low-temperature phase of models with an N-vector order parameter—they include XY systems that correspond to $N=2$—is very different from that of the Ising model, because of the presence of Goldstone modes at the coexistence curve. The singular behavior of the longitudinal susceptibility χ_L for $t<0$ and $H\to 0$ is governed by the zero-temperature infrared-stable fixed point.[21–23] In d dimensions, this leads to the prediction

$$f(x)\approx c_f(1+x)^{2/(d-2)} \quad \text{for} \quad x\to -1, \tag{27}$$

where $x\propto tM^{-1/\beta}$ and $f(x)$ is the scaling function introduced in Eq. (3) (as usual, $x=-1$ corresponds to the coexistence curve). This behavior at the coexistence curve has been verified in the framework of the large-N expansion to $O(1/N)$ (i.e., next-to-leading order).[21,24]

The nature of the corrections to the behavior (27) is less clear. Setting $\omega=1+x$ and $y=HM^{-\delta}$, it has been conjectured that ω has the form of a double expansion in powers of y and $y^{(d-2)/2}$ near the coexistence curve,[25,26,23] i.e., for $y\to 0$

$$\omega\equiv 1+x=c_1 y+c_2 y^{1-\epsilon/2}+d_1 y^2+d_2 y^{2-\epsilon/2}+d_3 y^{2-\epsilon}+\cdots, \tag{28}$$

where $\epsilon=4-d$. This expansion has been derived essentially from an ϵ-expansion analysis.[27] Note that in three dimensions this conjecture predicts an expansion in powers of $y^{1/2}$, or equivalently an expansion of $f(x)$ in powers of ω for $\omega\to 0$.

The asymptotic expansion of the d-dimensional equation of state at the coexistence curve has been computed analytically in the framework of the large-N expansion,[28] using the $O(1/N)$ formulas reported in Ref. 21. It turns out that the expansion (28) does not strictly hold for values of the dimension d such that

$$2<d=2+\frac{2m}{n}<4 \quad \text{for} \quad 0<m<n, \quad m,n\in\mathbb{N}. \tag{29}$$

In particular, in three dimensions one finds[28]

$$f(x)=\omega^2\left[1+\frac{1}{N}[f_1(\omega)+\ln\omega f_2(\omega)]+O(N^{-2})\right], \tag{30}$$

where the functions $f_1(\omega)$ and $f_2(\omega)$ have a regular expansion in powers of ω. Moreover,

$$f_2(\omega)=O(\omega^2), \tag{31}$$

so that logarithms appear in the expansion only at next-next-to-leading order.

This analysis indicates that Eq. (28) is not correct in three dimensions since it is not compatible with the exact large-N result. However, it is not clear how to modify it. A possibility is that Eq. (30) holds for all values of N. Note that the presence of logarithms in the expansion does not contradict the conjecture that the behavior near the coexistence curve is controlled by the zero-temperature infrared-stable Gaussian fixed point. In this case, logarithms are not indeed unexpected: for instance, they usually appear in reduced-temperature asymptotic expansions around Gaussian fixed points (see, e.g., Ref. 29).

C. Parametric representations

In order to obtain a representation of the critical equation of state that is valid in the whole critical region, one may use parametric representations, which implement in a simple way all scaling and analytic properties.[10–12] One may parametrize M and t in terms of R and θ according to

$$M=m_0 R^{\beta} m(\theta), \tag{32}$$

$$t=R(1-\theta^2),$$

$$H=h_0 R^{\beta\delta} h(\theta),$$

where h_0 and m_0 are normalization constants. The variable R is nonnegative and measures the distance from the critical point in the (t,H) plane; it carries the power-law critical singularities. The variable θ parametrizes the displacements along the line of constant R. The functions $m(\theta)$ and $h(\theta)$ are odd and regular at $\theta=0$ and at $\theta=1$. The constants m_0 and h_0 can be chosen so that $m(\theta)=\theta+O(\theta^3)$ and $h(\theta)=\theta+O(\theta^3)$. The smallest positive zero of $h(\theta)$, which should satisfy $\theta_0>1$, represents the coexistence curve, i.e., $T<T_c$ and $H\to 0$.

