Spin modes in polarized ^{3}He clusters

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The transverse-spin modes in clusters of ^{3}He are considered. These excitations correspond to a gapless Goldstone mode which comes along due to the broken symmetry in the spin space, and can exhibit themselves in polarized ^{3}He. These modes can be interpreted as weakly damped fluctuations of the transverse spin components, which propagate through ^{3}He. The dispersion law of such spin waves is calculated by means of a semiclassical variational method and compared with the results obtained on the basis of the Fermi-liquid theory. The finite sizes of the clusters lead to quantization of the spectra and result in a series of excited states. The number of long-living excited states is strongly affected by the Landau damping. The orthogonality relations and energy-weighted sum rule are formulated. The collective spin modes for $l=0$ and several cluster sizes are computed.

I. INTRODUCTION

A system of interacting fermions may exhibit a great variety of collective Bose excitations. These collective modes occur as a result of the interaction between fermions and are absent in a perfect noninteracting gas. The interaction plays a very important role when calculating spectra or formulating the existence criteria of weakly damped excitations. The situation is quite different in the case of systems with a broken symmetry, as, for instance, spin polarized Fermi systems, which exhibit a gapless Goldstone mode as a result of the broken symmetry. A collective magnetic excitation exists in any interacting Fermi system independently of the actual interaction between the particles. The propagation of spin fluctuations in spin polarized Fermi systems has been discussed in Refs. 1–4.

The problem addressed here is how the actual size of a finite system influences the collective magnetic modes which are known to exist in extended systems. We focus on the transverse spin excitations in small portions or droplets of polarized ^{3}He matter. There might be many branches of collective Bose excitations due to the spin degrees of freedom. To be specific we only consider transverse spin modes which have pure collective origin and will only occur in large enough drops. In the limit $N\to\infty$ they coincide with the spin waves in polarized liquid ^{3}He (bulk). These magnetic collective excitations are calculated on the basis of a semiclassical variational method which takes into account the finite size of the cluster and the associated boundary conditions, leading to the quantization of the spectrum and which is not restricted to a small polarization condition. The orthogonality relations and the energy-weighted sum rule are formulated and allow to predict the strength of the several modes associated with a prescribed transition operator. We compute the collective spin modes of ^{3}He clusters for several polarizations and cluster sizes, both in the presence and in the absence of the polarizing magnetic field.

II. FERMI-LIQUID APPROACH

Before we study the propagation of spin fluctuations in finite systems, using the semiclassical variational method developed in Refs. 5 and 6, we will give a short presentation of the Fermi-liquid approach and calculate the magnon spectrum in infinite matter. Dynamics and collective properties of an infinite Fermi fluid at the zero temperature can be described in terms of the collisionless quasiclassical transport equation[7] for the density matrix (distribution function) in spin space $\hat{n}$:

$$\partial_t\hat{n}+\frac{1}{2}[\partial_{\mathbf{p}}\hat{\epsilon}\cdot\nabla\hat{n}+\nabla\hat{n}\cdot\partial_{\mathbf{p}}\hat{\epsilon}]-\frac{1}{2}[\nabla\hat{\epsilon}\cdot\partial_{\mathbf{p}}\hat{n}+\partial_{\mathbf{p}}\hat{n}\cdot\nabla\hat{\epsilon}]+i[\hat{\epsilon},\hat{n}]=0. \tag{1}$$

Here $\hat{\epsilon}$ is the self-consistent single-particle excitation energy. For small perturbations of the density matrix $\delta\hat{n}$, the self-consistent energy can be expressed as a linear functional of $\delta\hat{n}$:

$$\delta\hat{\epsilon}_\sigma(\mathbf{p})=\mathrm{Tr}_{\sigma'}\sum_{p'}\hat{f}_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')\,\delta\hat{n}_{\sigma'}(\mathbf{p}'), \tag{2}$$

where σ labels different spin states. The function $\hat{f}$ describes the interaction between two quasiparticles (^{3}He-^{3}He) and is the main quantitative feature of the Fermi-liquid theory. In an isotropic polarized Fermi fluid the interaction function takes the form

$$\hat{f}_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')=\psi(\mathbf{p},\mathbf{p}')\hat{I}\hat{I}'+\zeta(\mathbf{p},\mathbf{p}')\hat{\sigma}\cdot\hat{\sigma}'. \tag{3}$$

Only the term with the function ζ in Eq. (3) contributes to the spectrum of transverse modes.

Both $\hat{n}$ and $\hat{\epsilon}$ are, indeed, linear functions of the spin operator, i.e., of the Pauli matrices $\hat{\sigma}_i$, $i=1,2,3$. In equilibrium the density matrix $\hat{n}$ is diagonal in the spin space:

$$\hat{n}^{(0)}(\mathbf{p})=\frac{n_+ + n_-}{2}\hat{I}+\frac{n_+ - n_-}{2}\hat{\sigma}_3, \tag{4}$$

where $\hat{I}$ is the unity matrix, and n_+ and n_- are the Fermi distribution functions for particles with spin up and spin down, respectively:

$$n_+=\theta(\mu_+-\epsilon_+), \quad n_-=\theta(\mu_--\epsilon_-). \tag{5}$$

Here, μ_+ and μ_- are the Fermi energies for the spin-up and spin-down components. The single-particle energy spectra, ϵ_+ and ϵ_- determine the self-consistent energy $\hat{\epsilon}^{(0)}$ in equilibrium:

$$\hat{\epsilon}^{(0)}(\mathbf{p})=\frac{\epsilon_+ + \epsilon_-}{2}\hat{I}+\frac{\epsilon_+ - \epsilon_-}{2}\hat{\sigma}_3=a(\mathbf{p})\hat{I}+b(\mathbf{p})\hat{\sigma}_3. \tag{6}$$

The number of exceeding spin up (or spin down), i.e., the effective degree of polarization α, is normally small, $|\alpha| \ll 1$. (Moreover, the standard Fermi-liquid theory does not hold to describe the transverse spin modes in a strongly polarized system,[8] $|\alpha| \leqslant 1$. Formally carrying out such a calculation would require integrating over the Fermi sea, which is in an obvious contradiction with the *cornerstone* of the Fermi-liquid approach.) We will, therefore, consider

$$\alpha \ll 1. \tag{7}$$

Keeping the linear in α term one can easily obtain

$$n_+ + n_- \approx n_0=\theta(\mu_0-\epsilon_0), \tag{8}$$

$$n_+ - n_- \approx \frac{4}{3}\frac{\partial n_0}{\partial \epsilon_0}\epsilon_F \alpha, \quad \epsilon_F=\frac{p_F^2}{2m^*}, \tag{9}$$

where the Fermi momentum, $p_F=(3\pi^2\rho_0)^{1/3}$, has been introduced. Here ρ_0 is the density of particles, m^* is the ^{3}He effective mass, and the index ''0'' refers to the characteristics of the unpolarized system, i.e., where $\alpha=0$. On the Fermi surface (with radius p_F) the interaction function $\hat{f}_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')$ depends only on the angle γ between vectors $\mathbf{p}$ and $\mathbf{p}'$ as $|\mathbf{p}|=|\mathbf{p}'|=p_F$. It is convenient to define the dimensionless interaction function $g_0\hat{f}$, where g_0 is the density of states on the Fermi surface, and expand it in a series of the Legendre polynomials $P_n(\cos\gamma)$ as usual, namely,

$$\frac{p_F m^*}{\pi^2}\zeta(\gamma)=\sum_{n=0}^{\infty} Z_n P_n(\cos\gamma), \tag{10}$$

$$\frac{p_F m^*}{\pi^2}\psi(\gamma)=\sum_{n=0}^{\infty} F_n P_n(\cos\gamma). \tag{11}$$

Combining Eqs. (2)–(11) one can calculate the spin-dependent term in the self-consistent energy of a quasiparticle in equilibrium [see Eq. (6)]:

$$b(\mathbf{p})=\frac{\Omega_{int}}{2}, \quad \Omega_{int}=\frac{4}{3}\alpha\epsilon_F Z_0. \tag{12}$$

