Density-wave states of nonzero angular momentum

Chetan Nayak
Physics Department, University of California, Los Angeles, CA 90095–1547
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We study the properties of states in which particle-hole pairs of nonzero angular momentum condense. These states generalize charge- and spin-density-wave states, in which *s*-wave particle-hole pairs condense. We show that the *p*-wave spin-singlet state of this type has Peierls ordering, while the *d*-wave spin-singlet state is the staggered flux state. We discuss model Hamiltonians which favor *p*- and *d*-wave density-wave order. There are analogous orderings for pure spin models, which generalize spin-Peierls order. The spin-triplet density-wave states are accompanied by spin-1 Goldstone bosons, but these excitations do not contribute to the spin-spin correlation function. Hence they must be detected with NQR or Raman-scattering experiments. Depending on the geometry and topology of the Fermi-surface, these states may admit gapless fermionic excitations. As the Fermi-surface geometry is changed, these excitations disappear at a transition which is third order in mean-field theory. The singlet *d*-wave- and triplet *p*-wave density-wave states are separated from the corresponding superconducting states by zero-temperature $O(4)$-symmetric critical points.

I. INTRODUCTION

In recent years, a number of materials have been uncovered in which the competition between effective attractive interaction and short-range repulsion appears to lead to the formation of superconducting states in which the Cooper pairs have nonzero relative angular momentum. In this paper, we suggest that such competition can also lead to density-wave states formed by the condensation of *particle-hole* pairs of nonzero relative angular momentum. These states generalize familiar charge- and spin-density-wave states, in which *s*-wave particle-hole pairs condense. We discuss several different possible ordering schemes, the types of interactions which favor them, their physical properties, and their possible relevance to experiments.

Several such states are already commonly known by other names, as we will show below. The singlet $l=1$ density-wave state is simply the Peierls state (or bond-ordered wave), while the singlet $l=2$ density-wave state is known as the staggered flux state of Refs. 1–3. However, triplet analogs of these states have not been discussed. Since the triplet analogs of these states break spin-rotational invariance, they have $S=1$ Goldstone boson excitations. However, the ground state does not have a nonzero expectation value for the spin at any wave vector. Hence, as we will see, these Goldstone bosons cannot be detected in experiments which couple simply to the spin density, such as neutron scattering or NMR. Instead, Raman scattering or NQR are necessary to couple to the Goldstone bosons of these more subtle types of ordering. More generaly, *s*-wave probes cannot couple directly to the orders discussed here; instead, local probes or those which couple to higher powers of the order parameter are necessary.

The *p*- and *d*-wave density-wave states are favored by the same types of interactions which favor the *s*-wave state—i.e., the charge-density wave (CDW). However, they evade interactions which disfavor CDW order. Similarly, they are favored by superconductivity-favoring pair-hopping terms,[4–7] while evading interactions which disfavor superconductivity. Hence, they are rather natural candidates for systems with competing repulsive interactions.

As in the case of higher angular momentum superconducting states, there is the possibility of gapless excitations, since the order parameter can have nodes on the Fermi surface. To consider one consequence of this, suppose that the shape of the Fermi surface is such that the nodes of the order parameter do not lie on the Fermi surface. Let us distort the shape of the Fermi surface by, say, changing the anisotropy between the hopping parameters, which can be done by applying uniaxial pressure. At the mean-field level, a third-order phase transition can occur at which gapless excitations appear. After this point, the system remains critical as a result of the node.

The analogy with superconductivity can be taken a step further by combining density wave order with superconducting order in a pseudospin SU(2) triplet following Yang[8] and Zhang.[9] At a critical point between the two types of order, this pseudospin SU(2) could become exact, giving—together with SU(2) spin symmetry—an $O(4)$-invariant critical point. We discuss the possible relevance of such a critical point to the pseudogap regime of the cuprate superconductors.

Particle-hole condensates with nonzero angular momentum were considered in the context of excitonic insulators by Halperin and Rice.[10] They were rediscovered in the context of the mean-field instabilities of extended Hubbard models by Schulz[11] and Nersesyan and co-workers.[12,13] At around the same time, Kotliar[1] and Marston and Affleck[2] found the staggered flux state as a mean-field solution of the Hubbard model. However, it was apparently not recognized that the singlet $d_{x^2-y^2}$ density-wave state is the same as the staggered flux state. More recently, this state,[14–16] and a related variant[17,18] which does not break translational symmetry, have been discussed in the context of the cuprate superconductors. A version of the $d_{x^2-y^2}$ density-wave state (see the comments in Sec. VII) has appeared in mean-field analyses of an SU(2) mean-field theory of the *t-J* model.[19,20] The nodal liquid state of Refs. 21–23 also bears a family resemblance to the staggered flux state; we will return to the rela-

tionship between these states in Sec. VII.

II. ORDER PARAMETERS AND BROKEN SYMMETRIES

We define the different possible density-wave orderings by analogy with the more familiar superconducting case. Consider a system of electrons on a square lattice of side a. A superconductor is defined by a nonvanishing expectation value of

$$\langle \psi_\alpha(k,t)\psi_\beta(-k,t)\rangle. \tag{1}$$

A triplet superconductor is characterized by the expectation value

$$\langle \psi_\alpha(k,t)\psi_\beta(-k,t)\rangle = \vec{\Delta}(p)\cdot\vec{\sigma}_\alpha^\gamma \epsilon_{\gamma\beta}. \tag{2}$$

Fermi statistics requires that $\vec{\Delta}(p)$ be odd in $\vec{p}$. p-wave superconductors can have the components of $\vec{\Delta}(p)$ chosen from $\sin k_x a$, $\sin k_y a$, or $\sin k_x a \pm i \sin k_y a$. For instance, a p_x superconductor with all spins polarized along the 3 direction will have $\Delta_1 + i\Delta_2 \neq 0$ and $\Delta_3 = \Delta_1 - i\Delta_2 = 0$:

$$\langle \psi_\alpha(k,t)\psi_\beta(-k,t)\rangle = \Delta_0(\sin k_x a)\sigma_\alpha^{+\gamma}\epsilon_{\gamma\beta}. \tag{3}$$

Spin-polarized p_y and $p_x + ip_y$ superconductors have $\sin k_x a$ replaced by $\sin k_y a$ and $\sin k_x a + i \sin k_y a$, respectively. The analog of the A' phase of ^{3}He has equal numbers of $\uparrow\uparrow$ and $\downarrow\downarrow$ pairs:

$$\langle \psi_\alpha(k,t)\psi_\beta(-k,t)\rangle = \Delta_0(\sin k_x a \sigma_\alpha^{1\gamma} + \sin k_y a \sigma_\alpha^{2\gamma})\epsilon_{\gamma\beta}. \tag{4}$$

An unpolarized p_x superconductor of $\uparrow\downarrow$ pairs has $\Delta_3 \neq 0$ and $\Delta_1 = \Delta_2 = 0$:

$$\langle \psi_\alpha(k,t)\psi_\beta(-k,t)\rangle = \Delta_0(\sin k_x a)\sigma_\alpha^{3\gamma}\epsilon_{\gamma\beta}. \tag{5}$$

As in the polarized case, unpolarized p_y and $p_x + ip_y$ superconductors have $\sin k_x a$ replaced by $\sin k_y a$ and $\sin k_x a + i \sin k_y a$, respectively. In principle, more complicated order parameters are possible, with all components of $\vec{\Delta}$ taking nonvanishing values. If any component of $\vec{\Delta}(p)$ is not real, time-reversal symmetry (T) is broken.