The parametric representation satisfies the requirements of regularity of the equation of state. Singularities can appear only at the coexistence curve (due for example to the logarithms discussed in Sec. III B), i.e., for $\theta=\theta_0$. Notice that the mapping (32) is not invertible when its Jacobian vanishes, which occurs when

$$Y(\theta)\equiv(1-\theta^2)m'(\theta)+2\beta\theta m(\theta)=0. \tag{33}$$

Thus, parametric representations based on the mapping (32) are acceptable only if $\theta_0<\theta_l$ where θ_l is the smallest posi-

tive zero of the function $Y(\theta)$. One may easily verify that the asymptotic behavior (27) is reproduced simply by requiring that

$$h(\theta) \approx (\theta_0 - \theta)^2 \quad \text{for} \quad \theta \to \theta_0 . \tag{34}$$

The relation among the functions $m(\theta)$, $h(\theta)$, and $F(z)$ is given by

$$z = \rho m(\theta)(1-\theta^2)^{-\beta}, \tag{35}$$

$$F[z(\theta)] = \rho(1-\theta^2)^{-\beta\delta} h(\theta), \tag{36}$$

where ρ is a free parameter.[9,8] In the exact parametric equation the value of ρ may be chosen arbitrarily but, as we shall see, when adopting an approximation procedure the dependence on ρ is not eliminated. In our approximation scheme we will fix ρ to ensure the presence of the Goldstone singularities at the coexistence curve, i.e., the asymptotic behavior (34). Since $z = \rho\,\theta + O(\theta^3)$, expanding $m(\theta)$ and $h(\theta)$ in (odd) powers of θ,

$$m(\theta) = \theta + \sum_{n=1} m_{2n+1} \theta^{2n+1}, \tag{37}$$

$$h(\theta) = \theta + \sum_{n=1} h_{2n+1} \theta^{2n+1},$$

and using Eqs. (35) and (36), one can find the relations among ρ, m_{2n+1}, h_{2n+1}, and the coefficients F_{2n+1} of the expansion of $F(z)$.

One may also write the scaling function $f(x)$ in terms of the parametric functions $m(\theta)$ and $h(\theta)$:

$$x = \frac{1-\theta^2}{\theta_0^2 - 1}\left[\frac{m(\theta_0)}{m(\theta)}\right]^{1/\beta}, \tag{38}$$

$$f(x) = \left[\frac{m(\theta)}{m(1)}\right]^{-\delta} \frac{h(\theta)}{h(1)}.$$

In Appendix A we report the definitions of some universal ratios of amplitudes that have been introduced in the literature and the corresponding expressions in terms of the functions $m(\theta)$ and $h(\theta)$.

D. Approximate polynomial representations

In order to construct approximate parametric representations we consider polynomial approximations of $m(\theta)$ and $h(\theta)$. This kind of approximation turned out to be effective in the case of Ising-like sistems.[9,8] The major difference with respect to the Ising case is the presence of the Goldstone singularities at the coexistence curve. In order to take them into account, at least in a simplified form which neglects the logarithms found in Eq. (30), we require the function $h(\theta)$ to have a double zero at θ_0 as in Eq. (34). Polynomial schemes may in principle reconstruct also the logarithms, but of course only in the limit of an infinite number of terms.

In order to check the accuracy of the results, it is useful to introduce two distinct schemes of approximation. In the first one, which we denote as (A), $h(\theta)$ is a polynomial of fifth order with a double zero at θ_0, and $m(\theta)$ a polynomial of order $(1+2n)$:

$$\text{scheme (A):} \quad m(\theta) = \theta\left(1 + \sum_{i=1}^{n} c_i \theta^{2i}\right), \tag{39}$$

$$h(\theta) = \theta(1 - \theta^2/\theta_0^2)^2.$$

In the second scheme, denoted by (B), we set

$$\text{scheme (B):} \quad m(\theta) = \theta, \tag{40}$$

$$h(\theta) = \theta(1-\theta^2/\theta_0^2)^2\left(1 + \sum_{i=1}^{n} c_i \theta^{2i}\right).$$

Here $h(\theta)$ is a polynomial of order $5+2n$ with a double zero at θ_0. In both schemes the parameter ρ is fixed by the requirement (34), while θ_0 and the n coefficients c_i are determined by matching the small-field expansion of $F(z)$. This means that, for both schemes, in order to fix the n coefficients c_i we need to know $n+1$ values of r_{2j}, i.e., $r_6, \ldots r_{6+2n}$. Note that for the scheme (B)

$$Y(\theta) = 1 - \theta^2 + 2\beta\theta^2, \tag{41}$$

independently of n, so that $\theta_l = (1-2\beta)^{-1}$. Concerning the scheme (A), we note that the analyticity of the thermodynamic quantities for $|\theta| < \theta_0$ requires the polynomial function $Y(\theta)$ not to have complex zeroes closer to origin than θ_0.

In Appendix B we present a more general discussion on the parametric representations.

E. Results

As input parameters for the determination of the parametric representations, we use the best available estimates of the critical exponents, which are $\alpha = -0.01285(38)$ (from the experiment of Ref. 1), $\eta = 0.0381(3)$ (from the high-temperature analysis of Ref. 2). Moreover we use the following estimates of r_{2j}: $r_6 = 1.96(2)$ which is compatible with all the estimates of r_6 reported in Table II, and $r_8 = 1.40(15)$ which takes somehow into account the differences among the various estimates.