A small deviation of $\hat{n}$ from its equilibrium value (4) is a linear function of $\hat{\sigma}$ as well, and will be sought in the form:

$$\delta\hat{n}(\mathbf{p})=\hat{n}(\mathbf{p})-\hat{n}^{(0)}(\mathbf{p})=\nu(\mathbf{p})\hat{I}+\sum_{k=1}^{2}\lambda_k(\mathbf{p})\hat{\sigma}_k. \tag{13}$$

The extra terms in the self–consistent energy due to the fluctuations of the density matrix can be calculated from Eqs. (2), (3), and (13) and read

$$\hat{\epsilon}-\hat{\epsilon}^{(0)}=\delta\hat{\epsilon}(\mathbf{p})=\hat{I}\sum_{p'}\psi(\mathbf{p},\mathbf{p}')\nu(\mathbf{p}') + \sum_{i=1}^{2}\hat{\sigma}_i\sum_{p'}\zeta(\mathbf{p},\mathbf{p}')\lambda_i(\mathbf{p}'). \tag{14}$$

Now we have a complete set of equations in order to calculate the spectra of magnons. When substituting Eqs. (4)–(14) in the linearized transport Eq. (1) we get two equations for λ_1 and λ_2, which describe the dynamics of the off-diagonal components of the density matrix, i.e., the transverse spin modes in which we are interested. Instead of λ_1 and λ_2 it is more convenient to use circular variables: $\lambda_\pm=\lambda_1\pm i\lambda_2$. In the case of a weakly polarized system, $|\alpha| \ll 1$, it is natural to seek a solution in the form

$$\lambda_\pm=\frac{\partial n_0}{\partial\epsilon_0}\eta_\pm(\theta,\varphi), \tag{15}$$

where θ and φ are the polar angle and azimuth in momentum space, i.e., on the Fermi sphere. We will be seeking the eigenvalues of the transport equation for the Fourier components: $\eta_\pm \propto \exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$. After some manipulations of Eqs. (4)–(15), we get

$$(\omega+\Omega_{int}-\mathbf{k}\cdot\mathbf{v})\,\eta_+(\theta,\varphi) - \left(\mathbf{k}\cdot\mathbf{v}+\frac{\Omega_{int}}{Z_0}\right)\int Z(\theta,\varphi,\theta',\varphi')\,\eta_+(\theta',\varphi')\frac{do'}{4\pi}=0. \tag{16}$$

Here $\mathbf{v}=p_F\mathbf{n}(\theta,\varphi)/m^*$ and $\mathbf{n}$ is a unit vector on the Fermi sphere. The equation (and solution) for η_- can be obtained from Eq. (16) simply by replacing $\omega\to-\omega$ and $\mathbf{k}\to-\mathbf{k}$.

Expanding the quantities ω and ν in a power series of the small wave vector, $kv/|\Omega_{int}| \ll 1$, (similar to the procedure used in Refs. 7 and 8) one obtains the following dispersion law:

$$\omega=\frac{k^2}{2M}, \quad M=m^*\alpha\frac{Z_0-\frac{1}{3}Z_1}{(1+Z_0)(1+\frac{1}{3}Z_1)}. \tag{17}$$

Here, the quantity M can be considered as the effective mass of the new elementary excitation, the magnon. These collective modes correspond to fluctuations of the transverse components of the spin. Spectrum (17) are the spin waves in a magnetized Fermi liquid obtained in Ref. 8.

Microscopically a collective mode of the Fermi-liquid type can be interpreted as a bound state of a particle and a hole in the Fermi sea. If the collective excitation moves too fast, it turns out to be unstable with respect to decay into a particle-hole pair. The threshold, above which the Landau damping comes into effect and kills the spin waves, can easily be seen from Eq. (16). The collective modes will survive if the condition

$$\omega+\Omega_{int}-\mathbf{k}\cdot\mathbf{v}>0, \tag{18}$$

is satisfied. In the long-wavelength limit where the frequency ω is given by Eq. (17), this criterion reduces approximately to: $kv\ll|\Omega_{int}|$.

III. COLLECTIVE EXCITATIONS: VARIATIONAL APPROACH

In this section we apply the semiclassical variational method developed in Refs. 5 and 6 to the propagation of spin fluctuations in finite systems such as droplets of ^{3}He matter. For an infinite system, the variational method is equivalent to the collisionless Landau-Boltzmann transport equation, Eq. (1). For a finite system, it leads also to appropriate boundary conditions.

A cluster of ^{3}He is described by a spin-dependent matrix-valued distribution function, depending not only on the momentum but also on the position,

$$\hat{n}=\hat{n}(\mathbf{p},\mathbf{r},t).$$

In analogy with Eq. (4), the equilibrium distribution function reads

$$\hat{n}=\begin{pmatrix} n_+ & 0 \\ 0 & n_- \end{pmatrix}=\frac{\hat{I}+\hat{\sigma}_3}{2}n_+ +\frac{\hat{I}-\hat{\sigma}_3}{2}n_- = n_0\hat{I}+n_3\hat{\sigma}_3, \tag{19}$$

with $n_+=\theta(\mu_+-\epsilon_+)$, $n_-=\theta(\mu_--\epsilon_-)$, $n_0=\frac{1}{2}(n_++n_-)$, $n_3=\frac{1}{2}(n_+-n_-)$, where $\epsilon_+=\epsilon_+(\mathbf{p},\mathbf{r},t)$, $\epsilon_-=\epsilon_-(\mathbf{p},\mathbf{r},t)$, are the equilibrium self-consistent one-body (effective) Hamiltonians, which are position, momentum and time dependent, respectively, for particles with spin up $(+)$ and spin down $(-)$. The interaction function is a local one of the form

$$\hat{f}_{\sigma\sigma'}(\mathbf{p},\mathbf{r},\mathbf{p}',\mathbf{r}')=\hat{f}_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')\,\delta(\mathbf{r}-\mathbf{r}'), \tag{20}$$

where $\hat{f}_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')$ has the general form defined in Eq. (3). In the present approach, we will consider

$$\psi(\mathbf{p},\mathbf{p}')=a+c(\mathbf{p}-\mathbf{p}')^2,\quad \zeta(\mathbf{p},\mathbf{p}')=b+d(\mathbf{p}-\mathbf{p}')^2, \tag{21}$$

where the coupling constants a,b,c,d will be chosen so that the Landau parameters F_0,F_1,Z_0,Z_1 derived from experiment,[7–10] are reproduced. The equilibrium self-consistent hamiltonian (6) with interaction (20) and (21) is given by

$$\begin{aligned}\hat{\epsilon}(1)&=\frac{\mathbf{p}^2}{2m}\hat{I}(1)-\beta\mathcal{H}\hat{\sigma}_3(1)+\mathrm{Tr}_2\int d\Gamma_2\,\delta(\mathbf{r}_1-\mathbf{r}_2)\\ &\quad\times[\psi(\mathbf{p}_1,\mathbf{p}_2)\hat{I}(2)\hat{I}(1)+\zeta(\mathbf{p}_1,\mathbf{p}_2)\hat{\boldsymbol{\sigma}}(2)\cdot\hat{\boldsymbol{\sigma}}(1)]\hat{n}(2)\\ &=\left(\frac{\mathbf{p}^2}{2m^*}+a\rho_0+c\tau_0\right)\hat{I}+[(\mathbf{p}^2d+b)\rho_3-\beta\mathcal{H}+d\tau_3]\hat{\sigma}_3,\end{aligned} \tag{22}$$

where $\mathcal{H}$ is the external magnetic field, $d\Gamma_2=dr_2^3dp_2^3/(2\pi)^3$, m is the mass of the ^{3}He atom, β is the magnetic moment of ^{3}He, $[2\beta=2.0378\times10^4\ \ (\mathrm{G\,s})^{-1}]$ and

$$\rho_0=2\int\frac{d^3p}{(2\pi)^3}n_0=\frac{p_{F+}^3+p_{F-}^3}{6\pi^2},$$

$$\rho_3=2\int\frac{d^3p}{(2\pi)^3}n_3=\frac{p_{F+}^3-p_{F-}^3}{6\pi^2},$$

$$\tau_0=\frac{(p_{F+}^5+p_{F-}^5)}{10\pi^2},\quad \tau_3=\frac{p_{F+}^5-p_{F-}^5}{10\pi^2},\quad \frac{1}{2m^*}=\frac{1}{2m}+c\rho_0.$$