A d-wave superconductor must be a spin-singlet superconductor. A $d_{x^2-y^2}$ superconductor has

$$\langle \psi_\alpha(k,t)\psi_\beta(-k,t)\rangle = \Delta_0(\cos k_x a - \cos k_y a)\epsilon_{\alpha\beta}, \tag{6}$$

while a d_{xy} superconductor has $\cos k_x a - \cos k_y a$ replaced by $\sin k_x a \sin k_y a$. A $d_{x^2-y^2} + id_{xy}$ superconductor breaks T with the order parameter:

$$\langle \psi_\alpha(k,t)\psi_\beta(-k,t)\rangle = \Delta_0(\cos k_x a - \cos k_y a + i \sin k_x a \sin k_y a)\epsilon_{\alpha\beta}. \tag{7}$$

We can define analogous orders for density-wave states. However, the spin structures will no longer be determined by Fermi statistics. Let us first consider the singlet orderings. A singlet s-wave density wave is simply a charge-density wave:

$$\langle \psi^{\alpha\dagger}(k+Q,t)\psi_\beta(k,t)\rangle = \Phi_Q \delta_\beta^\alpha \tag{8}$$

(An extended s wave is also possible.) A singlet p_x density-wave state has ordering

$$\langle \psi^{\alpha\dagger}(k+Q,t)\psi_\beta(k,t)\rangle = \Phi_Q \sin k_x a\, \delta_\beta^\alpha. \tag{9}$$

The singlet $p_x + ip_y$ density-wave states are defined by

$$\langle \psi^{\alpha\dagger}(k+Q,t)\psi_\beta(k,t)\rangle = \Phi_Q(\sin k_x a + i \sin k_y a)\delta_\beta^\alpha. \tag{10}$$

Similarly, the singlet $d_{x^2-y^2}$ density-wave states have

$$\langle \psi^{\alpha\dagger}(k+Q,t)\psi_\beta(k,t)\rangle = \Phi_Q(\cos k_x a - \cos k_y a)\delta_\beta^\alpha, \tag{11}$$

while the singlet $d_{x^2-y^2} + id_{xy}$ density-wave states have

$$\langle \psi^{\alpha\dagger}(k+Q,t)\psi_\beta(k,t)\rangle = \Phi_Q(\cos k_x a - \cos k_y a + i \sin k_x a \sin k_y a)\delta_\beta^\alpha. \tag{12}$$

These states belong to a class of states of the form

$$\langle \psi^{\alpha\dagger}(k+Q,t)\psi_\beta(k,t)\rangle = \Phi_Q f(k)\delta_\beta^\alpha. \tag{13}$$

$f(k)$ is an element of some representation of the space group of the vector $\vec{Q}$ in the square lattice. In this paper, we will focus primarily on the cases $f(k) = \sin k_x a$ and $f(k) = \cos k_x a - \cos k_y a$, but $f(k)$ could be an element of some larger representation. The s-wave (or extended s-wave) cases $f(k) = |f(k)|$ are the usual charge-density-wave states.

Q is the wave vector at which the density-wave ordering takes place. It may be commensurate or incommensurate. (In this paper, we will take ''commensurate'' to mean the situation in which $2\vec{Q}$ is a reciprocal-lattice vector. The term ''incommensurate'' will actually include higher-order commensurability.) For commensurate ordering such that $2Q$ is a reciprocal-lattice vector, e.g., $Q = (\pi/a, 0)$ or $Q = (\pi/a, \pi/a)$, we can take the Hermitian conjugate of the order parameter:

$$\langle \psi^{\dagger\beta}(k,t)\psi_\alpha(k+Q,t)\rangle = \Phi_Q^* f^*(k)\delta_\beta^\alpha,$$

$$\langle \psi^{\beta\dagger}(k+Q+Q,t)\psi_\alpha(k+Q,t)\rangle = \Phi_Q^* f^*(k)\delta_\beta^\alpha, \tag{14}$$

$$\Phi_Q f(k+Q)\delta_\beta^\alpha = \Phi_Q^* f^*(k)\delta_\beta^\alpha$$

Therefore, for Q commensurate,

$$\frac{f(k+Q)}{f^*(k)} = \frac{\Phi_Q^*}{\Phi_Q}. \tag{15}$$

Hence, if $f(k+Q) = -f^*(k)$, Φ_Q must be imaginary. For singlet p_x ordering, this will be the case if $Q = (\pi/a, 0)$ or $Q = (\pi/a, \pi/a)$. For singlet $d_{x^2-y^2}$ ordering, this will be the case if $Q = (\pi/a, \pi/a)$. If $f(k+Q) = f^*(k)$, Φ_Q must be real. For singlet p_x ordering, this will be the case if $Q = (0, \pi/a)$. For singlet d_{xy} ordering, this will be the case if $Q = (\pi/a, \pi/a)$.

For incommensurate ordering, Φ_Q can have arbitrary phase: the phase of Φ_Q is the Goldstone boson of broken translational invariance, i.e., the sliding density-wave mode.

Impurities will pin this mode—at second order in the impurity potential, as in the case of a spin-density wave—so we will not consider it further.

All of these states break translational and rotational invariance. To further analyze the symmetries of these states, it is instructive to write these orderings in real space. The singlet p_x density waves have a nonvanishing expectation value:

$$\langle \psi^{\dagger\alpha}(\vec{x},t)\psi_\beta(\vec{x}+a\hat{x},t)\rangle = \text{const} - \frac{i}{2}(\Phi_Q e^{i\vec{Q}\cdot\vec{x}} + \Phi_{-Q} e^{-i\vec{Q}\cdot\vec{x}}) \times \delta^\alpha_\beta(\delta_{\vec{y},\vec{x}+a\hat{x}} - \delta_{\vec{y},\vec{x}-a\hat{x}}). \quad (16)$$

We have only written the modulated term; const refers to the uniform contribution coming from the Fourier transform of $\psi^\dagger(k)\psi(k)$.

Let us consider the commensurate and incommensurate cases separately. The incommensurate singlet p_x density-wave states completely break the translational and rotational symmetries. If $\Phi_Q = -\Phi^*_{-Q}$, T is preserved; otherwise, it is broken. The singlet $p_x + ip_y$ density-wave states always break T. Commensurate states, on the other hand, break translation by one lattice spacing; translation by two lattice spacings is preserved. From Eq. (15), a commensurate singlet p_x density-wave state with $Q=(\pi/a,0)$ must have imaginary Φ_Q:

$$\langle \psi^{\dagger\alpha}(\vec{x},t)\psi_\beta(\vec{y},t)\rangle = \text{const} + |\Phi_Q| e^{i\vec{Q}\cdot\vec{x}} \times \delta^\alpha_\beta(\delta_{\vec{y},\vec{x}+a\hat{x}} - \delta_{\vec{y},\vec{x}-a\hat{x}}). \quad (17)$$

The singlet state of this type breaks no other symmetries; it is usually called the *Peierls state* or bond order wave. If $Q=(0,\pi/a)$, Φ_Q must be real:

$$\langle \psi^{\dagger\alpha}(\vec{x},t)\psi_\beta(\vec{y},t)\rangle = \text{const} - i|\Phi_Q| e^{i\vec{Q}\cdot\vec{x}} \times \delta^\alpha_\beta(\delta_{\vec{y},\vec{x}+a\hat{x}} - \delta_{\vec{y},\vec{x}-a\hat{x}}). \quad (18)$$

As a result of the *i*, the $Q=(0,\pi/a)$ singlet p_x density-wave states break T. However, the combination of T and translation by an odd number of lattice spacings remains unbroken. The same is true of the commensurate singlet $p_x + ip_y$ density-wave states. Examples of commensurate and incommensurate singlet p_x and $p_x + ip_y$ density-wave states are depicted in Fig. 1.

The commensurate $Q=(\pi/a,\pi/a)$ singlet $d_{x^2-y^2}$ density-wave states must have imaginary Φ_Q:

$$\langle \psi^{\dagger\alpha}(\vec{x},t)\psi_\beta(\vec{y},t)\rangle = \text{const} + \frac{i}{2}|\Phi_Q| e^{i\vec{Q}\cdot\vec{x}} \delta^\alpha_\beta(\delta_{\vec{y},\vec{x}+a\hat{x}} + \delta_{\vec{y},\vec{x}-a\hat{x}} - \delta_{\vec{y},\vec{x}+a\hat{y}} - \delta_{\vec{y},\vec{x}-a\hat{y}}). \quad (19)$$

As a result of *i*, the singlet $d_{x^2-y^2}$ density wave breaks T, as well as translational and rotational invariance. The combination of time reversal and a translation by one lattice spacing is preserved by this ordering. The commensurate $Q=(\pi/a,\pi/a)$ singlet $d_{x^2-y^2}$ density-wave state is often called the *staggered flux state*. There is also a contribution to this correlation function coming from $\psi^\dagger(k)\psi(k)$ which is uniform in space (the ...); as a result, the phase of the above bond correlation function—and, therefore, the flux through each plaquette—is alternating. The commensurate $Q=(\pi/a,\pi/a)$ singlet d_{xy} must have real Φ_Q; therefore, it does not break T.