The case $n=0$ of the two schemes (A) and (B) is the same, and requires the knowledge of α, η, and r_6. Unfortunately this parametrization does not satisfy the consistency condition $\theta_0^2 < \theta_l^2 = (1-2\beta)^{-1}$. Both schemes give acceptable approximations for $n=1$, using r_8 as an additional input parameter. The numerical values of the relevant parameters and the resulting estimates of universal amplitude ratios (see the appendix for their definition) are shown in Table III. The errors reported are related to the errors of the input parameters only. They do not take into account possible systematic errors due to the approximate procedure we are employing. We will return on this point later.

In Figs. 1 and 2 we show respectively the scaling functions $F(z)$ and $f(x)$, as obtained from the approximate representations given by the schemes (A) and (B) for $n=1$, using the input values $\alpha = -0.01285$, $\eta = 0.0381$, $r_6 = 1.96$, and $r_8 = 1.4$. The two approximations of $F(z)$ are practically indinstinguishable in Fig. 1. This is also numerically confirmed by the estimates of the universal costant F_0^∞ (reported in Table III), which is related to the large-z behavior of $F(z)$, see Eq. (26):

TABLE III. Results for the parameters and the universal amplitude ratios using the scheme (A), see Eq. (39), and the scheme (B), see Eq. (40). The label above each column indicates the scheme, the number of terms in the corresponding polynomial, and the input parameters employed, in addition to the critical exponents α and η. Note that the quantities reported in the first four lines do not have a physical meaning, but are related to the particular parametric representation employed. Numbers marked with an asterisk are inputs, not predictions.

	[(A) $n=1$; r_6, r_8]	[(B) $n=1$; r_6, r_8]	[(B) $n=2$; r_6, r_8, r_{10}]	[(B) $n=2$; $r_6, r_8, A^+/A^-$]
ρ	2.22(3)	2.07(2)	2.04(5)	2.01(4)
θ_0^2	3.84(10)	2.97(10)	2.8(4)	2.5(2)
c_1	−0.024(7)	0.72(12)	0.10(6)	0.15(5)
c_2	0	0	0.01(2)	0.02(1)
r_{10}	−9.6(1.1)	−11(2)	*−13(7)	−7(5)
A^+/A^-	1.055(3)	1.057(2)	1.055(3)	*1.054(1)
R_c	0.123(8)	0.113(3)	0.118(7)	0.123(6)
R_4	7.5(3)	7.9(2)	7.8(3)	7.6(2)
R_ξ^+	0.353(3)	0.350(2)	0.352(3)	0.354(3)
R_χ	1.38(9)	1.51(3)	1.47(8)	1.41(6)
F_0^∞	0.0303(3)	0.0301(3)	0.0302(3)	0.0304(3)
c_f	5(4)	62(41)		15(10)

$$F(z) \approx F_0^\infty z^\delta \quad \text{for} \quad z \to \infty. \tag{42}$$

This agreement is not trivial since the small-z expansion has a finite convergence radius given by $|z_0| = R_4^{1/2} \simeq 2.8$. Therefore, the determination of $F(z)$ on the whole positive real axis from its small-z expansion requires an analytic continuation, which turns out to be effectively performed by the approximate parametric representations we have considered. We recall that the large-z limit corresponds to the critical theory $t=0$, so that positive real values of z describe the high-temperature phase up to $t=0$. Instead, larger differences between the approximations given by the schemes (A) and (B) for $n=1$ appear in the scaling function $f(x)$, especially in the region $x<0$ corresponding to $t<0$ (i.e., the region which is not described by real values of z). Note that the apparent differences for $x>0$ are essentially caused by the normalization of $f(x)$, which is performed at the coexistence curve $x=-1$ and at the critical point $x=0$ requiring $f(-1)=0$ and $f(0)=1$. Although the large-x region corresponds to small z, the difference between the two approximate schemes does not decrease in the large-x limit due to their slightly different estimates of R_χ (see Table III). Indeed, for large values of x, $f(x)$ has an expansion of the form

$$f(x) = x^\gamma \sum_{n=0} f_n^\infty x^{-2n\beta} \tag{43}$$

with $f_0^\infty = R_\chi^{-1}$.