The Fermi momenta for particles with spin up and spin down, respectively, $p_{F+}(\mathbf{r},t)$ and $p_{F-}(\mathbf{r},t)$, are constant inside the cluster and vanish outside. Taking into account assumption (7), which in terms of p_{F+} and p_{F-} is given by

$$\alpha=\frac{p_{F+}^3-p_{F-}^3}{p_{F+}^3+p_{F-}^3}\ll1,$$

we have, up to first order in the polarization α, $\rho_3=\alpha\rho_0$, $\tau_3=\alpha p_F^2\rho_0$, $\upsilon_3=\alpha p_F^4\rho_0$, $\tau_0=p_F^5/(5\pi^2)$ with $\rho_0=p_F^3/(3\pi^2)$. Here and in the sequel we will use p_F and ϵ_F, respectively, for the Fermi momentum and Fermi energy of the unpolarized cluster. We stress, however, that $\alpha\ll1$ is not a necessary assumption in the present formalism.

We fix the parameters of the model in the following way:

$$\begin{aligned}F_0&=\frac{m^*p_F}{\pi^2}(a+2cp_F^2)=9.28,\\ F_1&=-\frac{m^*p_F}{\pi^2}(2cp_F^2)=5.39,\end{aligned} \tag{23}$$

$$\begin{aligned}Z_0&=\frac{m^*p_F}{\pi^2}(b+2dp_F^2)=-0.696,\\ Z_1&=-\frac{m^*p_F}{\pi^2}(2dp_F^2)=-0.54,\end{aligned} \tag{24}$$

with $p_F=1.55\times10^3$ eV/c, $m^*/m=2.80$.

In terms of the Landau parameters the spin-dependent term in the self-consistent energy of a quasiparticle in equilibrium ϵ_3, at the Fermi surface, reads

$$\epsilon_3=\frac{\Omega_{int}}{2}-\beta\mathcal{H},\quad \Omega_{int}=\frac{4}{3}\alpha\epsilon_FZ_0,$$

in accordance with Eq. (12).

In order to proceed, it is convenient to introduce the semiclassical limit of the commutator, which we denote by $[\cdot,\cdot]_{SC}$. Let Δ,Λ be operators in the spin space and let f,g be either operators in configuration space or their Wigner transforms, depending on the context. We define

$$[f\Delta,g\Lambda]_{SC}=\frac{i}{2}(\Delta\Lambda+\Lambda\Delta)\{f,g\}+fg[\Delta,\Lambda]. \quad (25)$$

In the right-hand side, f,g are functions of $\mathbf{r},\mathbf{p}$. In the left-hand side, they are operators. By $\{\cdot,\cdot\}$, we denote the Poisson bracket.

The self-consistency condition, which may be expressed as

$$[\hat{\epsilon},\hat{n}]_{SC}=0, \quad (26)$$

with $\hat{n}$ defined by Eq. (19) and $\hat{\epsilon}$ by Eq. (22), clearly holds. Indeed, we have

$$[\hat{\epsilon},\hat{n}]_{SC}=\{\epsilon_+,n_+\}\frac{\hat{I}+\hat{\sigma}_3}{2}+\{\epsilon_-,n_-\}\frac{\hat{I}-\hat{\sigma}_3}{2}=0. \quad (27)$$

In Ref. 5 an action principle was established for the collisionless Boltzmann transport equation. Let us consider the set of all trial distribution functions $\hat{n}(\mathbf{p},\mathbf{r})$ related by a canonical transformation to a fixed distribution function $\hat{n}_f(\mathbf{p},\mathbf{r})$. If, for all times t, the time-dependent distribution function $\hat{n}(\mathbf{p},\mathbf{r},t)$ belongs to this set a generating operator $\hat{F}(\mathbf{p},\mathbf{r},t)$ may be found such that

$$\frac{\partial\hat{n}}{\partial t}+i[\hat{n},\hat{F}]_{SC}=0. \quad (28)$$

The action principle may now be written as

$$\delta S=\delta\int L\,dt=0, \quad (29)$$

where

$$L=\mathrm{Tr}\int d\Gamma\,\hat{n}(\mathbf{p},\mathbf{r},t)\hat{F}(\mathbf{p},\mathbf{r},t)-E[\hat{n}], \quad (30)$$

with $E[\hat{n}]$ the energy functional

$$E[\hat{n}]=\mathrm{Tr}\int d\Gamma\frac{p^2}{2m}\hat{n}(\mathbf{p},\mathbf{r},t)+\frac{1}{2}\mathrm{Tr}\,\mathrm{Tr}'\int d\Gamma\,d\Gamma'\hat{f}_{\sigma\sigma'}(\mathbf{p},\mathbf{r},\mathbf{p}',\mathbf{r}')\hat{n}(\mathbf{p},\mathbf{r},t)\times\hat{n}(\mathbf{p}',\mathbf{r}',t).$$

In the present study we will only consider small fluctuations of the distribution function and it will be enough to consider the second-order Lagrangian on the infintesimal generator of the flucutations. In the sequel, we denote by $\hat{S}$ the generator of spin fluctuations of the distribution function. We focus on *transverse* modes which are associated with a generator of the form

$$\hat{S}=S_1\hat{\sigma}_1+S_2\hat{\sigma}_2, \quad (31)$$

where S_1 and S_2 are c-number varitional functions of $\mathbf{r}$, $\mathbf{p}$, and t. If the cluster on its ground state has total spin Σ and three-component $m_\Sigma=\Sigma$, this generator gives rise to excitations with quantum numbers Σ', $m_{\Sigma'}=\Sigma\pm1$.

The dynamics of small amplitude fluctuations is determined by the second-order Lagrangian:

$$L^{(2)}=\frac{i}{2}\mathrm{Tr}\int d\Gamma\,\hat{n}[\partial_t\hat{S},\hat{S}]_{SC}+\frac{1}{2}\mathrm{Tr}\int d\Gamma[\hat{S},\hat{\epsilon}]_{SC}[\hat{S},\hat{n}]_{SC}-\frac{1}{2}\mathrm{Tr}_1\,\mathrm{Tr}_2\int d\Gamma_1 d\Gamma_2[\hat{S},\hat{n}]_{SC}(1)\times[\hat{S},\hat{n}]_{SC}(2)\zeta(\mathbf{p}_1,\mathbf{p}_2)\hat{\boldsymbol{\sigma}}(1)\cdot\hat{\boldsymbol{\sigma}}(2)\delta(\mathbf{r}_1-\mathbf{r}_2), \quad (32)$$

where $\hat{\epsilon}(\mathbf{r},\mathbf{p},t)$ was defined in Eq. (22). The fluctuation of the distribution function is

$$\delta\hat{n}=i[\hat{S},\hat{n}]_{SC}. \quad (33)$$

Its time evolution is determined by the Lagrangian (32).