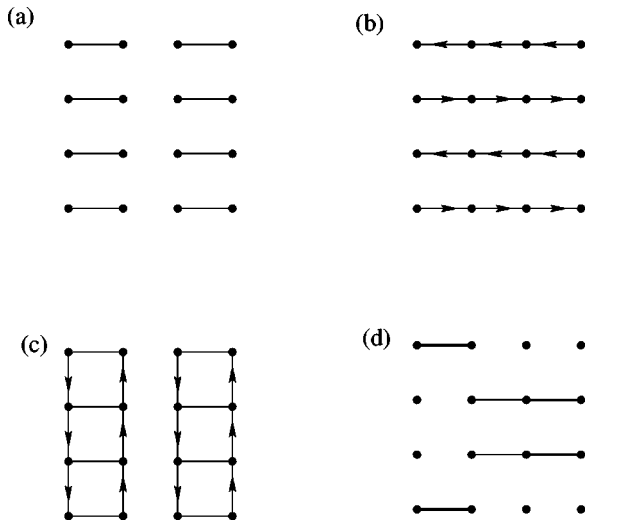

FIG. 1. (a) $Q=(\pi/a,0)$ p_x density-wave state. (b) $Q=(0,\pi/a)$ p_x density-wave state. (c) $Q=(\pi/a,0)$ p_x+ip_y density-wave state. (d) Incommensurate p_x density-wave state. Arrowless lines are bonds where the kinetic energy is large but there is no net current. The line thickness indicates the bond strength. Arrowed lines denote currents.

$$\langle \psi^{\dagger\alpha}(\vec{x},t)\psi_\beta(\vec{y},t)\rangle = \text{const} + \frac{1}{2}\Phi_Q e^{i\vec{Q}\cdot\vec{x}} \delta^\alpha_\beta(\delta_{\vec{y},\vec{x}+a\hat{x}+a\hat{y}} + \delta_{\vec{y},\vec{x}-a\hat{x}-a\hat{y}} - \delta_{\vec{y},\vec{x}-a\hat{x}+a\hat{y}} - \delta_{\vec{y},\vec{x}+a\hat{x}-a\hat{y}}) \quad (20)$$

On the other hand, the singlet $d_{x^2-y^2} + id_{xy}$ state does break T:

$$\langle \psi^{\dagger\alpha}(\vec{x},t)\psi_\beta(\vec{y},t)\rangle = \text{const} + \Phi_Q e^{i\vec{Q}\cdot\vec{x}} \delta^\alpha_\beta \Big[\frac{1}{2}(\delta_{\vec{y},\vec{x}+a\hat{x}} + \delta_{\vec{y},\vec{x}-a\hat{x}} - \delta_{\vec{y},\vec{x}+a\hat{y}} - \delta_{\vec{y},\vec{x}-a\hat{y}}) - \frac{i}{4}(\delta_{\vec{y},\vec{x}+a\hat{x}+a\hat{y}} + \delta_{\vec{y},\vec{x}-a\hat{x}-a\hat{y}} - \delta_{\vec{y},\vec{x}-a\hat{x}+a\hat{y}} - \delta_{\vec{y},\vec{x}+a\hat{x}-a\hat{y}}) \Big]. \quad (21)$$

Note that the nodeless commensurate singlet $d_{x^2-y^2} + id_{xy}$ density-wave state does not break more symmetries than the commensurate singlet $d_{x^2-y^2}$ density-wave state, in contrast to the superconducting case. Examples of singlet $d_{x^2-y^2}$, d_{xy}, and $d_{x^2-y^2} + id_{xy}$ density-wave states are depicted in Fig. 2.

If $\vec{Q}$ is not along one of the axes, the associated incommensurate density-wave states will mix *s*, *p*, *d*, etc. density-wave orders. For $\vec{Q}=(\pi/a,\pi/a)+\vec{q}$ with $|\vec{q}|$ small, there are

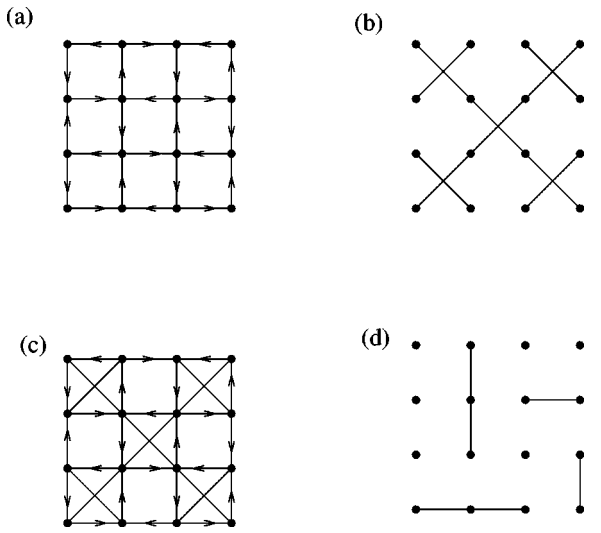

FIG. 2. (a) Commensurate $d_{x^2-y^2}$ density-wave state. (b) Commensurate d_{xy} density-wave state. (c) Commensurate $d_{x^2-y^2}+id_{xy}$ density-wave state. (d) Incommensurate, T-preserving $d_{x^2-y^2}$. Arrowless lines are bonds where the kinetic energy is large but there is no net current. Line thickness indicates bond strength. Arrowed lines denote currents.

states which are primarily $d_{x^2-y^2}$ density-wave states with small admixtures, proportional to $\vec{q}$, of *real* $d_{x^2-y^2}$, p_x, and p_y states,

$$f(k)=\cos k_x a-\cos k_y a-\frac{1}{2}(1+e^{iQ_x a})e^{ik_x a}+\frac{1}{2}(1-e^{\frac{i}{2}(Q_x-Q_x)a})e^{ik_y a}+\frac{1}{2}(1+e^{\frac{i}{2}(Q_x+Q_x)a})\times e^{-ik_y a}, \tag{22}$$

which satisfies Eq. (15) with Φ_Q imaginary. For $|\vec{q}|$ small, this is

$$f(k)=\cos k_x a-\cos k_y a+\frac{i}{2}q_x a(\cos k_x a-\cos k_y a)-\frac{1}{2}q_x a\sin k_x a-\frac{1}{2}q_y a\sin k_y a. \tag{23}$$

We now consider the triplet density-wave states. Triplet states all break spin-rotational invariance and, therefore, have Goldstone boson excitations. We will discuss the experimental consequences of these Goldstone bosons later. The triplet s-wave density-wave state is simply a spin-density wave. The triplet p- and d-wave states are characterized by

$$\langle\psi^{\alpha\dagger}(k+Q,t)\psi_\beta(k,t)\rangle=\vec{\Phi}_Q(k)\cdot\vec{\sigma}^\alpha_\beta, \tag{24}$$

with the components of $\vec{\Phi}_Q(k)$ chosen from, $\sin k_x a$, $\sin k_y a$, $\sin k_x a\pm i\sin k_y a$, $\cos k_x a-\cos k_y a$, $\sin k_x a\sin k_y a$, and $\cos k_x a-\cos k_y a\pm i\sin k_x a\sin k_y a$, respectively.

A state in which the particle-hole pairs are polarized, which is the most direct analog of a spin-density wave has $\vec{\Phi}_Q(p)$ of the form $\Phi^3_Q\neq 0$, $\Phi^1_Q=\Phi^2_Q=0$,

$$\langle\psi^{\alpha\dagger}(k+Q,t)\psi_\beta(k,t)\rangle=\Phi_Q f(k)\sigma^{3\alpha}_\beta, \tag{25}$$

where $f(k)$ is chosen from the above set. Alternatively, the particle-hole pairs can be unpolarized, e.g.,

$$\langle\psi^{\alpha\dagger}(k+Q,t),\psi_\beta(k,t)\rangle=\Phi_Q(\sin k_x a\,\sigma^{1\alpha}_\beta+i\sin k_y a\,\sigma^{2\alpha}_\beta). \tag{26}$$

As in the superconducting case, more complicated order parameters are possible, with all components of $\vec{\Delta}$ taking nonvanishing values. For commensurate ordering, we can follow the same logic as in Eq. (14). The phases of the components of $\vec{\Phi}_Q(p)$ are constrained in the same way as the singlet order parameters, as illustrated by the i in front of the second term in Eq. (26).