We also considered the case $n=2$, using the estimate $r_{10} = -13(7)$. In this case the scheme (A) was not particularly useful because it turned out to be very sensitive to r_{10}, whose estimate has a relatively large error. Combining the consistency condition $\theta_0 < \theta_l$ (which excludes values of $r_{10} \lesssim -10$ when using the central values of α, η, r_6, and r_8) with the IHT estimate of r_{10}, we found a rather good result for A^+/A^-, i.e., $A^+/A^- = 1.053(4)$. On the other hand, the

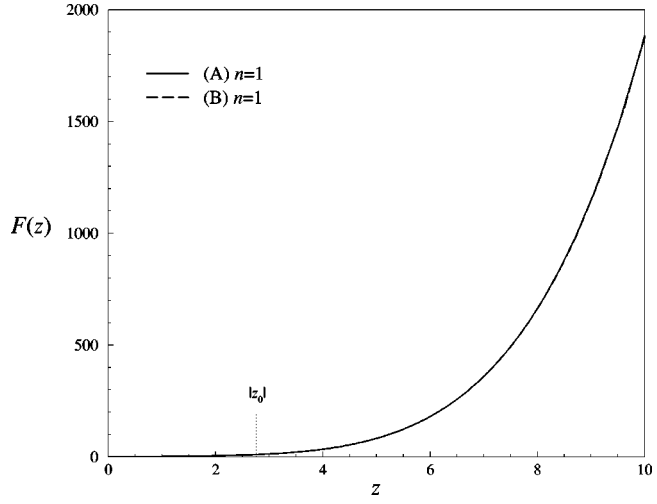

FIG. 1. The scaling function $F(z)$ vs z. The convergence radius of the small-z expansion is expected to be $|z_0| = R_4^{1/2} \simeq 2.8$.

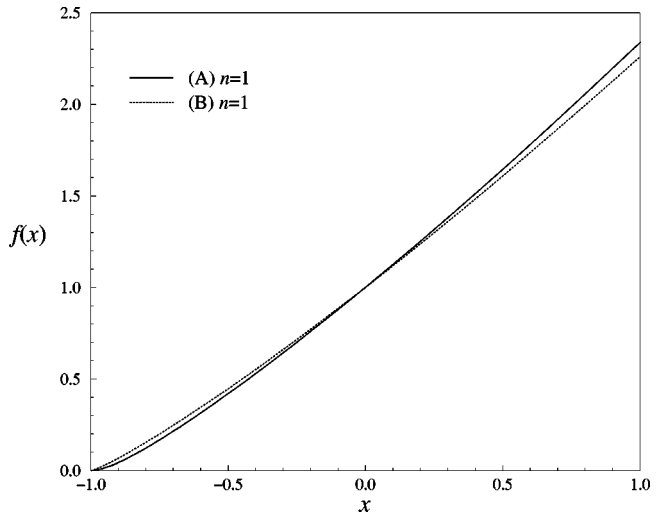

FIG. 2. The scaling function $f(x)$ vs x.

TABLE IV. Estimates of universal ampitude ratios obtained in different approaches. The ϵ-expansion estimates of R_c and R_χ have been obtained by setting $\epsilon=1$ in the $O(\epsilon^2)$ series calculated in Refs. 30.

	IHT-PR	HT	$d=3$ exp.	ϵ exp.	experiments
A^+/A^-	1.055(3)		1.056(4) (Ref. 31)	1.029(13) (Ref. 32)	1.054(1) (Ref. 1) 1.058(4) (Ref. 33) 1.067(3) (Ref. 34) 1.088(7) (Ref. 35)
R_ξ^+	0.353(3)	0.361(4), 0.362(4) (Ref. 36)	0.3606(20) (Ref. 29,37)	0.36 (Ref. 38)	
R_c	0.12(1)		0.123(3) (Ref. 39)	0.106	
R_χ	1.4(1)			1.407	

results for the other universal amplitude ratios considered, such as R_χ, R_c, R_4, etc., although consistent, turned out to be much more imprecise than those obtained for $n=1$, for example, $R_\chi=1.2(3)$. This fact may be explained noting that, except for the small interval $-10\lesssim r_{10}\lesssim -9$ (this interval corresponds to the central values of the other input parameters), the function $Y(\theta)$, see Eq. (33), has zeroes in the the complex plane which are closer to the origin than θ_0. Therefore, the parametric function $g_2(\theta)$ related to the magnetic susceptibility (see Appendix A2), and higher-order derivatives of the free-energy, have poles within the disk $|\theta|<\theta_0$. On the other hand, in the case $n=1$, θ_0^2 was closer to the origin than the zeroes of $Y(\theta)$ for the whole range of values of the input parameters.

For these reasons, for $n=2$, we present results only for the scheme (B). Combining the consistency condition $\theta_0<\theta_l$, which restricts the acceptable values of r_{10} (for example using the central estimates of the other input parameters it excludes values $|r_{10}|\gtrsim 12$), with the IHT estimate $r_{10}=-13(7)$, we arrive at the results reported in Table III (third column of data). The reported value has been obtained using $r_{10}\approx -9$.