IV. DISPERSION RELATIONS

In order to proceed and obtain the equations of motion for the clusters, we need to make an ansatz for the varitional functions $S_k(\mathbf{p},\mathbf{r},t)$, $k=1,2$, introduced in Eq. (31). The simplest ansatz,[5] since each one of the functions should have a time-even component, which describes the dynamical behavior of the system, and a time-odd part, which describes the static behavior of the system, is

$$S_k(\mathbf{p},\mathbf{r},t)=\phi_k(\mathbf{r},t)+\mathbf{p}\cdot\boldsymbol{\psi}_k(\mathbf{r},t), \quad k=1,2. \quad (34)$$

This approximation preserves the Goldstone mode and has the effect of replacing the Landau damped modes by a discrete set of modes. The Lagrangian which determines the time evolution of the fields $\phi_j,\boldsymbol{\psi}_j$ is derived from Eq. (32) and reduces to

$$L^{(2)}=\int d^3r\times\left\{\frac{\rho_0}{2}(\partial_\alpha\dot{\phi}_1\psi_{1\alpha}+\partial_\alpha\dot{\phi}_2\psi_{2\alpha}-\partial_\alpha\phi_1\dot{\psi}_{1\alpha}-\partial_\alpha\phi_2\dot{\psi}_{2\alpha})+\rho_3(\phi_2\dot{\phi}_1-\phi_1\dot{\phi}_2)+\frac{\tau_3}{3}(\boldsymbol{\psi}_2\cdot\dot{\boldsymbol{\psi}}_1-\boldsymbol{\psi}_1\cdot\dot{\boldsymbol{\psi}}_2)-\mathcal{E}(\phi_1,\phi_2,\boldsymbol{\psi}_1,\boldsymbol{\psi}_2)\right\}, \quad (35)$$

where the energy density $\mathcal{E}$ is given by

$$\mathcal{E}(\phi_1,\phi_2,\boldsymbol{\psi}_1,\boldsymbol{\psi}_2)=\rho_0\left(\frac{1}{2m^*}-d\rho_0\right)[(\nabla\phi_1)^2+(\nabla\phi_2)^2]+\frac{\tau_0}{6m^*}(\partial_\alpha\psi_{1\beta}\partial_\alpha\psi_{1\beta}+\partial_\alpha\psi_{1\beta}\partial_\beta\psi_{1\alpha}+\partial_\alpha\psi_{2\beta}\partial_\alpha\psi_{2\beta}$$
$$+\partial_\alpha\psi_{2\beta}\partial_\beta\psi_{2\alpha})$$
$$+2\left(\frac{\tau_3}{3m^*}+\beta H\rho_0-\frac{2}{3}d\rho_0\tau_3\right)(\boldsymbol{\psi}_1\cdot\nabla\phi_2-\boldsymbol{\psi}_2\cdot\nabla\phi_1)+\left(\frac{\tau_0}{6m^*}+\frac{b}{2}\rho_0^2+\frac{5d}{3}\rho_0\tau_0\right) \tag{36}$$
$$\times[(\nabla\cdot\boldsymbol{\psi}_1)^2+(\nabla\cdot\boldsymbol{\psi}_2)^2]-\frac{2}{3}\left(b\rho_3\tau_3+d\rho_3 v_3+\frac{5}{3}d\tau_3^2-\beta H\tau_3\right)(\boldsymbol{\psi}_1\cdot\boldsymbol{\psi}_1+\boldsymbol{\psi}_2\cdot\boldsymbol{\psi}_2)$$
$$+2\beta\mathcal{H}\rho_3(\phi_1^2+\phi_2^2)+\left(\frac{\tau_3}{m}+\beta H\rho_0+b\rho_0\rho_3+d\rho_0\tau_3+\frac{5}{3}d\rho_3\tau_0\right)\nabla\cdot(\phi_1\boldsymbol{\psi}_2-\phi_2\boldsymbol{\psi}_1). \tag{37}$$

The Euler-Lagrange equations read

$$2\rho_3\dot{\phi}_2-\rho_0\nabla\cdot\bar{\boldsymbol{\psi}}_1=-\frac{\delta\mathcal{E}}{\delta\phi_1}, \tag{38}$$

$$2\rho_3\dot{\phi}_1+\rho_0\nabla\cdot\bar{\boldsymbol{\psi}}_2=\frac{\delta\mathcal{E}}{\delta\phi_2}, \tag{39}$$

$$\frac{2\tau_3}{3}\bar{\boldsymbol{\psi}}_1+\rho_0\nabla\dot{\phi}_2=-\frac{\delta\mathcal{E}}{\delta\boldsymbol{\psi}_1}, \tag{40}$$

$$\frac{2\tau_3}{3}\bar{\boldsymbol{\psi}}_2-\rho_0\nabla\dot{\phi}_1=\frac{\delta\mathcal{E}}{\delta\boldsymbol{\psi}_2}. \tag{41}$$

In Appendix A we discuss these equations in terms of the more familiar equations for the time evolution of the transition density and transition current density, which arise in the Fermi-liquid approach. Equations (38) and (39) correspond to Eq. (A3) the continuity equation for the density, and Eqs. (40) and (41) are the equations of motion for the density current (A6).

The boundary conditions at $r=R$ (the cluster radius) are also derived from the Lagrangian and read

$$\mathbf{r}\cdot[\rho_0\dot{\boldsymbol{\psi}}_1+2\rho_0c_1\nabla\phi_1+(c_4-\beta\mathcal{H}\rho_0)\boldsymbol{\psi}_2]=0, \tag{42}$$

$$\mathbf{r}\cdot[\rho_0\dot{\boldsymbol{\psi}}_2+2\rho_0c_1\nabla\phi_2-(c_4-\beta\mathcal{H}\rho_0)\boldsymbol{\psi}_1]=0, \tag{43}$$

$$x_\alpha\left[\left(\frac{2\tau_3}{3m^*}+c_4+\beta\mathcal{H}\rho_0\right)\phi_1+c_3\nabla\cdot\boldsymbol{\psi}_2\right]$$
$$+x_\beta\frac{\tau_0}{3m^*}(\partial_\alpha\psi_\beta+\partial_\beta\psi_{2\alpha})=0, \tag{44}$$

$$x_\alpha\left[-\left(\frac{2\tau_3}{3m^*}+c_4+\beta\mathcal{H}\rho_0\right)\phi_2+c_3\nabla\cdot\boldsymbol{\psi}_1\right]$$
$$+x_\beta\frac{\tau_0}{3m^*}(\partial_\alpha\psi_{1\beta}+\partial_\beta\psi_{1\alpha})=0. \tag{45}$$

Notice that $\boldsymbol{\psi}_1,\phi_2$ behave as coordinates and $\boldsymbol{\psi}_2,\phi_1$ as momenta, or vice versa. The first two boundary equations correspond, in the Fermi-liquid approach, to Eqs. (A4) and (A5) given in Appendix A, and the last two equations correspond to Eqs. (A9) and (A10). We have introduced the quantities

$$c_1=\left(\frac{1}{2m^*}-d\rho_0\right)=\frac{1}{2m^*}\left(1+\frac{Z_1}{3}\right), \tag{46}$$

$$c_2=\left(b\rho_3\tau_3+d\rho_3 v_3+\frac{5}{3}d\tau_3^2\right)=\frac{2}{3}\alpha^2\epsilon_F\rho_0P_F^2\left(Z_0-\frac{Z_1}{3}\right), \tag{47}$$

$$c_3=\left(\frac{\tau_0}{m^*}+b\rho_0^2+\frac{10}{3}d\rho_0\tau_0\right)=\frac{4}{5}\epsilon_F\rho_0\left(\frac{3}{2}+\frac{5}{6}Z_0\right), \tag{48}$$

$$c_4=-\frac{\tau_3}{3m^*}+b\rho_0\rho_3+\frac{7}{3}d\rho_0\tau_3+\frac{5}{3}d\rho_3\tau_0$$
$$=\frac{2}{3}\epsilon_F\rho_3\left(-1+Z_0-\frac{2}{3}Z_1\right). \tag{49}$$

Now, we summarize the determination of the normal modes. In a first step we disregard the boundary conditions and determine a complete set of local solutions of the Euler-Lagrange equations. In a second step we construct, with the help of the boundary conditions, the correct linear combinations of the local solutions associated with a given frequency ω. The Euler-Lagrange equations admit two types of solutions for the variational scalar and vector funtions ϕ_i and $\boldsymbol{\psi}_i$, namely, irrotational and solenoidal solutions.