The orders discussed here can be generalized to other 2D and 3D lattices. The orbital wave functions $\sin k_x a$, etc., will be replaced by representations of the point groups of these other lattices.

To each T-preserving singlet ordering, we can associate an ordering of a pure spin model, in the same way that spin-Peierls ordering is related to Peierls ordering:

$$\langle\psi^{\alpha\dagger}(k+Q,t)\psi_\alpha(k,t)\rangle\rightarrow\langle\vec{S}(k+Q,t)\cdot\vec{S}(k,t)\rangle. \tag{27}$$

These spin orderings are states in which the exchange energies are large along preferred directions. These preferred diractions oscillate from one lattice point to the next with spatial frequency $\vec{Q}$. The simplest case is spin-Peierls ordering, in which the spins form dimers. Another example is d_{xy} ordering of a spin model, which takes the form

$$\langle\vec{S}(\vec{x},t)\vec{S}(\vec{y},t)\rangle=-\frac{1}{4}(\Phi_Q e^{i\vec{Q}\cdot\vec{x}}+\Phi_{-Q}e^{-i\vec{Q}\cdot\vec{x}})(\delta_{\vec{y},\vec{x}+a\hat{x}+a\hat{y}}+\delta_{\vec{y},\vec{x}-a\hat{x}-a\hat{y}}-\delta_{\vec{y},\vec{x}-a\hat{x}+a\hat{y}}-\delta_{\vec{y},\vec{x}+a\hat{x}-a\hat{y}}),$$

in analogy with Eq. (20).

III. MODEL HAMILTONIANS

We are primarily concerned in this paper with the universal properties of the states introduced above. We will not attempt to show that particular realistic models of interacting electrons have p- or d-wave density-wave ground states. Rather, we will content ourselves with discussing the types of interactions which favor such orders, and showing that they lead to energetically favorable trial variational wave functions for some idealized Hamiltonians.

The analog of the BCS reduced Hamiltonian for singlet density-wave order is

$$H=\int\frac{d^2k}{(2\pi)^2}\epsilon(k)\psi^{\alpha\dagger}(k)\psi_\alpha(k)-g\int\frac{d^2k}{(2\pi)^2}\frac{d^2k'}{(2\pi)^2}[f(k)f(k')\psi^{\alpha\dagger}\times(k+Q)\psi_\alpha(k)\psi^{\beta\dagger}(k')\psi_\beta(k'+Q)]. \tag{28}$$

In the triplet case, we replace the four-fermion operator of Eq. (28) by

$$\psi^{\alpha\dagger}(k+Q)\sigma_{\alpha}^{a\beta}\psi_{\beta}(k)\psi^{\gamma\dagger}(k')\sigma_{\gamma}^{a\delta}\psi_{\delta}(k'+Q). \quad (29)$$

We now introduce the variational wave function

$$|\Psi\rangle=\prod_{k,\alpha}\left[u_{k,\alpha}\psi^{\alpha\dagger}(k)+v_{k,\alpha}\psi^{\alpha\dagger}(k+Q)\right]|0\rangle. \quad (30)$$

Its energy can be minimized if we take

$$\overline{u}_{k,\alpha}v_{k,\alpha}=\frac{g\Phi_Q f(k)}{\sqrt{[\epsilon(k)-\epsilon(k+Q)]^2+4g^2|\Phi_Q|^2[f(k)]^2}} \quad (31)$$

in the singlet case and

$$\overline{u}_{k,\alpha}\vec{\sigma}_{\alpha}^{\beta}v_{k,\beta}=\frac{g\vec{\Phi}_Q f(k)}{\sqrt{[\epsilon(k)-\epsilon(k+Q)]^2+4g^2|\Phi_Q|^2[f(k)]^2}} \quad (32)$$

in the triplet case, and require Φ_Q to satisfy the gap equation:

$$g\int\frac{d^2k}{(2\pi)^2}\frac{(f(k))^2}{\sqrt{[\epsilon(k)-\epsilon(k+Q)]^2+4g^2|\Phi_Q|^2[f(k)]^2}}=1 \quad (33)$$

The reduced Hamiltonian has long-ranged interactions, so the variational wave function is essentially correct. We will now show that short-ranged Hamiltonians will include terms of the form (28), and that the trial wave function (30) is reasonable for these short-ranged Hamiltonians.

Consider, then, the following lattice model of interacting electrons:

$$\begin{aligned}H=&-t\sum_{\langle i,j\rangle}(c_{i\sigma}^{\dagger}c_{j\sigma}+\text{H.c.})+U\sum_i n_{i\uparrow}n_{i\downarrow}+V\sum_{\langle i,j\rangle}n_i n_j\\&-t_{c1}\sum_{\langle i,j\rangle,\langle i',j\rangle,i\neq i'}c_{i\sigma}^{\dagger}c_{j\sigma}c_{j\sigma}^{\dagger}c_{i'\sigma}-t_{c2}\sum_i\left[(c_{i+\hat{x},\sigma}^{\dagger}c_{i\sigma}\right.\\&\left.-c_{i\sigma}^{\dagger}c_{i+\hat{x},\sigma})(c_{i+\hat{x}+\hat{y},\sigma}^{\dagger}c_{i+\hat{y},\sigma}-c_{i+\hat{y},\sigma}^{\dagger}c_{i+\hat{x}+\hat{y},\sigma})+x\rightarrow y\right].\end{aligned} \quad (34)$$

The first two terms are the usual hopping t and on-site repulsion U of the Hubbard model. The third term is a nearest-neighbor repulsion V. The third and fourth terms lead to the correlated motion of pairs of electrons. t_{c1} hops an electron from i' to j when j is vacated by an electron hopping to i. t_{c2} hops nearest-neighbor pairs in the same direction. Terms of this general form were discussed in Refs. 4–7 as a mechanism for superconductivity. As we will see below, they not only favor superconductivity, but p- and d-wave-density wave order as well.

Fourier transforming the interaction terms into momentum space, we see that terms of the form of the reduced interaction (28) are, indeed, present:

$$\begin{aligned}H_{\text{int}}=&\int\frac{d^2k}{(2\pi)^2}\left(U\psi^{\uparrow\dagger}(k_1)\psi_{\uparrow}(k_2)\psi^{\downarrow\dagger}(k_3)\psi_{\downarrow}(k_4)\right.\\&+\{2V[\cos(k_3^x-k_4^x)a+\cos(k_3^y-k_4^y)a]\\&-2t_{c1}[\cos(k_1^x-k_4^x)a+\cos(k_1^y-k_4^y)a+2\cos k_1^x a\\&\times\cos k_4^x a+2\cos k_1^y a\cos k_4^y a]-2t_{c2}(\sin k_1^x a\sin k_4^x a\\&\left.+\sin k_1^y a\sin k_4^y a)\}\psi^{\alpha\dagger}(k_1)\psi_{\alpha}(k_2)\psi^{\beta\dagger}(k_3)\psi_{\beta}(k_4)\right).\end{aligned} \quad (35)$$

Let us now consider various candidate orderings and the terms which favor or penalize them. Antiferromagnetic order is favored by U but penalized by V. Charge-density-wave order is favored by V but penalized by U. p- and d-wave superconductivities are favored by t_{c2} and t_{c1}, respectively and penalized by V. p- and d-density-wave orders are favored by t_{c2} and t_{c1}, respectively, and are both favored by V. The density-wave states can be favored over the others by taking V large. The p- or d-wave states can be favored by taking t_{c2} or t_{c1} large. To be more precise, the mean-field equations for various ordered states read

$$\lambda\int\frac{d^2k}{(2\pi)^2}\frac{(f(k))^2}{\sqrt{[\epsilon(k)-\epsilon(k+Q)]^2+4\lambda^2|\Phi_Q|^2[f(k)]^2}}=1, \quad (36)$$

where

$$\begin{aligned}\lambda_{dDW}&=8V+96t_{c1},\\\lambda_{pDW}&=4V+16t_{c1}+16t_{c2},\\\lambda_{CDW}&=16V+96t_{c1}+16t_{c2}-2U,\\\lambda_{AF}&=2U.\end{aligned} \quad (37)$$

Hence the singlet $d_{x^2-y^2}$ density-wave state will be the ground state if

$$\begin{aligned}8t_{c1}&<U-4V<48t_{c1},\\8t_{c2}&<2V+40t_{c1},\end{aligned} \quad (38)$$

while the singlet p_x density-wave state will be the ground state if

$$\begin{aligned}6V+40t_{c1}&<U<2V+8t_{c1}+8t_{c2},\\2V+40t_{c1}&<8t_{c2},\end{aligned} \quad (39)$$

assuming that the van Hove singularities are at the antinodes of the order parameters. Otherwise, the p- and d-wave density-wave states will be favored over somewhat smaller regions of parameter space. By including spin-dependent interactions such as $J\vec{S}_i\cdot\vec{S}_i$, we can favor the triplet p- or d-wave states. Hence it appears that the orderings discussed in this paper are viable. The detailed energetics at large coupling strengths—which surely hold in physically interesting systems—are beyond the scope of this paper.