The coefficients c_i, reported in Table III, turn out to be relatively small in both schemes, and decrease rapidly, supporting our choice of the approximation schemes. The results of the various approximate parametric representations are in reasonable agreement. Their comparison is useful to get an idea of the systematic error due to the approximation schemes. There is a very good agreement for A^+/A^-, which is the experimentally most important quantity. Our final estimate is

$$A^+/A^- = 1.055(3), \tag{44}$$

obtained by considering our best approximations: scheme (A) with $n=1$ and scheme (B) with $n=2$. The tricky point is setting the error. Indeed, there are two sources of error. One has to consider the error due to the uncertainty of the input parameters—this is the one reported in Table III for each estimate—and the systematic error due to the polynomial truncation. The latter type of uncertainty is very difficult to estimate since we do not have results for many values of n. Comparing the results obtained with the two different schemes, no systematic discrepancy is observed. This seems to indicate that the error due to the polynomial truncation is somewhat smaller than the error due to the input parameters. Therefore, the error we quote is simply the latter. This is not a conservative error estimate. However, we believe it has the same reliability of the uncertainties quoted in Monte Carlo and experimental works, which claim that the correct estimate lies within *two* error bars—the so-called 95% confidence interval.

We mention that approximately one half of the error on A^+/A^- is due to the uncertainty on the critical exponent α, which unfortunately can be hardly improved by present theoretical means. Finally, to further check our results, we applied again the scheme (B) for $n=2$, using, instead of r_{10}, the precise experimental estimate $A^+/A^-=1.054(1)$ as input parameter. In practice we fix the coefficient c_2 in such a way to obtain the experimental estimate of A^+/A^-. The idea is to use the quantities known with the highest precision to determine, within our scheme of approximation, the equation of state and the corresponding universal amplitude ratios. The results are reported in the last column of Table III. They are in good agreement with those obtained before.

Averaging our best estimates, scheme (A) with $n=1$ and scheme (B) with $n=2$, we obtain the final estimates

$$R_\xi^+ \equiv (A^+)^{1/3} f^+ = 0.353(3), \tag{45}$$

$$R_c \equiv \frac{\alpha A^+ C^+}{B^2} = 0.12(1), \tag{46}$$

$$R_\chi \equiv \frac{C^+ B^{\delta-1}}{(\delta C^c)^\delta} = 1.4(1), \tag{47}$$

$$R_4 \equiv -\frac{C_4^+ B^2}{(C^+)^3} = |z_0|^2 = 7.6(4), \tag{48}$$

$$F_0^\infty \equiv \lim_{z\to\infty} z^{-\delta} F(z) = 0.0303(3), \tag{49}$$

$$0 < c_f \lesssim 20. \tag{50}$$

The determination of c_f, see Eq. (27), turns out to be rather unstable, indicating that the approximate parametric representation we have constructed are still relatively inaccurate in the region very close to the coexistence curve. The constant c_f is very sensitive to the values of the coefficients r_{2j}. Improved estimates of r_{2j} would be important especially for c_f. The approximation schemes (A) and (B) with $n=1$ provide also estimates of r_{10}. We obtain $r_{10}=-10(3)$, which is agreement with the IHT result $r_{10}=-13(7)$.

In Table IV we compare our results (denoted by IHT-PR) with the available estimates obtained from other theoretical

approaches and from experiments (for a review see, e.g., Ref. 40). Here we report results obtained from the analysis of high-temperature series (HT), of fixed-dimension perturbative expansions ($d=3$ exp.) and of expansions in $\epsilon=4-d$ (ϵ exp.). We also mention the estimate of Ref. 41, $A^+/A^-=1.08$, obtained from the phenomenological relation $A^+/A^- \approx 1-4\alpha$ and the HT estimate of α available at the time (using our estimate of α we would obtain $A^+/A^- = 1.06$, while using the experimental value we would get $A^+/A^- = 1.05$). Our estimate of A^+/A^- is as precise as the estimate reported in Ref. 31, obtained in the field-theoretic framework of minimal renormalization without ϵ expansion, which is a perturbative expansion at fixed dimension $d=3$. The agreement with the experimental result of Ref. 1 is very good.

F. Conclusions

Starting from the small-field expansion of the effective potential in the high-temperature phase, we have constructed approximate representations of the critical equation of state valid in the whole critical region. We have considered two approximation schemes based on polynomial representations that satisfy the general analytic properties of the equation of state (Griffith's analyticity) and take into account the Goldstone singularities at the coexistence curve. The coefficients of the truncated polynomials are determined by matching the small-field expansion in the high-temperature phase, which has been studied by lattice high-temperature techniques. The schemes considered can be systematically improved by increasing the order of the polynomials. However, such possibility is limited by the number of known coefficients r_{2j} of the small-field expansion of the effective potential. We have shown that the knowledge of the first few r_{2j} already leads to satisfactory results, for instance for the specific-heat amplitude ratio. Through the approximation schemes we have presented in this paper, the determination of the equation of state may be improved by a better determination of the coefficients r_{2j}, which may be achieved by extending the high-temperature expansion. We hope to return to this issue in the future.