The solenoidal solutions (such that $\nabla\cdot\boldsymbol{\psi}_1=\phi_1=\nabla\cdot\boldsymbol{\psi}_2=\phi_2=0$) may be expressed in terms of the quantities $\mathbf{G}_1=\nabla\times\boldsymbol{\psi}_1$ and $\mathbf{G}_2=\nabla\times\boldsymbol{\psi}_2$. Then, $\mathbf{G}_\pm=\mathbf{G}_1\pm i\mathbf{G}_2$ obey the equations

$$\mp i\dot{\mathbf{G}}_\pm=\frac{2}{\tau_3}(c_2-\beta\mathcal{H}\tau_3)\mathbf{G}_\pm+\frac{\tau_0}{2m^*\tau_3}\nabla^2\mathbf{G}_\pm.$$

Making use of $\nabla^2\mathbf{G}_i=-k^2\mathbf{G}_i$ we obtain

$$\omega=\pm\left[2\beta\mathcal{H}+\frac{3}{5}\frac{k^2}{2m^*\alpha}-\frac{4}{3}\alpha\epsilon_F\left(Z_0-\frac{1}{3}Z_1\right)\right]. \tag{50}$$

For $k=0$ we get the energy of homogeneous spin waves in infinite matter in the presence of a uniform magnetic field,[1,11] associated with the components $\hat{\sigma}_+$ and $\hat{\sigma}_-$ of the spin polarization.

In order to describe the irrotational normal modes (such that $\nabla\times\boldsymbol{\psi}_1=\nabla\times\boldsymbol{\psi}_2=0$) it is convenient to define the scalar functions $F_i=\nabla\cdot\boldsymbol{\psi}_i$, $i\in\{1,2\}$ and to make the local ansatz $\phi_1=\phi_{1n}\cos\omega_n t$, $\boldsymbol{\psi}_2=\boldsymbol{\psi}_{2n}\cos\omega_n t$, $\phi_2=\phi_{2n}\sin\omega_n t$, $\boldsymbol{\psi}_1=\boldsymbol{\psi}_{1n}\sin\omega_n t$. In terms of $\phi_\pm=\phi_1\pm\phi_2$ and $F_\pm=F_2\pm F_1$, the local equations of motion become

$$(\pm\omega_n+2\beta\mathcal{H})(2\rho_3\phi_{\pm n}+\rho_0F_{\mp n})=-2c_1\left(\rho_0k^2\phi_{\pm n}+\frac{2\tau_3}{3}F_{\mp n}\right), \tag{51}$$

$$(\pm\omega_n+2\beta\mathcal{H})\left(\frac{2\tau_3}{3}F_{\mp n}+\rho_0k^2\phi_{\pm n}\right)=-\frac{4\tau_3}{3}c_1k^2\phi_{\pm n}+\left(\frac{4}{3}c_2-c_3k^2\right)F_{\mp n}. \tag{52}$$

Notice that F_{+n} is coupled to ϕ_{-n} and F_{-n} is coupled to ϕ_{+n}. Assuming $\phi_{1n}=-\phi_{2n}$ and $F_{1n}=F_{2n}$, the functions ϕ_-,F_+ are associated with the modes excited by an operator of the form [see Eqs. (31) and (34)]

$$\hat{S}=S_-\hat{\sigma}_-e^{i\omega t}+S_+\hat{\sigma}_+e^{-i\omega t}, \tag{53}$$

with $S_+=(S_-)^*$ and

$$S_-=\frac{1}{2}(\phi_1-i\mathbf{p}\cdot\boldsymbol{\psi}_1).$$

The operators $\hat{\sigma}_-$ and $\hat{\sigma}_+$ excite states with oposite frequencies. The corresponding dispersion relation is

$$k^2\left(\rho_0\Omega_n-\frac{4\tau_3}{3}c_1\right)^2=2(\rho_3\Omega_n-\rho_0c_1k^2)\times\left(\frac{2\tau_3}{3}\Omega_n+\frac{4}{3}c_2-c_3k^2\right), \tag{54}$$

with $\Omega_n=\omega_n-2\beta\mathcal{H}$.

Assuming $\phi_{1n}=\phi_{2n}$ and $F_{1n}=-F_{2n}$, another solution is obtained. This corresponds to

$$\hat{S}=(S_-)^*\hat{\sigma}_-e^{-i\omega t}+(S_+)^*\hat{\sigma}_+e^{i\omega t}. \tag{55}$$

The dispersion relation in this case is equivalent to changing ω_n into $-\omega_n$ in Eq. (54).

For small k and $\beta\mathcal{H}=0$ one of the roots behaves like

$$\omega_n=\pm\delta k^2,\qquad \delta=-\left(\frac{\rho_0}{\rho_3}c_1+\frac{2\tau_3^2c_1^2}{3\rho_3c_2}\right)=-\frac{(1+Z_1/3)(1+Z_0)}{2m^*\alpha(Z_0-Z_1/3)}. \tag{56}$$

This is the Goldstone mode obtained already in Eq. (17) using the Fermi-liquid approach. The present variational approach gives the same long-wave behavior as the traditional Fermi-liquid approach.

V. ORTHOGONALITY RELATIONS AND SUM RULE

When writing, for the normal modes, $\phi_{1n}(\mathbf{r},t)=\phi_{1n}(\mathbf{r})\cos\omega_n t$, $\boldsymbol{\psi}_{2n}(\mathbf{r},t)=\boldsymbol{\psi}_{2n}(\mathbf{r})\cos\omega_n t$, $\phi_{2n}(\mathbf{r},t)=\phi_{2n}(\mathbf{r})\times\sin\omega_n t$, $\boldsymbol{\psi}_{1n}(\mathbf{r},t)=\boldsymbol{\psi}_{1n}(\mathbf{r})\sin\omega_n t$, the modes may be normalized so that the following orthonormality relations hold:

$$\int d^3r\left[\rho_0(\partial_\alpha\phi_{1n}\psi_{1m\alpha}-\partial_\alpha\phi_{2m}\psi_{2n\alpha})+2\rho_3(\phi_{1n}\phi_{2m})-\frac{2}{3}\tau_3(\boldsymbol{\psi}_{2n}\cdot\boldsymbol{\psi}_{1m})\right]=\pm\delta_{mn}\frac{\omega_n}{|\omega_n|}. \tag{57}$$

Arbitrary fields (initial conditions) $\phi_2(\mathbf{r}),\boldsymbol{\psi}_1(\mathbf{r}),\phi_1(\mathbf{r}),\boldsymbol{\psi}_2(\mathbf{r})$ may be expanded in normal modes. If we consider at $t=0$, $\phi_1(\mathbf{r})=\boldsymbol{\psi}_2(\mathbf{r})=0$ we have

$$\phi_2(\mathbf{r})=\sum_n C_n\phi_{2n}(\mathbf{r}),\quad \boldsymbol{\psi}_1(\mathbf{r})=\sum_n C_n\boldsymbol{\psi}_{1n}(\mathbf{r}),$$

$$0=\sum_n C_n\phi_{1n}(\mathbf{r}),\quad 0=\sum_n C_n\boldsymbol{\psi}_{2n}(\mathbf{r}),$$

where

$$C_n=\pm\int d^3r\left[\rho_0(\partial_\alpha\phi_{1n}\psi_{1\alpha}-\partial_\alpha\phi_2\psi_{2n\alpha})+2\rho_3(\phi_{1n}\phi_2)-2\frac{\tau_3}{3}\boldsymbol{\psi}_{2n}\cdot\boldsymbol{\psi}_1\right]. \tag{58}$$

The sign + holds for modes with positive norm. The sign − holds for modes with negative norm. Similarly,

$$\phi_1(\mathbf{r})=\sum_n D_n\phi_{1n}(\mathbf{r}),\quad \boldsymbol{\psi}_2(\mathbf{r})=\sum_n D_n\boldsymbol{\psi}_{2n}(\mathbf{r}),$$

$$0=\sum_n D_n\phi_{2n}(\mathbf{r}),\quad 0=\sum_n D_n\boldsymbol{\psi}_{1n}(\mathbf{r}),$$

where

$$D_n=\pm\int d^3r\left(-\rho_0(\partial_\alpha\phi_{2n}\psi_{2\alpha}-\partial_\alpha\phi_1\psi_{1n\alpha})+2\rho_3(\phi_{2n}\phi_1)-2\frac{\tau_3}{3}\boldsymbol{\psi}_{1n}\cdot\boldsymbol{\psi}_2\right). \tag{59}$$

The following sum rule is satisfied,

$$\sum_{n(+)}\omega_nC_n^2=2\mathcal{E}(0,\phi_2,\boldsymbol{\psi}_1,0). \tag{60}$$

The sum extends over modes with positive norm. The quantities C_n^2 and $\omega_nC_n^2$ should be regarded as the transition probability and the transition strength, respectively.