IV. EXPERIMENTAL SIGNATURES

We now turn to the question of the experimental signatures of such states. Since the order parameter changes sign as the Fermi surface is circled, there is no net CDW or spin-density-wave order which could be measured in, for instance, neutron scattering. When another symmetry—in addition to translational invariance—is broken, this is easier. (One can expect, on general grounds, that incommensurate

singlet or triplet p- or d-wave density-wave order at wave vector $\vec{Q}$ will induce CDW order at $2\vec{Q}$, since a term of the form $\Phi_Q^2\rho_{2Q}$ or $\vec{\Phi}_Q\cdot\vec{\Phi}_Q\rho_{2Q}$ is allowed by symmetry in the effective action. Nevertheless, we may wish to distinguish such a state from one which has only CDW order.)

Let us first consider broken time-reversal symmetry. The commensurate singlet $d_{x^2-y^2}$ density wave state—or staggered flux state[1–3]—breaks T; there is an alternating pattern of currents circulating about each plaquette of the lattice. These currents produce an alternating magnetic field measurable by μSR and, in principle, by neutron scattering.[3] The magnitude of the current along a link of the lattice will be

$$j=\frac{et}{\hbar}\Phi_Q\sim 10^{-5}\ \mathrm{A}\times\Phi_Q. \tag{40}$$

Now Φ_Q is related to the maximum of the gap according to $\Delta_0=g\Phi_Q$, where g is the appropriate coupling constant. Let us suppose, for the purposes of illustration, that the formation of the ordered state is driven by λ_{dDW}. Then, for λ_{dDW} small, $\Phi_Q\sim(t/\lambda_{dDW})e^{-(\mathrm{const})t/\lambda_{dDW}}$. Alternatively, we may take the high-T_c context as a guideline: observed gaps are $\sim$100–300 K, while interactions such are $\sim$1 eV. In this case, we expect $\Phi_Q\sim 10^{-2}$. This translates to a magnetic field at the center of each plaquette on the order of 10 G. The muons in a μSR experiment might see a lowwer field if they sit at points of high symmetry or away from the plane. The orbital magnetic moments are likely to be dwarfed by local spin moments.[3] Incommensurate ordering may or may not break T; if it does, the above analysis applies.

In the $\Phi_Q^3\neq 0$, $\Phi_Q^1=\Phi_Q^2=0$ triplet $d_{x^2-y^2}$ density-wave state, there are counter-circulating currents of up- and down-spin electrons. These currents cancel, so there is no net current circulating about each plaquette, but there is an alternating pattern of spin currents circulating about each plaquette. The checkerboard pattern of spin currents will generate, via the spin-orbit coupling,

$$H_{SO}=\int\frac{d^2k}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\vec{E}(q)\cdot(2\vec{k}+\vec{q})\psi^{\dagger\alpha}(k+q)\vec{\sigma}_\alpha^\beta\psi_\beta(k), \tag{41}$$

a quadrupolar electric field which is, in principle, measurable in NQR experiments. With the above estimate of the current, a nucleus with a nonzero quadrupole moment would have an induced splitting of order 10 Hz.

We now turn to broken spin-rotational invariance, characteristic of the triplet states. Since it transforms nontrivially under the point group of the square lattice, the triplet order parameter $\vec{\Phi}_Q$ will not couple to photons, neutrons, or nuclear spins according to $\vec{\Phi}_Q\cdot\vec{F}$, where $\vec{F}$ is $\vec{B}$, $\vec{S}_N$, or $\vec{I}$, respectively. Stated more physically, the triplet ordered states do not have anomalous expectation values for the spin density but, rather, for spin currents; spin currents do not couple simply to these probes. However, the order parameter $\vec{\Phi}_Q$ will couple to such probes at second order, since its square transforms trivially under the point group. Such a coupling will be of the form

$$H_{\mathrm{probe}}=\int d^2x[2(\vec{F}\cdot\vec{\Phi}_Q)(\vec{F}\cdot\vec{\Phi}_Q^*)-|\vec{\Phi}_Q|^2F^2]. \tag{42}$$

In the case of photons, this will lead to two-magnon Raman scattering. The coupling to nuclear spins couples directly to the nuclear quadrupole moment, and will lead to a shift in the nuclear quadrupole resonance frequencies.

In the presence of disorder, rotational symmetry will be broken. Hence there will be a small coupling, proportional to the disorder strength, of the Goldstone bosons to s-wave probes such as NMR and neutron scattering.

V. GAPLESS FERMIONIC EXCITATIONS

The mean-field Hamiltonian is

$$H=\int_{\mathrm{B.Z.}}\frac{d^2k}{(2\pi)^2}[\epsilon(k)\psi^{\alpha\dagger}(k)\psi_\alpha(k)+g\Phi_Q f(k)\psi^{\alpha\dagger}(k+Q)\psi_\alpha(k)]. \tag{43}$$

If we define the four component object $\chi_{A\alpha}$ according to

$$\begin{pmatrix}\chi_{1\alpha}\\ \chi_{2\alpha}\end{pmatrix}=\begin{pmatrix}\psi_\alpha(k)\\ \psi_\alpha(k+Q)\end{pmatrix}, \tag{44}$$

then the mean-field Hamiltonian can be written in the form

$$H=\int_{\mathrm{RBZ}}\frac{d^2k}{(2\pi)^2}\chi^{\alpha\dagger}(k)\left(\frac{1}{2}[\epsilon(k)-\epsilon(k+Q)]\tau_z+\Delta(k)\tau_x+\frac{1}{2}[\epsilon(k)+\epsilon(k+Q)]\right)\chi_\alpha \tag{45}$$

The integral is over the reduced Brillouin zone (RBZ). The τ's are Pauli matrices; the ''flavor'' index $A=1$ and 2 on which they act has been suppressed. $\Delta(k)$ is defined by

$$\Delta(k)\equiv\Delta_0 f(k)\equiv g\Phi_Q f(k). \tag{46}$$

The single-quasiparticle energies are

$$E_\pm(k)=\frac{1}{2}[\epsilon(k)+\epsilon(k+Q)]\pm\frac{1}{2}\sqrt{[\epsilon(k)-\epsilon(k+Q)]^2+4\Delta^2(k)}. \tag{47}$$

Let us consider the situation in which there is a node, i.e., when the argument of the square root vanishes (we discuss below the conditions under which this occurs). For simplicity, we will consider the commensurate $\vec{Q}=(\pi/a,\pi/a)$ singlet p_x density-wave state in a model with anisotropic nearest-neighbor hopping,

$$\epsilon(k)=-2t(r\cos k_xa+\cos k_ya), \tag{48}$$

with $r<1$. The mean-field quasiparticle energies are

$$E(k)=\pm\sqrt{4t^2(r\cos k_xa+\cos k_ya)^2+\Delta_0^2\sin^2k_xa}. \tag{49}$$

There is a node at $k_x=0$, $k_ya=\arccos(-r)$. Expanding about this node,

$$E(q) = \pm \sqrt{v_x^2 q_x^2 + v_y^2 q_y^2}, \tag{50}$$

with momenta $\vec{q}$ now measured from the node, and

$$v_x = \Delta_0 a, \quad v_y = 2ta\sqrt{1-r^2}. \tag{51}$$

The effective Lagrangian for the quasiparticles near the nodes can be written

$$\mathcal{L}_{\text{eff}} = \chi^{\alpha\dagger}(\partial_\tau - \tau_z v_y i\partial_y - \tau_x v_x i\partial_x)\chi_\alpha. \tag{52}$$