Finally, we mention that the approximation schemes which we have proposed can be applied to other N-vector models. Physically relevant values are $N=3$ and $N=4$. The case $N=3$ describes the critical phenomena in isotropic ferromagnets.[42] The case $N=4$ is interesting for high-energy physics: it should describe the critical behavior of finite-temperature QCD with two flavors of quarks at the chiral-symmetry restoring phase transition.[43]

APPENDIX A: UNIVERSAL RATIOS OF AMPLITUDES

1. Notations

Universal ratios of amplitudes characterize the critical behavior of thermodynamic quantities that do not depend on the normalizations of the external (e.g., magnetic) field, order parameter (e.g., magnetization), and temperature. Amplitude ratios of zero-momentum quantities can be derived from the critical equation of state. We consider several amplitudes derived from the singular behavior of the specific heat

$$C_H = A^{\pm} |t|^{-\alpha}, \tag{A1}$$

of the magnetic susceptibility in the high-temperature phase

$$\chi = \frac{1}{2} C^+ t^{-\gamma}, \tag{A2}$$

of the zero-momentum four-point connected correlation function in the high-temperature phase

$$\chi_4 = \frac{8}{3} C_4^+ t^{-\gamma-2\beta\delta}, \tag{A3}$$

of the second-moment correlation length in the high-temperature phase

$$\xi = f^+ t^{-\nu}, \tag{A4}$$

and of the spontaneous magnetization on the coexistence curve

$$M = B|t|^{\beta}. \tag{A5}$$

Using the above-reported normalizations for the amplitudes, the zero-momentum four-point coupling g_4, see Eq. (20), can be written as

$$g_4 = -\frac{C_4^+}{(C^+)^2 (f^+)^3}. \tag{A6}$$

In addition, we consider the amplitude derived from the critical behavior of the longitudinal susceptibility

$$\chi_L = C^c |H|^{-\gamma/\beta\delta}, \tag{A7}$$

along the critical isotherm.

2. Universal ratios of amplitudes from the parametric representation

In the following we report the expressions of the universal ratios of amplitudes in terms of the parametric representation (32) of the critical equation of state. The singular part of the free energy per unit volume can be written as

$$\mathcal{F} = h_0 m_0 R^{2-\alpha} g(\theta), \tag{A8}$$

where $g(\theta)$ is the solution of the first-order differential equation

$$(1-\theta^2) g'(\theta) + 2(2-\alpha)\theta g(\theta) = Y(\theta) h(\theta) \tag{A9}$$

that is regular at $\theta=1$. The function $Y(\theta)$ has been defined in Eq. (33). The longitudinal magnetic susceptibility can be written as

$$\chi_L^{-1} = \frac{h_0}{m_0} R^{\gamma} g_2(\theta), \quad g_2(\theta) = \frac{2\beta\delta\theta h(\theta) + (1-\theta^2) h'(\theta)}{Y(\theta)}. \tag{A10}$$

In order to reproduce the predicted Goldstone singularities, the function $g_2(\theta)$ must vanish at θ_0 according to

$$g_2(\theta) \sim \theta_0 - \theta \quad \text{for} \quad \theta \to \theta_0. \tag{A11}$$

From Eq. (A10) we see that $g_2(\theta)$ satisfies this condition if $h(\theta) \sim (\theta_0 - \theta)^2$ for $\theta \to \theta_0$.

From the equation of state one can derive universal amplitude ratios of quantities defined at zero momentum, i.e., integrated in the volume. We consider

$$A^+/A^- = (\theta_0^2-1)^{2-\alpha}\frac{g(0)}{g(\theta_0)}, \tag{A12}$$

$$R_c \equiv \frac{\alpha A^+ C^+}{B^2} = -\alpha(1-\alpha)(2-\alpha)(\theta_0^2-1)^{2\beta} \times [m(\theta_0)]^{-2} g(0), \tag{A13}$$

$$R_4 \equiv -\frac{C_4^+ B^2}{(C^+)^3} = |z_0|^2 = \rho^2 [m(\theta_0)]^2 (\theta_0^2-1)^{-2\beta}, \tag{A14}$$

$$R_\chi \equiv \frac{C^+ B^{\delta-1}}{(\delta C^c)^\delta} = (\theta_0^2-1)^{-\gamma}[m(\theta_0)]^{\delta-1}[m(1)]^{-\delta} h(1). \tag{A15}$$

Using Eqs. (35) and (36) one can compute $F(z)$ and obtain the small-z expansion coefficients r_{2j} of the effective potential. The constant F_0^∞, which is related to the behavior of $F(z)$ for $z\to\infty$, see Eq. (26), is given by

$$F_0^\infty \equiv \lim_{z\to\infty} z^{-\delta} F(z) = \rho^{1-\delta}[m(1)]^{-\delta} h(1). \tag{A16}$$