VI. DROPLET SOLUTIONS

The droplet solutions correspond to the correct linear combinations of the local solutions associated with a given frequency ω, which verify the boundary conditions. In Appendix B we present the solution of the equations involving

$\phi_1 = -\phi_2$ and $\psi_1 = \psi_2$. The solution of the equations involving $\phi_1 = \phi_2$ and $\psi_1 = -\psi_2$ follows analogous steps. In the next section we present the results obtained for a particular excitation operator.

We discuss briefly the different modes obtained. Beyond the expected Goldstone mode, associated with the limit $k \to 0$ in Eq. (54), other modes arise. The replacement of the continuum of Landau damped modes by a discrete set of modes was a consequence of the polynomial structure imposed on the generators. The boundary condition mixes the modes, so that, in the presence of the boundary, undamped modes may have an admixture of modes which are damped in the absence of the boundary. Landau damping is not a sign of instability but the result of an interference effect which is quenched by the boundary.[12] The actual normal modes are linear combinations of local modes associated with the same value ω of the frequency. The coefficients in the linear combination and the allowed values for ω are determined by the boundary equations.

In the present problem, the energy spectrum has positive and negative energies. In situations most frequently met one finds positive random-phase approximation (RPA) energies with positive norm and negative RPA energies with negative norm. Now we find in addition some positive RPA energies with negative norm and negative RPA energies with positive norm.[13] Although this feature is unusual in RPA calculations, it is not necessarily the sign of an instability. Analogous phenomena occur in other spin systems. This interesting circumstance is a consequence of the fact that the energy of the cluster increases with the polarization. In fact, the self-consistent Eq. (26) will define the minimum energy of the system for a given polarization $E(\alpha)$. For a zero magnetic field we have $E(\alpha) < E(\alpha')$ if $\alpha < \alpha'$. It happens, as already said, that the excitations generated by Eq. (31) belong to states with a different polarization of the initial state.

From a formal point of view, the presence of negative ω_n values is permitted by the identity

$$\sum_n (E_n - E_i)|\langle \phi_n|D|\phi_i\rangle|^2 = \frac{1}{2}|\langle \phi_i|[D,[H,D]]|\phi_i\rangle|,$$

where E_n and $|\phi_n\rangle$ are, respectively, the eigenvalues and the eigenvectors of H. The identity holds even if $|\phi_i\rangle$ is not the groundstate of H. In this case, the energy differences $E_n - E_i$ are not necessarily positive.

In the rest of this section we present a simplified approach which considers the cluster as a spherical droplet. On the basis of the Goldstone mode for extended ^{3}He matter,[3,6] a mode with energy given by

$$E_1 = -1.9 \cdot \alpha^{-1} N^{-2/3} K, \tag{61}$$

is expected.[6,13] The minus sign in Eq. (61) reflects the fact that a state with $\alpha \neq 0$ and zero magnetic field is not a state of minimum energy. This is the first level of a band with angular momentum $l = 0$, whose successive members correspond to increasing numbers of radial nodes. In fact, let us consider a spherical droplet of ^{3}He. The boundary conditions for the macroscopic spin Σ_i, $i = 1,2,3$,

$$\Sigma_i = \mathrm{Tr} \sum_p \hat{\sigma}_i \hat{n}(\mathbf{p}),$$

reduce to the criterion that there should be no spin distribution outside the surface of the droplet. The equation of motion for the circular component, $\Sigma_+ = \Sigma_1 + i\Sigma_2$,

$$i\partial_t \Sigma_+ = -\frac{1}{2M}\nabla^2 \Sigma_+, \quad M = \frac{m^* \alpha |Z_0 - Z_1/3|}{(1 + Z_1/3)(1 + Z_0)},$$

coincides, in fact, with the Schrödinger equation for a particle with mass M. The solutions describing standing spin waves in a spherical droplet correspond to the eigenfunctions of the stationary states, which have the usual form

$$\Sigma_+(r,\theta,\phi) = A j_l(kr) Y_{lm}(\theta,\phi), \quad k^2 = 2ME.$$

Here, the spherical coordinates r, θ, ϕ are used, A is the normalization factor, and E denotes the energy of the stationary states. The eigenvalues E are determined by the boundary conditions, $j_l(kR) = 0$, where R is the radius of the droplet. In other words, the energy spectrum of the spin excitation is quantized and may be described in terms of two quantum numbers l and n_l:

$$E(n_l, l) = E_{nl} = \frac{1}{2MR^2} z_{ln}^2,$$

where z_{ln} are the zeros of the Bessel function, $j_l(z_{ln}) = 0$, and the index n_l labels the zeros of j_l at a given l. In the above expression (61) E_1 is just E_{10}.

As discussed in Sec. II the collective mode will not be killed by Landau damping if condition (50) is satisfied. In the long-wavelength limit where the frequency ω is given by Eq. (56), this criterion reduces approximately to: $kv \leqslant |\Omega_{int}|$, which is equivalent to the following condition:

$$\frac{z_{ln} v}{R} \leqslant \frac{4}{3} |\alpha Z_0| \epsilon_F.$$

One can easily conclude that the transverse spin mode (n_l, l) exists if the following inequality is fulfilled:

$$\frac{2}{3 z_{ln}} p_F R |\alpha Z_0| > 1.$$

For $R = r_0 N^{1/3}$, $r_0 = 2.44$ Å and $z_{ln} = z_{01} = \pi$, we get

$$N \geqslant \frac{3\pi^2}{2|\alpha Z_0|^3} \approx \frac{44}{|\alpha|^3}. \tag{62}$$

VII. RESULTS AND CONCLUSIONS

In this section we give some numerical results obtained for the energy spectrum with the excitation operator $D(\mathbf{r}) = r^2 Y_{00}$, and discuss the behavior of the energy levels with N, the number of particles in the cluster. In Table I, the eigenfrequencies for $l = 0$ and $\alpha = 0.01$ and the respective fraction of the energy weighted sum rule defined as

$$f_n = \frac{\omega_n |C_n|^2}{\Sigma_n \omega_n |C_n|^2},$$

TABLE I. Eigenfrequencies of the $l=0$ modes for $\alpha=0.01$ in the absence/presence of the polarizing field $\mathcal{H}$. The transition strengths refer to the excitation operator $D(\mathbf{r})=r^2Y_{00}$.