Terms which break the nesting of the Fermi surface, such as the chemical potential or next-neighbor hopping, open hole pockets at the nodes:

$$\mathcal{L}_\mu = -\mu\chi^{\alpha\dagger}\chi_\alpha. \tag{53}$$

We now turn to the question of when a *p*- or *d*-wave density wave will have nodal excitations. Let us again begin with the commensurate $\vec{Q} = (\pi/a, \pi/a)$ singlet p_x density-wave state,

$$\langle \psi^{\alpha\dagger}(k+Q,t)\psi_\beta(k,t)\rangle = i|\Phi_Q|\sin k_x a\,\delta^\alpha_\beta, \tag{54}$$

in a system in which the Fermi surface is nested at $\vec{Q}$. This state will have gapless excitations if the nodal line $k_x = 0$ crosses the Fermi surface. For an open Fermi surface, this need not be the case. In an anisotropic nearest-neighbor tight-binding model [Eq. (48)], with $r > 1$, the Fermi surface at half-filling is an open Fermi surface which does not cross the line $k_x = 0$. Consequently, there are no gapless excitations. For $r < 1$, however, the Fermi surface does cross the line $k_x = 0$, and there are gapless excitations.

Are there any thermodynamic singularities at the transition at which gapless excitations occur? To answer this question, let us consider the mean-field ground-state energy

$$E_0 = \int_{\text{RBZ}} \frac{d^2k}{(2\pi)^2} E(k) = \int_{\text{RBZ}} \frac{d^2k}{(2\pi)^2} \sqrt{\epsilon^2(k) + \Delta^2(k)}. \tag{55}$$

The first and second derivatives of E_0 are continuous. However, the third derivative of the ground state energy with respect to r contains a term of the form

$$\frac{\partial^3 E_0}{\partial r^3} = \int_{\text{RBZ}} \frac{d^2k}{(2\pi)^2} \frac{8\epsilon(k)t^3\cos^3 k_x a}{[\epsilon^2(k) + \Delta^2(k)]^{3/2}} + \cdots. \tag{56}$$

This term diverges if there is a node on the Fermi surface, but is finite otherwise. Hence the phase with a node on the Fermi surface is a *critical line* with a singular third derivative of the ground-state energy. We will call this phase the ''critical phase'' of the p_x density wave. Note that the second derivative of the ground-state energy is everywhere continuous, but nowhere differentiable in the critical phase. It is separated by a third-order phase transition from the phase with no gapless excitations, the noncritical phase of the p_x density wave.

How does this observation generalize (a) away from half-filling and to non-nested Fermi surfaces; and (b) to *d*-wave and/or incommensurate ordering? To answer (a), let us change the chemical potential in order to move away from half-filling. Now

$$\frac{\partial^3 E_0}{\partial r^3} = \int_{E(k)<\mu} \frac{d^2k}{(2\pi)^2} \frac{8\epsilon(k)t^3\cos^3 k_x a}{[\epsilon^2(k) + \Delta^2(k)]^{3/2}} + \cdots. \tag{57}$$

Below half-filling, the denominator never diverges. Hence the system is always in the noncritical phase, despite the fact that there are gapless excitations. As the chemical potential is increased, the system crosses a third-order phase transition and enters the critical phase. Above half-filling, it is always in the critical phase.

Suppose, now, that we allow next-nearest-neighbor hopping t', thereby spoiling nesting. The ground-state energy is given by

$$\begin{aligned} E_0 &= \int_{E(k)<\mu} \frac{d^2k}{(2\pi)^2} E(k) \\ &= \int_{E(k)<\mu} \frac{d^2k}{(2\pi)^2} \left[\frac{1}{2}[\epsilon(k) + \epsilon(k+Q)]\right. \\ &\quad \left. - \frac{1}{2}\sqrt{[\epsilon(k) - \epsilon(k+Q)]^2 + 4\Delta^2(k)}\right] \end{aligned} \tag{58}$$

and

$$\frac{\partial^3 E_0}{\partial r^3} = \int_{E(k)<\mu} \frac{d^2k}{(2\pi)^2} \frac{4\epsilon(k)t^3\cos^3 k_x a}{\{[\epsilon(k) - \epsilon(k+Q)]^2 + 4\Delta^2(k)\}^{3/2}} + \cdots.$$

This diverges if the nodal line of $\Delta(k)$ crosses the curve $\epsilon(k) = \epsilon(k+Q)$, and this crossing point lies below the chemical potential. In such a case, the system is in the critical phase. Regardless of the details of the band structure, the curve $\epsilon(k) = \epsilon(k+Q)$ is determined by symmetry for commensurate $\vec{Q}$: it is the set of points for which $\vec{k}$ and $\vec{k}+\vec{Q}$ are related by a symmetry of the square lattice. For $\vec{Q} = (\pi/a, 0)$, $\epsilon(k) = \epsilon(k+Q)$ if $k_x = \pm\pi/2a$. For $\vec{Q} = (\pi/a, \pi/a)$, $\epsilon(k) = \epsilon(k+Q)$ if $k_x \pm k_y = \pm\pi/a$. A *d*-wave density wave will always have nodal lines which cross the curve $\epsilon(k) = \epsilon(k+Q)$; a *p*-wave density wave may or may not. If there is no crossing point, or the crossing point is not below the chemical potential, the system is in the noncritical phase. Again, the noncritical phase can have gapless excitations.

In the case of incommensurate ordering, similar considerations hold. Let us suppose that the Fermi surface is nested at incommensurate $\vec{Q}$; i.e., if $\vec{k}$ is on the Fermi surface, then $\vec{k}+\vec{Q}$ is as well, and $\epsilon(k) = \epsilon(k+Q) = \mu$. If the Fermi surface intersects the nodal lines of $\Delta(k)$, then there will be gapless nodal excitations. If the chemical potential is now lowered or the hopping parameters are changed, so that the Fermi surface is no longer perfectly nested, then the nodes will open into hole pockets. Again, as the nesting condition is approached, a third-order phase transition will occur, as in the commensurate case.

In summary, the system will be in a ''critical'' state if nodal points are at or below the Fermi surface. Otherwise, the system will be ''noncritical,'' whether or not there other gapless excitations. The transition between these two states and the entire critical phase is characterized, in mean-field-

theory, by a diverent third derivative of the ground-state energy. There is no reason to mistrust mean-field theory, since there are no strong order-parameter fluctuations which might destabilize out calculations.

VI. TRANSITIONS TO SUPERCONDUCTING STATES

As Zhang[9] recently emphasized, enhanced symmetry can be dynamically generated at a critical point between two different ordered electronic states. The focus of that work was a critical point between an antiferromagnet and a $d_{x^2-y^2}$ superconductor. In earlier work, Yang identified an SU(2) symmetry [which, together with SU(2) spin-rotational symmetry, trivially forms an $O(4)=\mathrm{SU}(2)\times\mathrm{SU}(2)\times Z_2$] which is an exact symmetry of the Hubbard model at half-filling with $\mu=U/2$. This symmetry would be dynamically generated at a critical point between a CDW and an *s*-wave superconductor. We now consider the modification of this idea to *p*- and *d*-wave ordering.