Using relations (38), one can easily obtain the constant c_f, which is related to the behavior of $f(x)$ for $x\to -1$, see Eq. (27),

$$c_f \equiv \lim_{x\to -1} (1+x)^{-2} f(x). \tag{A17}$$

We consider also the universal amplitude ratio $R_\xi^+ \equiv (A^+)^{1/3} f^+$ which can be obtained from the estimates of R_4, R_c, and g_4:

$$R_\xi^+ \equiv (A^+)^{1/3} f^+ = \left(\frac{R_4 R_c}{g_4}\right)^{1/3}. \tag{A18}$$

We mention that in the case of superfluid helium it is customary to define a different hyperuniversal combination

$$R_\xi^{\mathrm{T}} \equiv (A^-)^{1/3} f_{\mathrm{T}}^-, \tag{A19}$$

where f_{T}^- is the amplitude of a transverse correlation length ξ_{T} defined from the stiffness constant ρ_s, i.e., $\xi_{\mathrm{T}} = \rho_s^{-1}$. R_ξ^{T} can be determined directly from experiments below T_c (see, e.g. Ref. 40).

APPENDIX B: GENERAL DISCUSSION ON THE PARAMETRIC REPRESENTATIONS

A wide family of parametric representations was introduced a long time ago,[10–12] in forms that can be related to our Eqs. (32). Since the application of parametric representations in practice requires some approximation scheme, one may explore the freedom left in these representations and understand how this freedom may be exploited in order to optimize the approximation. The parametric form of the equation of state forces relations between the two functions $\bar{m} \equiv \rho m(\theta)$ and $h(\theta)$, but it is easy to get convinced that one of the two functions can be chosen arbitrarily. For definiteness, let us take $\bar{m}$ to be arbitrary and find the constraints that must be satisfied by $h(\theta)$ as a consequence of the equation of state.

It is convenient for our purposes to establish these constraints by imposing the formal independence of the function $F(z)$ from the parametrization adopted for $\bar{m} \equiv \rho m(\theta)$, which we may symbolically write in the form of a functional equation

$$\frac{\delta}{\delta \bar{m}}\left[\frac{\rho h(\theta)}{(1-\theta^2)^{\beta\delta}}\right] = 0, \tag{B1}$$

keeping z fixed. By expanding $m(\theta)$ according to Eq. (37) and treating the coefficients ρ and $c_n \equiv m_{2n+1}$ as variational parameters, we may turn the above equation into a set of partial differential equations (keeping z fixed)

$$\frac{d}{d\rho}\left[\frac{\rho h(\theta)}{(1-\theta^2)^{\beta\delta}}\right] = 0, \tag{B2}$$

$$\frac{d}{dc_i}\left[\frac{\rho h(\theta)}{(1-\theta^2)^{\beta\delta}}\right] = 0, \tag{B3}$$

which must be satisfied exactly for all i by the function $h(\theta)$. Simple manipulations lead to the following explicit form:

$$Y(\theta)\left(h + \rho\frac{\partial h}{\partial \rho}\right) = m(\theta)\left[(1-\theta^2)\frac{\partial h}{\partial \theta} + 2\beta\delta\theta h\right], \tag{B4}$$

$$Y(\theta)\frac{\partial h}{\partial c_i} = \theta^{2i+1}\left[(1-\theta^2)\frac{\partial h}{\partial \theta} + 2\beta\delta\theta h\right], \tag{B5}$$

where $Y(\theta)$ is defined in Eq. (41). In turn, by expanding

$$h(\theta,\rho,c_i) = \theta + \sum_{n=1}^{\infty} h_{2n+1}(\rho,c_i)\,\theta^{2n+1}, \tag{B6}$$

and substituting into the above equations one obtains an infinite set of linear differential recursive equations for the coefficients h_{2n+1}, which generalize the relations found in Ref. 8, where the case $m(\theta) = \theta$ was analyzed. A typical approximation to the exact parametric equation of state amounts to a truncation of h to a polynomial form. We may in this case refine the approximation by reinterpreting the first recursion equations involving a coefficient h_{2t+1} which is forcefully set equal to zero as stationarity conditions, which force the parameters ρ and c_i into the values minimizing the unwanted dependence of the truncated $F(z)$ on the parameters themselves, i.e., on the choice of the function $\bar{m} \equiv \rho m(\theta)$. The above procedure implies global (i.e., θ independent) stationarity, and as a consequence all physical amplitudes turn out to be stationary with respect to variations of ρ and c_i.