	$\mathcal{H}=0$				$\mathcal{H}=\mathcal{H}_0$			
N	ω_- (K)	f_n ($\times 100$)	ω_+ (K)	f_n ($\times 100$)	ω_- (K)	f_n ($\times 100$)	ω_+ (K)	f_n ($\times 100$)
10^4	-0.0018	-0.56			5×10^{-7}	3×10^{-3}		
	0.096	48.40	0.085	51.37	0.103	43.70	0.078	55.23
	0.276	0.20	0.282	0.39	0.283	0.09	0.275	0.57
10^5	-0.0018	-2.70			1×10^{-5}	0.03		
	0.048	47.44	0.037	54.27	0.055	36.94	0.031	61.06
	0.127	0.11	0.134	0.49	0.135	2×10^{-4}	0.127	0.89
10^6	-0.0020	-14.34			3×10^{-4}	3.02		
	0.026	46.44	0.015	66.20	0.033	24.65	0.010	64.46
	0.059	2×10^{-4}	0.069	0.53	0.066	0.36	0.062	1.00
10^7	-0.0029	-121.90			0.003	36.74		
	0.017	52.16	0.007	163.78	0.022	17.84	0.011	36.00
	0.018	0.96	0.057	0.14	0.029	2.12	0.050	0.18
0.5×10^8	-0.0018	-36.74			0.0053	58.08		
	0.00015	6.56	0.012	73.26	0.0075	10.97	0.013	17.48
	0.013	32.78	0.055	0.06	0.021	2.91	0.048	0.05
10^8					0.0057	71.14		
					0.0066	0.03	0.014	13.5
					0.013	4.27	0.048	0.02
10^9					0.0060	55.40		
					0.0066	10.98	0.016	5.85
					0.0070	17.80	0.047	0.001

with C_n defined in Eq. (58), are displayed for increasing particle numbers, both in the absence and in the presence of the polarizing field $\beta\mathcal{H}_0=\frac{2}{3}\epsilon_F\alpha(1+Z_0)$. The modes excited by the operatores $\sigma_\pm$ are designated by $\omega_\pm$ and given in separate columns.

We will first analyze the results obtained in the absence of the magnetic field. If the number of particles is large enough, an appreciable amount of the total strength is concentrated on the negative energy level ω_0. The absolute value of the energy of this mode increases slowly until a maximum value ~ 0.003 K is attained for $N\sim 10^7$ particles and starts decreasing only for larger clusters. Between $N=2\times10^7$ and $N=2\times10^8$ it behaves approximately with $N^{-11/21}$. For larger clusters at least two competing negative energy modes exist and this kind of analysis is not possible anymore. The energy of all the other modes decrease with an increasing size of the cluster. Notice that the results with no magnetic field displayed in Table I correspond to particle numbers below or equal to the lower limit given by Eq. (62).

We consider now the results with the equilibrium magnetic field turned on. The mode corresponding to the negative energy mode ω_0' carries almost no strength for small clusters. As the particle number increases, its energy increases until a saturation value, ~ 0.006 K, is reached. For $N>10^7$ this mode carries the largest fraction of the total strength. On the contrary, the energy of the other modes decrease with an increase of the particle number until they reach a saturating value. The only exception is the lowest energy state excited by σ_+. This state for small clusters carries the largest fraction of the total strength and its energy decreases with N. As the cluster increases it loses strength to ω_0' and its energy will increase slightly with N until reaching a saturating value.

We find that the bulk Goldstone mode (61) is in agreement with the negative energy modes. In Table II we give, for different polarizations and for clusters with a number of particles defined by the lower limit of Eq. (62), the energy $E_1(N)$ [Eq. (61)] and the lowest negative mode obtained by the variational method. We see that they agree very well. In small drops, the Goldstone mode is not at all easily excited. As the size of the cluster increases, new modes suddenly emerge in the negative energy half plane, one by one, carrying, at first, little strength. The new mode soon starts removing strength from the remaining modes. When it approximately crosses the smooth curve $E_1(N)$, most of the strength of the negative energy modes is concentrated in it or nearby. This is the Goldstone mode. It is important to notice that the negative energy mode carries the signature of finite-size ef-

TABLE II. For a given polarization α and for clusters with a number of particles defined by the lower limit of Eq. (62), the energy E_1 [Eq. (61)] and the lowest negative mode obtained by the variational method ω_0.

α	N	$E_1(K)$	$\omega_0(K)$
0.1	4.4×10^4	-0.0184	-0.0192
0.075	1.0×10^5	-0.0138	-0.0144
0.05	3.5×10^5	-0.0092	-0.0096
0.025	2.8×10^6	-0.0046	-0.0048
0.01	4.4×10^7	-0.00184	-0.00192
0.0075	1.0×10^8	-0.00138	-0.00144
0.005	3.5×10^8	-0.00092	-0.00096

fects in the cluster. Only for large enough clusters will it behave approximately as $N^{-11/21}$ for a fixed polarization. In particular for $\alpha=0.01$ the energy of this mode increases until $N=10^7$ and only for larger clusters will the energy start decreasing.

It may also be noticed that the energy spectrum scales with $\alpha^{-1}N^{-1/3}$. Varying α and N in such a way that $\alpha N^{1/3}$ is kept fixed, the energy multiplied by $N^{1/3}$ and the strength distribution do not change noticeably.

In the present work we have calculated the energy of the transverse spin-wave modes and respective fractions of the energy weighted sum rule for clusters of ^{3}He of different sizes. It was verified that the mode corresponding to the Goldstone mode will only carry a non-negligible fraction of the total strength for large enough clusters. It was shown that as far as the dependence of the energy on N was concerned, the behavior of the so-called Goldstone mode is different from the other modes. It was also found that the energy multiplied by $N^{1/3}$ and the strength distribution scale with $\alpha N^{1/3}$.

The modes obtained for a finite system are essentially not different from the modes of the bulk, but they are coupled, behave and show up in a different way. For instance, as the size shrinks some modes gradually disappear. For other modes the energy gradually increase.

In Ref. 14, spin transverse modes in droplets of ^{3}He, with a radius ~0.5–0.7 mm, have been detected at a finite temperature (~12–10 mK) and large polarization (four times the equilibrium polarization). We do not expect, however, that finite-size effects may be detected in so large droplets. In the present calculation finite-size effects are observed for droplets with polarization $\alpha=0.01$ if $N<10^7$ which corresponds to $R\sim500$ Å. For larger polarizations we expect this limit to be even smaller since our results scale with $\alpha N^{1/3}$. In a future work finite temperature, larger polarizations and modes with different angular momentum will be considered.

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APPENDIX A

In this Appendix a connection is presented between our variational approach and the more familiar fluid-dynamical formalism.[11]

The transition density $\delta\hat{n}$ is given by

$$\begin{aligned}\delta\hat{n}&=2\int\frac{d^3p}{(2\pi\hbar)^3}i[\hat{S},\hat{n}]_{SC}\\&=-\hat{\sigma}_1(2\rho_3\phi_2-\rho_0\nabla\cdot\boldsymbol{\psi}_1)+\hat{\sigma}_2(2\rho_3\phi_1+\rho_0\nabla\cdot\boldsymbol{\psi}_2),\\&=\delta n_1\hat{\sigma}_1+\delta n_2\hat{\sigma}_2\end{aligned}\tag{A1}$$

and the transition current by

$$\begin{aligned}\hat{\mathbf{j}}(\mathbf{r},t)&=2\int\frac{d^3p}{(2\pi\hbar)^3}(\nabla_p\hat{\epsilon}\delta\hat{n}+\hat{n}\nabla_p\delta\hat{\epsilon})\\&=\frac{1}{m^*}\left(1+\frac{Z_1}{3}\right)\left[\hat{\sigma}_1\left(-\frac{2}{3}\tau_3\boldsymbol{\psi}_2+\rho_0\nabla\phi_1\right)\right.\\&\quad\left.+\hat{\sigma}_2\left(\frac{2}{3}\tau_3\boldsymbol{\psi}_1+\rho_0\nabla\phi_2\right)\right]\\&=\mathbf{j}_1\hat{\sigma}_1+\mathbf{j}_2\hat{\sigma}_2.\end{aligned}\tag{A2}$$

It is easily seen that Eqs. (38) and (39) are precisely the continuity equation corresponding to

$$\dot{\delta\hat{n}}+\nabla\cdot\hat{\mathbf{j}}-2i\int\frac{d^3p}{(2\pi)^3}[\beta\mathcal{H}\hat{\sigma}_3,\delta\hat{n}]=0.\tag{A3}$$