We first consider a transition at half-filling between a singlet commensurate $d_{x^2-y^2}$ density-wave and a $d_{x^2-y^2}$ superconductor. We group the two order parameters into a vector:

$$\Phi_{\underline{i}}(q)f(k)=\begin{pmatrix}\sqrt{2}\,\mathrm{Re}\left\{\left\langle\psi_\uparrow^\dagger\left(k+\frac{q}{2}\right)\psi_\downarrow^\dagger\left(-k+\frac{q}{2}\right)\right\rangle\right\}\\ \sqrt{2}\,\mathrm{Im}\left\{\left\langle\psi_\uparrow^\dagger\left(k+\frac{q}{2}\right)\psi_\downarrow^\dagger\left(-k+\frac{q}{2}\right)\right\rangle\right\}\\ i\left\langle\psi^{\alpha\dagger}\left(k+Q+\frac{q}{2}\right)\psi_\alpha\left(k-\frac{q}{2}\right)\right\rangle\end{pmatrix}. \tag{59}$$

[We will use underlined lowercase Roman letters such as $\underline{i}=1,2$, and 3 to denote pseudospin triplet indices and uppercase Roman letters to denote peudospin doublet indices $A=1$ and 2. Lowercase Roman indices $a=1,2$, and 3 will be vector indices (i.e., real spin triplet indices) and Greek letters $\alpha=1$ and 2 will be used for real spin SU(2) spinor indices. Pauli matrices $\tau^{\underline{i}}$ will be used for pseudospin, while σ^a will be reserved for spin.] If, following Yang,[8] we introduce the following SU(2) generators which we will call pseudospin SU(2)

$$O^3=\int_{\mathrm{RBZ}}\frac{d^2k}{(2\pi)^2}[\psi^{\alpha\dagger}(k)\psi_\alpha(k)\ \psi^{\alpha\dagger}(k+Q)\psi_\alpha(k+Q)],$$

$$O^+=\int_{\mathrm{RBZ}}\frac{d^2k}{(2\pi)^2}i\psi_\uparrow^\dagger(k)\psi_\downarrow^\dagger(-k+Q) \tag{60}$$

$$O^-=\int_{\mathrm{RBZ}}\frac{d^2k}{(2\pi)^2}i\psi_\uparrow(k)\psi_\downarrow(-k+Q),$$

then the order parameters form a triplet under this SU(2):

$$\begin{pmatrix}\Phi_+(q)f(k)\\ \Phi_0(q)f(k)\\ \Phi_-(q)f(k)\end{pmatrix}=\begin{pmatrix}-\left\langle\psi_\uparrow^\dagger\left(k+\frac{q}{2}\right)\psi_\downarrow^\dagger\left(-k+\frac{q}{2}\right)\right\rangle\\ i\left\langle\psi^{\alpha\dagger}\left(k+Q+\frac{q}{2}\right)\psi_\alpha\left(k-\frac{q}{2}\right)\right\rangle\\ \left\langle\psi_\uparrow\left(k+\frac{q}{2}\right)\psi_\downarrow\left(-k+\frac{q}{2}\right)\right\rangle\end{pmatrix}. \tag{61}$$

There is a small but important difference between our pseudospin SU(2) and Yang's[8]: the factors of i in the definitions of $O^\pm$. These are necessary since a commensurate $d_{x^2-y^2}$ density wave breaks T, while a $d_{x^2-y^2}$ superconductor does not. Consequently, our pseudospin SU(2) does not commute with T, which is an inversion followed by a rotation by π about the 3 axis.

The electron fields transform as a doublet under the pseudospin SU(2) as well as the spin SU(2). We will group them into four-component objects $\Psi_{A\alpha}$, where A is the pseudospin index, $A=1$ and 2, and α is the spin index, $\alpha=\uparrow,\downarrow$:

$$\begin{pmatrix}\Psi_{1\alpha}\\ \Psi_{2\alpha}\end{pmatrix}=\begin{pmatrix}\psi_\alpha(k)\\ i\epsilon_{\alpha\beta}\psi^{\beta\dagger}(-k+Q)\end{pmatrix}. \tag{62}$$

A ''microscopic'' Hamiltonian which is $O(4)$ invariant can be written as

$$H=H_0+H_{int}, \tag{63}$$

$$H_0=\int_{\mathrm{RBZ}}\frac{d^2k}{(2\pi)^2}\epsilon(k)\Psi^{A\alpha\dagger}\Psi_{A\alpha}, \tag{64}$$

if H_{int} is given by

$$\begin{aligned}H_{int}=\int\frac{d^2q}{(2\pi)^2}[&u^{(0,0)}\lambda^{(0,0)}(q)\lambda^{(0,0)}(q)\\&+u^{(1,0)}\lambda_{\underline{i}}^{(1,0)}(q)\lambda_{\underline{i}}^{(1,0)}(q)+u^{(1,0)}\lambda_{\underline{i}Q}^{(1,0)}(q)\lambda_{\underline{i}Q}^{(1,0)}(q)\\&+u^{(0,1)}\lambda_a^{(0,1)}(q)\lambda_a^{(0,1)}(q)+u_Q^{(0,1)}\lambda_{aQ}^{(0,1)}(q)\lambda_{aQ}^{(0,1)}(q)\\&+u^{(1,1)}\lambda_{\underline{i}a}^{(1,1)}(q)\lambda_{\underline{i}a}^{(1,1)}(q)+u_Q^{(1,1)}\lambda_{\underline{i}aQ}^{(1,1)}(q)\lambda_{\underline{i}aQ}^{(1,1)}(q)],\end{aligned}$$

where

$$\lambda^{(0,0)}=\int\frac{d^2k}{(2\pi)^2}f(k)\Psi^{A\alpha\dagger}\left(k+\frac{q}{2}\right)\Psi_{A\alpha}\left(k-\frac{q}{2}\right),$$

$$\lambda_{\underline{i}}^{(1,0)}=\int\frac{d^2k}{(2\pi)^2}f(k)\Psi^{A\alpha\dagger}\left(k+\frac{q}{2}\right)\tau_{\underline{i}A}^B\Psi_{B\alpha}\left(k-\frac{q}{2}\right),$$

$$\lambda_{\underline{i}Q}^{(1,0)}=\int\frac{d^2k}{(2\pi)^2}f(k)\epsilon^{\alpha\beta}\Psi_{C\alpha}\left(k+\frac{q}{2}\right)\epsilon^{CA}\tau_{\underline{i}A}^B\Psi_{B\beta}\left(-k+\frac{q}{2}\right),$$

$$\lambda_a^{(0,1)}=\int\frac{d^2k}{(2\pi)^2}f(k)\Psi^{A\alpha\dagger}\left(k+\frac{q}{2}\right)\sigma_{\underline{i}\alpha}^\beta\Psi_{A\beta}\left(k-\frac{q}{2}\right), \tag{65}$$

$$\lambda^{(0,1)}_{aQ} = \int \frac{d^2k}{(2\pi)^2} f(k) \epsilon^{AB} \Psi_{A\gamma}\left(k+\frac{q}{2}\right) \epsilon^{\gamma\beta} \sigma^{\beta}_{\underline{i}\alpha} \Psi_{B\beta}\left(-k+\frac{q}{2}\right),$$

$$\lambda^{(1,1)}_{\underline{i}a} = \int \frac{d^2k}{(2\pi)^2} f(k) \Psi^{A\alpha\dagger}\left(k+\frac{q}{2}\right) \tau^{B}_{\underline{i}A} \sigma^{\beta}_{\underline{i}\alpha} \Psi_{B\beta}\left(k-\frac{q}{2}\right),$$

$$\lambda^{(1,1)}_{\underline{i}aQ} = \int \frac{d^2k}{(2\pi)^2} f(k) \Psi_{C\gamma}\left(k+\frac{q}{2}\right) \epsilon^{CA} \tau^{B}_{\underline{i}A} \times \epsilon^{\gamma\beta} \sigma^{\beta}_{\underline{i}\alpha} \Psi_{B\beta}\left(-k+\frac{q}{2}\right).$$

[We have only written down the quartic terms; higher-order $O(4)$ invariants also exist, but they are irrelevant at weak coupling.] These microscopic Hamiltonians describe electrons at half-filling with a nested Fermi surface and interactions which favor density-wave and superconducting order equally. In other words, they describe a critical point at half-filling between a $d_{x^2-y^2}$ density wave and a $d_{x^2-y^2}$ superconductor. Near the critical point, we can focus on the low-energy degrees of freedom: the Goldstone modes and the nodal fermionic excitations. We can write down an $O(4)$ invariant action for this:

$$\begin{aligned} S_{\rm eff} = & \int d\tau \frac{d^2k}{(2\pi)^2} \Psi^{A\alpha\dagger}(k)[\partial_\tau - \epsilon(k)] \Psi_{A\alpha}(k) \\ & + ig \int d\tau \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \Phi_{\underline{i}}(q) f(k) \\ & \times \Bigg[\epsilon^{\alpha\beta} \Psi_{C\alpha}\left(k+\frac{q}{2}\right) \epsilon^{CA} \tau^{\underline{i}B}_{A} \Psi_{B\beta}\left(-k+\frac{q}{2}\right) \\ & + \epsilon_{\alpha\beta} \Psi^{A\alpha\dagger}\left(k+\frac{q}{2}\right) \tau^{\underline{i}B}_{A} \epsilon^{BC} \Psi^{B\beta\dagger}\left(-k+\frac{q}{2}\right) \Bigg] \\ & + \int d\tau d^2x \left((\partial_\mu \Phi_{\underline{i}})^2 + \frac{1}{2} r \Phi_{\underline{i}} \Phi_{\underline{i}} + \frac{1}{4!} u (\Phi_{\underline{i}} \Phi_{\underline{i}})^2 \right). \end{aligned} \tag{66}$$

In this Lagrangian, we have rescaled all of the velocities and stiffnesses to 1. In general, these quantities will be different—breaking the $O(4)$ symmetry—and this cannot be done. Symmety-breaking terms will be briefly addressed below.