These statements are fairly general, but it is certainly interesting to consider the first few nontrivial examples. For the lowest order truncation of h we may adopt the parametrization

$$h(\theta)=\theta\left(1-\frac{\theta^2}{\theta_0^2}\right)^p, \tag{B7}$$

which includes both the Ising model in three dimensions ($p=1$) and general $O(N)$ symmetric models with Goldstone bosons in d dimensions [$p=2/(d-2)$]. It is easy to recognize that the following relationship must then hold:

$$\frac{1}{6}\rho^2+c_1=\gamma-\frac{p}{\theta_0^2}. \tag{B8}$$

Global stationarity implies that the stability conditions may be extracted from the variation of any physical quantity. In particular we may concentrate on the universal zero of $F(z)$, z_0, noting that $z_0=z(\theta_0)$, see Eq. (35). As a consequence of the above results, the simplest models can all be described by the parametrization

$$\frac{z_0}{\sqrt{6}}=\frac{\sqrt{(\gamma-c_1)\theta_0^2-p}}{(1-\theta_0^2)^\beta}(1+c_1\theta_0^2). \tag{B9}$$

Let us first consider the case $c_1=0$. The minimization procedure leads to

$$\rho^2=\frac{6\gamma(\gamma-p)}{\gamma-2p\beta},$$

$$\theta_0^2=\frac{\gamma-2p\beta}{(1-2\beta)\gamma}. \tag{B10}$$

Setting $p=1$ one immediately recognizes the linear parametric model representation for the Ising model.[10–12] Unfortunately when $p=2$ the solution is not physically satisfactory, because it gives $\theta_0^2<0$ for all $N\geqslant 2$, and therefore the scheme is useless for models with Goldstone singularities.

Let us now include c_1. Requiring z_0 to be stationary with respect to variations of both parameters, we obtain

$$\rho^2=\frac{6\gamma(\gamma-p+1)}{3\gamma-2(p-1)\beta},$$

$$\theta_0^2=\frac{3\gamma-2(p-1)\beta}{(3-2\beta)\gamma},$$

$$c_1=\gamma\frac{2(\gamma-p)+(2\beta-1)}{3\gamma-2(p-1)\beta}, \tag{B11}$$

implying also

$$\frac{|z_0|}{\sqrt{6}}=2\left(\frac{\gamma-p+1}{3-2\beta}\right)^{3/2-\beta}\left(\frac{\gamma}{2\beta}\right)^\beta. \tag{B12}$$

In the Ising model the above solution reduces to

$$\rho^2=2\gamma, \tag{B13}$$

$$\theta_0^2=\frac{3}{3-2\beta},$$

$$c_1=\frac{2}{3}(\beta+\gamma)-1.$$

Note that, substituting the physical values of the critical exponents β and γ for the $N=1$ model, c_1 turns out to be a very small number ($c_1=0.04256$) and the predicted numerical value of z_0 is 2.8475, consistent within 1% with the linear parametric model prediction.[8] It is fair to say that in the Ising case the above solution has a status which is comparable to the linear parametric model, both conceptually and in terms of predicting power. It is therefore possible to take it as the starting point of an alternative approximation scheme whose higher-order truncations might prove quite effective.

Unfortunately when we consider the XY system, setting $p=2$ and choosing the values of the exponents pertaining to $N=2$, the value of c_1 becomes too large for the approximation to be sensible. Indeed we get $c_1=-0.6762$ and all testable predictions turn out to be far away from the corresponding physical values. It is, however, worth exploring the features of this approach because, as we shall show, it has formal properties which might prove useful when considering parametric representations of the equation of state for higher values of N. Let us indeed consider the function $g_2(\theta)$ entering the parametric representation of the magnetic susceptibility. We know that this function will in general show singularities in the complex θ plane corresponding to the zeroes of the function $Y(\theta)$. However, when substituting the expressions of $h(\theta)$ and $m(\theta)$ obtained from the saddle-point evaluation of the parameters ρ, θ_0, and c_1, after some simple manipulations, we find out that all singularities cancel and

$$g_2(\theta)=\left(1-\frac{\theta^2}{\theta_0^2}\right)^{p-1}. \tag{B14}$$

This fact was already observed in the case $c_1=0$ for all values of the truncation order t. Therefore, the stationarity prescription is a way to ensure a higher degree of regularity in the parametric representation of thermodynamic functions.

Finally, let us observe that the stationary solution can be applied to the large-N limit of $O(N)$ models in any dimension $2<d<4$. In this limit $\beta=\frac{1}{2}$, $\gamma=p=2/(d-2)$. As a consequence we obtain from our previous results $\rho^2=12/(d+2)$, $\theta_0^2=(d+2)/4$, $c_1=0$, implying also

$$h(\theta)=\theta\left(1-\frac{4}{d+2}\theta^2\right)^{2/(d-2)},$$

$$g_2(\theta)=\left(1-\frac{4}{d+2}\theta^2\right)^{(4-d)/(d-2)}. \tag{B15}$$

We therefore obtained an exact parametrization of the equation of state in the large-N limit for all d. Thus, for sufficiently large values of N, the scheme we have defined may be a sensible starting point for the parametric representation of the thermodynamical functions in the critical domain.

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