The scalar boundary condition can be written in the form

$$\mathbf{r}\cdot[\rho_0\dot{\boldsymbol{\psi}}_1+\mathbf{j}_1+\rho_0\beta(\mathcal{H}_0-\mathcal{H})\boldsymbol{\psi}_2]=0,\tag{A4}$$

$$\mathbf{r}\cdot[\rho_0\dot{\boldsymbol{\psi}}_2+\mathbf{j}_2-\rho_0\beta(\mathcal{H}_0-\mathcal{H})\boldsymbol{\psi}_1]=0,\tag{A5}$$

where $\beta\mathcal{H}_0=\frac{2}{3}\epsilon_F\alpha(1+Z_0)$. When ρ_3 assumes the equilibrium value in the magnetic field

$$\rho_3=\frac{N(0)}{1+Z_0}\beta\mathcal{H}_0,\quad N(0)=\frac{3\rho_0}{2\epsilon_F},$$

the scalar boundary condition reduces to

$$\mathbf{r}\cdot(\rho_0\dot{\boldsymbol{\psi}}_1+\mathbf{j}_1)=0,\quad \mathbf{r}\cdot(\rho_0\dot{\boldsymbol{\psi}}_2+\mathbf{j}_2)=0.$$

The local Eqs. (40) and (41) are the equations of motion for $\hat{\mathbf{j}}$:

$$\frac{\partial\hat{\mathbf{j}}}{\partial t}+\frac{\partial}{\partial x_j}\int\frac{d^3p}{(2\pi)^3}\hat{\mathbf{j}}\left(\delta\hat{n}\frac{\partial}{\partial p_j}\hat{\epsilon}_0+n_0\frac{\partial}{\partial p_j}\delta\hat{\epsilon}\right)+\frac{2}{m^*}\left(1+\frac{Z_1}{3}\right)\int\frac{d^3p}{(2\pi)^3}n_0\nabla\delta\hat{\epsilon}+i\int\frac{d^3p}{(2\pi)^3}\times\hat{\mathbf{j}}([\delta\hat{\epsilon},\hat{n}]+[\hat{\epsilon},\delta\hat{n}])=0. \tag{A6}$$

The ''spin-stress'' tensor $\hat{\Pi}_\sigma=\Pi_{\sigma_1}\hat{\sigma}_1+\Pi_{\sigma_2}\hat{\sigma}_2$ is

$$(\hat{\Pi}_\sigma)_{ik}=(\hat{\Pi}^0_\sigma)_{ik}+\frac{2}{5}\rho_0\frac{\epsilon_F}{m}\left(1+\frac{Z_1}{3}\right)\times\left(\partial_i\psi_{1k}+\partial_k\psi_{1i}-\frac{2}{3}\nabla\cdot\boldsymbol{\psi}_1\delta_{ik}\right)\hat{\sigma}_1+\frac{2}{5}\rho_0\frac{\epsilon_F}{m}\left(1+\frac{Z_1}{3}\right)\left(\partial_i\psi_{2k}+\partial_k\psi_{2i}-\frac{2}{3}\nabla\cdot\boldsymbol{\psi}_2\delta_{ik}\right)\hat{\sigma}_2 \tag{A7}$$

$$(\hat{\Pi}^0_\sigma)_{ik}=\frac{2\epsilon_F}{3m^*}\delta_{ik}\left(1+\frac{Z_1}{3}\right)(1+Z_0)\delta\hat{n}. \tag{A8}$$

The boundary conditions in terms of the spin-stress tensor take the simple form

$$x_\beta\left[(\Pi_{\sigma_1})_{\alpha\beta}-\frac{1}{m^*}\left(1+\frac{Z_1}{3}\right)\rho_0\frac{2\beta}{2}(\mathcal{H}-\mathcal{H}_0)\phi_2\delta_{\alpha\beta}\right]=0, \tag{A9}$$

$$x_\beta\left[(\Pi_{\sigma_2})_{\alpha\beta}+\frac{1}{m^*}\left(1+\frac{Z_1}{3}\right)\rho_0\beta(\mathcal{H}-\mathcal{H}_0)\phi_1\delta_{\alpha\beta}\right]=0. \tag{A10}$$

APPENDIX B

In this Appendix we present the solution of the equations involving $\phi_1=-\phi_2$ and $\boldsymbol{\psi}_1=\boldsymbol{\psi}_2$. The solution of the equations involving $\phi_1=\phi_2$ and $\boldsymbol{\psi}_1=-\boldsymbol{\psi}_2$ follows analogous steps. We describe the computation of the energy eigenvalues. Let

$$\phi_1=-\phi_2=X_1 j_l(k_1r)Y_{l0}+X_2 j_l(k_2r)Y_{l0},$$

$$\boldsymbol{\psi}_1=\boldsymbol{\psi}_2=Y_1\nabla[j_l(k_1r)Y_{l0}]+Y_2\nabla[j_l(k_2r)Y_{l0}]+Y_5\nabla\times(\nabla\times\mathbf{r})[j_l(k_5r)Y_{l0}].$$

Here, $j_l(kr)$ are the spherical Bessel functions, $Y_{lm}(\theta,\phi)$ are the spherical harmonics, $k_5=\sqrt{2m^*\tau_3[\Omega_n+(2/\tau_3)c_2]/\tau_0}$ is one of the roots of Eq. (42). Moreover, k_1 and k_2 are the roots of Eq. (54), namely, $\begin{smallmatrix}k_1\\k_2\end{smallmatrix}=\sqrt{(f_b\pm\sqrt{f_b^2-f_af_c})/f_a}$, with $f_a=2\rho_0c_1c_3$, $f_b=\rho_0c_1(\frac{2}{3}\tau_3\Omega_n+\frac{4}{3}c_2)+\rho_3c_3\Omega_n+\frac{1}{2}(\rho_0\Omega_n-\frac{4}{3}\tau_3c_1)^2$, $f_c=2\rho_3\Omega_n(\frac{2}{3}\tau_3\Omega_n+\frac{4}{3}c_2)$. We observe that k_1, k_2, k_5 may be complex. The relations between Y_1, X_1, Y_2, and X_2 are

$$Y_1=\frac{2\rho_3\Omega_n-2\rho_0c_1k_1^2}{k_1^2\left(\rho_0\Omega_n-\frac{4\tau_3}{3}c_1\right)}X_1,$$

$$Y_2=\frac{\rho_0\Omega_n-\frac{4\tau_3}{3}c_1}{\frac{2\tau_3}{3}\Omega_n-\frac{4}{3}c_2+c_3k_2^2}X_2.$$

Trivial algebraic manipulations lead to the following set of equations which determines X_1, Y_1, X_2, Y_2, and Y_5,

$$\sum_{\kappa=1}^{2}[2\rho_0c_1X_\kappa+(\rho_0\Omega_n+c_4+\beta\mathcal{H}\rho_0)Y_\kappa]r\frac{df_{\kappa l}}{dr}+Y_5(\rho_0\Omega_n+c_4+\beta\mathcal{H}\rho_0)l(l+1)f_{5l}=0,$$

$$\sum_{\kappa=1}^{2}\left[2Y_\kappa\left(r\frac{df_{\kappa l}}{dr}-f_{\kappa l}\right)\right]+Y_5\left\{r^2\frac{d^2f_{5l}}{dr^2}+[l(l+1)-2]f_{5l}\right\}=0,$$

$$\sum_{\kappa=1}^{2}\left[\left(c_4+\beta\mathcal{H}\rho_0+\frac{4\tau_3}{3}c_1\right)X_\kappa f_{\kappa l}+\left(c_3-\frac{2\tau_0}{3m^*}\right)Y_\kappa\nabla^2f_{\kappa l}+\frac{2\tau_0}{3m^*}Y_\kappa\frac{d^2f_{\kappa l}}{dr^2}\right]+\frac{2\tau_0}{3m^*}l(l+1)Y_5\frac{d(f_{5l}/r)}{dr}=0,$$

where $f_{nl}(r)=j_l(k_nr)$, $n=1,2,5$.

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