The transition between the $d_{x^2-y^2}$ density wave and the $d_{x^2-y^2}$ superconductor is driven by a pseudospin-2 symmetry-breaking field, which we will call u:

$$\mathcal{L}_u = u(\Phi_0^2 + \Phi_+ \Phi_-) = u(\Phi_{\underline{3}}^2 - \Phi_{\underline{1}}^2 - \Phi_{\underline{2}}^2). \tag{67}$$

For $u<0$, the 3 axis is an easy axis, and the $d_{x^2-y^2}$ density-wave state is favored; for $u>0$, the 1-2 plane is an easy plane, and the $d_{x^2-y^2}$ superconducting state is favored.

We can move away from a nested Fermi surface by tuning the chemical potential or a next-neighbor hopping parameter. Such effects are encapsulated by a pseudospin-1 symmetry-breaking term:

$$S_\mu = \mu O^3 = \int d\tau d^2x (\epsilon_{\underline{ij}} \Phi_{\underline{i}} \partial_\tau \Phi_{\underline{j}} + \Psi^\dagger \tau^3 \Psi), \tag{68}$$

where O^3 is the pseudospin SU(2) generator defined above. If $u=0$, μ will immediately force the pseudospin into the 1-2 plane, i.e., the superconductor will be favored. If $u<0$, the $d_{x^2-y^2}$ density-wave state will be favored until $\mu_c \propto (\sqrt{-u})$. At this point, a first-order phase transition—the pseudo-spin-flip transition—will occur at which the pseudospin switches from an easy-axis phase to an easy-plane phase. If we allow Φ_0 to have a different velocity than $\Phi_\pm$, then this first order phase transition can become two second-order phase transitions. Depending on the values of these parameters and the strength of quantum fluctuations, the intervening phase can either have both types of order or neither.

The critical point occurs when the jump in Φ is tuned to zero. Hence, it is a tricritical point. At such a critical point, $O(4)$-breaking terms can scale to zero. The critical point and the quantum critical region[24,25] are described by the physics of the critical fluctuations coupled to nodal fermionic excitations. By arguments similar to those of Ref. 21, the nodal fermions are neutral, spin-$\frac{1}{2}$ objects. A more detailed analysis will be given elsewhere.[26]

Similar conclusions can be drawn for d_{xy} and $d_{x^2-y^2} + i d_{xy}$ transitions; the latter case is particularly simple since there are no fermions. In the case of transitions between p_x-wave density-wave and superconducting states, the order parameters are both pseudospin *and* spin triplets. Hence the effective-field theory for such a transition takes the form

$$\begin{aligned} S_{\rm eff} = & \int d\tau \frac{d^2k}{(2\pi)^2} \Psi^{A\alpha\dagger}[\partial_\tau - \epsilon(k)] \Psi_{A\alpha} \\ & + ig \int d\tau \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \Phi^{a}_{\underline{i}}(q) f(k) \Bigg[\epsilon^{\gamma\alpha} \sigma^{a\beta}_{\alpha} \Psi_{C\gamma} \\ & \times \left(k+\frac{q}{2}\right) \epsilon^{CA} \tau^{aB}_{A} \Psi_{B\beta}\left(-k+\frac{q}{2}\right) + \sigma^{a\beta}_{\alpha} \epsilon_{\beta\gamma} \Psi^{A\alpha\dagger} \\ & \times \left(-k+\frac{q}{2}\right) \tau^{aB}_{A} \epsilon^{BC} \Psi^{B\gamma\dagger}\left(-k+\frac{q}{2}\right) \Bigg] + \int d\tau d^2x \\ & \times \left((\partial_\mu \Phi_{\underline{i}})^2 + \frac{1}{2} \Phi^{a}_{\underline{i}} \Phi^{a}_{\underline{i}} + \frac{1}{4!} u (\Phi^{a}_{\underline{i}} \Phi^{a}_{\underline{i}})^2 \right), \end{aligned}$$

where

$$\begin{pmatrix} \Phi^{a}_{\underline{1}} \\ \Phi^{a}_{\underline{2}} \\ \Phi^{a}_{\underline{3}} \end{pmatrix} = \begin{pmatrix} \sqrt{2}\,\mathrm{Re}\{\langle \psi^\dagger_\gamma(k) \epsilon^{\gamma\alpha} \sigma^{a\beta}_{\alpha} \psi^\dagger_\beta(-k) t \rangle\} \\ \sqrt{2}\,\mathrm{Im}\{\langle \psi^\dagger_\gamma(k) \epsilon^{\gamma\alpha} \sigma^{a\beta}_{\alpha} \psi^\dagger_\beta(-k) \rangle\} \\ i \langle \psi^{\alpha\dagger}(k+Q) \sigma^{a\beta}_{\alpha} \psi_\beta(k) \rangle \end{pmatrix}. \tag{69}$$

VII. DISCUSSION

In this paper, we have discussed the properties of ordered states in which particle-hole pairs with nonzero angular momentum condense. These states generalize charge- and spin-density-wave states in the same way that p- and d-wave superconductors generalize s-wave superconductivity. However, unlike in the superconducting case—where the Meiss-

ner effect follows directly from the broken symmetry, irrespective of the pairing channel—the angular variation of the condensate makes *p*- and *d*-wave density-wave ordering difficult to detect. Experiments seeking to uncover such order must (a) be sensitive to spatial variations of kinetic energy or currents, or (b) measure higher-order correlations of the charge or spin density. We explained how μSR, neutron scattering, NQR, and Raman scattering can be used in this regard. Impurities, which break rotational invariance, would cause an admixture of *p*- or *d*-wave ordering with *s*-wave ordering. It is natural to wonder whether experiments which appear to detect SDW order should be re-examined to see if they have actually uncovered *p*- or *d*-wave order which, as a result of impurities, is masquerading as *s*-wave order.

As in the superconducting case, the nontrivial pairing symmetry can lead to the existence of nodal excitations. As parameters such as the chemical potential or next-neighbor hopping are varied, nodal excitations appear at a transition which is third order in mean-field theory. The ''phase'' with nodal excitations is always critical.

The analogies between *p*- and *d*-wave density-wave ordering and *p*- and *d*-wave superconductivity begs the question: what is the nature of a phase transition between such states? In answering this question, we are led to one of the motivations of this work. The pseudogap regime of the cuprate superconductors exhibits some properties which can be associated with $d_{x^2-y^2}$ ordering. One explanation is that some features of the $d_{x^2-y^2}$ superconducting state have been inherited. However, it is natural to inquire whether the physics of this regime could also be determined in part by proximity to a $d_{x^2-y^2}$ density-wave state or the transition between the density-wave and superconducting states. In other words, we ask whether the physics of the pseudogap regime should be described by a theory which incorporates fluctuations between $d_{x^2-y^2}$ density-wave and superconducting states. In this connection, we note that the physics of the critical point between $d_{x^2-y^2}$ density-wave and superconducting states bears a rough resemblance to that of the SU(2) mean-field theory of the *t*-*J* model.[19,20] In that theory, the gauge field parametrizes fluctuations between the $d_{x^2-y^2}$ density-wave and superconducting states, a role played in our analysis by the Goldstone bosons of the $O(4)$ effective theory. We also note that the nodal liquid state[21–23] shares many features of the $d_{x^2-y^2}$ density-wave and superconducting states. In both the nodal liquid and SU(2) gauge theories, the quasiparticle spectra of spin-charge-separated phases follow from their proximity to a $d_{x^2-y^2}$ density wave. These issues and their possible relevance to the cuprates will be further explored elsewhere.[26]

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